Testing epidemic changes of infinite dimensional parameters

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Abstract

To detect epidemic change in the mean of a sample of size n of random elements in a Banach space, we introduce new test statistics DI based on weighted increments of partial sums. We obtain their limit distributions under the null hypothesis of no change in the mean. Under alternative hypothesis our statistics can detect very short epidemics of length $\log^\gamma n$, $\gamma > 1$. We present applications to detect epidemic changes in distribution function or characteristic function of real valued observations as well as changes in covariance matrixes of random vectors.

Some keywords: change point, epidemic alternative, functional central limit theorem, Hölder norm, partial sums processes

1 Introduction

A central question in the area of change point detection is testing for changes in the mean of a sample. Indeed many change point problems may be reduced to this basic setting, see e.g. Brodsky and Darkhovsky [1]. Here we present a new illustration of this general approach. Starting from the detection of epidemic changes in the mean of Banach space valued random elements, we construct new tests to detect changes in the distribution function or the characteristic function of real valued observations as well as changes in covariance matrixes of random vectors.

Let $\mathbb{B}$ be a separable Banach space with a norm $\|x\|$ and dual space $\mathbb{B}'$ with duality denoted by $f(x)$, $f \in \mathbb{B}'$, $x \in \mathbb{B}$. Suppose that $X_1, \dots, X_n$ are random elements in $\mathbb{B}$ with means $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ respectively. We want to test the standard null hypothesis of a constant mean

(H_0): $\mathfrak{m}_1 = \cdots = \mathfrak{m}_n$

against the so called epidemic alternative

(H_A): *there are integers* $1 < k^* < m^* < n$ *such that*
$\mathfrak{m}_1 = \mathfrak{m}_2 = \cdots = \mathfrak{m}_{k^*} = \mathfrak{m}_{m^*+1} = \cdots = \mathfrak{m}_n$,
$\mathfrak{m}_{k^*+1} = \cdots = \mathfrak{m}_{m^*}$ *and* $\mathfrak{m}_{k^*} \neq \mathfrak{m}_{k^*+1}$.

The study of epidemic change models (with $\mathbb{B} = \mathbb{R}$) goes back to Levin and Kline [8] who proposed test statistics based on partial sums. Using a maximum likelihood approach, Yao [14] suggested some weighted versions of these statistics which may be viewed as some discrete Hölder norm of the partial sums process, see also Csörgő and Horváth [3] and the references therein.

In [12], we study a large class of statistics obtained by discretizing Hölder norms of the partial sums process (with $\mathbb{B} = \mathbb{R}$). One important feature of Hölderian weighting is the detection of short epidemics. Roughly speaking, the use of Hölderian tests allows the detection of epidemics whose length $l^* := m^* - k^*$ is at least of the order of $\ln^\gamma n$ with $\gamma > 1$, while the same test statistics without Hölderian weight detects only epidemics such that $n^{-1/2} l^*$ goes to infinity. Among the test statistics suggested in [12], the statistics $\mathrm{DI}(n, \rho)$ built on the dyadic increments of partial sums are of special interest because their limiting distribution is explicitly computable. The aim of this new contribution is to investigate asymptotical behavior of $\mathrm{DI}(n, \rho)$ in the setting of $\mathbb{B}$ valued random elements X_i's.

The paper is organized as follows. Section 2 introduce the dyadic increment test statistics and present general results on their asymptotical behavior under (H_0) or under (H_A). Several applications are proposed in Section 3, involving epidemic changes in distribution function (through Kolmogorov-Smirnov or Cramér-von Mises versions of $\mathrm{DI}(n, \rho)$), in characteristic function, in the covariance of a random vector. Section 4 gathers the relevant background on Hölderian functional central limit theorem and the proofs.

2 The class of DI test statistics

Let us denote by D_j the set of dyadic numbers in $[0, 1]$ of level j, i.e.

$$\mathrm{D}_0 = \{0, 1\}, \qquad \mathrm{D}_j = \left\{(2l - 1)2^{-j};\ 1 \le l \le 2^{j-1}\right\}, \quad j \ge 1.$$

Write for $r \in \mathrm{D}_j$, $j \ge 0$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

For a function $x : [0, 1] \to \mathbb{B}$, we shall denote

$$\lambda_r(x) := x(r) - \frac{1}{2}(x(r^+) + x(r^-)), \quad r \in \mathrm{D}_j,\ j \ge 1 \tag{1}$$

and $\lambda_r(x) = x(r)$ in the special case $r = 0, 1$. Consider partial sums

$$S(0) = 0, \quad S(u) = \sum_{k \le u} X_k, \quad 0 < u < \infty.$$

We define also

$$S_n(a, b) := S(nb) - S(na) = \sum_{na < k \le nb} X_k, \quad 0 \le a < b \le 1$$

and

$$S_n(t) = S_n(0, t) = \sum_{k \le nt} X_k, \quad 0 \le t \le 1.$$

Dyadic increments statistics $\mathrm{DI}(n, \rho)$ depend on a weight function $\rho : [0, 1] \to \mathbb{R}$ and are defined by

$$\begin{aligned} \mathrm{DI}(n, \rho) &:= \max_{1 \le j \le \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \|\lambda_r(S_n)\| \\ &= \frac{1}{2} \max_{1 \le j \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \Big\| \sum_{nr^- < k \le nr} X_k - \sum_{nr < k \le nr^+} X_k \Big\|. \end{aligned} \tag{2}$$

In this paper, we write "log" for the logarithm with basis 2 ($\log(2^j) = j$) and "ln" for the natural logarithm ($\ln(\mathrm{e}^t) = t$). Throughout we assume that ρ belongs to the following class $\mathcal{R}$.

Definition 1. Let $\mathcal{R}$ be the class of non decreasing functions $\rho : [0, 1] \to \mathbb{R}$, positive on $(0, 1]$, such that $\rho(0) = 0$ and satisfying

i) for some $0 < \alpha \le 1/2$, and some function L which is normalized slowly varying at infinity,

$$\rho(h) = h^\alpha L(1/h), \quad 0 < h \le 1; \tag{3}$$

ii) $\theta(t) := t^{1/2}\rho(1/t)$ is C^1 on $[1, \infty)$;

iii) there is a $\beta > 1/2$ and some $a > 0$, such that $\theta(t) \ln^{-\beta}(t)$ is non decreasing on $[a, \infty)$.

For a random element X in a separable Banach space $\mathbb{B}$ such that for every $f \in \mathbb{B}'$, $\mathbf{E}\, f(X) = 0$ and $\mathbf{E}\, f^2(X) < \infty$, its covariance operator $Q = Q(X)$ is the linear bounded operator from $\mathbb{B}'$ to $\mathbb{B}$ defined by $Qf = \mathbf{E}\,\big(f(X)X\big)$, $f \in \mathbb{B}'$. A random element $X \in \mathbb{B}$ (or covariance operator Q) is said to be *pregaussian* if there exists a mean zero Gaussian random element $Y \in \mathbb{B}$ with the same covariance operator as X, i.e. for all $f, g \in \mathbb{B}'$, $\mathbf{E}\, f(X)g(X) = \mathbf{E}\, f(Y)g(Y)$. Since the distribution of a centered Gaussian random element is defined by its covariance structure, we denote by Y_Q a zero mean Gaussian random element with covariance operator Q.

For any pregaussian covariance Q there exists a $\mathbb{B}$-valued Brownian motion W_Q with parameter Q, i.e. a centered Gaussian process indexed by $[0,1]$ with independent increments such that $W_Q(t) - W_Q(s)$ has the same distribution as $|t-s|^{1/2} Y_Q$.

A random element X is said to satisfy the central limit theorem in $\mathbb{B}$ (denoted $X \in \mathrm{CLT}(\mathbb{B})$) if the sequence $n^{-1/2}(X_1 + \cdots + X_n)$ converges in distribution in $\mathbb{B}$, where $X_1, \ldots, X_n$ are independent copies of X. Necessarily then (see [7]), X is mean zero, pregaussian and satisfies the moment condition

$$\lim_{t\to\infty} t\,\mathbf{P}\big(\|X_1\| > t^{1/2}\big) = 0. \tag{4}$$

However the central limit theorem for X cannot be characterized in general in terms of the only integrability of X because the geometry of $\mathbb{B}$ is involved in the problem.

