*Lithuanian Mathematical Journal, Vol.X, No. x, 20xx, **Author's Version** December 4, 2009*

On limit theorems for Banach space valued linear processes

A. Račkauskas

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-2006 Vilnius, Lithuania
Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania
(e-mail: Alfredas.Rackauskas@mif.vu.lt)

Ch. Suquet

Laboratoire P. Painlevé, UMR 8524 CNRS Université Lille I, Bât M2, Cité Scientifique,
F-59655 Villeneuve d'Ascq Cedex, France
(e-mail: Charles.Suquet@math.univ-lille1.fr)

Received

Abstract. Let $(\epsilon_i)_{i\in\mathbb{Z}}$ be i.i.d. random elements in the separable Banach space $\mathbb{E}$ and $(a_i)_{i\in\mathbb{Z}}$ be continuous linear operators from $\mathbb{E}$ to the Banach space $\mathbb{F}$, such that $\sum_{i\in\mathbb{Z}}\|a_i\|$ is finite. We prove that the linear process $(X_n)_{n\in\mathbb{Z}}$ defined by $X_n := \sum_{i\in\mathbb{Z}} a_i(\epsilon_{n-i})$ inherits from $(\epsilon_i)_{i\in\mathbb{Z}}$ the central limit theorem as well as functional central limit theorems in various Banach spaces of $\mathbb{F}$ valued functions, including Hölder spaces.

MSC: 60F17, 60B12.

Keywords: Central limit theorem in Banach spaces, Hölder space, functional central limit theorem, linear process, partial sum proces.

1 INTRODUCTION

Linear process $(\sum_{j\in\mathbb{Z}} a_j\epsilon_{k-j}, k\in\mathbb{Z})$, where the innovations ϵ_i are random variables is an intensively studied model in statistics. In the context of the recent and quick expansion of the field of functional data analysis, it seems useful to consider extensions of this model to the case where the innovations are random elements in an infinite dimensional space, replacing then the coefficients a_j by linear operators. This paper is devoted to the study of central limit theorems and functional central limit theorems for such Banach space valued linear processes. To avoid measurability complications, all the Banach space considered in this paper are supposed to be separable. For a Banach space $\mathbb{E}$ with a norm $\|\cdot\|$, $\mathbb{E}'$ denotes its topological dual. The notations $\mathbb{E}$, $\mathbb{F},\dots$ with indexes or without are reserved for Banach spaces. We write $\mathcal{L}(\mathbb{E},\mathbb{F})$ for the space of bounded linear operators $a:\mathbb{E}\to\mathbb{F}$ endowed with the norm $\|a\| = \sup_{\|x\|\le 1}\|a(x)\|$. For $a\in\mathcal{L}(\mathbb{E},\mathbb{F})$, a^* denotes its conjugate.

We study a stationary linear process

$$X_k = \sum_{j\in\mathbb{Z}} a_j(\epsilon_{k-j}),\quad k\in\mathbb{Z}, \tag{1.1}$$

where $(\epsilon_k,\ k\in\mathbb{Z})$ are $\mathbb{E}$-valued *innovations* and $(a_k,\ k\in\mathbb{Z})\subset\mathcal{L}(\mathbb{E},\mathbb{F})$ is a set of bounded linear operators called *linear filter*. When for each $k\in\mathbb{Z}$, the series (1.1) converge a.s. in the norm topology of $\mathbb{F}$, $(X_k, k\in\mathbb{Z})$ constitutes a set of $\mathbb{F}$-valued random elements. A recent reference for Banach space valued linear processes is Bosq [4]. Roughly speaking our aim is to obtain a way to transfer some functional convergence of partial sum process built on the innovations to a similar result (with the same normalisation) for the partial sum process built on the linear process. A motivating work for this problem is the paper by Phillips and Solo [13], where it is established that the validity of the purely algebraic Beveridge-Nelson decomposition of the linear filter into long-run and transistory components preserves the limit behavior of innovations for the corresponding linear process. Moreover in this paper, a large class of real valued innovations including independent identically distributed random variables and a class of martingale differences is considered. Also a wide spectrum of limit results including strong law of large numbers, law of iterated logarithm, central limit theorem and invariance principles is presented.

We consider functional limit behavior for innovations that can be formulated as follows. Let $\{(\xi_n^{(\epsilon)}(t), t \in [0,1]),\ n \in \mathbb{N}\}$, be a sequence of stochastic processes constructed from partial sums of innovations $\epsilon = (\epsilon_k, k \in \mathbb{Z})$ (polygonal line process). Next we choose a suitable separable Banach space of $\mathbb{E}$-valued functions, say $\mathcal{F}(\mathbb{E})$ as paths space for $(\xi_n(t), t \in [0,1])$, $n \in \mathbb{N}$ and assume that the sequence $(\xi_n^{(\epsilon)})$ converges in distribution in this space. We define the corresponding partial sum process $\xi_n^{(X)}$ just by substituting the ϵ_k's by the X_k's in the definition of $\xi_n^{(\epsilon)}$ and consider $\xi_n^{(X)}$ as a random element in some separable function space $\mathcal{F}(\mathbb{F})$ corresponding to $\mathcal{F}(\mathbb{E})$. To be more precise, the norm endowing $\mathcal{F}(\mathbb{F})$ is obtained by substituting $\| \ \|_F$ to $\| \ \|_E$ whenever this one appears in the definition of the norm endowing $\mathcal{F}(\mathbb{E})$. This can be done e.g. with the L^p spaces, spaces of continuous functions, spaces of Hölderian functions, ...

Now the question is: *Assuming that $\epsilon = (\epsilon_k, k \in \mathbb{Z})$ is any sequence of random elements from a given class and that $b_n^{-1}\xi_n^{(\epsilon)}$ converges in distribution in $\mathcal{F}(\mathbb{E})$, under what conditions on the linear filter $(a_k, k \in \mathbb{Z})$, does $b_n^{-1}\xi_n^{(X)}$ converges in distribution in $\mathcal{F}(\mathbb{F})$?* In case of positive answer, we shall say that the linear process $(X_k, k \in \mathbb{Z})$ "inherits" its functional limit behavior from the innovations $(\epsilon_k, k \in \mathbb{Z})$. Examples of paths spaces $\mathcal{F}(\mathbb{E})$ considered in this paper include Hölder spaces $\mathrm{H}_\rho^o(\mathbb{E})$ (precise definitions of these spaces are given in subsequent sections) as well as more classical function spaces such as the space $\mathrm{C}(\mathbb{E})$ of continuous functions with values in $\mathbb{E}$. Our main result (Th.4 below) establishes that under some restrictions on the space $\mathcal{F}(\mathbb{E})$, the condition

$$\sum_k \|a_k\| < \infty, \tag{1.2}$$

is sufficient for the linear process $(X_k, k \in \mathbb{Z})$ to inherit functional limit behavior from independent identically distributed innovations $(\epsilon_k, k \in \mathbb{Z})$.

Beyond the theoretical interest, there is also some practical motivation for investigating functional limit theorems in non classical paths spaces like Hölder ones. For instance some test statistics based on Hölder norms of partial sum process were recently shown to be very useful in the problem of detecting short epidemic changes [17, 18].

A recent survey of functional central limit theorems in $\mathrm{C}[0,1]$ or $D[0,1]$ for linear processes may be found in [11]. The central limit theorem for Hilbert space valued linear processes was studied in 1997 by Merlevède, Peligrad and Utev [10] in the case of i.i.d. innovations. This was completed in 2003 by Dedecker and Merlevède [5] who obtained a conditional CLT and FCLT for Hilbert space valued linear processes built on strictly stationary sequences of innovations. The first Hölderian FCLT for linear processes are given in [7] where the innovations are real valued. Depending on the rate of convergence of the series of the filter coefficients, the linear processes obtained has short or long memory and the limiting processes is either standard or fractional Brownian motion. The first Hölderian FCLT for Hilbert space valued linear processes appears in [14].

The paper is organized as follows. In Section 2 we present a key lemma which is in a sense an analogue to Beveridge-Nelson decomposition. In section 3 we give some central limit like theorems for Banach space valued linear processes. In section 4, the convergence of partial sum processes is investigated for a class of abstract Banach function spaces with applications to some classical function spaces.

Throughout the paper a lot of various norms are used. We shall often take the freedom to denote them simply by $\|x\|$, if the context is clear enough to dispel doubts on the precise meaning. This way, various occurences of the notation $\| \ \|$ in the same formula may have different meanings.

