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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR SUMS OF NEARLY NONSTATIONARY PROCESSES[1]

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Abstract. We study some Hölderian functional central limit theorems for the polygonal line partial sum processes built on a first order autoregressive process $y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k$ with ϕ_n converging to 1 and i.i.d. centered square integrable innovations. In the case where $\phi_n = \mathrm{e}^{\gamma/n}$ with a negative constant γ, we prove that the limiting process is an integrated Ornstein-Uhlenbeck one. In the case where $\phi_n = 1 - \gamma_n/n$, with γ_n tending to infinity slower than n, the convergence to Brownian motion is established in Hölder space in terms of the rate of γ_n and the integrability of the ε_k's.

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1 Introduction

In this paper we investigate asymptotic behavior of the first-order autoregressive process $(y_{n,k} : k = 1, \dots, n; n = 1, 2, \dots)$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \tag{1.1}$$

where $\phi_n < 1$, $\phi_n \to 1$, as $n \to \infty$, (ε_k) is a sequence of independent identically distributed random variables with $\mathbb{E}\varepsilon_k = 0$ and $y_{n,0}$ is random. Despite the fact that $(y_{n,k})$ is a triangular array, for simplicity, we shall omit the index n and we shall write $y_k = \phi_n y_{k-1} + \varepsilon_k$. The process (y_k) when $\phi_n \to 1$, as $n \to \infty$, is called *nearly nonstationary*.

In this paper we focus on polygonal line processes built on the y_k's:

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{k-1} + (nt - [nt]) y_{[nt]}, \quad t \in [0,1], \quad n \geq 1. \tag{1.2}$$

Remark 1. The definition of S_n^{pl} is quite unusual with a general term y_{k-1} where one would expect y_k. This definition is more convenient from the technical point of view. However, asymptotic results proved in this paper remain true with y_{k-1} replaced by y_k as well.

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Our aim is to investigate the asymptotic behavior of S_n^{pl} in $\mathrm{C}[0,1]$ and in a class of Hölder spaces considering the following two cases

- *Case 1*: $\phi_n = \mathrm{e}^{\gamma/n}$ ($\gamma < 0$ is a negative constant);
- *Case 2*: $\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \infty$ slower than n.

The central limit theorem for the sums $S_n^{\mathrm{pl}}(1) = \sum_{k=1}^n y_{k-1}$, $n \geq 1$ is proved by Phillips [7] in case 1 under normalization $n^{-3/2}$ and Giraitis and Phillips [3] in case 2 under normalization $n^{-1/2}(1-\phi_n)$. Phillips [7] and Cumberland and Sykes [2] found that the sequence of normalized processes $(n^{-1/2}y_{[nt]})$ converges weakly to an Ornstein-Uhlenbeck process in the classical Skorohod space $\mathrm{D}[0,1]$ in case 1.

The weak convergence of a sequence of stochastic processes in some functions space F provides results about the asymptotic distribution of functionals of the paths which are continuous with respect to the topology of F. Since the Hölder spaces are topologically embedded in $\mathrm{C}[0,1]$, they support more continuous functionals. From this point of view, the alternative framework of Hölder spaces gives functional limit theorems of a wider scope (see more in [5]).

In case 1 we show that the sequence $(n^{-3/2}S_n^{\mathrm{pl}})$ converges weakly in either $\mathrm{C}[0,1]$ or in some Hölder space to an integrated Ornstein-Uhlenbeck process, if the invariance principle for innovations holds in the space under consideration (Theorem 1). In case 2 we obtain the convergence in distribution of $(n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}})$ to a standard Brownian motion in $\mathrm{C}[0,1]$ under square integrability of the innovations and provide conditions to obtain the convergence in distribution in Hölder spaces (Theorems 2 and 3).

The paper is organized as follows: section 2 introduces notations and some necessary background, section 3 is devoted to case 2 whereas section 4 to case 1. Finally section 5 contains some technical parts of some proofs and some supplementary results.

2 Preliminaries

By $\|f\|_\infty$ we denote the uniform norm of $f \in \mathrm{C}[0,1]$. For $\alpha \in [0,1)$ the Hölder space

$$\mathrm{H}_\alpha^o[0,1] := \left\{ f \in C[0,1] : \lim_{\delta \to 0} \omega_\alpha(f,\delta) = 0 \right\},$$

endowed with the norm $\|f\|_\alpha := |f(0)| + \omega(f,1)$, where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t \in [0,1] \\ 0 < t-s < \delta}} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

is a separable Banach space. In the special case where $\alpha = 0$, H_0^o is equal to $C[0,1]$ and $\|f\|_0$ is equivalent to $\|f\|_\infty$.

The polygonal line process built from i.i.d. random variables (ε_j) is

$$W_n^{\mathrm{pl}}(t) = \sum_{j=1}^{[nt]} \varepsilon_j + (nr - [nt])\varepsilon_{[nt]+1}, \quad t \in [0,1]. \tag{2.1}$$

Throughout the paper $W = (W(t), t \in [0,1])$ is a standard Brownian motion. By the classical Levy's result on the modulus of continuity of W, $W \in \mathrm{H}_\alpha^o[0,1]$ with probability one for every $0 \leq \alpha < 1/2$.

In what follows $\xrightarrow[n\to\infty]{\mathrm{E}}$ signifies convergence in distribution in the metric space E. Accordingly, the classical convergence in distribution of a sequence of random variables is denoted by $\xrightarrow[n\to\infty]{\mathbb{R}}$ and convergence in probability is denoted by $\xrightarrow[n\to\infty]{\mathrm{P}}$.

