PUB. IRMA, LILLE 2005
Vol. 64, N° VII

Hölderian random functions*

Antoine AYACHE Philippe HEINRICH
Laurence MARSALLE Charles SUQUET

Laboratoire P. PAINLEVÉ, CNRS UMR 8524,
Bât. M2, Université Lille 1, Cité Scientifique
59655 Villeneuve d'Ascq Cedex, France.
Antoine.Ayache@math.univ-lille1.fr

Abstract

Hölder regularity which plays a key rôle in fractal geometry raises an increasing interest in probability and statistics. In this paper we discuss various aspects of local and global regularity for stochastic processes and random fields. As a main result we show the invariability of the pointwise Hölder exponent of a continuous and nowhere differentiable random field which has stationary increments and satisfies a zero-one law. We also survey some recent uses of Hölder spaces in limit theorems for stochastic processes and statistics.

Résumé

La régularité hölderienne qui joue un rôle clé en géométrie fractale suscite un intérêt grandissant en porbabilités et statistique. Dans cette contribution nous discutons divers aspects de la régularité hölderienne locale et globale pour les processus stochastiques et les champs aléatoires. Notre résultat principal est la constance temporelle et déterministe de l'exposant de Hölder ponctuel d'un champ aléatoire continu et nulle part différentiable, à accroissements indépendants et vérifiant une loi du zéro-un. Nous donnons aussi un panorama de quelques utilisations récentes des espaces de Hölder dans les théorèmes limites pour les processus stochastiques et en statistique.

Keywords : change point detection, empirical process, epidemic model, functional central limit theorem, Hölder space, local Hölder exponent, multifractional Brownian motion, pointwise Hölder exponent, self-normalization, zero-one law.

Mathematics Subject Classifications (2000): Primary 60G17, Secondary 60-02, 60B12, 60F17, 60F20, 62G10.

*Preprint

1 Introduction

The concept of Hölder regularity is quite important in fractal geometry, signal and image processing, finance, statistics and telecommunications [8]. Hölder exponents have been used frequently to measure the roughness of a curve or of a surface [10]; applications in signal and image processing are numerous and include interpolation, segmentation [26] and denoising [27]. They are closely related to other fractal indices such as fractal dimensions, self-similarity parameters and multifractal spectra (see e.g. [12, 18, 44, 45]). On the other hand, the Hölder spaces provide a functional framework for limit theorems in the theory of stochastic processes. The use of Hölder topologies leads to more precise results than the classical framework of continuous functions spaces.

This paper discuss both uses of Hölder regularity in the study of stochastic processes.

1.1 Hölder exponents

When looking for random fields modeling some roughness, it is quite natural to investigate the pointwise Hölder regularity of various extensions of the well known Brownian motion.

Recall that $\{B_H(t), t \in \mathbb{R}^d\}$, the fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is the real-valued, self-similar and stationary increments continuous Gaussian field defined for every $t \in \mathbb{R}^d$ as the Wiener integral

$$B_H(t) = \int_{\mathbb{R}^d} \frac{e^{it.\xi} - 1}{|\xi|^{H+d/2}} \, d\widehat{W}(\xi), \tag{1}$$

where $d\widehat{W}$ is a complex-valued white noise. This field was first introduced by Kolmogorov [20] for generating Gaussian "spirals" in a Hilbert space. Later, the seminal article of Mandelbrot and Van Ness [30] emphasized its relevance for the modeling of natural phenomena (hydrology, finance,...) and thus greatly contributed to make it popular. Since then, this field turns out to be a very powerful tool in modeling. The monograph of Doukhan, Oppenheim and Taqqu [11] offers a systematic treatment of fBm, as well as an overview of different areas of applications.