2 BASIC AUXILIARY RESULT

The following key lemma is essentially an adaptation of Lemma 1 of Peligrad and Utev [12]. For our aim and for possible future applications to random fields it is convenient to state it in the setting of summable collections of vectors indexed by an infinite set I. We recall here the basic facts about such a summability theory and refer to L. Schwartz [19, Chap. XIV] for more information. A collection $(x_i, i \in I)$ of elements in the normed vector space $(\mathbb{V}, \| \ \|)$ is said to be summable in $\mathbb{V}$ with sum S if, for every $\varepsilon > 0$, there exists a finite set $J \subset I$ such that for every finite set $K \subset I$ containing J, $\|S_K - S\| < \varepsilon$, where $S_K := \sum_{i \in K} x_i$. When it exists, such a S is unique and one define $\sum_{i \in I} x_i := S$. If $(x_i, i \in I)$ is summable, the set I' of indexes $i \in I$ for which $x_i \neq 0$ is at most countable (so we could restrict without loss of generality to the case where I is countable, but this provide no real simplification in fact). In the special case where $\mathbb{V} = \mathbb{R}$ and the x_i's are non negative real numbers, $(x_i, i \in I)$ is summable if and only if $M < \infty$, where M is the supremum of the S_K's over all finite subsets K of I; in this case $S = M$. Note that we can always define $\sum_{i \in I} x_i$ as the supremum M, finite or not, in the case of non negative x_i's. When the vector space $(\mathbb{V}, \| \ \|)$ is complete, the summability of $(\|x_i\|, i \in I)$ in $\mathbb{R}$ implies the summability of $(x_i, i \in I)$ in the space $(\mathbb{V}, \| \ \|)$.

Lemma 1. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two separable Banach spaces and let $(a_i)_{i \in I}$ be a collection of continuous linear operators $a_i : \mathbb{E}_1 \to \mathbb{E}_2$, satisfying for some $0 < p \leq 1$*

$$\sum_{i \in I} \|a_i\|^p < \infty. \tag{2.1}$$

Then $(a_i)_{i \in I}$ is summable in $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ and

$$A := \sum_{i \in I} a_i \tag{2.2}$$

defines a continuous linear operator $\mathbb{E}_1 \to \mathbb{E}_2$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and assume that $(U_{n,i}, n \in \mathbb{N}, i \in I)$ is a collection of random elements $\Omega \to \mathbb{E}_1$ satisfying

$$\sup_{n \in \mathbb{N}, i \in I} \mathbf{E} \|U_{n,i}\|^p < \infty, \tag{2.3}$$

with the same p as in (2.1). Then one can define for $n \in \mathbb{N}$ a random element $Y_n : \Omega \to \mathbb{E}_2$ such that

$$Y_n = \sum_{i \in I} a_i(U_{n,i}) \quad \textit{almost surely.} \tag{2.4}$$

Assume moreover that for every fixed i, j in I,

$$\|U_{n,i} - U_{n,j}\| \xrightarrow[n \to \infty]{\Pr} 0. \tag{2.5}$$

Then for every index $e \in I$, the following convergence

$$\|Y_n - A(U_{n,e})\| \xrightarrow[n \to \infty]{\Pr} 0 \tag{2.6}$$

holds.

Proof. As $0 < p \leq 1$, (2.1) gives $\sum_{i \in I} \|a_i\| < \infty$ which entails the summability of $(a_i)_{i \in I}$ in $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ and justifies the definition of A. Moreover, the set I' of indexes $i \in I$ for which $a_i \neq 0$ is at most countable, so all the sums over I involving the a_i's in this proof are reduced to sums over I', avoiding measurability concerns.

We legitimate the existence of Y_n by noting that from (2.1) and (2.3), for $0 < p \leq 1$,

$$\mathbf{E} \sum_{i \in I} \|a_i\|^p \|U_{n,i}\|^p = \sum_{i \in I} \|a_i\|^p \mathbf{E} \|U_{n,i}\|^p < \infty,$$

whence $\sum_{i \in I} \|a_i\|^p \|U_{n,i}\|^p$ as well as $\sum_{i \in I} \|a_i\| \|U_{n,i}\|$ are almost surely finite, so $\sum_{i \in I} a_i(U_{n,i})$ is almost surely convergent in $\mathbb{E}_2$.

Now let us prove the convergence to zero of $P_{n,\varepsilon} := P(\|Y_n - AU_{n,e}\| > \varepsilon)$ for arbitrary $\varepsilon > 0$. Once fixed such an ε, the summability of $\sum_{i \in I} \|a_i\|^p$ combined with (2.3) provides for any positive δ a *finite* subset K of I, such that for every $n \geq 1$,

$$\sum_{i \in I \setminus K} \|a_i\|^p \mathbf{E} \|U_{n,i} - U_{n,e}\|^p < \delta \varepsilon^p.$$

Starting from the splitting

$$Y_n - AU_{n,e} = \sum_{i \in I} a_i(U_{n,i} - U_{n,e}) = \sum_{i \in I \setminus K} a_i(U_{n,i} - U_{n,e}) + \sum_{i \in K} a_i(U_{n,i} - U_{n,e}),$$

we easily obtain

$$\begin{aligned} P_{n,\varepsilon} &\leq \frac{2^p}{\varepsilon^p} \sum_{i \in I \setminus K} \|a_i\|^p \mathbf{E} \|U_{n,i} - U_{n,e}\|^p + P\left(\sum_{i \in K} \|a_i\|^p \|U_{n,i} - U_{n,e}\|^p > \frac{\varepsilon^p}{2^p} \right) \\ &\leq 2^p \delta + P\left(\sum_{i \in K} \|a_i\|^p \max_{j \in K} \|U_{n,j} - U_{n,e}\|^p > \frac{\varepsilon^p}{2^p} \right) \\ &\leq 2^p \delta + \sum_{j \in K} P\left(\|U_{n,j} - U_{n,e}\| > \frac{\varepsilon}{2^{1/p} \tau} \right), \end{aligned}$$

where we put $\tau^p := \sum_{i \in I} \|a_i\|^p$, recalling that $\tau > 0$. Now from (2.5) and the finiteness of K we obtain :

$$\limsup_{n \to \infty} P_{n,\varepsilon} \leq 2^p \delta.$$

As this limsup does not depend on the arbitrary positive δ, it is in fact a null limit, which was to be proved.

3 LIMIT THEOREMS FOR SUMS

We discuss now the asymptotic distributional behavior of sums of Banach space valued linear processes. Proposition 1 below provides the general scheme leading to central limit theorem or convergence to α-stable distribution.

Consider innovations $(\epsilon_k, k \in \mathbb{Z})$ consisting of random elements with values in a *separable* Banach space $\mathbb{E}$ and corresponding linear processes $(X_k, k \in \mathbb{Z})$ defined by (1.1) where $(a_k, k \in \mathbb{Z}) \subset \mathcal{L}(\mathbb{E}, \mathbb{F})$, the Banach space $\mathbb{F}$ being also separable. For $p > 0$, we shall write $(a_k, k \in \mathbb{Z}) \in \ell^p(\mathbb{E}, \mathbb{F})$ provided

$$\sum_{k \in \mathbb{Z}} \|a_k\|^p < \infty. \tag{3.1}$$

As already observed above, the membership of $(a_k, k \in \mathbb{Z})$ in $\ell^p(\mathbb{E}, \mathbb{F})$ with $0 < p \leq 1$ yields the convergence in $\mathcal{L}(\mathbb{E}, \mathbb{F})$ of $\sum_{k\in\mathbb{Z}} a_k$, legitimating the definition of the operator

$$A := \sum_{k\in\mathbb{Z}} a_k \tag{3.2}$$

as an element of $\mathcal{L}(\mathbb{E}, \mathbb{F})$.

By the argument already used at the begining of Lemma's 1 proof, a sufficient condition on the innovations for the existence of the linear process $(X_k, k \in \mathbb{Z})$ associated to a filter in $\ell^p(\mathbb{E}, \mathbb{F})$ is that

$$\sup_{j\in\mathbb{Z}} \mathbf{E}\, \|\epsilon_j\|^p < \infty. \tag{3.3}$$

In what follows, we put for $k \leq l$,

$$S_{k,l}^{(X)} := \sum_{i=k}^{l} X_i, \quad S_{k,l}^{(\epsilon)} := \sum_{i=k}^{l} \epsilon_i$$

and abbreviate $S_{1,n}^{(X)}$ in $S_n^{(X)}$, $S_{1,n}^{(\epsilon)}$ in $S_n^{(\epsilon)}$ for $n \geq 1$. We also set $S_0^{(\epsilon)} = S_0^{(X)} = 0$.

Let $\mathcal{I}^{\mathrm{sta}}(\mathbb{E})$ be the set of *stationary* $\mathbb{Z}$-indexed sequences of $\mathbb{E}$-valued innovations $(\epsilon_k, k \in \mathbb{Z})$ and let $\mathcal{I}^{\mathrm{iid}}(\mathbb{E})$ denotes its subset of sequences of independent identically distributed innovations.