In Račkauskas and Suquet [8] it is proved that for $0 < \alpha < 1/2$ the convergence

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o[0,1]} W \tag{2.2}$$

holds if and only if

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)}P(|\varepsilon_1| \geq t) = 0. \tag{2.3}$$

Condition (2.3) provides precise relation between the strength of the convergence (2.2) and the integrability of summands. Compared with the classical Donsker invariance principle, it shows the price to be paid for functional convergence in a stronger topology. When $\alpha > 0$, condition (2.3) implies that $\mathbb{E}\,|\varepsilon_1|^p < \infty$ for $p < (1/2-\alpha)^{-1}$ and in particular $\mathbb{E}\varepsilon_1^2 < \infty$. We note also that condition (2.3) with $\alpha = 0$ does not imply the convergence of $n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}}$ to W in $C[0,1]$.

Throughout the paper we work with random polygonal lines and study their asymptotic behaviour in Hölder topology. As a polygonal line is characterized by its vertices, it is useful to know how its Hölderian asymptotic behaviour depends on the control of its vertices. To explain this, it is convenient here to represent a polygonal line π_n with vertices $(l/n, V_l)$, $0 \leq l \leq n$, $V_0 = 0$, under the form:

$$\pi_n(t) = (1 - \{nt\})V_{[nt]} + \{nt\}\, V_{[nt]+1}, \quad 0 \leq t \leq 1, \tag{2.4}$$

where $\{nt\} = nt - [nt]$ is the fractional part of nt. We claim that the Hölder norm of such a line is reached at two vertices, that is

$$\|\pi_n\|_\alpha = \max_{0\leq j<k\leq n} \frac{|V_k - V_j|}{(\frac{k}{n} - \frac{j}{n})^\alpha}. \tag{2.5}$$

For a proof of this fact, see Lemma 3 in section 5. From (2.5) we immediatly deduce that

$$\|\pi_n\|_\alpha \leq 2n^\alpha \max_{1\leq l\leq n} |V_l|\,. \tag{2.6}$$

The estimate (2.6) enables us to reduce the investigation of the asymptotic behaviour of the random polygonal line S_n^{pl} (properly normalized) to the case where the initialization in (1.1) is given by $y_{n,0} = 0$. Indeed let us associate to each autoregressive process $(y_{n,k})$ satisfying (1.1), the process $(y'_{n,k})$ defined by

$$y'_{n,k} = y_{n,k} - \phi_n^k y_{n,0}. \tag{2.7}$$

Then $(y'_{n,k})$ satisfies (1.1) with initialization $y'_{n,0} = 0$ and the above mentionned reduction may be formulated as follows.

Proposition 1. *Let $S_n^{\mathrm{pl}\prime}$ be the polygonal line process obtained by substituting in* (1.2) *the $y_{n,j}$'s by the $y'_{n,j}$'s. Assume that $c_nS_n^{\mathrm{pl}\prime}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$, where the c_n's are some positive normalizing constants. Then $c_nS_n^{\mathrm{pl}}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$ to the same limit provided that*

$$\frac{c_nn^\alpha}{1-\phi_n} y_{n,0} \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{2.8}$$

Proof . The stochastic process $\pi_n = c_n S_n^{\mathrm{pl}} - c_n S_n^{\mathrm{pl}\prime}$ is a random polygonal line with vertices $(l/n, V_l)$, $0 \le l \le n$, $V_0 = 0$, where

$$V_l = \sum_{j=0}^{l-1} c_n \phi_n^l y_{n,0} = \frac{1-\phi_n^l}{1-\phi_n} c_n y_{n,0}.$$

Applying (2.6) and recalling that $0 < \phi_n < 1$ for n large enough, we obtain

$$\|\pi_n\|_\alpha \le 2n^\alpha c_n |y_{n,0}| \max_{0\le k\le n} \frac{1-\phi_n^k}{1-\phi_n} \le \frac{2c_n n^\alpha}{1-\phi_n} |y_{n,0}| .$$

Then under (2.8), $\left\| c_n S_n^{\mathrm{pl}} - c_n S_n^{\mathrm{pl}\prime} \right\|_\alpha$ goes in probability to 0 and the result follows by Slutsky's lemma.

3 Functional limit theorems in Case 1

In this section we study the process (1.1) in the case where $\phi_n = \mathrm{e}^{\gamma/n}$ with a constant $\gamma < 0$. Note that instead of putting any direct assumption on the ε_j's, we assume rather some functional weak convergence of W_n^{pl} to W. This extends the scope of the result far beyond the case where the ε_j's are i.i.d. (for some Hölderian invariance principles, in the case of weakly dependent random variables, see Hamadouche [4]).

Theorem 1. *In the case 1 where (y_k) is generated by (1.1) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$, suppose that the sequence of polygonal lines $(n^{-1/2} W_n^{\mathrm{pl}})$ converges weakly to the standard Brownian motion W either in $\mathrm{C}[0,1]$ or in $\mathrm{H}_\alpha^o[0,1]$ for some $0 < \alpha < 1/2$. Suppose moreover that $y_{n,0} = o_P(n^{1/2})$ or $y_{n,0} = o_P(n^{1/2-\alpha})$ according to the function space considered. Then $n^{-3/2} S_n^{\mathrm{pl}}$ converges weakly, as $n \to \infty$, in the space under consideration to the integrated Ornstein-Uhlenbeck process J defined by:*

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{3.1}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Proof . Since the Banach spaces $(C[0,1], \|\,\|_\infty)$ and $(\mathrm{H}_0^o, \|\,\|_0)$ are isomorphic, the unified proof proposed here for the spaces $\mathrm{H}_\alpha^o[0,1]$, $0 \le \alpha < 1/2$, includes the special case of the space $C[0,1]$. By Proposition 1 and our assumption $y_{n,0} = o_P(n^{1/2-\alpha})$, it is enough to give the proof in the case where $y_{n,0} = 0$.

The idea is to approximate the polygonal line $n^{-3/2} S_n^{\mathrm{pl}}$ by some linear interpolation of a smooth process J_n which is a functional of $n^{-1/2} W_n^{\mathrm{pl}}$, continuous in Hölder topology, with $\left\| n^{-3/2} S_n^{\mathrm{pl}} - J_n \right\|_\alpha = o_P(1)$.