The field $\{B_H(t), t \in \mathbb{R}^d\}$ is a natural generalization of the Wiener process ($\{B_{1/2}(t), t \in \mathbb{R}\}$ is a Wiener process) and shares many nice properties with it. However, one of the main advantages of fBm with respect to Wiener process is that its increments are correlated and can even display long range dependence. Still, fBm is not always a realistic model. Indeed, its pointwise Hölder exponent remains constant all along its trajectory which can be a serious drawback in several applications (see for example [28, 2, 3, 4]). Generally speaking, a multifractional field is a field with continuous trajectories that extends fBm and whose pointwise Hölder exponent is allowed to change from one point to another. Recall that $\{\alpha_X(t), t \in T\}$ the pointwise Hölder exponent of a continuous and nowhere differentiable field $\{X(t), t \in T\}$ is defined for every $t \in T$

as

$$\alpha_X(t) = \sup\Big\{\alpha;\ \limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{|h|^\alpha} = 0\Big\}. \tag{2}$$

A paradigmatic example of a multifractional field is multifractional Brownian motion (mBm). It was introduced independently in [28] and in [7] but the denomination multifractional Brownian motion is due to Lévy Véhel. MBm is obtained by substituting to the Hurst parameter in the harmonizable representation (1) of fBm a continuous function $t \mapsto H(t)$ with values in $(0,1)$. When the function $H(\cdot)$ is smooth enough (typically when it is a C^1 function), the pointwise Hölder exponent of mBm satisfies for any $t \in T$, almost surely $\alpha_{\mathrm{mBm}}(t) = H(t)$, which means that it can change from one point to another.

In [5] it has been proved that when the increments of any order of mBm are stationary the function $H(\cdot)$ is constant (i.e. mBm reduces to an fBm). Moreover, no example of a multifractional field with stationary increments has been constructed yet. *This is why it seems natural to wonder whether there exists a continuous, nowhere differentiable and stationary increments field $\{X(t), t \in T\}$ whose pointwise Hölder exponent changes from one point to another.* In Section 2 we show that the answer is negative when we impose in addition to $\{X(t), t \in T\}$ to satisfy a zero-one law.

1.2 Hölder spaces as a functional framework

In many situations some uniform control on the regularity is needed. For instance let us consider the following statistical problem. Having observed the random variables $X_1, \dots, X_n$, we need to test the null hypothesis that $X_1, \dots, X_n$ have the same expectation μ_0 against the alternative of a change from μ_0 to μ_1 between the unknown instants k^* and m^* with going back to μ_0 after m^*. This is known as the *epidemic model*. It is quite natural, see [41] for a step by step explanation, to use here the weighted test statistics

$$\mathrm{UI}(n,a) := \max_{1\le i<j\le n} \frac{\big|S(j) - S(i) - S(n)(t_j - t_i)\big|}{|t_j - t_i|^a}$$

where $S(n) := \sum_{1\le i\le n} X_i$, $t_i := i/n$ and $0 < a < 1/2$. The asymptotic distribution of $\mathrm{UI}(n,a)$ follows from the weak convergence of a partial sums process ξ_n in some Hölder space $\mathcal{H}^a$ (a precise definition of Hölder spaces is given in Section 3). The practical interest of the exponent a here lies in the sensitivity of the test. Detecting the shortest epidemics requires to take the biggest possible a and this leads to investigate weak convergence of ξ_n in $\mathcal{H}^a$.

In Section 4 we survey some recent advances in the asymptotic theory of sequences of stochastic processes considered as random elements in some Hölder space $\mathcal{H}$. The issues addressed may be classified along the following two main directions.

A) Classical limit theorems for normalized sums $b_n^{-1} S_n$ of independent random elements X_i in $\mathcal{H}$: laws of large number, central limit theorems, see e.g. [31], [32], [35], [40].

B) Weak convergence in $\mathcal{H}$ of sequences of random elements ξ_n of the form

$$t \longmapsto \xi_n(t) = G_n(X_1, \dots, X_n, t),$$

where $X_1, \dots, X_n$ is usually a sample of i.i.d. random variables or random elements in some Banach space and G_n a function smooth enough to ensure the membership in $\mathcal{H}$ of ξ_n .