Proposition 1. *Assume that* $(\epsilon_k, k \in \mathbb{Z}) \in \mathcal{I}^{\mathrm{sta}}(\mathbb{E})$ *and* $(a_k, k \in \mathbb{Z}) \in \ell^p(\mathbb{E}, \mathbb{F})$ *for some* $p \in (0, 1]$. *If for a norming sequence of positive numbers* $(b_n, n \geq 1)$ *going to infinity and a centering sequence* $(c_n, n \geq 1) \subset \mathbb{E}$ *one has*

$$b_n^{-1} S_n^{(\epsilon)} - c_n \xrightarrow[n\to\infty]{\mathbb{E}} Y \tag{3.4}$$

and

$$\sup_n \mathbf{E}\, \|b_n^{-1} S_n^{(\epsilon)}\|^p < \infty, \tag{3.5}$$

then the linear process $(X_k, k \in \mathbb{Z})$ *defined by* (1.1) *satisfies*

$$b_n^{-1} S_n^{(X)} - A(c_n) \xrightarrow[n\to\infty]{\mathbb{F}} A(Y).$$

Proof. It follows from (3.5) that $\mathbf{E}\, \|\epsilon_1\|^p < \infty$. By stationarity (3.3) is hence satisfied, which insures here the existence of the linear process $(X_k, k \in \mathbb{Z})$. The continuity of A and continuous mapping theorem provides the following convergence in distribution:

$$A\big(b_n^{-1} S_n^{(\epsilon)} - c_n\big) \xrightarrow[n\to\infty]{\mathbb{F}} A(Y).$$

Now writing

$$\begin{aligned} b_n^{-1} S_n^{(X)} - A(c_n) &= \sum_{i\in\mathbb{Z}} a_i\big(b_n^{-1} S_{1-i,n-i}^{(\epsilon)}\big) - A(c_n) \\ &= R_n + A\big(b_n^{-1} S_n^{(\epsilon)} - c_n\big), \end{aligned}$$

where

$$R_n = \sum_{i\in\mathbb{Z}} a_i\big(b_n^{-1} S_{1-i,n-i}^{(\epsilon)}\big) - A\big(b_n^{-1} S_n^{(\epsilon)}\big)$$

and using the extension to Banach space of Slutsky lemma (see Th.4.1 in [2]), we just have to check that R_n converges in probability to zero.

To this end we apply Lemma 1 with $\mathbb{E}_1 = \mathbb{E}$, $\mathbb{E}_2 = \mathbb{F}$, $U_{n,i} = b_n^{-1} S^{(\epsilon)}_{1-i,n-i}$. The assumption (2.3) is satisfied in view of (3.5) and the stationarity of $\big(S^{(\epsilon)}_{1-i,n-i}\big)_{i\in\mathbb{Z}}$. The elementary estimate

$$\mathbf{E}\,\|b_n^{-1} S^{(\epsilon)}_{1-i,n-i} - b_n^{-1} S^{(\epsilon)}_{1-j,n-j}\|^p \le \frac{2|j-i|\mathbf{E}\,\|\epsilon_1\|^p}{b_n^p}$$

enables us to check (2.5) and to complete the proof.

For a random element Y in a separable Banach space $\mathbb{E}$ such that for every $f \in \mathbb{E}'$, $\mathbf{E}\,\langle f, Y\rangle = 0$ and $\mathbf{E}\,\langle f, Y\rangle^2 < \infty$ its covariance operator $Q = Q(Y)$ is the linear bounded operator from $\mathbb{E}'$ to $\mathbb{E}$ defined by $Qf = \mathbf{E}\,(\langle f, Y\rangle Y), f \in \mathbb{E}'$. A random element $Y \in \mathbb{E}$ (or covariance operator Q) is said to be pregaussian if there exists a mean zero Gaussian random element $G \in \mathbb{E}$ with the same covariance operator as Y, i.e. for all $f, g \in \mathbb{E}'$, $\mathbf{E}\,\langle f, Y\rangle\langle g, Y\rangle = \mathbf{E}\,\langle f, G\rangle\langle g, G\rangle$. Since the distribution of a centered Gaussian random element is defined by its covariance structure, we denote by G_Q a zero mean Gaussian random element with covariance operator Q.

Following the terminology adopted in [9], a random element Y in $\mathbb{E}$ is said to satisfy the central limit theorem in $\mathbb{E}$ (denoted $Y \in \mathrm{CLT}(\mathbb{E})$) if the sequence $n^{-1/2}(Y_1 + \cdots + Y_n), n \ge 1$ converges in distribution in $\mathbb{E}$, where the Y_i's, are independent copies of Y.

It is well-known that the central limit theorem in $\mathbb{E}$ is not a direct extension of the finite dimensional case. Depending on the geometry of the space $\mathbb{E}$, one can even find some bounded random element ϵ_1 which does not satisfies the central limit theorem, see e.g. [9]. So in the general case, no integrability condition on ϵ_1 will ensure that ϵ_1 satisfies the central limit theorem in $\mathbb{E}$. In so called type 2 spaces (e.g., any Hilbert space, L_p with $p \ge 2$) $\mathbf{E}\,\|Y\|^2 < \infty$ implies $Y \in \mathrm{CLT}(\mathbb{E})$.

If $Y \in \mathrm{CLT}(\mathbb{E})$, then (see Ledoux and Talagrand [9]) Y is necessarily pregaussian; the limit G is a Gaussian random element in $\mathbb{E}$ with the same covariance structure as Y; Y has mean zero and satisfy

$$\lim_{t\to\infty} t^2 P(\|Y\| > t) = 0, \tag{3.6}$$

in particular $\mathbf{E}\,\|Y\|^p < \infty$ for every $0 < p < 2$. Moreover for every $0 < p < 2$,

$$\sup_{n\ge 1} \mathbf{E}\,\|n^{-1/2}(Y_1 + \cdots + Y_n)\|^p < \infty. \tag{3.7}$$

Next we prove that *independent on the geometry of the Banach space* $\mathbb{E}$, any linear filter $(a_i, i \in \mathbb{Z}) \in \ell^1(\mathbb{E}, \mathbb{F})$ generates linear processes inheriting central limit property from i.i.d. innovations.

Theorem 1. *Assume that* $(\epsilon_k, k \in \mathbb{Z}) \in \mathcal{I}^{\mathrm{iid}}$ *and* $(a_k, k \in \mathbb{Z}) \in \ell^1(\mathbb{E}, \mathbb{F})$. *Then*

$$n^{-1/2}(\epsilon_1 + \cdots + \epsilon_n) \xrightarrow[n\to\infty]{\mathbb{E}} G_\epsilon \tag{3.8}$$

yields

$$n^{-1/2}(X_1 + \cdots + X_n) \xrightarrow[n\to\infty]{\mathbb{F}} G_X. \tag{3.9}$$

Moreover G_ϵ *and* G_X *are mean zero Gaussian random elements with covariances respectively* $Q(\epsilon_1)$ *and* $AQ(\epsilon_1)A^*$.

Proof. The convergence (3.9) follows from proposition 1 since condition (3.5) is satisfied by (3.7). As discussed above the random element G_ϵ in (3.8) is necessarily Gaussian and its covariance operator is $Q(G_\epsilon) = Q(\epsilon_1)$. As A is also linear, $A(G_\epsilon)$ is a Gaussian random element in $\mathbb{F}$. It is classical to check that covariance operator of $A(X)$ is $AQ(\epsilon_1)A^*$.

An example provided by Merlevède, Peligrad and Utev [10] shows also, that the condition $(a_k, k \in \mathbb{Z}) \in \ell^1(\mathbb{E}, \mathbb{F})$ cannot be relaxed. They constructed an example of i.i.d. innovations $(\epsilon_k, k \in \mathbb{Z})$ in a separable Hilbert space $\mathbb{H}$ and a linear filter $(a_k, k \in \mathbb{Z}) \subset \mathcal{L}(\mathbb{H}, \mathbb{H})$ such that ϵ_1 satisfies the central limit theorem, $\sum_k \|a_k\| = \infty$ and $n^{-1/2} S_n^{(X)}$ is not tight.

Let us remark that, if the space $\mathbb{E}$ is of type 2 (respect. of cotype 2) (we refer to Ledoux and Talagrand [9] for definitions) then $\mathbf{E}\,\|\epsilon_1\|^2 < \infty$ (respect. ϵ_1 is pregaussian) yields the central limit theorem for innovations and therefore for linear processes $(X_k, k \in \mathbb{Z})$ with absolutely summable linear filters. This was established first by Denis'evskiĭ [6] for the type 2 case.

Let $\alpha \in (0, 2]$. Let us recall that an $\mathbb{E}$-valued random element G_α is said to be stable with index α (α-stable for short) if for every $n \geq 1$ there exists $c_n \in \mathbb{E}$ such that $n^{-1/\alpha} \sum_{j=1}^n G_{\alpha,j} - c_n$ has the same distribution as G_α, where $G_{\alpha,j}, j \geq 1$ are independent copies of G_α. We refer to Araujo and Giné [1] for details concerning stable laws in Banach spaces and their domains of attraction.