The first step is to approximate $\pi_{n,1} := n^{-3/2} S_n^{\mathrm{pl}}$ by successive the polygonal lines $\pi_{n,2}$, $\pi_{n,3}$, $\pi_{n,4}$ where the later has vertices $(l/n, V_{l,4})$ given by

$$V_{l,4} = \int_0^{l/n} n^{-1/2} W_n^{\mathrm{pl}}(s)\,\mathrm{d}s + \gamma \int_0^{l/n} \int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r)\,\mathrm{d}r\,\mathrm{d}s, \tag{3.2}$$

and satisfies

$$\left\| n^{-3/2} S_n^{\mathrm{pl}} - \pi_{n,4} \right\|_\alpha = o_P(1). \tag{3.3}$$

See Annex 5.1 for the details of this approximation and the proof of (3.3).

Next we note that $\pi_{n,4}$ is exactly the linear interpolation at the points $t_{n,l} = l/n$ of the random function:

$$J_n(t) := \int_0^t n^{-1/2} W_n^{\mathrm{pl}}(s)\,\mathrm{d}s + \gamma \int_0^t \int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r)\,\mathrm{d}r\,\mathrm{d}s.$$

By an elementary chaining argument, the interpolation error is controlled by

$$\|J_n - \pi_{n,4}\|_\alpha \le 4\omega_\alpha\Big(J_n, \frac{1}{n}\Big),$$

which converges in probability to zero, provided that J_n converges weakly in $\mathrm{H}_\alpha^o[0,1]$, see Theorem 4.

Now, it only remains to check that J_n converges weakly to J in $\mathrm{H}_\alpha^o[0,1]$. As the functional

$$\mathrm{H}_\alpha^o[0,1] \to \mathrm{H}_\alpha^o[0,1] \quad : \quad x \longmapsto \int_0^\bullet x(s)\,\mathrm{d}s + \gamma \int_0^\bullet \int_0^s \mathrm{e}^{\gamma(s-r)} x(r)\,\mathrm{d}r\,\mathrm{d}s$$

is continuous on $\mathrm{H}_\alpha^o[0,1]$, this last convergence follows from the convergence of $n^{-1/2}W_n^{\mathrm{pl}}$ to W (see (2.2)). This ends the proof of Theorem 1.

Taking into account the classical Donsker-Prohorov invariance principle and the functional central limit theorem proved in [8] we have the following corollary of Theorem 1 in the classical case of i.i.d. innovations.

Corollary 1. *Assume that (y_k) is generated by (1.1) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$ and that the ε_k's are i.i.d. and centered. Then the weak convergence of $n^{-3/2}S_n^{\mathrm{pl}}$ to J holds*

- *in $\mathrm{C}[0,1]$ provided that $\mathbb{E}\varepsilon_1^2 < \infty$ and $y_{n,0} = o_P(n^{1/2})$;*
- *in $\mathrm{H}_\alpha^o[0,1]$ for $0 < \alpha < 1/2$ under condition (2.3) and $y_{n,0} = o_P(n^{1/2-\alpha})$.*

4 Functional limit theorems in Case 2

In this section we shall investigate the polygonal line process S_n^{pl} built on the y_k's, as defined by (1.2), where $\phi_n = 1 - \gamma_n/n$ and $\gamma_n \to \infty$ slower than n.

A key point in all the following limit theorems is to keep a good control on the asymptotic behavior of $\max_{1\le k\le n} |y_k|$. This is provided by the following lemma which may be of independent interest.

Lemma 1. *Suppose the process (y_k) is generated by (1.1) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of non negative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Suppose moreover that $y_{n,0} = 0$. Let $p \ge 2$. Assume that the innovations (ε_k) satisfy*

$$\begin{array}{ll} \lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0, & \text{if } p > 2 \\ \mathbb{E}\varepsilon_0^2 < \infty, & \text{if } p = 2 \end{array} \tag{4.1}$$

For $p \ge 2$, put $\alpha = 1/2 - 1/p$. Then

$$n^{-1/2}\gamma_n^\alpha \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{4.2}$$

The proof of this lemma can be found in section 5.2.

We start with asymptotic behavior of S_n^{pl} in the space $\mathrm{C}[0,1]$.

Theorem 2. *Suppose the process (y_k) is generated by* (1.1) *and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of non negative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Assume also that the innovations (ε_k) are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$ and that $y_{n,0} = o_P(n^{1/2})$. Then the following convergence holds.*

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W. \tag{4.3}$$

Proof. Using Proposition 1 and the assumption $y_{n,0} = o_P(n^{1/2})$ it suffices to prove the result when $y_{n,0} = 0$. To prove (4.3), in view of the Donsker-Prohorov invariance principle (see [1]), it is enough to show that

$$\varDelta_n = \|\xi_n\|_\infty \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{4.4}$$

where

$$\xi_n = \frac{1-\phi_n}{n^{1/2}} S_n^{\mathrm{pl}} - n^{-1/2} W_n^{\mathrm{pl}}.$$

We observe that ξ_n is a polygonal line with vertices at the points $t_{n,k} = k/n$, $0 \le k \le n$. Its supremum norm is reached at one of its vertices. Hence

$$\varDelta_n = \sup_{0\le t\le 1} \left| \frac{1-\phi_n}{n^{1/2}} S_n^{\mathrm{pl}}(t) - n^{-1/2} W_n^{\mathrm{pl}}(t) \right| = n^{-1/2} \max_{1\le k\le n} \left| (1-\phi_n) \sum_{j=1}^k y_{j-1} - \sum_{j=1}^k \varepsilon_j \right|.$$

For every $k \ge 1$, it follows from (1.1) that $\sum_{j=1}^k y_j = \phi_n \sum_{j=1}^k y_{j-1} + \sum_{j=1}^k \varepsilon_j$, whence

$$(1-\phi_n) \sum_{j=1}^k y_{j-1} = y_k + \sum_{j=1}^k \varepsilon_j, \tag{4.5}$$

so $\varDelta_n$ reduces to

$$\varDelta_n = n^{-1/2} \max_{1\le k\le n} |y_k|.$$

By the particular case where $p = 2$ in Lemma 1, the convergence (4.2) holds true with $\alpha = 0$. Hence $n^{-1/2} \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0$ and (4.4) follows. The proof of the theorem is complete.