Problem A) is directly connected to the Probability Theory in Banach Spaces. It is well known in this area that the limit theorems for a sequence of random elements $b_n^{-1}S_n$ in some separable Banach space $\mathbb{B}$ involve the geometry of $\mathbb{B}$. For instance, if the X_i's are i.i.d. with null expectation, then the square integrability of $\|X_1\|$ gives the asymptotic normality of $n^{-1/2}S_n$ when $\mathbb{B}$ is of type 2 (e.g. $\mathbb{B}$ is a Hilbert space or a L^p space with $2 \le p < \infty$). But if $\mathbb{B} = c_0$, the classical space of sequences converging to 0, we can find a *bounded* random element X_1 in c_0 which does not satisfy the CLT, i.e. the corresponding sequence $n^{-1/2}S_n$ is not asymptotically Gaussian. This makes hopeless characterizing the CLT in a general $\mathbb{B}$ in terms of integrability properties of X_1 only. Because all the Hölder spaces $\mathcal{H}$ under consideration here contain a subspace isomorphic to c_0 and are concrete function spaces, they provide an interesting framework to study the asymptotic behavior of $b_n^{-1}S_n$ in a context where the geometry of the Banach space is "bad".

Problem B) is more oriented to statistical applications. Indeed, the weak convergence $\xi_n \xrightarrow{\mathcal{D}} \xi$ in some function space E means

$$\mathbf{E}\, g(\xi_n) \xrightarrow[n\to\infty]{} \mathbf{E}\, g(\xi), \tag{3}$$

for every continuous and bounded function $g : E \to \mathbb{R}$. By the continuous mapping theorem, this implies that for every functional $f : E \to \mathbb{R}$, continuous with respect to the strong topology of E,

$$f(\xi_n) \xrightarrow[n\to\infty]{} f(\xi), \quad \text{in distribution.} \tag{4}$$

The classical functional frameworks for such convergence $\xi_n \xrightarrow{\mathcal{D}} \xi$ are the Skorohod space when ξ_n has jumps and some space $\mathcal{C}$ of continuous functions when ξ_n has continuous paths. The interest in replacing, whenever possible, $\mathcal{C}$ by $\mathcal{H}$ is that this strenghtening of the topology on the paths space enlarges the set of continuous functionals f. Usually, the random functions ξ_n share more smoothness than their weak limit ξ. For instance in the Hölderian version of the invariance principle for partial sums processes, the paths of ξ_n are random polygonal lines, while ξ is a Brownian motion. In such cases the global smoothness of ξ put a natural bound in the choice of the "best" space $\mathcal{H}$. The example of the Brownian motion W shows here that the classical ladder of Hölder spaces $\mathcal{H}^a$ is not rich enough. Indeed $\mathcal{H}^a$ is the space of functions x whose increments $x(t+h) - x(t)$, $h \ge 0$ are $O(h^a)$ uniformly in t. Due to Lévy's result on W's modulus of uniform continuity [25, Th. 52,2], it seems desirable to consider also the spaces of functions x such that $x(t+h)-x(t)=O\big(h^{1/2}\ln^b(1/h)\big)$. In more generality, this leads to introduce a ladder of Hölder spaces $\mathcal{H}^\rho$, where membership of x in $\mathcal{H}^\rho$ is equivalent to the uniform estimate $x(t+h)-x(t)=O(\rho(h))$, for some weight function ρ.

This raises a third problem which in some sense is also preliminary to Problem A):

C) Given a stochastic process $X=\{X(t),\, t\in T\}$, find conditions in terms of its finite dimensional distributions so that X admits a version with paths in the Hölder space $\mathcal{H}^\rho$.

2 Critical Exponents

2.1 Zero-One laws and Exponents

Let $X=\{X(t), t\in T\}$ be a real process with, say, separable and metric time set T. We can view X as a random element in $\mathbb{R}^T$ endowed with product σ-field $\mathcal{B}(\mathbb{R})^{\otimes T}$ where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field of $\mathbb{R}$. The kind of zero-one law we shall focus on can be stated as follows:

Definition 1. We will say that X *satisfies a zero-one law* if for each measurable linear subspace V of $\mathbb{R}^T$,

$$\mathbf{P}\,(X\in V)=0 \text{ or } 1. \tag{5}$$

It is known that Gaussian, stable and some infinitely divisible (without Gaussian component) processes satisfy (5). Their associated finite-order chaos processes do as well. We refer the reader to the paper [43] by Rosinski and Samorodnitsky and the references therein. One classical application of such a zero-one law is to establish regularity properties of paths. For instance, neglecting measurability questions at first glance, V could be:

1. the space of bounded functions on T,
2. the space of continuous functions on T, if T is compact,
3. the space of uniformly continuous functions on T,
4. the space of Lipschitz functions on T,
5. the space of a-Hölderian functions on T,
6. the space of absolutely continuous functions on T, if T is an interval in $\mathbb{R}$.