Theorem 2. *Let $0 < \alpha \leq 2$. Assume that $(\epsilon_k, k \in \mathbb{Z}) \in \mathcal{I}^{\mathrm{iid}}$ and (2.1) is satisfied with some $p < \min\{\alpha, 1\}$. Then if for a norming sequence (b_n) and a centering sequence $(c_n) \subset \mathbb{E}$ it holds*

$$b_n^{-1}(\epsilon_1 + \cdots + \epsilon_n) - c_n \xrightarrow[n\to\infty]{\mathbb{E}} G_\alpha \tag{3.10}$$

we have also

$$b_n^{-1}(X_1 + \cdots + X_n) - A(c_n) \xrightarrow[n\to\infty]{\mathbb{F}} A(G_\alpha). \tag{3.11}$$

Proof. The convergence (3.11) follows from proposition 1 since the condition (3.5) can be easily checked in the same way as in one dimensional case (see Araujo and Giné [1], Ex.9, Ch.2, Sec.6). Note in passing that necessarily in (3.10), $\mathbf{E}\,\|\epsilon_1\|^p$ is finite and b_n goes to infinity.

4 FUNCTIONAL LIMIT THEOREMS

Consider $\mathbb{E}$-valued innovations $(\epsilon_k, k \in \mathbb{Z})$ and corresponding linear processes $(X_k, k \in \mathbb{Z})$ defined by (1.1) where $(a_k, k \in \mathbb{Z}) \subset \mathcal{L}(\mathbb{E}, \mathbb{F})$. In this section, we use *polygonal partial sum processes* built on the sequence $(\epsilon_k, k \in \mathbb{Z})$ or on $(X_k, k \in \mathbb{Z})$, represented by the following formula defining the partial sum processes $\xi_n^{(\epsilon)}$, $n \geq 1$, the processes $\xi_n^{(X)}$ being defined similarly just substituting the ϵ_i's by the X_i's :

$$\xi_n^{(\epsilon)}(t) = \sum_{i=1}^n \epsilon_i e_{n,i}(t), \quad t \in [0, 1] \tag{4.1}$$

where the function $e_{n,i}$ is defined on $\mathbb{R}$ by

$$e_{n,i}(t) = \begin{cases} 0 & \text{if} \quad t < (i-1)/n \\ tn - (i-1) & \text{if} \quad (i-1)/n \leq t \leq i/n \\ 1 & \text{if} \quad t > i/n. \end{cases}$$

For a reason which will be clarified later, we complete these definitions by putting $e_{n,n+1}(t) := e_{n,n}(t - 1/n)$, $t \in \mathbb{R}$.

4.1 General result

For the functional central limit theorems we have in view, the above defined partial sum processes will be considered as random elements of some function spaces $\mathcal{F}(\mathbb{G})$, where $\mathbb{G} = \mathbb{E}$ for the partial sums processes built on the ϵ_i's and $\mathbb{G} = \mathbb{F}$ for those built on the X_i's. We assume that $\mathcal{F}(\mathbb{G})$ is a *separable* Banach space of functions $f : [0, 1] \to \mathbb{G}$ when endowed with the norm $\|f\|_{\mathcal{F}(\mathbb{G})}$. Let us agree here that the formal definition of the norms $\|f\|_{\mathcal{F}(\mathbb{E})}$ and $\|f\|_{\mathcal{F}(\mathbb{F})}$ are the same up to the substitution of $\| \, \|_{\mathbb{E}}$ by $\| \, \|_{\mathbb{F}}$ whenever it appears in the definition of $\|f\|_{\mathcal{F}(\mathbb{E})}$. Well known examples of such situation are the spaces of $\mathbb{G}$-valued continuous function or the spaces of $\mathbb{G}$-valued Hölderian functions built on some given weight function ρ (see the definition in subsection 4.2.2). All the assumptions made on functions spaces in this section are stated for $\mathcal{F}(\mathbb{E})$ because they appear naturally on this form when trying to establish our results. Note still that they implicitly induce some restrictions on $\mathcal{F}(\mathbb{F})$ due to the above assumption on the definition of the norms $\|f\|_{\mathcal{F}(\mathbb{E})}$ and $\|f\|_{\mathcal{F}(\mathbb{F})}$.

To insure the membership of $\xi_n^{(\epsilon)}$ and $\xi_n^{(X)}$ in the relevant function space $\mathcal{F}(\mathbb{G})$ for each $n \geq 1$, let us assume once for all and without further mention that

(A0) the space $\mathcal{F}(\mathbb{E})$ contains functions $f = eg$ where $e \in \mathbb{E}$ and $g : [0, 1] \to \mathbb{R}$ is any polygonal function.

Let us note that both versions imply that $\mathcal{F}(\mathbb{E})$ contain the constant functions $t \mapsto e$ where e is any fixed element in $\mathbb{E}$. At some places we shall need also the following property.

(A1) There is a constant c_1 such that for every constant function $e : [0, 1] \to \mathbb{E}$, $t \mapsto e$,

$$\|e\|_{\mathbb{E}} \leq c_1 \|e\|_{\mathcal{F}(\mathbb{E})}.$$

Now we can ask the following question where $(b_n)_{n\geq1}$ is a norming sequence of positive real numbers.

If $\left(b_n^{-1}\xi_n^{(\epsilon)}\right)_{n\geq1}$ *converge in distribution in* $\mathcal{F}(\mathbb{E})$*, under what conditions does* $\left(b_n^{-1}\xi_n^{(X)}\right)_{n\geq1}$ *converge in distribution in* $\mathcal{F}(\mathbb{F})$*?*

To deal with the filter $(a_i, i \in \mathbb{Z})$, we shall need the following assumption.

(A2) There is a constant c_2 such that for every $a \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ and every $f \in \mathcal{F}(\mathbb{E})$,

$$\|a \circ f\|_{\mathcal{F}(\mathbb{F})} \leq c_2 \|a\| \cdot \|f\|_{\mathcal{F}(\mathbb{E})}.$$

Before stating other properties of the function space $\mathcal{F}(\mathbb{E})$ involved in this investigation, it is convenient to introduce some definitions.

For any $h \in (-1, 1)$ and any function $f : [0, 1] \to \mathbb{E}$, define the *pseudo-translation* $T_h f : [0, 1] \to \mathbb{E}$ of f by

$$T_h f(t) := \begin{cases} f(0) & \text{if } t + h < 0 \\ f(t + h) & \text{if } 0 \leq t + h \leq 1 \\ f(1) & \text{if } t + h > 1. \end{cases}$$

For any interval $[u, v] \subset [0, 1]$ and any function $f : [0, 1] \to \mathbb{E}$, define the *pseudo-restriction* of f on the interval $[u, v]$, denoted $R_u^v f$, by

$$(R_u^v f)(t) := \begin{cases} f(u) & \text{if } t < u \\ f(t) & \text{if } u \leq t \leq v \\ f(v) & \text{if } t > v. \end{cases}$$

(A3) $\mathcal{F}(\mathbb{E})$ contains all the pseudo-translations of its elements and satisfies for some constant c_3,

$$\|T_h f\|_{\mathcal{F}(\mathbb{E})} \le c_3 \|f\|_{\mathcal{F}(\mathbb{E})}, \quad h \in (-1,1),\ f \in \mathcal{F}(\mathbb{E}).$$

(A4) $\mathcal{F}(\mathbb{E})$ contains all the pseudo-restrictions of its elements and satisfies for some constant c_4,

$$\|R_u^v f\|_{\mathcal{F}(\mathbb{E})} \le c_4 \|f\|_{\mathcal{F}(\mathbb{E})}, \quad 0 \le u \le v \le 1,\ f \in \mathcal{F}(\mathbb{E}).$$

It is worth noticing that the special case $u = v = s$ in (A4) provides the control by $\|f\|_{\mathcal{F}(\mathbb{E})}$ of the pointwise evaluations $\delta_s f = f(s)$ considered as constant functions :

$$\|\delta_s f\|_{\mathcal{F}(\mathbb{E})} \le c_4 \|f\|_{\mathcal{F}(\mathbb{E})}, \quad s \in [0,1],\ f \in \mathcal{F}(\mathbb{E}). \tag{4.2}$$

Combined with (A1), this gives

$$\|f(s)\|_{\mathbb{E}} \le c_1 c_4 \|f\|_{\mathcal{F}(\mathbb{E})}, \quad s \in [0,1],\ f \in \mathcal{F}(\mathbb{E}). \tag{4.3}$$

With this property the weak convergence in $\mathcal{F}(\mathbb{E})$ implies the convergence of finite dimensional distributions.

Properties (A3) and (A4) may be expressed in term of operators by saying that pseudo-translations T_h and pseudo-restrictions R_u^v map $\mathcal{F}(\mathbb{E})$ into itself and that the families of pseudo restrictions, pseudo-translations, viewed as families of linear operators are equicontinuous (or equivalently uniformly bounded for the relevant operator norm). When (4.2) is satisfied, this equicontinuity holds also for the family of pointwise evaluations. Among classical spaces of $\mathbb{E}$-valued function sharing these properties, we can mention the space of continuous functions as well as the Hölder spaces.