Next we extend Theorem 2 by proving convergence of S_n^{pl} in the Hölder space $\mathrm{H}_\beta^o[0,1]$, $0 < \beta < \alpha$, of course under stronger condition on (ε_k) than finiteness of the second moment. An extra restriction on the rate of divergence of (γ_n) seems to be necessary, but we have no answer to this.

Theorem 3. *Suppose (y_k) is generated by* (1.1) *and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of non negative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Assume also that the innovations (ε_k) are i.i.d. and satisfy condition* (4.1) *for some $p > 2$. Put $\alpha = \frac{1}{2} - \frac{1}{p}$. Then for $0 < \beta < \alpha$,*

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^o[0,1]} W, \tag{4.6}$$

provided that $y_{n,0} = o_P(n^{1/2-\beta})$ and

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{\beta}{\alpha}} > 0. \tag{4.7}$$

Proof . By [8], condition (4.1) gives the weak convergence of $n^{-1/2}W_n^{\mathrm{pl}}$, defined by (2.1), to the standard Brownian motion in the space $\mathrm{H}^o_\alpha[0,1]$. By continuous embedding of Hölder spaces, the same convergence remains true in $\mathrm{H}^o_\beta[0,1]$ for $0<\beta<\alpha$. Therefore it is enough to show that

$$D_{n,\beta} := \|\zeta_n\|_\beta \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{4.8}$$

where

$$\zeta_n := n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}}.$$

Note that ζ_n is a polygonal line with vertices at the points $t_{n,k}=k/n$, $0\le k\le n$. According to Lemma 3, the Hölderian norm of such a polygonal line is reached at two vertices, so

$$\left\| n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}}\right\|_\beta \le \max_{1\le j<k\le n}\frac{\left|n^{-1/2}(y_k-y_j)\right|}{|k/n-j/n|^\beta} \le 2n^{\beta-\frac{1}{2}}\max_{1\le k\le n}|y_k|\,.$$

Using Proposition 1 and the assumption $y_{n,0}=o_P(n^{1/2-\beta})$ it suffices to prove (4.8) when $y_{n,0}=0$. Then, by Lemma 1, $\max_{1\le k\le n}|y_k| = o_P(n^{1/2}\gamma_n^{-\alpha})$, so the convergence (4.8) is satisfied provided that

$$\limsup_{n\to\infty}\frac{n^\beta}{\gamma_n^\alpha}<\infty,$$

which is equivalent to our assumption (4.7).

5 Annex

5.1 Completion of the proof of Theorem 1

We explicit here the approximation of $n^{-3/2}S_n^{\mathrm{pl}}$ by the polygonal line defined by (3.2). To control the distance in Hölder norm between polygonal lines, we use the following property. Let π_n a polygonal line with representation (2.4). As a consequence of (2.6), if we approximate each V_l by some $\widetilde{V}_l$ in such a way that $|V_l-\widetilde{V}_l| = o_P(n^{-\alpha})$, *uniformly* in $1\le l\le n$, then the corresponding polygonal line $\widetilde{\pi_n}$ satisfies $\|\pi_n-\widetilde{\pi_n}\|_\alpha = o_P(1)$.

In what follows, we will denote the successive polygonal lines approximating $n^{-3/2}S_n^{\mathrm{pl}}$ by $\pi_{n,i}$ and their vertices by $(l/n, V_{l,i})$, $i=1,2,3,4$. At each step we will use the following facts

$$\left\| n^{-1/2}W_n^{\mathrm{pl}}\right\|_\infty \quad \text{is stochastically bounded} \tag{5.1}$$

and

$$\omega_\alpha\Big(n^{-1/2}W_n^{\mathrm{pl}}, \frac{1}{n}\Big) \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{5.2}$$

by tightness in $\mathrm{H}^o_\alpha[0,1]$, $0\le\alpha<1/2$, see Theorem 4 below.

We start with $\pi_{n,1}=n^{-3/2}S_n^{\mathrm{pl}}$ for which

$$V_{l,1}=Y_l=n^{-3/2}\sum_{k=1}^{l} y_{k-1}.$$

We express y_k in terms of innovations

$$y_k = \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n} \varepsilon_j.$$

Noting that $\varepsilon_j = W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{j-1}{n}\right)$, we obtain

$$\begin{aligned} y_k &= \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n} \left(W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{j-1}{n}\right)\right) \\ &= W_n^{\mathrm{pl}}\left(\frac{k}{n}\right) + \sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n}(1-\mathrm{e}^{-\gamma/n}) W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) \\ &= W_n^{\mathrm{pl}}\left(\frac{k}{n}\right) + \frac{\gamma}{n}\sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n} W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) + \frac{\gamma^2 u_n}{2n^2}\sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n} W_n^{\mathrm{pl}}\left(\frac{j}{n}\right), \end{aligned}$$

where $u_n = 2n^2\gamma_n^{-2}\left(1-\mathrm{e}^{-\gamma/n} - \gamma n^{-1}\right)$. As

$$\mathrm{e}^{-\gamma/n} = 1 - \frac{\gamma}{n} + \frac{\gamma^2}{2n^2} + o\left(\frac{1}{n^2}\right),$$

it follows

$$u_n = -1 + \frac{2n^2}{\gamma^2} o\left(\frac{1}{n^2}\right) \to -1, \quad \text{as} \quad n \to \infty.$$