If for some countable dense subset S of T, we have

$$\mathbf{P}\big(\forall t\in T,\, \exists (s_n)_{n\ge 1}\subset S,\, s_n\to t,\, X(s_n)\to X(t)\big)=1,$$

then the measurability of $\{X\in V\}$, for the V's displayed above, can be established provided the underlying probability space $(\Omega,\mathcal{F},\mathbf{P})$ is complete (as can be always assumed). Indeed, the events $\{X \in V\}$ may then be expressed, up to negligeable sets, as ones involving only the restriction of X to S. The following result illustrates how useful this zero-one law can be.

Theorem 1. *Assume that T is an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$). Let $X = \{X(t),\ t \in T\}$ be a continuous, nowhere differentiable process satisfying the zero-one law (5). Then, for all $t \in T$, the pointwise Hölder exponent of X at t is almost surely deterministic. In other words, for all $t \in T$, there exists a number $H(t) \in [0,1]$ such that*

$$\mathbf{P}\big(\alpha_X(t) = H(t)\big) = 1.$$

Proof. This result has already been established by Ayache and Taqqu for Gaussian processes, see [6]. Their proof is based on the same key-argument (zero-one law), but the one we give here uses it more directly and explicitely.

We set $S = \mathbb{Q}^d$. Let t be some arbitrary point of the open set T and choose $\eta > 0$ such that the ball $B(t,\eta)$ be in T. Since X has almost all continuous and nowhere differentiable paths on T, we know that $\alpha_X(t,\omega)$ belongs to $[0,1]$ for almost all $\omega \in \Omega$. We can thus define

$$\begin{aligned} u_*(t) &:= \sup\big\{u \in \mathbb{R}; \mathbf{P}(\alpha_X(t) \le u) = 0\big\}, \\ u^*(t) &:= \inf\big\{u \in \mathbb{R}; \mathbf{P}(\alpha_X(t) \le u) = 1\big\}. \end{aligned}$$

By definition, the interval $[u_*(t), u^*(t)]$ is the support of the distribution function of $\alpha_X(t)$. To prove that $\alpha_X(t)$ is almost surely deterministic, we only need to check that $u^*(t) \le u_*(t)$. Let $u < u^*(t)$, we thus have $\mathbf{P}(\alpha_X(t) > u) > 0$. On the event $\{\alpha_X(t) > u\}$, we have $\limsup_{h\to 0} |h|^{-u}|X(t+h) - X(t)| = 0$ which implies that $h \mapsto h^{-u}(X(t+h) - X(t))$ is bounded on the countable bounded subset $\{h \in S;\ |h| < \eta\}$. It follows, by inclusion, that

$$0 < \mathbf{P}\big(\alpha_X(t) > u\big) \le \mathbf{P}\left(\sup_{\substack{|h|<\eta \\ h\in S}} \frac{|X(t+h) - X(t)|}{|h|^u} < \infty\right).$$

We shall prove that this last probability is equal to 1, using the zero-one law (5). To this end, note that the event

$$\left\{\sup_{\substack{|h|<\eta \\ h\in S}} \frac{|X(t+h) - X(t)|}{|h|^u} < \infty\right\}$$

can clearly be written $\{X \in V\}$ for some linear subspace V, which is $\mathcal{B}(\mathbb{R})^{\otimes T}$-measurable since it involves only countable many projections $x \mapsto x(t)$ from $\mathbb{R}^T$ to $\mathbb{R}$. The zero-one law ensures consequently that

$$\mathbf{P}\left(\sup_{|h|<\eta} \frac{|X(t+h) - X(t)|}{|h|^u} < \infty\right) = 1,$$

where we skipped the restriction $h \in S$ in the supremum thanks to the continuity of X. But now, a simple inclusion of events yields

$$\mathbf{P}\left(\limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{|h|^u} < \infty\right) = 1,$$

which can be read as $\mathbf{P}\left(u \leq \alpha_X(t)\right) = 1$ or, equivalently, as $\mathbf{P}(\alpha_X(t) < u) = 0$. This means that $u \leq u_*(t)$ which gives $u^*(t) \leq u_*(t)$, since u is arbitrary in $(-\infty, u^*(t))$. Besides, by definition $u_*(t) \leq u^*(t)$, whence $u_*(t) = u^*(t)$. We call $H(t)$ this common value. We just have proved that the distribution function of $\alpha_X(t)$ jumps from 0 to 1 at $H(t)$ in other words $\mathbf{P}(\alpha_X(t) = H(t)) = 1$. □