Theorem 3. *Let the innovations $(\epsilon_k, k \in \mathbb{Z})$ belong to $\mathcal{I}^{\mathrm{sta}}(\mathbb{E})$ and suppose that for some $0 < p \le 1$,* (3.1) *is satisfied and $\mathbf{E}\,\|\epsilon_1\|^p$ is finite. Assume that for some normalizing sequence $(b_n)_{n\ge1}$ going to infinity the following convergence holds*

$$b_n^{-1} \xi_n^{(\epsilon)} \xrightarrow[n\to\infty]{\mathcal{F}(\mathbb{E})} Y, \tag{4.4}$$

together with

$$\sup_{n\ge1} \mathbf{E}\,\|b_n^{-1}\xi_n^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})}^p < \infty. \tag{4.5}$$

Futhermore let the function space $\mathcal{F}(\mathbb{E})$ possess the properties (A2)*,* (4.2)*,* (A3) *and be such that for each $i \ge 1$ and any $x \in \mathbb{E}$*

$$\|x e_{n,i}\|_{\mathcal{F}(\mathbb{E})} = \|x\| o(b_n). \tag{4.6}$$

Assume finally that the distribution of the limiting process Y is supported by some subspace $\mathcal{V}$ of $\mathcal{F}(\mathbb{E})$ on which the pseudo-translations operate continuously, which means

$$\lim_{h\to0} \|T_h f - f\|_{\mathcal{F}(\mathbb{E})} = 0, \quad f \in \mathcal{V}. \tag{4.7}$$

Then

$$\xi_n^{(X)} \xrightarrow[n\to\infty]{\mathcal{F}(\mathbb{F})} AY, \tag{4.8}$$

where A is the operator defined by (3.2).

Proof. In view of (3.1), we can write for every $t \in [0,1]$ the expansion

$$\begin{aligned}\xi_n^{(X)}(t) &= \sum_{k=1}^n X_k e_{n,k}(t) = \sum_{k=1}^n \sum_{i\in\mathbb{Z}} a_i(\epsilon_{k-i}) e_{n,k}(t)\\ &= \sum_{i\in\mathbb{Z}} a_i \left(\sum_{k=1}^n \epsilon_{k-i} e_{n,k}(t) \right),\end{aligned}$$

which leads naturally to introduce the partial sum processes

$$\xi_{n,i}^{(\epsilon)}(t) := \sum_{k=1}^n \epsilon_{k-i} e_{n,k}, \quad t \in [0,1],\ i \in \mathbb{Z},\ n \geq 1.$$

Then the above pointwise expansion can be rewriten under the functional form

$$b_n^{-1}\xi_n^{(X)} = \sum_{i\in\mathbb{Z}} \tilde{a}_i(b_n^{-1}\xi_{n,i}^{(\epsilon)}), \tag{4.9}$$

where for each $i \in \mathbb{Z}$, $\widetilde{a}_i$ is an operator mapping $\mathcal{F}(\mathbb{E})$ to $\mathcal{F}(\mathbb{F})$ defined by

$$\widetilde{a}_i f = a_i \circ f, \quad f \in \mathcal{F}(\mathbb{E}).$$

A priori the series of functions (4.9) converges pointwise on $[0,1]$, but this convergence holds also in the norm topology of $\mathcal{F}(\mathbb{F})$, which is usually stronger than the pointwise convergence, at least when (4.3) is satisfied with $\mathcal{F}(\mathbb{F})$ instead of $\mathcal{F}(\mathbb{E})$. Indeed according to (A2) we have $\|\widetilde{a}_i\| \leq c_2\|a_i\|$ for each $i \in \mathbb{Z}$, so the convergence of the series (4.9) holds almost surely in $\mathcal{F}(\mathbb{E})$ due to (3.1) and subject to

$$\sup_{n\geq 1, i\in\mathbb{Z}} \mathbf{E}\,\|b_n^{-1}\xi_{n,i}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})}^p < \infty, \tag{4.10}$$

which in turn follows from the assumption (4.5) by stationarity.

Now, on the ground of the functional representation (4.9), we are in a position to prove the convergence (4.8) for $(b_n^{-1}\xi_n^{(X)})$ through Slutsky's lemma in $\mathcal{F}(\mathbb{F})$ and Lemma 1 applied with $\mathbb{E}_1 = \mathcal{F}(\mathbb{E})$, $\mathbb{E}_2 = \mathcal{F}(\mathbb{F})$, with the a_i's substituted by the $\tilde{a}_i$'s and with $U_{n,i} = b_n^{-1}\xi_{n,i}^{(\epsilon)}$.

In view of (4.10), it only remains to check condition (2.5) of Lemma 1, which via an obvious chaining argument is reduced here in proving that for arbitrarily fixed i,

$$b_n^{-1}\|\xi_{n,i}^{(\epsilon)} - \xi_{n,i+1}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \xrightarrow[n\to\infty]{\Pr} 0. \tag{4.11}$$

By stationarity, we can as well take $i = 0$. Now we have

$$\xi_{n,0}^{(\epsilon)} - \xi_{n,1}^{(\epsilon)} = \sum_{k=1}^{n-1} \epsilon_k(e_{n,k} - e_{n,k+1}) - \epsilon_0 e_{n,1} + \epsilon_n e_{n,n}.$$

Observing that for every $t \in [0,1]$, $e_{n,k+1}(t + n^{-1}) = e_{n,k}(t)$ and $e_{n,n+1}(t) = 0$, we can recast the above equality as

$$\xi_{n,0}^{(\epsilon)} - \xi_{n,1}^{(\epsilon)} = \Delta_{-1/n}(\xi_n^{(\epsilon)}) - \epsilon_o e_{n,1},$$

where $\Delta_h f := T_h f - f$. It follows by (4.6) that

$$\|\xi_{n,0}^{(\epsilon)} - \xi_{n,1}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \leq \|\epsilon_0\| \cdot o(b_n) + \omega_{1/n}(\xi_n^{(\epsilon)}),$$

where for every $0 < \delta < 1$ the functional ω_δ is defined by

$$\omega_\delta(f) = \sup_{|h| \leq \delta} \|\Delta_h f\|_{\mathcal{F}(\mathbb{E})}, \quad f \in \mathcal{F}(\mathbb{E}).$$

At this stage, it only remains to prove the convergence in probability to zero of $b_n^{-1}\omega_{1/n}(\xi_n^{(\epsilon)})$. As the functional ω_δ is clearly subadditive on $\mathcal{F}(\mathbb{E})$, it satisfies $|\omega_\delta(f) - \omega_\delta(g)| \leq \omega_\delta(f - g) \leq (1 + c_3)\|f - g\|_{\mathcal{F}(\mathbb{E})}$, using (A3). Hence ω_δ is a *continuous* functional on $\mathcal{F}(\mathbb{E})$. From this and the convergence assumption (4.4), we deduce that for arbitrary positive δ and τ

$$\limsup_{n\to\infty} P(b_n^{-1}\omega_{1/n}(\xi_n^{(\epsilon)}) \geq \tau) \leq \limsup_{n\to\infty} P(b_n^{-1}\omega_\delta(\xi_n^{(\epsilon)}) \geq \tau) \leq P(\omega_\delta(Y) \geq \tau).$$

By (4.7), the last probability tends to zero with δ, so the proof is complete.

The weak convergence of $(b_n^{-1}\xi_n^{(\epsilon)})$ in $\mathcal{F}(\mathbb{E})$ implies its tightness and also the weaker property of stochastic boundedness in $\mathcal{F}(\mathbb{E})$. Let us recall here that a sequence $(\zeta_n)_{n\geq1}$ of random elements in some vector space $(B, \| \ \|_B)$ is said stochastically bounded if $\sup_{n\geq1} P(\|\zeta_n\|_B > r)$ goes to zero when r goes to infinity. When the innovations are i.i.d., it is possible to relax the assumptions on the sequence $(b_n^{-1}\xi_n^{(\epsilon)})$ in Theorem 3 by proving that its stochastic boundedness in $\mathcal{F}(\mathbb{E})$ implies (4.5). That is the aim of Proposition 2 where we restrict to the normalizing sequence $b_n^{1/\alpha}\ell(n)$, $0 < \alpha \leq 2$, with ℓ slowly varying. To motivate this choice, let us observe that if $\mathcal{F}(\mathbb{E})$ satisfies (A1), the $\mathcal{F}(\mathbb{E})$ convergence (4.4) implies the weak convergence in $\mathbb{E}$ of $b_n^{-1}\xi_n^{(\epsilon)}(1) = b_n^{-1}S_n^{(\epsilon)}$ to $Y(1)$ and that normalizing constants b_n are necessarily of the above form in such a convergence for i.i.d. ϵ_i's. Let us recall here that any positive slowly varying function ℓ admits a representation:

$$\ell(t) = \kappa(t) \exp\Big(\int_1^t \varepsilon(s)\frac{\mathrm{d}s}{s}\Big), \tag{4.12}$$

where $\kappa(t)$ tends to $c > 0$ and $\varepsilon(t)$ tends to zero as t tends to infinity.