To complete these preliminaries, we recall here the following classical estimate (see [7], (3.5) p.59) for the tail of a mean zero Gaussian random element Y in $\mathbb{B}$.

$$\mathbf{P}(\|Y\| \geq u) \leq 4 \exp\left(\frac{-u^2}{8\mathbf{E}\,\|Y\|^2}\right), \quad u > 0. \tag{5}$$

In order to investigate the asymptotic behavior of the statistics $\mathrm{DI}(n,\rho)$ we consider a null hypothesis a little bit stronger then H_0. Namely,

(H_0'): *$X_1, \ldots, X_n$ are i.i.d. mean zero pregaussian random elements in $\mathbb{B}$ with covariance Q.*

Let $(Y_Q, Y_{Q,r}, r \in \mathrm{D})$ be a collection of non degenerate independent mean zero Gaussian random elements with covariance Q. Set with θ as in Definition 1 ii)

$$\mathrm{DI}(\rho, Q) := \sup_{j\geq 1} \frac{1}{\sqrt{2}\theta(2^j)} \max_{t\in \mathrm{D}_j} \|Y_{Q,r}\|. \tag{6}$$

Due to the definition of the class $\mathcal{R}$, it can be shown (see Theorem 13 below) that the random variable $\mathrm{DI}(\rho, Q)$ is well defined for any $\rho \in \mathcal{R}$.

Theorem 2. *If under (H_0'), $X_1 \in \mathrm{CLT}(\mathbb{B})$ and for every $A > 0$*

$$\lim_{t\to\infty} t\,\mathbf{P}\big(\|X_1\| > A\theta(t)\big) = 0, \tag{7}$$

then

$$n^{-1/2}\mathrm{DI}(n,\rho) \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{DI}(\rho, Q). \tag{8}$$

As $\theta(t) = t^{1/2}\rho(1/t)$, Condition (7) is clearly stronger than (4) which is implicitly included in the assumption "$X_1 \in \mathrm{CLT}(\mathbb{B})$". If $\rho(h) = \rho_\alpha(h) = h^\alpha, h \in [0,1]$, where $0 < \alpha < 1/2$, then Condition (7) reads

$$\mathbf{P}\big(\|X_1\| \geq t\big) = o\big(t^{-p(\alpha)}\big) \quad \text{with} \quad p(\alpha) = \Big(\frac{1}{2} - \alpha\Big)^{-1}.$$

In the case where $\rho(h) = \rho_{\alpha,\beta}(h) = h^{1/2}\ln^{\beta}(c/h)$, $h \in [0,1]$ with $\beta > 1/2$, Condition (7) is equivalent to

$$\mathbf{E}\exp\{d\|X_1\|^{1/\beta}\} < \infty, \quad \text{for all } \ d > 0.$$

For a discussion of the condition "$X_1 \in \mathrm{CLT}(\mathbb{B})$", we refer to [7]. Let us mention simply here that if the space $\mathbb{B}$ is either of type 2 or has Rosenthal's property (e.g. $\mathbb{B}$ is any separable Hilbert space, $\mathbb{B} = L_p(S,\mu)$ with $2 \le p < \infty$ etc.) then Condition (7) yields $X_1 \in \mathrm{CLT}(\mathbb{B})$. If $\mathbb{B}$ is of cotype 2 (e.g. $\mathbb{B} = L_p(S,\mu)$, $1 \le p \le 2$), then "$X_1 \in \mathrm{CLT}(\mathbb{B})$" follows from X_1 being pregaussian.

Due to the independence of $Y_{Q,r}$ the limiting distribution function $F_{Q,\rho}(u)$ of the dyadic increments statistic is completely specified by the distribution function

$$\Phi_Q(u) := \mathbf{P}\big(\|Y_{Q,1}\| \le u\sqrt{2}\big), \quad u \ge 0.$$

Namely, we have

Proposition 3. *If $\rho \in \mathcal{R}$, then the distribution of $\mathrm{DI}(\rho, Q)$ is absolutely continuous and its distribution function $F_{Q,\rho}$ is given by*

$$F_{Q,\rho}(u) := \mathbf{P}\big(\mathrm{DI}(\rho,Q) \le u\big) = \prod_{j=1}^{\infty}\big[\Phi_Q\big(\theta(2^j)u\big)\big]^{2^{j-1}}, \quad u \ge 0. \tag{9}$$

The convergence in (9) is uniform on any interval $[\varepsilon, \infty)$, $\varepsilon > 0$.

For practical applications we sumarized in the next proposition some estimates of the tail of distribution function $F_{Q,\rho}$. Denote for $J \ge 0$

$$F_{Q,\rho}^{(J)}(u) = \prod_{j=1}^{J}\big[\Phi_Q\big(\theta(2^j)u\big)\big]^{2^{j-1}}, \quad u \ge 0$$

and

$$c_J := \inf_{\gamma > 0}\Big\{\gamma + 8\sum_{j=J+1}^{\infty}\frac{2^j}{\theta^2(2^j)}\exp\Big(\frac{-\gamma\theta^2(2^j)}{4}\Big)\Big\}. \tag{10}$$

Proposition 4. *For each $\rho \in \mathcal{R}$, $F_{Q,\rho}$ satisfies the following estimates.*

i) For each $u > 0$

$$1 - F_{Q,\rho}(u) \le 4\exp\left(\frac{-u^2}{8c_0\mathbf{E}\,||Y_Q||^2}\right). \tag{11}$$

ii) For each $J \ge 1$ and $u > 0$

$$\left[1 - 4\exp\left(\frac{-u^2}{8c_J\mathbf{E}\,||Y_Q||^2}\right)\right]F_{Q,\rho}^{(J)}(u) \le F_{Q,\rho}(u) \le F_{Q,\rho}^{(J)}(u). \tag{12}$$

We end this section by examining a consistency of rejecting (H_0') versus the epidemic alternative (H_A') for large values of $\mathrm{DI}(n,\rho)$, where

$$(H_A') \qquad X_k = \begin{cases} \mathfrak{m}_c + X_k' & \text{if } k \in \mathbb{I}_n := \{k^*+1,\dots,m^*\} \\ X_k' & \text{if } k \in \mathbb{I}_n^c := \{1,\dots,n\} \setminus \mathbb{I}_n \end{cases}$$

where $\mathfrak{m}_c \neq 0$ may depend on n and the X_k''s satisfy (H_0').

Theorem 5. *Let $\rho \in \mathcal{R}$. Under (H_A'), write $l^* := m^* - k^*$ for the length of epidemics and assume that*

$$\lim_{n\to\infty} n^{1/2} \frac{h_n \|\mathfrak{m}_c\|}{\rho(h_n)} = \infty, \quad \textit{where} \quad h_n := \min\Big\{\frac{l^*}{n}; 1 - \frac{l^*}{n}\Big\}. \tag{13}$$

Then

$$n^{-1/2}\mathrm{DI}(n,\rho) \xrightarrow[n\to\infty]{\mathrm{pr}} \infty.$$

To discuss Condition (13), consider for simplicity the case where $\mathfrak{m}_c$ does not depend on n. When $\rho(h) = h^\alpha$, (13) allows us to detect *short epidemics* such that $l^* = o(n)$ and $l^* n^{-\delta} \to \infty$, where $\delta = (1-2\alpha)(2-2\alpha)^{-1}$. Symmetrically one can detect *long epidemics* such that $n - l^* = o(n)$ and $(n-l^*)n^{-\delta} \to \infty$.

When $\rho(h) = h^{1/2}\ln^\beta(c/h)$ with $\beta > 1/2$, (13) is satisfied provided that $h_n = n^{-1}\ln^\gamma n$, with $\gamma > 2\beta$. This leads to detection of short epidemics such that $l^* = o(n)$ and $l^* \ln^{-\gamma} n \to \infty$ as well as of long ones verifying $n - l^* = o(n)$ and $(n-l^*)\ln^{-\gamma} n \to \infty$.

3 Examples

3.1 Testing change of distribution function

As an example of applications of Theorem 2, here we consider change-point problem for distribution function of a random sample in $\mathbb{R}$ under epidemic alternative. Let $Z_1,\dots,Z_n$ be real valued random variables with distribution functions $F_1,\dots,F_n$ respectively. Consider the null hypothesis

$$(H_0): \quad F_1 = \dots = F_n = F$$

and the following epidemic alternative:

(H_A): *there are integers $1 < k^* < m^* < n$ such that*
$F_1 = F_2 = \dots = F_{k^*} = F_{m^*+1} = \dots = F_n$,
$F_{k^*+1} = \dots = F_{m^*}$ *and* $F_{k^*} \neq F_{k^*+1}$.

Tests constructed in Lombard [9] and Gombay [5] are based on rank statistics. Theorem 2 suggests tests based on the dyadic increments of empirical process. For simplicity we consider only the case of continuous function F. Then we can restrict to the uniform empirical process built on the sample $(U_1,\dots,U_n)$, where $U_k = F(Z_k)$.