Remark 1. Other exponents may be defined to characterize the regularity of a function, for instance the *local Hölder exponent* at time t

$$\tilde{\alpha}_X(t) = \sup\left\{\alpha; \exists \eta > 0 \sup_{u,v\in B(t,\eta)} \frac{|X(u)-X(v)|}{|u-v|^\alpha} < \infty\right\},$$

where $B(t,\eta)$ denotes the open ball centered at t and of radius η, and the *global Hölder exponent* on a compact set $K \subset T$

$$\beta_X = \sup\left\{\beta; \sup_{u,v\in K} \frac{|X(u)-X(v)|}{|u-v|^\beta} < \infty\right\}.$$

When X is a continuous nowhere differentiable process, satisfying a zero-one law, the same property as in Theorem 1 holds. More precisely, for all compact subset K of T, β_X is almost surely deterministic, and for all $t \in T$, $\tilde{\alpha}_X(t)$ is almost surely deterministic. The proof of both results is the same as for Theorem 1, the only change concerns the measurable subspace of $\mathbb{R}^T$ involved in the zero-one law. In the case of $\tilde{\alpha}_X(t)$, we use

$$\tilde{V} := \bigcup_{n\in\mathbb{N}^*} \left\{x \in \mathbb{R}^T; \sup_{u,v\in B(t,1/n)\cap S} \frac{|x(u)-x(v)|}{|u-v|^\alpha} < \infty\right\},$$

where S is a countable dense subset of T, and in the case of β_X, we define

$$W := \left\{x \in \mathbb{R}^T; \sup_{u,v\in K\cap S} \frac{|x(u)-x(v)|}{|u-v|^\alpha} < \infty\right\}.$$

2.2 Processes with Stationary Increments

Throughout all this paragraph, T, the set of times, can be taken to be equal to any non-empty open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$).

Let $X = \{X(t), t \in T\}$ denote a continuous nowhere differentiable process, for which a zero-one law holds (see (5)). Thanks to Subsection 2.1, we know that the pointwise Hölder exponent of X at t is deterministic, but depends on t. Now, we assume besides that X has stationary increments. This means that

$(X(s_2)-X(s_1),\dots,X(s_n)-X(s_1))$ and $(X(s_2+t)-X(s_1+t),\dots,X(s_n+t)-X(s_1+t))$ are identically distributed for any $s_1,\dots,s_n,s_1+t,\dots,s_n+t\in T$ and any integer $n\geq 2$. Since the pointwise Hölder exponent is defined by means of increments, we can show that it doesn't depend anymore on t.

Theorem 2. *Let $X=\{X(t),t\in T\}$ be a continuous nowhere differentiable process, with stationary increments. We assume that a zero-one law holds for X. Then there exists $H\in[0,1]$ such that for all $t\in T$*

$$\mathbf{P}\left(\alpha_X(t)=H\right)=1.$$

Proof. The scheme of the proof is the following: since T is a non-empty set, it contains at least one element, that we will denote 0 for the sake of simplicity. Using an equivalent definition of the pointwise Hölder exponent, we prove that $\alpha_X(t)$ and $\alpha_X(0)$ have the same law, for all $t\in T$. Thanks to Theorem 1, we know that there exists $H=H(0)\in[0,1]$ such that the law of $\alpha_X(0)$ is a Dirac mass at point H. Consequently, for all $t\in T$, the law of $\alpha_X(t)$ is a Dirac mass at point H.