Proposition 2. *Assume that the innovations $(\epsilon_i, i \in \mathbb{Z})$ are i.i.d. and that the function space $\mathcal{F}(\mathbb{E})$ satisfies* (A4) *and*

(A5) *there is some positive constant c_5 such that for every $g \in \mathcal{F}(\mathbb{E})$ and every $0 \leq u < v \leq 1$, putting $f(t) = g(u + (v - u)t)$, $0 \leq t \leq 1$, the function f belongs to $\mathcal{F}(\mathbb{E})$ and satisfies*

$$\|f\|_{\mathcal{F}(\mathbb{E})} \leq c_5\|R_u^v g\|_{\mathcal{F}(\mathbb{E})}.$$

For $0 < \alpha \leq 2$, let $b_n^{1/\alpha}\ell(n)$ with ℓ slowly varying. If the sequence $(b_n^{-1}\xi_n^{(\epsilon)}, n \geq 1)$ is stochastically bounded in the space $\mathcal{F}(\mathbb{E})$ then for $0 < p < \alpha$,

$$\sup_{n\geq1} \mathbf{E}\,\|b_n^{-1}\xi_n^{(\epsilon)}\|_{\mathcal{F}}^p < \infty. \tag{4.13}$$

Proof. For any integers $m, n \geq 1$, let us decompose the process $\xi_{mn}^{(\epsilon)}$ as

$$\xi_{nm}^{(\epsilon)} = \sum_{k=1}^{m} \Big(R_{s_{k-1}}^{s_k} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_{k-1}) \Big), \quad s_k = \frac{k}{m},$$

where the summands are independent random functions in $\mathcal{F}(\mathbb{E})$, but not identically distributed (because the intervals where these random functions are constant are not the same). Denote by $\xi_{n;k}^{(\epsilon)}$, $1 \leq k \leq m$, the partial sum processes built on the innovations $\big(\epsilon_i,\ (k-1)n < i \leq kn\big)$ which are i.i.d. copies of $\xi_n^{(\epsilon)} = \xi_{n;1}^{(\epsilon)}$. Using (A5) together with the invariance of $\mathbb{E}$-valued constant functions by the linear operators of pseudo-restriction gives

$$\|\xi_{n;k}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \leq c_5 \Big\| R_{s_{k-1}}^{s_k} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_{k-1}) \Big\|_{\mathcal{F}(\mathbb{E})}, \quad k = 1, \dots, m,$$

from which we deduce that

$$\begin{aligned}\max_{1\leq k\leq m} \|\xi_{n;k}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} &\leq c_5 \max_{1\leq k\leq m} \Big\| R_{s_{k-1}}^{s_k} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_{k-1}) \Big\|_{\mathcal{F}(\mathbb{E})} \\ &\leq 2c_5 \max_{1\leq k\leq m} \Big\| \sum_{j=1}^{k} \Big(R_{s_{j-1}}^{s_j} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_{j-1}) \Big) \Big\|_{\mathcal{F}(\mathbb{E})}.\end{aligned}$$

Observing that

$$\sum_{j=k+1}^{m} \Big(R_{s_{j-1}}^{s_j} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_{j-1}) \Big) = R_{s_k}^{1} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_k)$$

and applying Ottaviani maximal inequality for the partial sums of independent random elements in the Banach space $\mathcal{F}(\mathbb{E})$, see e.g. Lemma 6.2 p.152 in [9], we obtain

$$P\Big(\max_{1\leq k\leq m} \|\xi_{n;k}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \geq 4c_5 u \Big) \leq \frac{P\Big(\|\xi_{nm}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \geq u \Big)}{1 - \max_{1\leq k<m} P\Big(\|R_{s_k}^{1} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_k)\|_{\mathcal{F}(\mathbb{E})} \geq u \Big)}.$$

Now recalling that (A4) implies (4.2), we obtain

$$\|R_{s_k}^{1} \xi_{nm}^{(\epsilon)} - \xi_{nm}^{(\epsilon)}(s_k)\|_{\mathcal{F}(\mathbb{E})} \leq 2c_4 \|\xi_{nm}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})},$$

whence

$$P\Big(\max_{1\leq k\leq m} \|\xi_{n;k}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \geq 4c_5 u \Big) \leq \frac{P\Big(\|\xi_{nm}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \geq u \Big)}{1 - P\Big(\|\xi_{nm}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \geq u(2c_4)^{-1} \Big)}.$$

Using this last estimate and the stochastic boundedness of $(n^{-1/2}\xi_n^{(\epsilon)})_{n\geq 1}$ in $\mathcal{F}(\mathbb{E})$, we can find a positive constant c such that

$$\sup_{m,n\geq 1} P\Big(\max_{1\leq k\leq m} \|b_n^{-1} \xi_{n;k}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})} \geq c \frac{m^{1/\alpha}\ell(mn)}{\ell(n)} \Big) \leq 1 - \exp(-1). \tag{4.14}$$

Here the special choice of the constant $1-\exp(-1)$ is just for convenience, but any $0<\varepsilon<1$ would suit instead.

From the representation formula (4.12), it is easily seen that with some positive constant c',

$$\frac{\ell(mn)}{\ell(n)} \le c' m^{\delta_n}, \quad m\ge 1, n\ge 1,$$

where $\delta_n = \sup_{t\ge n}|\varepsilon(t)|$ goes to 0 as n goes to infinity. Now fixing $0<p<r<\alpha$, there is an integer n_0 depending on r such that for every $n\ge n_0$, $m^{\delta_n+1/\alpha}\le m^{1/r}$. For $n<n_0$, using the fact that $\max_{1\le n<n_0}\ell(nt)$ is also slowly varying, one can find a constant c'' such that $m^{1/\alpha}\ell(mn)\ell^{-1}(n)\le c'' m^{1/r}$ for $m\ge 1$.

Going back to (4.14) with these estimates, we obtain with a possibly increased constant c:

$$P\Big(\max_{1\le k\le m}\|b_n^{-1}\xi_{n;k}^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})}\ge cm^{1/r}\Big)\le 1-\exp(-1), \quad m\ge 1, n\ge 1. \tag{4.15}$$

Recalling that the $\xi_{n;k}^{(\epsilon)}$, $1\le k\le m$ are i.i.d. copies of $\xi_n^{(\epsilon)}=\xi_{n;1}^{(\epsilon)}$, (4.15) can be recast after some elementary work as

$$P\Big(\|b_n^{-1}\xi_n^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})}\ge cm^{1/r}\Big)\le 1-\exp(-1/m)\le\frac{1}{m}, \quad m\ge 1, n\ge 1.$$

From this it is easy to see, that $t^rP(\|b_n^{-1}\xi_n\|_{\mathcal{F}(\mathbb{E})}>ct)\le 2$ for every $t\ge 1$, $n\ge 1$. A classical integration by part enables us to bound, $\mathbf{E}\,\|b_n^{-1}\xi_n^{(\epsilon)}\|_{\mathcal{F}(\mathbb{E})}^p$ by a constant depending on c, p, r, but not on n. This yields the result.

Combining Theorem 3 with Proposition 2 gives the following theorem which is the pattern for all the concrete examples which follow. It seems in order to recall here what we mean by a $\mathbb{E}$-valued Brownian motion. If ϵ is a centered *pregaussian* random element in the Banach space $\mathbb{E}$ with covariance $Q(\epsilon)$, there exists a Gaussian random element G in $\mathbb{E}$ with the same covariance. We denote then by $W_{Q(\epsilon)}$ a $\mathbb{E}$-valued Brownian motion modelled on this covariance structure, i.e. a centered $\mathbb{E}$-valued Gaussian process with independent increments and such that $W_{Q(\epsilon)}(t)-W_{Q(\epsilon)}(s)$ has the same distribution as $|t-s|^{1/2}G$. As the distribution of $W_{Q(\epsilon_i)}$ depends only on that of ϵ_i, we shall abbreviate in the sequel $W_{Q(\epsilon_i)}$ in $W_{Q(\epsilon)}$, denoting by ϵ any random element in $\mathbb{E}$ with the same distribution as the ϵ_i's. We note that if some partial sum process built on i.i.d. innovations $(\epsilon_i)_{i\ge 1}$ with normalization $n^{1/2}$ converges weakly to a Gaussian process on some space of $\mathbb{E}$-valued functions satisfying (4.3), then $\epsilon_1\in\mathrm{CLT}(\mathbb{E})$, so ϵ_1 is necessarily pregaussian and in particular $\mathbf{E}\,\|\epsilon_1\|$ is finite.