Now our first approximation consist in neglecting the last term in the sum above, which gives the polygonal line $\pi_{n,2}$ with

$$V_{l,2} = \frac{1}{n}\sum_{k=1}^{l} W_n\left(\frac{k-1}{n}\right) + \frac{\gamma}{n^2}\sum_{k=1}^{l}\sum_{j=1}^{k-2} \mathrm{e}^{(k-j-1)\gamma/n} W_n\left(\frac{j}{n}\right), \tag{5.3}$$

where $W_n := n^{-1/2} W_n^{\mathrm{pl}}$ for writing simplicity. For the approximation error, we have the following bound valid for $n \geq \gamma$:

$$|V_{l,2} - V_{l,1}| \leq \frac{\gamma^2 \mathrm{e}^{\gamma}}{2n} \|W_n\|_\infty .$$

Next, approximating Riemann sums by integrals in (5.3), we obtain the polygonal line $\pi_{n,3}$ with

$$V_{l,3} = \int_0^{l/n} W_n(s)\,\mathrm{d}s + \frac{\gamma}{n}\sum_{k=1}^{l} \mathrm{e}^{\gamma k/n}\int_0^{k/n} \mathrm{e}^{-\gamma r} W_n(r)\,\mathrm{d}r. \tag{5.4}$$

Let us estimate the error of approximation. For any $f \in \mathrm{C}[0,1]$,

$$\frac{1}{n}\sum_{j=1}^{k-k_0} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{k/n} f(s)\,\mathrm{d}s$$
$$= \sum_{j=1}^{k-k_0} \int_{(j-1)/n}^{j/n} \Big(f\Big(\frac{j+j_0}{n}\Big) - f(s)\Big)\,\mathrm{d}s - \int_{(k-k_0)/n}^{k/n} f(s)\,\mathrm{d}s,$$

whence

$$\left| \frac{1}{n}\sum_{j=1}^{k-k_0} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{k/n} f(s)\,\mathrm{d}s \right| \le \omega_0\Big(f, \frac{1+j_0}{n}\Big) + \|f\|_\infty \frac{k_0}{n}. \tag{5.5}$$

Moreover,

$$\text{if } f \in \mathrm{H}^o_\alpha[0,1], \quad \omega_0(f,\delta) \le \omega_\alpha(f,\delta)\delta^\alpha. \tag{5.6}$$

If $f(t) = g(t)h(t)$ with g of class C^1 and $h \in \mathrm{C}[0,1]$,

$$\omega(gh,\delta) \le \|g\|_\infty \omega(h,\delta) + \|g'\|_\infty \|h\|_\infty \delta. \tag{5.7}$$

Using (5.5)–(5.7), we obtain the uniform bound

$$|V_{l,3} - V_{l,2}| \le \frac{1+\gamma \mathrm{e}^\gamma}{n^\alpha}\omega_\alpha\Big(W_n, \frac{1}{n}\Big) + \frac{\gamma \mathrm{e}^\gamma(2+\gamma \mathrm{e}^\gamma)}{n}\|W_n\|_\infty .$$

Finally, we replace the last sum remaining in (5.4) by an integral of $f_n(s) := \mathrm{e}^{\gamma s}\int_0^s \mathrm{e}^{-\gamma r} W_n(r)\,\mathrm{d}r$, $s \in [0,1]$, noting that $|f_n'(s)| \le (1+\gamma\mathrm{e}^\gamma)\|W_n\|_\infty$ for each $s \in [0,1]$. This gives the polygonal line $\pi_{n,4}$ with vertices

$$V_{l,4} = \int_0^{l/n} W_n(s)\,\mathrm{d}s + \gamma\int_0^{l/n} \mathrm{e}^{\gamma s}\int_0^s \mathrm{e}^{-\gamma r} W_n(r)\,\mathrm{d}r\,\mathrm{d}s. \tag{5.8}$$

The approximation error is given by the uniform bound

$$|V_{l,4} - V_{l,3}| \le \frac{1+\gamma\mathrm{e}^\gamma}{n}\|W_n\|_\infty .$$

Noting that $\pi_{n,4}$ is exactly the polygonal line defined by (3.2), gathering all the estimate of errors above, recalling (2.6), we obtain finally with some positive constants C_γ and C'_γ:

$$\left\| n^{-3/2} S_n^{\mathrm{pl}} - \pi_{n,4} \right\|_\alpha \le C_\gamma \omega_\alpha\Big(W_n, \frac{1}{n}\Big) + C'_\gamma \|W_n\|_\infty n^{\alpha-1}. \tag{5.9}$$

Recalling (5.1) and (5.2), it follows that

$$\left\| n^{-3/2} S_n^{\mathrm{pl}} - \pi_{n,4} \right\|_\alpha \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

so (3.3) is proved.

5.2 Maximal inequality

Here we give a detailed proof of Lemma 1. It is convenient to start with the following weaker result which already contains the estimate $\max_{1\le k\le n}|y_k| = O_P(n^{1/2}\gamma_n^{-\alpha})$ if $\mathbb{E}\,|\varepsilon_0|^p < \infty$.

Lemma 2. *Let $(\eta_j)_{j\ge 0}$ be a sequence of i.i.d. random variables, with $\mathbb{E}\eta_0 = 0$ and $\mathbb{E}\,|\eta_0|^q < \infty$ for some $q \ge 2$. Suppose $\phi_n = 1 - \frac{\gamma_n}{n}$, where $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Define*

$$z_k = \sum_{j=1}^{k} \phi_n^{k-j}\eta_j. \tag{5.10}$$

Then there exists an integer $n_0(q) \ge 1$ depending on q only, such that for every $n \ge n_0(q)$, $\gamma_n > \gamma_{n_0}(q)$, and every $\lambda > 0$,

$$P\left(\max_{1\le k\le n}|z_k| > \lambda\right) \le \frac{4C_q \mathrm{e}^q \mathbb{E}\,|\eta_0|^q}{\lambda^q} n^{q/2}\gamma_n^{1-q/2}, \tag{5.11}$$

where C_q is the universal constant in the Rosenthal inequality of order q. Choosing $\lambda = n^{1/2}\gamma_n^{1/q-1/2}\tau$ for arbitrary $\tau > 0$ provides:

$$\max_{1\le k\le n}|z_k| = O_P\left(n^{1/2}\gamma_n^{1/q-1/2}\right).$$

The right hand side of (5.11) becomes smaller as q increases, subject to an optimal choice of λ. It seems difficult to say if the bound (5.11) is sharp. We can nevertheless remark that in the boundary case, where $\gamma_n = n$ and so the z_k's become i.i.d., our bound would lead to the estimate $\max_{1\le k\le n}|z_k| = O_{\mathrm{P}}(n^{1/q})$ which is optimal in this case.