As in the proof of Theorem 1, S denotes a countable dense subset of T. Let t be a fixed point of T. It can be easily shown that the pointwise Hölder exponent of X at t is given by:

$$\alpha_X(t)=\liminf_{h\to 0}\frac{\log|X(t+h)-X(t)|}{\log|h|},$$

with the usual convention that $\log 0=-\infty$. This definition reads as

$$\begin{aligned}\alpha_X(t)&=\sup_{R>0}\inf_{|h|<R}\frac{\log|X(t+h)-X(t)|}{\log|h|}\\&=\sup_{n\in\mathbb{N}}\inf_{\substack{|h|<1/n\\h\in S}}\frac{\log|X(t+h)-X(t)|}{\log|h|},\end{aligned}$$

the last equality coming from the monotonicity of the infimum with respect to R and from the continuity of the paths of X. To obtain the identity in law between $\alpha_X(t)$ and $\alpha_X(0)$, we introduce an increasing sequence $(S_k)_{k\geq 1}$ of *finite* sets such that $\cup_{k\geq 1}S_k=S$. Then, note that for $u\in\mathbb{R}$

$$\{\alpha_X(t)\leq u\}=\bigcap_{n\in\mathbb{N}}\!\downarrow\ \bigcap_{m\in\mathbb{N}}\!\downarrow\ \bigcup_{k\in\mathbb{N}}\!\uparrow\left\{\min_{\substack{|h|<1/n\\h\in S_k}}\frac{\log|X(t+h)-X(t)|}{\log|h|}<u+\frac{1}{m}\right\},$$

so that, for every $t\in T$ and $u\in\mathbb{R}$, by sequential monotonic continuity of $\mathbf{P}$,

$$\mathbf{P}\left(\alpha_X(t)\leq u\right)=\lim_n\lim_m\lim_k\mathbf{P}\left(\min_{\substack{|h|<1/n\\h\in S_k}}\frac{\log|X(t+h)-X(t)|}{\log|h|}<u+\frac{1}{m}\right).$$

The event mentioned in the last probability involves a *finite* number of increments of X, all of them based on point t. The stationarity of the increments implies that we can replace them with the analogue increments, based on point 0. It follows that for all $u \in \mathbb{R}$

$$\mathbf{P}(\alpha_X(0) \le u) = \mathbf{P}(\alpha_X(t) \le u),$$

which means that $\alpha_X(t)$ and $\alpha_X(0)$ have the same law, for all $t \in T$. We already know that the pointwise Hölder exponent of X at point 0 is deterministic, so that there exists $H \in [0,1]$ such that $\mathbf{P}(\alpha_X(0) = H) = 1$. The equality in law between $\alpha_X(t)$ and $\alpha_X(0)$ thus leads to $\mathbf{P}(\alpha_X(t) = H) = 1$, for all $t \in T$. □

3 Hölder spaces

Let us introduce the Hölder spaces by an informal description of the most familiar case. For fixed $0 < a < 1$, $\mathcal{H}^a$ is the set of functions $x : [0,1] \to \mathbb{R}$ such that $|x(t) - x(s)| \le K|t-s|^a$ for some constant K depending only on x and a. The best constant K in this uniform estimate defines a semi-norm on the vector space $\mathcal{H}^a$. By adding $|x(0)|$ to this semi-norm we obtain a norm $\|x\|_a$ which makes $\mathcal{H}^a$ a non separable Banach space. Clearly if $0 < a < b < 1$, $\mathcal{H}^b$ is topologically embedded in $\mathcal{H}^a$ and all these Hölder spaces are topologically embedded in the classical Banach space $\mathcal{C}$ of continuous functions $[0,1] \to \mathbb{R}$.

To remedy the non separability drawback of $\mathcal{H}^a$, one introduces its subspace $\mathcal{H}^{a,o}$ of functions x such that $|x(t) - x(s)| = o(|t-s|^a)$ uniformly. This subspace is *closed* (hence also a Banach space for the same norm $\|x\|_a$) and *separable.*

One interesting feature of the spaces $\mathcal{H}^{a,o}$ is the existence of a basis of triangular functions, see [9]. It is convenient to write this basis as a triangular array of functions, indexed by the dyadic numbers. Let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j, i.e.

$$\mathrm{D}_0 = \{0,1\}, \qquad \mathrm{D}_j = \left\{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\right\}, \quad j \ge 1.$$

Write for $r \in \mathrm{D}_j$, $j \ge 0$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