Theorem 4. *For innovations $(\epsilon_k, k\in\mathbb{Z})$ in $\mathcal{I}^{\mathrm{iid}}(\mathbb{E})$ and a filter $(a_k, k\in\mathbb{Z})$ in $\ell^1(\mathbb{E},\mathbb{F})$, let us assume that the following convergence holds*

$$n^{-1/2}\xi_n^{(\epsilon)}\xrightarrow[n\to\infty]{\mathcal{F}(\mathbb{E})}W_{Q_\epsilon}. \tag{4.16}$$

If moreover the space $\mathcal{F}(\mathbb{E})$ satisfies (A1)–(A5)*, (4.6) with $b_n^{1/2}$ and if there exists $\mathcal{V}$ such that (4.7) holds with $Y=W_{Q_\epsilon}$, then*

$$n^{-1/2}\xi_n^{(X)}\xrightarrow[n\to\infty]{\mathcal{F}(\mathbb{F})}W_{AQ(\epsilon)A^*}, \tag{4.17}$$

where the limiting process $W_{AQ(\epsilon)A^*}$ is a $\mathbb{F}$ valued Brownian motion.

4.2 Examples

Applying Theorem 4, we obtain functional central limit theorems in some classical function spaces for linear processes. Let us recall that we consider only separable Banach spaces $\mathbb{E}$, $\mathbb{F}$.

4.2.1 FCLT in the space of continuous functions

We write $\mathrm{C}(\mathbb{E})$ for the Banach space of continuous functions $f : [0,1] \to \mathbb{E}$ endowed with the supremum norm

$$\|f\|_\infty := \sup\{\|f(t)\|_{\mathbb{E}};\ t \in [0,1]\}. \tag{4.18}$$

Coupling Theorem 4 with invariance principle due to Kuelbs [8] we obtain the following invariance principle for partial sum polygonal line processes $(\xi_n^{(X)})$.

Theorem 5. *Assume that innovations $(\epsilon_i, i \in \mathbb{Z})$ are i.i.d. mean zero $\mathbb{E}$-valued random elements satisfying the central limit theorem. Then if $(a_k, k \in \mathbb{Z}) \in \ell_1(\mathbb{E}, \mathbb{F})$ it holds that*

$$n^{-1/2}\xi_n^{(X)} \xrightarrow[n\to\infty]{\mathrm{C}(\mathbb{F})} W_{AQ(\epsilon)A^*}.$$

Proof. As proved by Kuelbs [8], the assumptions on the innovations yield

$$n^{-1/2}\xi_n^{(\epsilon)} \xrightarrow[n\to\infty]{\mathrm{C}(\mathbb{E})} W_{Q(\epsilon)}.$$

It is elementary to check the properties (A0)–(A5) for $\mathcal{F}(\mathbb{E}) = \mathrm{C}(\mathbb{E})$. Condition (4.6) is trivially satisfied since $\|xe_{n,i}\|_\infty = \|x\|_E$ does not depend on n. As for (4.7), we can simply choose $\mathcal{V} = \mathrm{C}(\mathbb{E})$. Indeed this space inherits its separability from $\mathbb{E}$, supports any $\mathbb{E}$-valued Brownian motion and since every $f \in \mathrm{C}(\mathbb{E})$ is uniformly continuous on $[0,1]$, the translations operate continuously on $\mathrm{C}(\mathbb{E})$. So we conclude by applying Theorem 4.

As far as we know, the above version of FCLT in $\mathrm{C}(\mathbb{F})$ for linear processes is new.

4.2.2 FCLT in Hölder spaces

Let ρ be a real valued non decreasing function on $[0,1]$, null and right continuous at 0, positive on $(0,1]$. Put

$$\omega_\rho(f,\delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{\|f(t)-f(s)\|}{\rho(t-s)}.$$

We associate to ρ the Hölder spaces

$$\mathrm{H}_\rho(\mathbb{E}) := \{f \in \mathrm{C}(\mathbb{E});\ \omega_\rho(f,1) < \infty\}$$

and

$$\mathrm{H}_\rho^o(\mathbb{E}) := \{f \in \mathrm{C}(\mathbb{E});\ \lim_{\delta\to 0}\omega_\rho(f,\delta) = 0\},$$

both equiped with the norm

$$\|f\|_\rho := \|f(0)\| + \omega_\rho(f,1).$$

To discard triviality, we may assume that $\rho(h) \geq ch$ for some positive constant c. Then $\mathrm{H}_\rho^o(\mathbb{E})$ contains all the $\mathbb{E}$ valued polygonal lines indexed by $[0,1]$ and inherits the separability of $\mathbb{E}$

(see [15]). When $\rho(h) = h^\alpha$, $0 < \alpha < 1$, the corresponding Hölder spaces H_ρ and H_ρ^o will be denoted simply by H_α and H_α^o. As in [16], we shall restrict our study of the Hölderian FCLT to the case of weight functions ρ in the class $\mathcal{R}$ defined below.

Definition 1. *Let $\mathcal{R}$ be the class of non decreasing functions $\rho : [0,1] \to \mathbb{R}$, positive on $(0,1]$, such that $\rho(0) = 0$ and satisfying*

i) for some $0 < \alpha \leq 1/2$, and some positive function L which is normalized slowly varying at infinity,

$$\rho(h) = h^\alpha L(1/h), \quad 0 < h \leq 1; \tag{4.19}$$

ii) $\theta(t) = t^{1/2}\rho(1/t)$ is C^1 on $[1,\infty)$;

iii) there is a $\beta > 1/2$ and some $a > 1$, such that $\theta(t)\ln^{-\beta}(t)$ is non decreasing on $[a,\infty)$.

We say that a function is *ultimately* decreasing or increasing or non decreasing or non increasing if the corresponding monotonicity holds on some interval $[c,\infty)$. Let us recall that $L(t)$ is a positive continuous normalized slowly varying at infinity if and only if it belongs to the Zygmund class *i.e.* for every $\delta > 0$, $t^\delta L(t)$ is ultimately increasing and $t^{-\delta}L(t)$ is ultimately decreasing (Bojanic and Karamata [3, Th.1.5.5]). It follows that for some $0 < \tau \leq 1$, $h^\alpha L(1/h)$ is non decreasing on $[0,\tau]$. Here we assume for convenience that it is non decreasing *on the whole interval* $(0,1]$. This is not a real restriction since the Hölder norms generated by $\rho(h)$ and $\rho(\tau h)$ are easily seen to be equivalent.

Remark 1. Clearly $L(t)\ln^{-\beta}(t)$ is normalized slowly varying for any $\beta > 0$, so when $\alpha < 1/2$, $t^{1/2-\alpha}L(t)\ln^{-\beta}(t)$ is ultimately non decreasing and iii) is automatically satisfied.

The assumption ii) of C^1 regularity for θ is not a real restriction, since the function $\rho(1/t)$ being α-regularly varying at infinity is asymptoticaly equivalent to a C^∞ α-regularly varying function $\tilde{\rho}(1/t)$ (see [3]). Then the corresponding Hölderian norms are equivalent.

The following proposition is proved in [15].

Proposition 3. *For any ρ in $\mathcal{R}$, the space $\mathrm{H}_\rho^o(\mathbb{E})$ supports any $\mathbb{E}$-valued Brownian motion W_Q.*

In what follows, the weight function ρ belongs to $\mathcal{R}$ and we recall that

$$\theta(t) = t^{1/2}\rho(1/t), \quad t \geq 1.$$

Next we consider partial sum polygonal line processes $\xi_n^{(X)}, n \geq 1$.

Theorem 6. *Assume that innovations $(\epsilon_k, k \in \mathbb{Z})$ are i.i.d. and that $\epsilon_1 \in \mathrm{CLT}(\mathbb{E})$. Assume moreover that for every positive δ,*

$$\lim_{t\to\infty} tP\big(\|\epsilon_1\| > \delta\theta(t)\big) = 0. \tag{4.20}$$

If $(a_i, i \in \mathbb{Z}) \in \ell^1(\mathbb{E},\mathbb{F})$ then

$$n^{-1/2}\xi_n^{(X)}(t) \xrightarrow[n\to\infty]{\mathrm{H}_\rho^o(\mathbb{F})} W_{AQ(\epsilon)A^*}. \tag{4.21}$$

Proof. As proved in [16], the condition $\epsilon_1 \in \mathrm{CLT}(\mathbb{E})$ together with (4.20) gives

$$n^{-1/2}\xi_n^{(\epsilon)} \xrightarrow[n\to\infty]{\mathrm{H}_\rho^o(\mathbb{E})} W_{Q(\epsilon)}. \tag{4.22}$$

This enables us to obtain the convergence (4.21) by checking that the relevant assumptions of Theorem 4 are satisfied by $\mathcal{F}(\mathbb{E}) = \mathrm{H}^o_\rho(\mathbb{E})$. Condition (A0) is satisfied since $\rho(h) \geq ch$ for some positive constant c. Conditions (A1) and (A2) are obviously satisfied with $c_1 = c_2 = 1$. Next we observe that for $f \in \mathrm{H}^o_\rho(\mathbb{E})$,

$$\|f(t') - f(s')\|_\mathbb{E} \leq \rho(t' - s')\omega_\rho(f, t - s), \quad 0 \leq s \leq s' \leq t' \leq t \leq 1. \tag{4.23}$$