Define

$$\kappa_n(s,t) := \sum_{i \le ns} \big(\mathbf{1}_{\{U_i \le t\}} - t\big), \quad s,t \in [0,1].$$

For $r \in \mathrm{D}_j$, $j \ge 1$, define random functions $\lambda_r(\kappa_n)$ by

$$\lambda_r(\kappa_n)(t) := \kappa_n(r,t) - \frac{1}{2}\big(\kappa_n(r^-,t) + \kappa_n(r^+,t)\big), \quad t \in [0,1]$$

which for practical computations may be conveniently recast as

$$2\lambda_r(\kappa_n)(t) = \sum_{nr^- < i \le nr} \big(\mathbf{1}_{\{U_i \le t\}} - t\big) - \sum_{nr < i \le nr^+} \big(\mathbf{1}_{\{U_i \le t\}} - t\big), \quad t \in [0,1]. \tag{14}$$

3.1.1 Cramér-von Mises type DI statistics

The Cramér-von Mises type dyadic increments statistics are defined by

$$\mathrm{CMDI}(n,\rho) = \max_{1 \le j \le \log n} \frac{1}{\rho^2(2^{-j})} \max_{r \in \mathrm{D}_j} \big\|\lambda_r(\kappa_n)\big\|_2^2,$$

where

$$\big\|\lambda_r(\kappa_n)\big\|_2^2 := \int_0^1 \big|\lambda_r(\kappa_n)(t)\big|^2 \, \mathrm{d}t.$$

Its limiting distribution is completely defined by the limiting distribution of the classical Cramér-von Mises statistic, namely, by the distribution function

$$L_2(u) = \mathbf{P}\left\{ \int_0^1 B^2(t)\, \mathrm{d}t \le u \right\}, \quad u \ge 0,$$

where $B(t), t \in [0,1]$ is a standard Brownian bridge. The distribution function $L_2(u)$ is well known and several of its representations are available (see [13]).

Theorem 6. *If $\rho \in \mathcal{R}$ then*

$$\lim_{n\to\infty} \mathbf{P}\big\{n^{-1}\mathrm{CMDI}(n,\rho) \le u\big\} = L_{2,\rho}(u),$$

for each $u > 0$, where

$$L_{2,\rho}(u) = \prod_{j=1}^{\infty} \Big[L_2\big(2\theta^2(2^j)u\big)\Big]^{2^{j-1}}.$$

Proof. The result is a straightforward application of Theorem 2 to the random elements $X_1, \dots, X_n$ with values in the space $L_2(0,1)$ defined by

$$X_k(t) = \mathbf{1}_{\{U_k \le t\}} - t, \quad t \in [0,1], \quad k = 1, \dots, n. \tag{15}$$

It is well known that these random elements satisfy the central limit theorem with Brownian bridge as the limiting Gaussian element. The moment condition (7) is fulfiled since the X_k's are bounded. □

For practical uses, the following estimate of the tails of $L_{2,\rho}$ can be helpfull. Define for $u > 0$

$$L_{2,\rho}^{(J)}(u) = \prod_{j=1}^{J} \Big[L_2\big(2\theta^2(2^j)u\big)\Big]^{2^{j-1}}.$$

Proposition 7.

i) If $u > 0$ then

$$1 - L_{2,\rho}(u) \leq 4 \exp\Big(\frac{-3u}{4c_0}\Big).$$

where the constant c_0 is defined by (10).

ii) For each $u > 0$ and $J \geq 1$

$$\Big[1 - 4\exp\Big(\frac{-3u}{4c_J}\Big)\Big] L_{2,\rho}^{(J)}(u) \leq L_{2,\rho}(u) \leq L_{2,\rho}^{(J)}(u).$$

Proof. The result is a straightforward adaptation of Proposition 4, with $Y_Q = B$, a standard Brownian bridge, using the elementary fact that $\mathbf{E}\, \|B\|_2^2 = 1/6$. $\square$

3.1.2 Kolmogorov-Smirnov type DI statistics

The Kolmogorov-Smirnov type dyadic increments statistics are defined by

$$\mathrm{KSDI}(n, \rho) = \max_{1 \leq j \leq \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \big\| \lambda_r(\kappa_n) \big\|_\infty,$$

where

$$\big\| \lambda_r(\kappa_n) \big\|_\infty := \sup_{0 \leq t \leq 1} \big| \lambda_r(\kappa_n)(t) \big|.$$

Let $L_\infty(u)$ be the limiting distribution of the classical Kolmogorov-Smirnov statistic,

$$L_\infty(u) = \mathbf{P}\left\{ \max_{0 \leq t \leq 1} |B(t)| \leq u \right\}, \quad u \geq 0.$$

This distribution function is well known and has several representations (see [13]).

Theorem 8. *If $\rho \in \mathcal{R}$ then*

$$\lim_{n \to \infty} \mathbf{P}\big\{ n^{-1/2} \mathrm{KSDI}(n, \rho) \leq u \big\} = L_{\infty,\rho}(u),$$

for each $u > 0$, where

$$L_{\infty,\rho}(u) = \prod_{j=1}^{\infty} \Big[L_\infty\big(\sqrt{2}\theta(2^j)u\big)\Big]^{2^{j-1}}.$$

The proof of this theorem is not a straightforward corollary of Theorem 2 and is postponed to Subsection 4.5 below.

3.2 Testing change of characteristic function

Let $Z_1, Z_2, \dots, Z_n$ be independent random variables with characteristic functions $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n$ respectively. We want to test the standard null hypothesis of equal characteristic functions

(H_0) : $\quad \mathfrak{c}_1 = \mathfrak{c}_2 = \cdots = \mathfrak{c}_n = \mathfrak{c}$

against the epidemic alternative

(H_A): *there are integers* $1 < k^* < m^* < n$ *such that*
$\mathfrak{c}_1 = \mathfrak{c}_2 = \cdots = \mathfrak{c}_{k^*} = \mathfrak{c}_{m^*+1} = \cdots = \mathfrak{c}_n,$
$\mathfrak{c}_{k^*+1} = \cdots = c_{m^*}$ *and* $\mathfrak{c}_{k^*} \neq \mathfrak{c}_{k^*+1}$.

Define

$$c_n(s,t) := \sum_{1 \le k \le ns} \big(\exp(itZ_k) - \mathfrak{c}(t) \big), \quad s \in [0,1], \quad t \in \mathbb{R}$$

and define for $r \in \mathrm{D}_j$, $j \ge 1$,

$$\lambda_r(c_n)(t) := c_n(r,t) - \frac{1}{2}\big(c_n(r^-,t) + c_n(r^+,t)\big), \quad t \in [0,1].$$

With any probability measure μ on $\mathbb{R}$ set

$$C^2(n,r) := \int_{-\infty}^{\infty} \big|\lambda_r(c_n)(t)\big|^2 \mu(\mathrm{d}t)$$

and define the test statistics

$$\mathrm{CDI}(n,\rho) := \max_{1 \le j \le \log n} \frac{1}{\rho^2(2^{-j})} \max_{r \in \mathrm{D}_j} C^2(n,r).$$

The limiting distribution function of this statistics depends on the distribution function

$$G_2(u) = \mathbf{P}\left\{ \int_{\mathbb{R}} |Y_{\mathfrak{c}}(t)|^2 \mu(\mathrm{d}t) \le u \right\}, \quad u \ge 0,$$

where $Y_{\mathfrak{c}}$ is a complex Gaussian process with zero mean and covariance

$$\mathbf{E}\, Y_{\mathfrak{c}}(t)\overline{Y_{\mathfrak{c}}(s)} = \mathfrak{c}(t-s) - \mathfrak{c}(t)\mathfrak{c}(-s), \quad s,t \in \mathbb{R}.$$

Theorem 9. *If* $\rho \in \mathcal{R}$ *then*

$$\lim_{n\to\infty} \mathbf{P}\big\{ n^{-1}\mathrm{CDI}(n,\rho) \le u \big\} = G_{2,\rho}(u),$$

for each $u > 0$, *where*

$$G_{2,\rho}(u) = \prod_{j=1}^{\infty} \Big[G_2\big(2\theta^2(2^j)u\big) \Big]^{2^{j-1}}.$$

Proof. Consider the random processes $X_1, \dots, X_n$ defined by

$$X_k(t) = \exp(itZ_k) - \mathfrak{c}(t), \quad t \in \mathbb{R}, \quad k = 1, \dots, n.$$

Interpreting these processes as random elements in the complex Banach space $L_2(\mathbb{R},\mu)$ we can apply Theorem 2. Since the space $L_2(\mathbb{R},\mu)$ is of type 2 and evidently $\mathbf{E}\,||X_k||^2 < \infty$, the central limit theorem is satisfied. Condition (7) is fulfiled since the X_k's are bounded. ☐

3.3 Testing change of covariance matrix

Consider random vectors $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{R}^d$ with mean zero and covariance matrices $Q_1, \dots, Q_n$ respectively. We want to test the null hypothesis

$$(H_0): \quad Q_1 = \cdots = Q_n = Q,$$

against the epidemic alternative

(H_A): *there are integers* $1 < k^* < m^* < n$ *such that*
$Q_1 = Q_2 = \cdots = Q_{k^*} = Q_{m^*+1} = \cdots = Q_n$,
$Q_{k^*+1} = \cdots = c_{m^*}$ *and* $Q_{k^*} \neq Q_{k^*+1}$.