Proof. The idea of the proof relies on the following observation. For $a < k \le b$,

$$|z_k| = \phi_n^k \left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right| \le \phi_n^a \left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right|.$$

Here $\{\sum_{j=1}^{k}\phi_n^{-j}\eta_j,\ a < k \le b\}$ is a martingale adapted to its natural filtration and if we repeat this procedure with regularly spaced bounds a and b, we keep the structure of a geometric sum for the coefficients ϕ_n^a. To profit of these two features we are lead to the following splitting:

$$n = MK, \quad \max_{1\le k\le n}|z_k| = \max_{1\le m\le M}\ \max_{(m-1)K<k\le mK}|z_k|,$$

where M and K (not necessarily integers) depend on n in a way which will be precised later. Applying this splitting we obtain first:

$$P\left(\max_{1\le k\le n}|z_k| > \lambda\right) \le \sum_{1\le m\le M} P\left(\phi_n^{(m-1)K}\max_{1\le k\le mK}\left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right| > \lambda\right).$$

Then using Markov's and Doob's inequalities at order q gives

$$P\left(\max_{1\le k\le n}|z_k| > \lambda\right) \le \sum_{1\le m\le M}\frac{\phi_n^{q(m-1)K}T_m}{\lambda^q} \quad \text{where} \quad T_m := \mathbb{E}\left|\sum_{1\le j\le mK}\phi_n^{-j}\eta_j\right|^q. \tag{5.12}$$

To bound T_m, we treat separately the special case $q = 2$ with a simple variance computation and use Rosenthal inequality in the case $q > 2$. In both cases, the following elementary estimate is useful.

$$\sum_{1 \le j \le mK} \phi_n^{-jq} = \phi_n^{-[qmK]} \sum_{j=1}^{[mK]} \phi_n^{[mK]q-jq} = \phi_n^{-[qmK]} \sum_{j=0}^{[mK]-1} \phi_n^{jq}$$
$$\le \frac{\phi_n^{-[qmK]}}{1-\phi_n^q} \le \frac{\phi_n^{-qmK}}{1-\phi_n}$$

recalling that $0 < \phi_n < 1$, whence,

$$\sum_{1 \le j \le mK} \phi_n^{-jq} \le \frac{n}{\gamma_n} \phi_n^{-qmK}. \tag{5.13}$$

Now in the special case $q = 2$, we have

$$T_m = \operatorname{Var}\left(\sum_{j=1}^{k} \phi_n^{-j} \eta_j\right) = \mathbb{E}\eta_0^2 \sum_{1 \le j \le mK} \phi_n^{-2j},$$

so by (5.13),

$$T_m \le \frac{n}{\gamma_n} \phi_n^{-2mK} \mathbb{E}\eta_0^2. \tag{5.14}$$

When $q > 2$, we apply Rosenthal inequality which gives here

$$T_m \le C_q \left(\left(\mathbb{E}\eta_0^2\right)^{q/2} \left(\sum_{1 \le j \le mK} \phi_n^{-2j} \right)^{q/2} + \mathbb{E}\left|\eta_0\right|^q \sum_{1 \le j \le mK} \phi_n^{-jq} \right).$$

As $q > 2$, $\left(\mathbb{E}\eta_0^2\right)^{q/2} \le \mathbb{E}\left|\eta_0\right|^q$. Also we may assume without loss of generality that $\frac{n}{\gamma_n} \ge 1$, so $\frac{n}{\gamma_n} \le \left(\frac{n}{\gamma_n}\right)^{q/2}$. Then using (5.13), we obtain

$$T_m \le 2C_q \mathbb{E}\left|\eta_0\right|^q n^{q/2} \gamma_n^{-q/2} \phi_n^{-qmK}. \tag{5.15}$$

Note that (5.14) obtained in the special case $q = 2$ can be included in this formula by defining $C_2 := 1/2$.

Going back to (5.12) with this estimate, we obtain

$$P\left(\max_{1 \le k \le n} |z_k| > \lambda\right) \le 2C_q \mathbb{E}\left|\eta_0\right|^q n^{q/2} \gamma_n^{-q/2} \lambda^{-q} \sum_{1 \le m \le M} \phi_n^{-Kq}$$
$$\le 2C_q \mathbb{E}\left|\eta_0\right|^q n^{q/2} \gamma_n^{-q/2} \lambda^{-q} M \phi_n^{-Kq}.$$

Now, choosing $K = \frac{n}{\gamma_n}$, we see that ϕ_n^{-Kq} converges to e^q, so for $n \ge n_0(q)$, $\phi_n^{-Kq} \le 2\mathrm{e}^q$. Then (5.11) follows by pluging this upper bound in the inequality above and noting that $M = \gamma_n$.

Remark 2. Under assumptions of Lemma 2 there exists such constant c_q depending on q only, such that for every $n \geq 1$ and every $\lambda > 0$

$$P\left(\max_{1\leq k\leq n} |z_k| > \lambda\right) \leq \frac{c_q \mathbb{E}\,|\eta_0|^q}{\lambda^q} n^{q/2}\gamma_n^{1-q/2}.$$

Remark 3. The Lemma 2 can be proved by applying Hájek-Rényi type inequality (e.g. see Petrov [6] section III.5, paragraph 6). In authors' opinion, the method applied in the proof of Lemma 2 seems more suitable for generalization, e.g. for dependent innovations.