In particular, choosing $s = s' = 0$ and $t' = t$ and recalling that ρ is non decreasing on $[0, 1]$ gives

$$\|f(t)\|_\mathbb{E} \leq \|f(0)\|_\mathbb{E} + \rho(1)\omega_\rho(f, 1) \leq \max(1, \rho(1))\|f\|_\rho, \quad 0 \leq t \leq 1. \tag{4.24}$$

Now using (4.23) and (4.24), it is easy to see that (A3) and (A4) are satisfied with $c_3 = c_4 = 2\max(1, \rho(1))$. Condition (4.6) is satisfied since

$$\|xe_{n,i}\|_\rho = \frac{\|x\|_\mathbb{E}}{\rho(1/n)} = \|x\|_\mathbb{E}\, o\big(n^{1/2}\ln^{-1/2} n\big),$$

in view of condition iii) in Definition 1. To check (4.7), Proposition 3 allows us to take $\mathcal{V} = \mathrm{H}^o_\rho(\mathbb{E})$. To see that translations operate continuously on $\mathrm{H}^o_\rho(\mathbb{E})$, first we deduce from (4.23) that

$$\|\varDelta_h f(0)\|_\mathbb{E} \leq \rho(|h|)\omega_\rho(f, 1). \tag{4.25}$$

Next to control $\omega_\rho(\varDelta_h f, 1)$ we use (4.23) to bound differently the increment $\|\varDelta_h f(t) - \varDelta_h f(s)\|_\mathbb{E}$ according to the comparison of $t - s$ with $|h|$. If $|h| \leq t - s$,

$$\begin{aligned}\|\varDelta_h f(t) - \varDelta_h f(s)\|_\mathbb{E} &\leq \|T_h f(t) - f(t)\|_\mathbb{E} + \|T_h f(s) - f(s)\|_\mathbb{E}\\ &\leq 2\rho(h)\omega_\rho(f, |h|),\end{aligned}$$

whence by monotonicity of ρ,

$$\frac{\|\varDelta_h f(t) - \varDelta_h f(s)\|_\mathbb{E}}{\rho(t - s)} \leq 2\omega_\rho(f, |h|), \quad |h| \leq t - s. \tag{4.26}$$

If $0 \leq t - s < |h|$,

$$\begin{aligned}\|\varDelta_h f(t) - \varDelta_h f(s)\|_\mathbb{E} &\leq \|T_h f(t) - T_h f(s)\|_\mathbb{E} + \|f(t) - f(s)\|_\mathbb{E}\\ &\leq 2\rho(t - s)\omega_\rho(f, |h|),\end{aligned}$$

whence

$$\frac{\|\varDelta_h f(t) - \varDelta_h f(s)\|_\mathbb{E}}{\rho(t - s)} \leq 2\omega_\rho(f, |h|), \quad 0 < t - s < |h|. \tag{4.27}$$

Gathering (4.25), (4.26) and (4.27) gives

$$\|\varDelta_h f\|_\rho \leq \omega_\rho(f, 1)\rho(h) + 2\omega_\rho(f, |h|).$$

As f belongs to $\mathrm{H}^o_\rho(\mathbb{E})$ this upper bound goes to zero with h and this achieves the verification of (4.7). It is worth noticing that the same argument would fail with f in $\mathrm{H}_\rho(\mathbb{E})$ but not in $\mathrm{H}^o_\rho(\mathbb{E})$.

To complete the proof it remains to check (A5). To this end, let g be any function in $\mathrm{H}^o_\rho(\mathbb{E})$, fix an arbitrary pair $0 \leq u \leq v \leq 1$ and define $f : [0, 1] \to \mathbb{E}$ by $f(t) = g(u + (v - u)t)$. Then we have $f(0) = g(u) = R_u^v g(0)$ and the problem is reduced to bounding $\omega_\rho(f, 1)$ by $\omega_\rho(R_u^v g, 1)$, where $R_u^v g$ is the pseudo-restriction of g to $[u, v]$. We note here that since $R_u^v g$ is constant on each interval $[0, u]$ and $[v, 1]$ and ρ is non decreasing,

$$\omega_\rho(R_u^v g, 1) = \sup_{u \le x < y \le v} \frac{\|g(y) - g(x)\|_{\mathbb{E}}}{\rho(y - x)}.$$

Now we have for $0 \le s < t \le 1$,

$$\begin{aligned}\|f(t) - f(s)\|_{\mathbb{E}} &= \|g(u + (v - u)t) - g(u + (v - u)s)\|_{\mathbb{E}} \\ &\le \rho\big((v - u)(t - s)\big) \sup_{u \le x < y \le v} \frac{\|g(y) - g(x)\|_{\mathbb{E}}}{\rho(y - x)} \\ &\le \rho(t - s)\omega_\rho(R_u^v g, 1),\end{aligned}$$

which gives $\omega_\rho(f, 1) \le \omega_\rho(R_u^v g, 1)$. Recalling the value of $f(0)$ we conclude that (A5) is satisfied with $c_5 = 1$.

When $\mathbb{E} = \mathbb{H}$ is a separable Hilbert space, $\mathbb{F}$ being still any separable Banach space, we obtain the following simple corollary, extending the main result of [14] which was proved by another method in the special case $\mathbb{E} = \mathbb{F} = \mathbb{H}$.

Corollary 1. Assume that $\mathbb{E} = \mathbb{H}$ is a separable Hilbert space and that the innovations $(\epsilon_k, k \in \mathbb{Z})$ are i.i.d. and satisfy (4.20). If $(a_i, i \in \mathbb{Z}) \in \ell^1(\mathbb{E}, \mathbb{F})$ then the polygonal partial sum process $\xi_n^{(X)}(t)$ converges weakly in $\mathrm{H}_\rho^o(\mathbb{F})$ to the Brownian motion $W_{AQ(\epsilon)A^*}$.

Proof. Applying Theorem 6, we just have to check that $\epsilon_1 \in \mathrm{CLT}(\mathbb{E})$. Due to the Hilbertian structure of $\mathbb{E}$, this follows from the square integrability of ϵ_1, which in turm follows from (4.20).

Acknowledgment

Authors would like to thank the referee for precise suggestions that led to an improvement of the paper.

REFERENCES

1. A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York, 1980.

2. P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.

3. N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular variation*, Encyclopaedia of Mathematics and its Applications, Cambridge University Press, 1987.

4. D. Bosq, *Linear processes in function spaces*, Volume 149 of *Lecture Notes in Statistics*, Springer, 2000.

5. J. Dedecker and F. Merlevède, The conditional central limit theorem in Hilbert spaces, *Stochastic Process. Appl.*, **108**:229–262, 2003.

6. N.A. Denis'evskiĭ, A central limit theorem for a linear process in a Banach space (Russian), *Teor. Veroyatnost. i Mat. Statist.*, **40**:17–23, 1989. English translation in Theory Probab. Math. Statist. 40:19–25, 1990

7. M. Juodis, A. Račkauskas, and Ch. Suquet, Hölderian functional central limit theorems for linear processes, *ALEA Lat. Am. J. Probab. Math. Stat.*, **5**:1–18, 2009.

8. J. Kuelbs, The invariance principle for Banach space valued random variables, *J. Multivariate Analysis*, **3**:161–172, 1973.

9. M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Springer-Verlag, Berlin, Heidelberg, 1991.

10. F. Merlevède, M. Peligrad, and S. Utev, Sharp conditions for the CLT of linear processes in a Hilbert space, *J. Theoret. Probab.*, **10**(3):681–693, 1997.

11. F. Merlevède, M. Peligrad, and S. Utev, Recent advances in invariance principles for stationary sequences, *Probability Surveys*, **3**:1–36, 2006.

12. M. Peligrad and S. Utev, Invariance principle for stochastic processes with short memory, in *High Dimensional probability*, Volume 51 of *IMS Lecture Notes and Monograph Series*, pp. 18–32, IMS, 2006.

13. P.C.B. Philips and V. Solo, Asymptotics for linear processes, *Ann. Statist.*, **20**:971–1001, 1992.

14. A. Račkauskas and Ch. Suquet, Hölderian invariance principle for Hilbertian linear processes, *ESAIM Probab. Statist.*, **13**:261–275, 2008.

15. A. Račkauskas and Ch. Suquet, Hölder versions of Banach spaces valued random fields, *Georgian Mathematical Journal*, **8**(2):347–362, 2001.

16. A. Račkauskas and Ch. Suquet, Necessary and sufficient condition for the Hölderian functional central limit theorem, *Journal of Theoretical Probability*, **17**(1):221–243, 2004.

17. A. Račkauskas and Ch. Suquet, Hölder norm test statistics for epidemic change, *Journal of Statistical Planning and Inference*, **126**(2):495–520, 2004.

18. A. Račkauskas and Ch. Suquet, Testing epidemic changes of infinite dimensional parameters, *Statistical Inference for Stochastic Processes*, **9**:111–134, 2006.

19. L. Schwartz, *Topologie générale et analyse fonctionnelle*, Hermann, Paris, 1970.