Define

$$V_n(s) := \sum_{1 \le k \le ns} (\mathbf{e}_k^\tau \mathbf{e}_k - Q), \quad s \in [0,1],$$

where $\mathbf{e}^\tau$ means the transposition of a vector $\mathbf{e}$ and put for $r \in \mathrm{D}_j$, $j \ge 1$,

$$\lambda_r(V_n) := V_n(r) - \frac{1}{2}\big(V_n(r^-) + V_n(r^+)\big).$$

We denote by $\|\ \|$ the Euclidean norm in any $\mathbb{R}^l$, and use it here for vectors in $\mathbb{R}^d$ as well as for $d \times d$ matrices identified with vectors in $\mathbb{R}^{d^2}$. Consider the test statistic

$$\mathcal{V}(n, \rho) = \max_{1 \le j \le \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} ||\lambda_r(V_n)||.$$

Its limiting distribution is completely determined by the distribution function

$$\Psi_2(u) = \mathbf{P}\big\{||N(0,T)|| \le u\big\},$$

where $N(0,T)$ is a mean zero Gaussian random vector in $\mathbb{R}^{d^2}$ with the covariance $T = (t_{ijkl},\ i,j,k,l = 1, \dots, d)$,

$$t_{ijkl} = \mathbf{E}\,(\epsilon_i\epsilon_j\epsilon_k\epsilon_l) - \mathbf{E}\,(\epsilon_i\epsilon_j)\mathbf{E}\,(\epsilon_k\epsilon_l),$$

where the ϵ_i's are coordinates of $\mathbf{e}_1$, $\mathbf{e}_1^\tau = (\epsilon_1, \dots, \epsilon_d)$.

Theorem 10. *If* $\rho \in \mathcal{R}$ *and for each* $A > 0$

$$\lim_{t\to\infty} t\,\mathbf{P}\big\{||\mathbf{e}_1|| \ge A\theta^{1/2}(t)\big\} = 0$$

then

$$\lim_{n\to\infty} \mathbf{P}\big\{n^{-1}\mathcal{V}^2(n,\rho) \le u\big\} = \Psi_{2,\rho}(u),$$

for each $u > 0$, *where*

$$\Psi_{2,\rho}(u) = \prod_{j=1}^{\infty}\Big[\Psi_2\big(\theta(2^j)\sqrt{2u}\,\big)\Big]^{2^{j-1}}.$$

Proof. The result is a straightforward application of Theorem 2 to the random elements $X_1, \dots, X_n$ with values in the space $\mathbb{R}^{d^2}$ defined by

$$X_k = \mathbf{e}_k^\tau \mathbf{e}_k - Q, \quad k = 1, \dots, n.$$

□

3.4 Consistency

In the examples of Subsections 3.1 to 3.3, let us denote under (H_A) by G, φ and R respectively the distribution function, the characteristic function and the covariance matrix during the epidemics. All these parameters may depend on n. The following consistency results follow from an easy adaptation of the proof of Theorem 5.

Proposition 11. *With the notations of Theorem 5, let $u_n := n^{1/2} \frac{h_n}{\rho(h_n)}$. Under (H_A) in the examples of Subsections 3.1 to 3.3 and when n goes to infinity,*

i) if $u_n^2 \int_{\mathbb{R}} |G - F|^2 \,\mathrm{d}F \to \infty$, then $\mathrm{CMDI}(n, \rho) \to \infty$ in probability;

ii) if $u_n \|G - F\|_\infty \to \infty$, then $\mathrm{KSDI}(n, \rho) \to \infty$ in probability;

iii) if $u_n^2 \int_{\mathbb{R}} |\varphi - \mathfrak{c}|^2 \,\mathrm{d}\mu \to \infty$, then $\mathrm{CDI}(n, \rho) \to \infty$ in probability;

iv) if $u_n^2 \|R - Q\|_2^2 \to \infty$, then $n^{-1}\mathcal{V}^2(n, \rho) \to \infty$ in probability.

4 Proofs

The proof of Theorem 2 presented below in Subsection 4.2 is based on the invariance principle in Hölder spaces of Banach space valued functions. The relevant background is gathered in Subsection 4.1. The computation of the limiting distribution under null hypothesis is treated in Subsection 4.3. Subsection 4.4 contains the proof of consistency results and the convergence of $\mathrm{KSDI}(n, \rho)$ is established in Subsection 4.5.

4.1 Hölderian probabilistic background

We write $\mathcal{C}([0, 1], \mathbb{B})$ for the Banach space of continuous functions $x : [0, 1] \to \mathbb{B}$, endowed with the supremum norm $\|x\|_\infty := \sup\{||x(t)||;\ t \in [0, 1]\}$. We abbreviate $\mathcal{C}([0, 1], \mathbb{R})$ to $\mathcal{C}[0, 1]$. Let ρ be a real valued non decreasing function on $[0, 1]$, null and right continuous at 0. Put

$$\omega_\rho(x, \delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{|x(t) - x(s)|}{\rho(t - s)}.$$

We associate to ρ the separable Hölder space

$$\mathcal{H}_\rho^o([0, 1], \mathbb{B}) := \{x \in \mathcal{C}([0, 1]; \mathbb{B}) : \lim_{\delta\to 0} \omega_\rho(x, \delta) = 0\},$$

equipped with the norm

$$\|x\|_\rho := |x(0)| + \omega_\rho(x, 1).$$

Under some technical conditions allways fulfilled when $\rho \in \mathcal{R}$, this norm is equivalent (see e.g. [10]) to the sequence norm $\| \, \|_\rho^{\text{seq}}$

$$\|x\|_\rho \sim \|x\|_\rho^{\text{seq}} := \sup_{j \geq 0} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \|\lambda_r(x)\|, \quad x \in \mathcal{H}_\rho^o([0,1], \mathbb{B}),$$

with the λ_r's defined by (1).

Consider the classical Donsker-Prohorov polygonal process

$$\xi_n(t) = S(nt) - (nt - [nt]) X_{[nt]+1}, \quad t \in [0,1]. \tag{16}$$

The following result is proved in [11].

Theorem 12. *Let $\rho \in \mathcal{R}$ and let the hypothesis (H_0') be satisfied. Then the convergence*

$$n^{-1/2} \xi_n \xrightarrow{\mathcal{D}} W_Q$$

holds in the space $\mathcal{H}_\rho^o([0,1], \mathbb{B})$ if and only if $X_1 \in \mathrm{CLT}(\mathbb{B})$ and for every $A > 0$

$$\lim_{t \to \infty} t \mathbf{P}\big(||X_1|| > A\theta(t)\big) = 0.$$

We also shall exploit a particular representation of the $\mathbb{B}$-valued Wiener process W_Q, which generalizes to the framework of $\mathcal{H}_\rho^o([0,1], \mathbb{B})$ the classical Ciesielski-Kampé de Fériet representation of the Brownian motion (see [2], [6]) as a random series of triangular functions in $\mathcal{C}([0,1], \mathbb{R})$. Here we need some more notation. For $r \in \mathrm{D}_j$, $j \geq 1$, write H_r for the $L^2[0,1]$ normalized Haar function:

$$H_r(t) := \begin{cases} +(r^+ - r^-)^{-1/2} = +2^{(j-1)/2} & \text{if } t \in (r^-, r]; \\ -(r^+ - r^-)^{-1/2} = -2^{(j-1)/2} & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

In the special case $j = 0$, put $H_0(t) := -\mathbf{1}_{[0,1]}(t)$, $H_1(t) := \mathbf{1}_{[0,1]}(t)$. Denote by D^* the set of all dyadic numbers of $(0,1]$. It is well known that $\{H_r;\, r \in \mathrm{D}^*\}$ is an Hilbertian basis of $L^2[0,1]$.