Proof of Lemma 1. It is convenient to rewrite the assumption (4.1) as

$$P(|\varepsilon_0| > t) = \frac{f(t)}{t^p}, \quad f(t) \xrightarrow[t\to\infty]{} 0.$$

Moreover

$$f^*(b) := \sup_{t\geq b} f(t) \xrightarrow[b\to\infty]{} 0.$$

In the special case where $p = 2$, (4.1) is replaced by $\mathbb{E}\varepsilon_0^2 < \infty$, but the above representation of $P(|\varepsilon_0| > t)$ remains valid since $f(t) = t^2 P(|\varepsilon_0| > t) \leq \mathbb{E}(\varepsilon_0^2 \mathbf{1}_{\{|\varepsilon_0|>t\}})$ by Markov inequality and this upper bound goes to zero by dominated convergence theorem.

Let us fix arbitrary positive numbers δ and ϵ, and introduce the truncated random variables

$$\varepsilon'_j = \varepsilon_j \mathbf{1}_{\{|\varepsilon_j|\leq b_n\}} \qquad \widetilde{\varepsilon}'_j = \varepsilon'_j - \mathbb{E}\varepsilon'_j$$
$$\varepsilon''_j = \varepsilon_j \mathbf{1}_{\{|\varepsilon_j|> b_n\}} \qquad \widetilde{\varepsilon}''_j = \varepsilon''_j - \mathbb{E}\varepsilon''_j,$$

where the truncation level b_n goes to infinity at a rate which will be precised later. Since $\mathbb{E}\varepsilon_j = 0$, $\varepsilon_j = \widetilde{\varepsilon}'_j + \widetilde{\varepsilon}''_j$. Now let us recall that

$$y_k = \sum_{j=1}^{k} \phi_n^{k-j}\varepsilon_j = \widetilde{z}'_k + \widetilde{z}''_k,$$

where $\widetilde{z}'_k$ and $\widetilde{z}''_k$ are defined by substituting ε_j by $\widetilde{\varepsilon}'_j$ and $\widetilde{\varepsilon}''_j$ respectively in the definition of z_k, given by (5.10). Then for positive $\lambda = \lambda_n$, whose dependence on n will be precised later,

$$P\left(\max_{1\leq k\leq n} |y_k| > 2\lambda\right) \leq P'_n + P''_n, \tag{5.16}$$

where

$$P'_n := P\left(\max_{1\leq k\leq n} \left|\widetilde{z}'_k\right| > \lambda\right), \quad P''_n := P\left(\max_{1\leq k\leq n} \left|\widetilde{z}''_k\right| > \lambda\right).$$

To bound P'_n, applying Lemma 2 to $\widetilde{z}'_k$ gives for any $q > p$

$$P'_n \leq \frac{4\mathrm{e}^q C_q \mathbb{E}|\widetilde{\varepsilon}'_0|^q}{\lambda^q} n^{q/2}\gamma_n^{1-q/2} \leq \frac{2^{q+2}\mathrm{e}^q C_q \mathbb{E}|\varepsilon'_0|^q}{\lambda^q} n^{q/2}\gamma_n^{1-q/2},$$

since by elementary convexity inequalities, $\mathbb{E}|\widetilde{\varepsilon}_0'|^q \le 2^q \mathbb{E}|\varepsilon_0'|^q$. Now

$$\begin{aligned}\mathbb{E}|\varepsilon_0'|^q &= \int_0^\infty q t^{q-1} P\left(|\varepsilon_0|\,\mathbf{1}_{\{|\varepsilon_j|\le b_n\}} > t\right)\,\mathrm{d}t = \int_0^{b_n} q t^{q-1} P(t < |\varepsilon_0| \le b_n)\,\mathrm{d}t\\ &\le \int_0^{b_n} q t^{q-1} P(|\varepsilon_0| > t)\,\mathrm{d}t = \int_0^{b_n} q t^{q-1}\frac{f(t)}{t^p}\,\mathrm{d}t\\ &\le \frac{q\,\|f\|_\infty}{q-p} b_n^{q-p}.\end{aligned}$$

Going back to P_n' we find that

$$P_n' \le \frac{2^{q+2}\mathrm{e}^q q C_q \|f\|_\infty}{q-p}\cdot\frac{n^{q/2}\gamma_n^{1-q/2} b_n^{q-p}}{\lambda^q}.$$

Now we choose $\lambda = n^{1/2}\gamma_n^{1/p-1/2}\delta$, $q = p+1$ and

$$b_n = \delta^{p+1}\epsilon\gamma_n^{1/p} \tag{5.17}$$

with arbitrary $\epsilon > 0$. Recalling that γ_n goes to infinity, the same holds for b_n. This choice gives

$$P_n' = P\left(n^{-1/2}\gamma_n^\alpha \max_{1\le k\le n}\left|\widetilde{z}_k'\right| > \delta\right) \le C_p'\epsilon, \tag{5.18}$$

with $C_p' = 2^{p+3}\mathrm{e}^{p+1}(p+1)C_{p+1}\|f\|_\infty$.

To bound P_n'', we apply Lemma 2 with $z_k = \widetilde{z}_k''$ and $q = 2$ (keeping the above choices of λ and b_n which do not depend on q):

$$P_n'' \le \frac{8\mathrm{e}^2}{\delta^2}\gamma_n^{1-2/p}\mathbb{E}(\varepsilon_0'')^2.$$

In the special case where $p = 2$, this reduces to

$$P_n'' \le \frac{8\mathrm{e}^2}{\delta^2}\mathbb{E}(\varepsilon_0^2\mathbf{1}_{\{|\varepsilon_0|>b_n\}})$$

and this bound goes to zero by Lebesgue's dominated convergence theorem, since b_n defined by (5.17) goes to infinity. When $p > 2$, we estimate $\mathbb{E}(\varepsilon_0'')^2$ as follows.