For $r \in \mathrm{D}_j$, $j \geq 1$, the $\mathcal{C}[0,1]$ normalized triangular Faber-Schauder function Λ_r is continuous, piecewise affine with support $[r^-, r^+]$ and taking the value 1 at r:

$$\Lambda_r(t) := \begin{cases} 2^j(t - r^-) & \text{if } t \in (r^-, r]; \\ 2^j(r^+ - t) & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

In the special case $j = 0$, we just take the restriction to $[0,1]$ in the above formula, so $\Lambda_0(t) := 1 - t$, $\Lambda_1(t) := t$. The Λ_r's are linked to the H_r's in the general case $r \in \mathrm{D}_j$, $j \geq 1$ by

$$\Lambda_r(t) = 2^{(j+1)/2} \int_0^t H_r(s) \, \mathrm{d}s, \quad t \in [0,1] \tag{17}$$

and in the special case $j = 0$ by $\Lambda_0(t) = 1 + \int_0^t H_0(s) \, \mathrm{d}s$, $\Lambda_1(t) = \int_0^t H_1(s) \, \mathrm{d}s$.

Theorem 13. *Let $Q : \mathbb{B}' \to \mathbb{B}$ be a pregaussian covariance and let $\{Y_{Q,r}; r \in \mathrm{D}^*\}$ be a triangular array of i.i.d. zero mean Gaussian random elements in $\mathbb{B}$ with covariance Q. Then the series of $\mathbb{B}$-valued random functions*

$$W_Q := Y_{Q,1}\Lambda_1 + \sum_{j=1}^{\infty} \sum_{r \in \mathrm{D}_j} 2^{-(j+1)/2} Y_{Q,r}\Lambda_r, \tag{18}$$

converges a.s. in the space $\mathcal{H}^o_\rho([0,1],\mathbb{B})$ for any $\rho \in \mathcal{R}$. W_Q is a $\mathbb{B}$-valued Brownian motion started at 0. Removing the term $Y_{Q,1}\Lambda_1$ in (18) gives a Brownian bridge B_Q with sequential norm:

$$\|B_Q\|^{\mathrm{seq}}_\rho = 2^{-1/2} \sup_{j \geq 1} \frac{1}{\theta(2^j)} \max_{r \in \mathrm{D}_j} \|Y_{Q,r}\|. \tag{19}$$

Proof. According to Prop. 3 c) in [10], the series (18) converges almost surely in the space $\mathcal{H}^o_\rho([0,1],\mathbb{B})$ if

$$\lim_{j \to \infty} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \|2^{-j/2} Y_{Q,r}\| = 0, \text{ almost surely.} \tag{20}$$

A sufficient condition for the convergence (20) is that for every positive ε,

$$\sum_{j \geq 1} \mathbf{P}\Big\{ \max_{r \in \mathrm{D}_j} \|Y_{Q,r}\| \geq \varepsilon\theta(2^j) \Big\} < \infty. \tag{21}$$

By (5) and identical distribution of the $Y_{Q,r}$'s, (21) will follow in turn from

$$\sum_{j \geq 1} 2^j \exp\left(\frac{-\varepsilon^2\theta^2(2^j)}{8\mathbf{E}\,\|Y_{Q,1}\|^2} \right) < \infty.$$

To check this last condition we simply remark that from Definition 1 iii), we have $\theta(2^j) \geq bj^\beta$ for some positive constant b and with $\beta > 1/2$. It follows easily that for every positive c,

$$\sum_{j=1}^{\infty} 2^j \exp\bigl(-c\theta^2(2^j)\bigr) < \infty. \tag{22}$$

Therefore (20) is satisfied and the series (18) of $\mathbb{B}$-valued random functions converges almost surely in the space $\mathcal{H}^o_\rho([0,1],\mathbb{B})$ and defines a mean zero Gaussian random element W_Q in $\mathcal{H}^o_\rho([0,1],\mathbb{B})$.

This convergence together with the obvious continuity of the λ_r's considered as linear operators $\mathcal{H}^o_\rho([0,1],\mathbb{B}) \to \mathbb{B}$ legitimates the equality

$$\lambda_r(B_Q) = \sum_{j=1}^{\infty} \sum_{r' \in \mathrm{D}_j} 2^{-(j+1)/2} \lambda_r\bigl(Y_{Q,r'}\Lambda_{r'}\bigr), \quad \text{a.s.}$$

Allowing the meaning of the notation λ_r to depend on its argument (as operator $\mathcal{H}_\rho^o([0,1],\mathbb{B})\to\mathbb{B}$ or as functional $\mathcal{H}_\rho^o([0,1],\mathbb{R})\to\mathbb{R}$) and using Lemma 1 in [10] we get

$$\lambda_r\bigl(Y_{Q,r'}\Lambda_{r'}\bigr)=Y_{Q,r'}\lambda_r\bigl(\Lambda_{r'}\bigr)=Y_{Q,r'}\mathbf{1}_{\{r=r'\}},\quad r,r'\in \mathrm{D}^*.$$

Hence for $r\in\mathrm{D}_j$ $(j\geq 1)$, $\lambda_r(B_Q)=2^{-(j+1)/2}Y_{Q,r}$ and (19) follows.

To complete the proof, it remains to check that W_Q is a Brownian motion with parameter Q. It is convenient here to recast (17) as $2^{-(j+1)/2}\Lambda_r(t)=\langle H_r,\mathbf{1}_{[0,t]}\rangle$, where $\langle\ ,\ \rangle$ denotes the scalar product in the space $L^2[0,1]$. Now (18) implies clearly for each $t\in[0,1]$ that

$$W_Q(t)=\sum_{r\in\mathrm{D}^*}Y_{Q,r}\langle H_r,\mathbf{1}_{[0,t]}\rangle,$$

where the series converges almost surely in the strong topology of $\mathbb{B}$. It follows that for any $0\leq s<t\leq 1$ and $f\in\mathbb{B}'$,

$$f\bigl(W_Q(t)-W_Q(s)\bigr)=\sum_{r\in\mathrm{D}^*}f(Y_{Q,r})\langle H_r,\mathbf{1}_{(s,t]}\rangle.$$

This almost surely convergent series of independent mean zero Gaussian random variables converges also in quadratic mean, which legitimates the following covariance computation. For $0\leq s<t\leq 1$, $0\leq s'<t'\leq 1$, $f,g\in\mathbb{B}'$, put

$$K(f,g,s,t,s',t'):=\mathbf{E}\Bigl[f\bigl(W_Q(t)-W_Q(s)\bigr)g\bigl(W_Q(t')-W_Q(s')\bigr)\Bigr].$$

Then

$$\begin{aligned}K(f,g,s,t,s',t')&=\sum_{r\in\mathrm{D}^*}\mathbf{E}\Bigl[f(Y_{Q,r})g(Y_{Q,r})\Bigr]\langle H_r,\mathbf{1}_{(s,t]}\rangle\langle H_r,\mathbf{1}_{(s',t']}\rangle\\&=\mathbf{E}\bigl[f(Y_Q)g(Y_Q)\bigr]\langle\mathbf{1}_{(s,t]},\mathbf{1}_{(s',t']}\rangle,\end{aligned}\tag{23}$$

by identical distribution of the $Y_{Q,r}$'s and Parseval's identity for the Haar basis of $L^2[0,1]$. Whenever $(s,t]\cap(s',t']=\emptyset$, (23) gives the independence of the Gaussian random variables $f\bigl(W_Q(t)-W_Q(s)\bigr)$ and $g\bigl(W_Q(t')-W_Q(s')\bigr)$ for each pair $f,g\in\mathbb{B}'$, whence follows the independence of increments for the $\mathbb{B}$-valued process W_Q. Moreover (23) gives

$$K(f,g,s,t,s,t)=\mathbf{E}\bigl[f(Y_Q)g(Y_Q)\bigr]|t-s|=\mathbf{E}\bigl[f(|t-s|^{1/2}Y_Q)g(|t-s|^{1/2}Y_Q)\bigr],$$

from which it is clear that $W_Q(t)-W_Q(s)$ and $|t-s|^{1/2}Y_Q$ have the same distribution since both are mean zero Gaussian random elements in $\mathbb{B}$. □

4.2 Proof of Theorem 2

Consider the functionals $g_n,g:\mathcal{H}_\rho^o([0,1],\mathbb{B})\to\mathbb{R}$ defined by

$$g_n(x):=\max_{1\leq j\leq\log n}\max_{r\in\mathrm{D}_j}\frac{1}{\rho(2^{-j})}\|\lambda_r(x)\|,\quad g(x):=\sup_{j\geq 1}\max_{r\in\mathrm{D}_j}\frac{1}{\rho(2^{-j})}\|\lambda_r(x)\|.$$

Recalling (2) and (16) we see that

$$n^{-1/2}\mathrm{DI}(n,\rho) = g_n(n^{-1/2}\xi_n) + R_n, \tag{24}$$

where

$$|R_n| \le \frac{2}{n^{1/2}\rho(1/n)} \max_{1\le i\le n} \|X_i\| = \frac{2}{\theta(n)} \max_{1\le i\le n} \|X_i\|.$$

Assumption (7) together with the elementary bound

$$\mathbf{P}\Big\{\frac{1}{\theta(n)} \max_{1\le i\le n} \|X_i\| \ge \varepsilon\Big\} \le n\mathbf{P}\big\{\|X_1\| \ge \varepsilon\theta(n)\big\}$$

gives immediately that

$$R_n = o_{\mathrm{pr}}(1). \tag{25}$$

It is easily seen that the set of functionals $\{g, g_n; n \ge 1\}$ is *equicontinuous* on $\mathcal{H}^o_\rho([0,1]$. Moreover g_n converges pointwise to g. Combined with the tightness of $(n^{-1/2}\xi_n)_{n\ge 1}$, this leads (see [12] for the details) to

$$g_n\big(n^{-1/2}\xi_n\big) = g\big(n^{-1/2}\xi_n\big) + o_{\mathrm{pr}}(1). \tag{26}$$

Now the conclusion (8) follows from (24), (25), (26), Theorem 12, continuous maping theorem and (19), noting that $g(W_Q) = \|B_Q\|^{\mathrm{seq}}_\rho$.

4.3 Limiting distribution under null hypothesis

Proof of Proposition 3. Since $\mathrm{DI}(\rho, Q) = \|B_Q\|^{\mathrm{seq}}_\rho$, we know from Theorem 13 that $\mathrm{DI}(\rho, Q)$ is almost surely finite for any $\rho \in \mathcal{R}$, so $\mathrm{DI}(\rho, Q)$ is a non negative random variable. As the norm of the mean zero Gaussian random element B_Q in the *separable* Banach space $\mathcal{H}^o_\rho([0,1],\mathbb{B})$, it has an absolutely continuous distribution, see Prop. 12.1 in [4].

Introduce the random variables

$$M_J := \max_{1\le j\le J} \frac{1}{\sqrt{2}\theta(2^j)} \max_{r\in \mathrm{D}_j} \|Y_{Q,r}\|.$$

By independence and identical distribution of the $Y_{Q,r}$'s,

$$\mathbf{P}(M_J \le u) = \prod_{j=1}^{J} \big[\Phi_Q\big(\theta(2^j)u\big)\big]^{2^{j-1}} = F^{(J)}_{Q,\rho}(u), \quad u \ge 0.$$

As the sequence $(M_J)_{J\ge 1}$ is non decreasing and converges to $\|B\|^{\mathrm{seq}}_\rho$, we get

$$\mathbf{P}(M_J \le u) \downarrow \mathbf{P}(\|B\|^{\mathrm{seq}}_\rho \le u) = F_{Q,\rho}(u),$$

which establishes the infinite product representation (9) for $F_{Q,\rho}$.

The uniform convergence of this infinite product on any interval $[\varepsilon, \infty)$, $\varepsilon > 0$ follows clearly from the same convergence of the series

$$G(u) := \sum_{j=1}^{\infty} 2^j \big[1 - \Phi_Q\big(\theta(2^j)u\big)\big],$$

which in turn is easily deduced from (5) and (22). $\square$

Proof of Proposition 4. Let W_Q be a Brownian motion with representation (18) and define

$$W_{Q,J} := \sum_{j>J} \sum_{r \in \mathrm{D}_j} 2^{-(j+1)/2} Y_{Q,r} \Lambda_r, \quad J = 0, 1, \dots$$

It is clear from the proof of Theorem 13 that this series converges a.s. in the space $\mathcal{H}^o_\rho([0,1], \mathbb{B})$ for any $\rho \in \mathcal{R}$ and define a mean zero Gaussian random element of $\mathcal{H}^o_\rho([0,1], \mathbb{B})$. Puting

$$p_J(W_Q) := \|W_{Q,J}\|^{\mathrm{seq}}_\rho = 2^{-1/2} \sup_{j>J} \frac{1}{\theta(2^j)} \max_{r \in \mathrm{D}_j} \|Y_{Q,r}\|$$

and invoking (5) with the separable Banach space $\mathcal{H}^o_\rho$ instead of $\mathbb{B}$, we get

$$\mathbf{P}\big(p_J(W_Q) \geq u\big) \leq 4 \exp\left(\frac{-u^2}{8\mathbf{E}\, p_J(W_Q)^2} \right), \quad u > 0. \tag{27}$$

Clearly $\mathrm{DI}(\rho, Q) = p_0(W_Q)$ and $F_{Q,\rho}(u) = F_Q^{(J)}(u)\mathbf{P}(p_J(W_Q) \leq u)$, so all the estimates given in Proposition 4 rely on finding an upper bound for $\mathbf{E}\, p_J(W_Q)^2$. To this aim, let us note that for any positive τ,

$$\begin{aligned} \mathbf{E}\, p_J(W_Q)^2 &\leq \tau + \int_\tau^\infty \mathbf{P}\big(p_J(W_Q)^2 \geq u\big)\, \mathrm{d}u \\ &\leq \tau + \sum_{j>J} \sum_{r \in \mathrm{D}_j} \int_\tau^\infty \mathbf{P}\big(\|Y_{Q,r}\|^2 \geq 2\theta^2(2^j) u\big)\, \mathrm{d}u \\ &\leq \tau + 8\mathbf{E}\,\|Y_Q\|^2 \sum_{j>J} \frac{2^j}{\theta^2(2^j)} \exp\left(\frac{-\theta^2(2^j)\tau}{4\|Y_Q\|^2} \right), \end{aligned} \tag{28}$$

where we used again (5) before integration. Now puting $\tau = \gamma \mathbf{E}\,\|Y_Q\|^2$ and optimizing the bound (28) with respect to γ leads to $\mathbf{E}\, p_J(W_Q)^2 \leq c_J \mathbf{E}\,\|Y_Q\|^2$ with c_J defined by (10).

Going back to (27) with this estimate, we get

$$\mathbf{P}\big(p_J(W_Q) \leq u\big) \geq 1 - 4 \exp\left(\frac{-u^2}{8c_J \mathbf{E}\,\|Y_Q\|^2} \right), \quad u > 0,$$

which provides (11) and the lower bound in (12). The upper bound in (12) being obvious, the proof is complete. □

4.4 Proof of consistency

We only detail the case where $l^*/n \leq 1/2$ in Theorem 5. It is enough to prove that

$$n^{-1/2}\mathrm{DI}(n, \rho) \geq \frac{n^{1/2}(l^*/n)}{4\rho(4l^*/n)} \|\mathfrak{m}_c\| - n^{-1/2}\mathrm{DI}'(n, \rho), \tag{29}$$

where $\mathrm{DI}'(n, \rho)$ is computed by substituting the X'_k's for the X_k's in $\mathrm{DI}(n, \rho)$. Indeed by Theorem 2, $n^{-1/2}\mathrm{DI}'(n, \rho)$ is stochastically bounded while by (13) and the representation (3) of ρ in Definition 1, the factor of $\|\mathfrak{m}_c\|$ in (29) goes to infinity.

Let us denote by S'_n the sum obtained by substituting the X'_k's for the X_k's in S_n. Introducing the cardinals

$$a_{n,r} := \sharp\left(\mathbb{I}_n \cap (nr^-, nr]\right), \qquad b_{n,r} := \sharp\left(\mathbb{I}_n \cap (nr, nr^+]\right)$$

we can write

$$2\lambda_r(S_n) = 2\lambda_r(S'_n) + (a_{n,r} - b_{n,r})\mathfrak{m}_c,$$

which reduces the verification of (29) to that of

$$\max_{1\le j\le \log n} \max_{r\in \mathrm{D}_j} \frac{|a_{n,r} - b_{n,r}|}{\rho(2^{-j})} \ge \frac{l^*}{2\rho(4l^*/n)}. \tag{30}$$

Clearly it is enough to examine the configurations where 2^{-j} has the same order of magnitude as l^*/n and only one of the integers $a_{n,r}$ and $b_{n,r}$ is positive.