$$\begin{aligned}\mathbb{E}(\varepsilon_0'')^2 &= \int_0^\infty 2tP\left(|\varepsilon_0|\mathbf{1}_{\{|\varepsilon_0|>b_n\}} > t\right)\,\mathrm{d}t = \int_0^{b_n} 2tP(|\varepsilon_0| > b_n)\,\mathrm{d}t + \int_{b_n}^\infty 2tP(|\varepsilon_0| > t)\,\mathrm{d}t\\ &= b_n^2 P(|\varepsilon_0| > b_n) + \int_{b_n}^\infty 2t^{1-p} f(t)\,\mathrm{d}t \le f(b_n) b_n^{2-p} + \frac{2}{p-2} f^*(b_n) b_n^{2-p}\\ &\le \frac{p}{p-2}\delta^{(p+1)(2-p)}\epsilon^{2-p}\gamma_n^{2/p-1} f^*(b_n).\end{aligned}$$

Finally, we see that there is a constant $C_{\delta,\epsilon,p}''$ such that for $p \ge 2$,

$$P_n'' \le C_{\delta,\epsilon,p}'' f^*(b_n). \tag{5.19}$$

Going back to (5.16) with (5.18) and (5.19), we obtain

$$Q_n := P\left(n^{-1/2}\gamma_n^\alpha \max_{1\le k\le n} |y_k| > \delta\right) \le C'_p\epsilon + C''_{\delta,\epsilon,p} f^*(b_n).$$

This gives $\limsup_{n\to\infty} Q_n \le C'_p\epsilon$ and as ϵ is arbitrary, so (4.2) follows.

5.3 Supplementary results

The next theorem gives a characterization of the tightness of sequences of random elements in a Hölder space (see [9] theorem 13 for the case $0 < \alpha < 1$ and proposition 1 for $\alpha = 0$).

Theorem 4. *The sequence (ξ_n) of random elements in $\mathrm{H}^o_\alpha[0,1]$ ($0 \le \alpha < 1$) is tight if and only if*

(a) $\lim_{A\to\infty} \sup_{n\ge 1} \mathbb{P}\left(\|\xi_n\|_\infty > A\right) = 0$;

(b) $\forall\epsilon > 0$, $\lim_{\delta\to 0} \sup_{n\ge 1} \mathbb{P}\left(\omega_\alpha(\xi_n, \delta) \ge \epsilon\right) = 0$.

To prove that the Hölder norm of a polygonal line is reached at two vertices, it is convenient to generalize a bit by considering more general weight functions than $h \mapsto h^\alpha$.

Lemma 3. *Let $\rho : [0,1] \to \mathbb{R}$ be a weight function satisfying the following properties.*

i) ρ is concave.

ii) $\rho(0) = 0$ and ρ is positive on $(0,1]$.

iii) ρ is non decreasing on $[0,1]$.

Let $t_0 = 0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$ and f be a real valued polygonal line function on $[0,1]$ with vertices at the t_i's, i.e. f is continuous on $[0,1]$ and its restriction to each interval $[t_i, t_{i+1}]$ is an affine function. Define

$$R(s,t) := \frac{|f(t) - f(s)|}{\rho(t-s)}, \quad 0 \le s < t \le 1.$$

Then

$$\sup_{0\le s<t\le 1} R(s,t) = \max_{0\le i<j\le n} R(t_i, t_j). \tag{5.20}$$

Proof. Obviously (5.20) will be established if we prove that

$$R(s,t) \le \max_{0\le i<j\le n} R(t_i, t_j), \tag{5.21}$$

for every pair of reals s, t such that $0 \le s < t \le 1$. This in turn, is easily deduced from the following estimates where in each configuration considered, f is supposed to be affine on $[a,b]$.

$$R(s,t) \le \begin{cases} R(a,b) & \text{if } a \le s < t \le b, \\ \max\big(R(s,a), R(s,b)\big) & \text{if } s < a \le t \le b, \\ \max\big(R(a,t), R(b,t)\big) & \text{if } a \le s \le b < t. \end{cases}$$

In the first configuration,

$$f(t) - f(s) = \frac{f(b) - f(a)}{b-a}(t-s),$$

whence

$$R(s,t) = R(a,b)\frac{t-s}{\rho(t-s)}\frac{\rho(b-a)}{b-a}. \tag{5.22}$$

By concavity of ρ, the function $h \mapsto \rho(h)/h$ is non increasing on $(0,1]$, as the slope of the chord between 0 and h. So $\rho(t-s)/(t-s) \geq \rho(b-a)/(b-a)$, whence $\frac{t-s}{\rho(t-s)}\frac{\rho(b-a)}{b-a} \leq 1$ and (5.22) gives $R(s,t) \leq R(a,b)$.

In the second configuration, let us parametrize the segment $[a,b]$ by putting $t = (1-u)a + ub$, $u \in [0,1]$. Then $t - s = (1-u)(a-s) + u(b-s)$ and as $t \mapsto f(t) - f(s)$ is affine on $[a,b]$, $f(t) - f(s) = (1-u)(f(a) - f(s)) + u(f(b) - f(s))$. Now to estimate $R(s,t)$, using triangular inequality for the numerator and the concavity of ρ for the denominator gives:

$$R(s,t) \leq \frac{(1-u)\,|f(a) - f(s)| + u\,|f(b) - f(s)|}{(1-u)\rho(a-s) + u\rho(b-s)} = \frac{Au + B}{Cu + D} = A' + \frac{B'}{Cu + D},$$

where the constants $A, A', \dots, D$ depend on f, ρ, a, b and s (which is fixed here). As ρ is non decreasing, $(1-u)\rho(a-s) + u\rho(b-s) \geq \rho(a-s) > 0$, so $Cu + D$ remains positive when u varies between 0 and 1. It follows that the homographic function $A' + B'/(Cu+D)$ is monotonic on $[0,1]$ and hence reaches its maximum at $u = 0$ or at $u = 1$. This gives $R(s,t) \leq \max\big(R(s,a), R(s,b)\big)$.

The bound for $R(s,t)$ in the third configuration is obtained in a completely similar way, so we omit the details.

Remark 4. In the case of vector valued polygonal lines, the result and the proof are still valid, replacing $|f(t) - f(s)|$ by $\|f(t) - f(s)\|$ in the definition of $R(s,t)$.

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