Let us fix the level j by $2^{-j-1} < l^*/n \le 2^{-j}$ and denote by τ the middle of $[t_{k^*}, t_{m^*}]$, where $t_{k^*} := k^*/n$ and $t_{m^*} := m^*/n$. Then we get a unique $r_0 \in \mathrm{D}_j$ such that $r_0^- \le \tau < r_0^+$ and we have to consider the three following cases.

a) $r_0^- \le \tau < r_0^- + 2^{-j-1}$. Then $\tau + l^*/(2n) \le r_0^- + 2^{-j-1} + 2^{-j-1} = r_0$, so $[\tau, t_{m^*}] \subset [r_0^-, r_0]$.

b) $r_0 + 2^{-j-1} \le \tau < r_0^+$. Then $\tau - l^*/(2n) \ge r_0$, so $[t_{k^*}, \tau] \subset [r_0, r_0^+]$.

c) $r_0 - 2^{-j-1} \le \tau < r_0 + 2^{-j-1}$. Then $r_0^- \le t_{k^*} < t_{m^*} \le r_0^+$. Only one of both dyadics r_0^- and r_0^+ has the level $j-1$. If $r_0^- \in \mathrm{D}_{j-1}$, writing $r_1 := r_0^-$ we have $r_1^+ = r_0^+$ and $[t_{k^*}, t_{m^*}] \subset [r_1, r_1^+]$. Else $r_0^+ \in \mathrm{D}_{j-1}$ and with $r_1 := r_0^+$, $r_1^- = r_0^-$ so that $[t_{k^*}, t_{m^*}] \subset [r_1^-, r_1]$.

Now to obtain the lower bound (30), we simply observe that in the cases a) or b), we have $|a_{n,r_0} - b_{n,r_0}| \ge l^*/2$ and $2^{-j} < 2l^*/n$, while in the case c), $|a_{n,r_1} - b_{n,r_1}| = l^*$ and $2^{-(j-1)} < 4l^*/n$.

4.5 Proof of Theorem 8

Consider the random processes X_k, $k = 1, \dots, n$ as defined by (15). To get into a separable Banach space framework we use smoothing kernels K_ε, $\varepsilon > 0$, where

$$K_\varepsilon(t) = \varepsilon^{-1}K(t/\varepsilon)$$

and K is a fixed smooth probability kernel vanishing outside $[-1, 1]$. Put

$$\widehat{X_k}(t) = K_\varepsilon * X_k(t) = \int X_k(v)K_\varepsilon(t-v)\,\mathrm{d}v = \int X_k(t-v)K_\varepsilon(v)\,\mathrm{d}v,$$

where $0 \le t \le 1$ and $k = 1, \dots, n$. Obviously $\widehat{X_k} \in \mathcal{C}[0,1]$ for each $k = 1, \dots, n$. Moreover $\widehat{X_k}$ satisfies the deterministic Lipschitz condition

$$\left|\widehat{X_k}(t) - \widehat{X_k}(s)\right| \le C(\varepsilon)\|K'\|_\infty |t-s|,$$

which implies that $\widehat{X_1}$ satisfies the central limit theorem in $\mathbb{B} = \mathcal{C}[0,1]$, see e.g. [7, Th. 14.2, p.396]. An elementary computation shows that $\widehat{X_1}$ has the same covariance Q_ε as the Gaussian random element $K_\varepsilon * B$ of $\mathcal{C}[0,1]$ where B is a standard Brownian bridge. By boundedness of $\widehat{X_1}$, the moment condition (7) is automatically satisfied, hence Theorem 2 applies to the $\widehat{X_k}$'s with $Q = Q_\varepsilon$.

Let us denote by $\widehat{\kappa_n}$ and $\mathrm{KSDI}(n,\rho,\varepsilon)$ the quantities defined as κ_n and $\mathrm{KSDI}(n,\rho)$ respectively, but with X_i replaced by $\widehat{X_i}$. Let $L^{(\varepsilon)}_{\infty,\rho}$ be defined as $L_{\infty,\rho}$ replacing the Brownian bridge B by $K_\varepsilon * B$. This distribution function is continuous everywhere on $\mathbb{R}$, due to Proposition 3 and the separability of $\mathcal{C}[0,1]$. Set

$$\begin{aligned}
\Delta(u) &= \bigl|\mathbf{P}\bigl\{\mathrm{KSDI}(n,\rho) \le u\bigr\} - L_{\infty,\rho}(u)\bigr|,\\
\Delta_\varepsilon(u) &= \bigl|\mathbf{P}\bigl\{\mathrm{KSDI}(n,\rho,\varepsilon) \le u\bigr\} - L^{(\varepsilon)}_{\infty,\rho}(u)\bigr|,\\
I_1(\delta,\varepsilon) &= \mathbf{P}\bigl\{\bigl|\mathrm{KSDI}(n,\rho,\varepsilon) - \mathrm{KSDI}(n,\rho)\bigr| \ge \delta\bigr\},\\
I_2(\delta,\varepsilon) &= \bigl|L^{(\varepsilon)}_{\infty,\rho}(u+\delta) - L_{\infty,\rho}(u)\bigr|,\\
I_3(\delta,\varepsilon) &= \bigl|L^{(\varepsilon)}_{\infty,\rho}(u-\delta) - L_{\infty,\rho}(u)\bigr|.
\end{aligned}$$

Elementary computations give

$$\Delta(u) \le \max\{\Delta_\varepsilon(u-\delta); \Delta_\varepsilon(u+\delta)\} + I_1(\delta,\varepsilon) + I_2(\delta,\varepsilon) + I_3(\delta,\varepsilon). \tag{31}$$

By Theorem 2 and continuity of $L^{(\varepsilon)}_{\infty,\rho}$, it follows that for each v and each $\varepsilon > 0$,

$$\lim_{n\to\infty} \Delta_\varepsilon(v) = 0. \tag{32}$$

Estimation of $I_1(\delta,\varepsilon)$.

Put for any function $x : [0,1] \to \mathbb{R}$, $\omega(x,\varepsilon) = \sup\{|x(t) - x(u)|;\ |t-u| \le \varepsilon\}$ and note that if x is Lebesgue integrable,

$$\|x - K_\varepsilon * x\|_\infty = \sup_{t\in[0,1]} \Bigl|\int \bigl(x(t-u) - x(t)\bigr) K_\varepsilon(u)\,\mathrm{d}u\Bigr| \le \omega(x,\varepsilon),$$

since the support of the function K_ε is the interval $[-\varepsilon,\varepsilon]$. Applying this inequality when x is a linear combination of the random functions X_i's, we obtain for each dyadic r,

$$\|\lambda_r(\kappa_n) - \lambda_r(\widehat{\kappa_n})\|_\infty \le \omega\bigl(\lambda_r(\kappa_n),\, \varepsilon\bigr). \tag{33}$$

Set

$$p_n(x) := \max_{1\le j\le \log n} \frac{1}{\rho(2^{-j})} \max_{r\in \mathrm{D}_j} \|\lambda_r(x)\|_\infty.$$

Since p_n is a semi norm, we have $|p_n(x) - p_n(y)| \le p_n(x-y)$ for any bounded functions $x, y : [0,1] \to \mathbb{R}$. Applying this with $x = n^{-1/2}\kappa_n$ and $y = n^{-1/2}\widehat{\kappa_n}$ and taking into account (33), we see that

$$\begin{aligned}
I_1(\delta,\varepsilon) &\le \mathbf{P}\bigl\{n^{-1/2} p_n(\kappa_n - \widehat{\kappa_n}) \ge \delta\bigr\}\\
&\le \sum_{1\le j\le \log n} \sum_{r\in \mathrm{D}_j} \mathbf{P}\Bigl\{\omega\bigl(\lambda_r(n^{-1/2}\kappa_n),\, \varepsilon\bigr) \ge \delta\rho(2^{-j})\Bigr\}.
\end{aligned} \tag{34}$$

Introduce the integers

$$n'_r := \sharp\{i \in \mathbb{N};\ nr^- < i \le nr\}, \quad n''_r := \sharp\{i \in \mathbb{N};\ nr < i \le nr^+\},$$

the empirical processes

$$\xi'_{n,r}(t) := \frac{1}{\sqrt{n'_r}} \sum_{nr^- < i \le nr} (\mathbf{1}_{\{U_i \le t\}} - t), \quad \xi''_{n,r}(t) := \frac{1}{\sqrt{n''_r}} \sum_{nr < i \le nr^+} (\mathbf{1}_{\{U_i \le t\}} - t)$$

and set

$$P'_{n,r} := \mathbf{P}\Big\{\omega(\xi'_{n,r}, \varepsilon) \ge \frac{\delta}{2}\theta(2^j)\Big\}, \quad P''_{n,r} := \mathbf{P}\Big\{\omega(\xi''_{n,r}, \varepsilon) \ge \frac{\delta}{2}\theta(2^j)\Big\},$$

where $\theta(t) = t^{1/2}\rho(1/t)$ is as in Definition 1. Recalling (14), we observe that

$$\lambda_r(\kappa_n) = \frac{1}{2}\Big(\frac{n'_r}{n}\Big)^{1/2}\xi'_{n,r} - \frac{1}{2}\Big(\frac{n''_r}{n}\Big)^{1/2}\xi''_{n,r}.$$

As $n'_r, n''_r \le 1 + n2^{-j} < 4n2^{-j}$, it follows that

$$\omega\big(\lambda_r(\kappa_n), \varepsilon\big) \le 2^{-j/2}\omega(\xi'_{n,r}, \varepsilon) + 2^{-j/2}\omega(\xi''_{n,r}, \varepsilon),$$

which allows us to recast (34) as

$$I_1(\delta, \varepsilon) \le \sum_{1 \le j \le \log n} \sum_{r \in \mathrm{D}_j} \big(P'_{n,r} + P''_{n,r}\big).$$

We shall detail only the estimation of the sum $I'_1(\delta, \varepsilon)$ of the terms $P'_{n,r}$, the extension to the sum $I''_1(\delta, \varepsilon)$ of the $P''_{n,r}$'s being obvious.

In what follows, we will denote by c_i, $i = 1, 2, \dots$ positive constants which do not depend on ε, δ. By the Mason, Shorack, Wellner inequality (see e.g. [13], p.545), we have for $0 < \varepsilon \le 1/2$,

$$P'_{n,r} \le \frac{c_1}{\varepsilon} \exp\Big\{-c_2\frac{\delta^2}{\varepsilon}\theta^2(2^j)\psi\Big(\frac{\delta\theta(2^j)}{2\varepsilon\, {n'_r}^{1/2}}\Big)\Big\}, \tag{35}$$

where $\psi(t) = 2t^{-2}\big((1+t)\ln(1+t) - t\big)$. The relevant fact about ψ here is that $\psi(t)$ decreases from 1 to 0 when t goes from 0 to ∞. Let us remark that for $j = \log n$, $\theta(2^j){n'_r}^{-1/2} = \theta(n)\big(1 + o(1)\big)$, so the value taken by ψ in (35) cannot be bounded from below by a positive constant, uniformly in the range $1 \le j \le \log n$. This prevents us from exploiting directly (22).

To remedy this drawback, we shall split the sum $I'_1(\delta, \varepsilon)$, according to $j \le j_n$ or $j > j_n$, denoting respectively by $I'_{1,1}(\delta, \varepsilon)$ and $I'_{1,2}(\delta, \varepsilon)$ the corresponding sums. We choose j_n such that for $j \le j_n$, the argument of ψ in (35) is say, at most 1. Because $\theta(t) = o(t^{1/2})$ when t goes to infinity and $n'_r > n2^{-j+1}$, it is easily seen that a suitable choice is

$$j_n = \frac{1}{2}\log n + \log\big(c_3\delta/\varepsilon\big). \tag{36}$$

In view of (35), we have with $m := \inf_{t\geq 1}\theta(t)^2 > 0$,

$$\begin{aligned} I'_{1,1}(\delta,\varepsilon) &\leq \sum_{j\leq j_n} \frac{c_1}{\varepsilon} 2^j \exp\Big\{-\frac{c_2\delta^2\psi(1)}{\varepsilon}\theta^2(2^j)\Big\} \\ &\leq \frac{c_1}{\varepsilon}\exp\Big\{-\frac{c_2\delta^2\psi(1)m}{2\varepsilon}\Big\}\sum_{j=1}^{\infty} 2^j \exp\Big\{-\frac{c_2\delta^2\psi(1)}{2\varepsilon}\theta^2(2^j)\Big\}. \end{aligned}$$

Noting that the sum of this last series increases in ε and recalling (22), we see that for each $\delta > 0$,

$$\lim_{\varepsilon\to 0}\sup_{n\geq 1} I'_{1,1}(\delta,\varepsilon) = 0. \tag{37}$$

Next to estimate $I'_{1,2}(\delta,\varepsilon)$, we bound $\omega(\xi'_{n,r},\varepsilon)$ by $2\|\xi'_{n,r}\|_\infty$ and apply the Dvoretzky, Kiefer, Wolfowitz inequality (see e.g. [13], p.354) to obtain

$$I'_{1,2}(\delta,\varepsilon) \leq c_4 \sum_{j>j_n} 2^j \exp\big(-c_5\delta^2\theta^2(2^j)\big).$$

From (22) and (36), it follows now that for each $\varepsilon > 0$ and each $\delta > 0$,

$$\lim_{n\to\infty} I'_{1,2}(\delta,\varepsilon) = 0. \tag{38}$$

Estimation of $I_2(\delta,\varepsilon)$ and $I_3(\delta,\varepsilon)$.

Recalling (6), we get

$$L^{(\varepsilon)}_{\infty,\rho}(u) = \mathbf{P}\Big\{\sup_{j\geq 1}\frac{1}{\sqrt{2}\theta(2^j)}\max_{r\in D_j}\|K_\varepsilon * B_r\|_\infty \leq u\Big\},$$

where $(B_r(t), t\in[0,1], r\in \mathrm{D}_j, j\geq 1)$ is a triangular array of independent standard Brownian bridges. Introduce on the space of triangular arrays $\mathfrak{a} = (x_r, r\in \mathrm{D}_j, j\geq 1)$ of bounded measurables functions $x_r : [0,1]\to\mathbb{R}$, the seminorm $\|\mathfrak{a}\| := \sup_{j\geq 1}\frac{1}{\sqrt{2}\theta(2^j)}\max_{r\in D_j}\|x_r\|_\infty$ and put $\mathfrak{b} = (B_r, r\in \mathrm{D}_j, j\geq 1)$, $\widehat{\mathfrak{b}} = (K_\varepsilon * B_r, r\in \mathrm{D}_j, j\geq 1)$. With these notations, $L_{\infty,\rho}(u) = \mathbf{P}(\|\mathfrak{b}\|\leq u)$ and $L^{(\varepsilon)}_{\infty,\rho}(u) = \mathbf{P}(\|\widehat{\mathfrak{b}}\|\leq u)$ and it is elementary to check that

$$\mathbf{P}\big(\|\mathfrak{b}\|\leq u-\delta\big) - \mathbf{P}\big(\|\mathfrak{b}-\widehat{\mathfrak{b}}\|\geq\delta\big) \leq \mathbf{P}\big(\|\widehat{\mathfrak{b}}\|\leq u\big) \leq \mathbf{P}\big(\|\mathfrak{b}\|\leq u+\delta\big) + \mathbf{P}\big(\|\mathfrak{b}-\widehat{\mathfrak{b}}\|\geq\delta\big)$$

whence

$$\big|L^{(\varepsilon)}_{\infty,\rho}(u) - L_{\infty,\rho}(u+\delta)\big| \leq L_{\infty,\rho}(u+\delta) - L_{\infty,\rho}(u-\delta) + \mathbf{P}\big(\|\mathfrak{b}-\widehat{\mathfrak{b}}\|\geq\delta\big).$$

Now, using (6) and the classical estimate of the modulus of continuity of the Brownian motion, we see that

$$\begin{aligned} \mathbf{P}\big(\|\mathfrak{b}-\widehat{\mathfrak{b}}\|\geq\delta\big) &\leq \mathbf{P}\Big\{\sup_{j\geq 1}\frac{1}{\sqrt{2}\theta(2^j)}\max_{r\in D_j}\|K_\varepsilon * B_r - B_r\|_\infty \geq \delta\Big\} \\ &\leq \sum_{j=1}^{\infty} 2^j \mathbf{P}\big\{\omega(B,\varepsilon)\geq \delta\theta(2^j)\big\}. \end{aligned} \tag{39}$$

To control the tail of $\omega(B,\varepsilon)$, we use the representation $B(t) = W(t) - tW(1)$ where W is a standard Brownian motion, together with Inequality 1 in [13], p.536 to obtain

$$\mathbf{P}\big(\omega(B,\varepsilon) \geq u\big) \leq \frac{c_6}{u\sqrt{\varepsilon}} \exp\left(\frac{-c_7 u^2}{\varepsilon}\right) + \exp\left(\frac{-u^2}{8\varepsilon^2}\right), \quad u > 0.$$

Reporting this estimate into (39) and using again (22), the bounded convergence theorem gives then the convergence to zero of $\mathbf{P}\big(\|\mathfrak{b} - \widehat{\mathfrak{b}}\| \geq \delta\big)$ when ε goes to zero. Therefore, for each $\delta > 0$ we have

$$\limsup_{\varepsilon\to 0} I_2(\delta,\varepsilon) \leq L_{\infty,\rho}(u+\delta) - L_{\infty,\rho}(u-\delta).$$

By continuity of $L_{\infty,\rho}$, it follows

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} I_2(\delta,\varepsilon) = 0. \tag{40}$$

Clearly the same holds true for $I_2(\delta,\varepsilon)$. Now the proof is easily completed gathering (31), (32), (37), (38) and (40).

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