For $r \in \mathrm{D}_j$, $j \ge 1$, the triangular Faber-Schauder functions Λ_r are continuous, piecewise affine with support $[r^-, r^+]$ and taking the value 1 at r:

$$\Lambda_r(t) = \begin{cases} 2^j(t - r^-) & \text{if } t \in (r^-, r]; \\ 2^j(r^+ - t) & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

When $j = 0$, we just take the restriction to $[0,1]$ in the above formula, so

$$\Lambda_0(t) = 1 - t, \quad \Lambda_1(t) = t, \quad t \in [0,1].$$

The sequence $\{\Lambda_r; r \in \mathrm{D}_j,\ j \ge 0\}$ is a Schauder basis of $\mathcal{C}$. Each $x \in \mathcal{C}$ has a unique expansion

$$x = \sum_{j=0}^{\infty} \sum_{r \in \mathrm{D}_j} \lambda_r(x)\Lambda_r, \tag{6}$$

with uniform convergence on $[0,1]$. The Schauder scalar coefficients $\lambda_r(x)$ are given by

$$\lambda_r(x) = x(r) - \frac{x(r^+) + x(r^-)}{2}, \quad r \in \mathrm{D}_j,\ j \geq 1, \tag{7}$$

and in the special case $j = 0$ by

$$\lambda_0(x) = x(0), \quad \lambda_1(x) = x(1). \tag{8}$$

The partial sum $\sum_{j=0}^{n}$ in the series (6) gives the linear interpolation of x by a polygonal line between the dyadic points of level at most n.

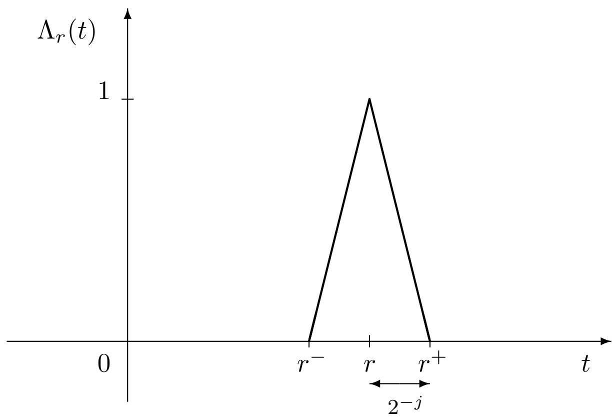

Figure 1: The Faber-Schauder triangular function Λ_r

Ciesielski [9] proved that $\{\Lambda_r; r \in \mathrm{D}_j,\ j \geq 0\}$ is also a Schauder basis of each space $\mathcal{H}^{a,o}$ (hence the convergence (6) holds in the $\mathcal{H}^a$ topology when $x \in \mathcal{H}^{a,o}$) and that the norm $\|x\|_a$ is equivalent to the following sequence norm :

$$\|x\|_a^{\mathrm{seq}} := \sup_{j \geq 0} 2^{ja} \max_{r \in \mathrm{D}_j} |\lambda_r(x)|.$$

This equivalence of norms provides a very convenient discretization procedure to deal with Hölder spaces and is extended in Račkauskas and Suquet [34] to the more general setting of Hölder spaces of Banach space valued functions x, with a modulus of continuity controlled by some weight function ρ.

Let $(\mathbb{B}, \|\ \|)$ be a separable Banach space. We write $\mathcal{C}_{\mathbb{B}}$ for the Banach space of continuous functions $x : [0,1] \to \mathbb{B}$ endowed with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\|\,;\ t \in [0,1]\}$. Let ρ be a real valued non decreasing function on $[0,1]$, null and right continuous at 0. Put

$$\omega_\rho(x,\delta) := \sup_{\substack{s,t \in [0,1],\\ 0<t-s<\delta}} \frac{\|x(t) - x(s)\|}{\rho(t-s)}.$$

Denote by $\mathcal{H}^{\rho}_{\mathbb{B}}$ the set of continuous functions $x : [0,1] \to \mathbb{B}$ such that $\omega_\rho(x,1) < \infty$. The set $\mathcal{H}^{\rho}_{\mathbb{B}}$ is a Banach space when endowed with the norm

$$\|x\|_\rho := \|x(0)\| + \omega_\rho(x,1).$$

Define

$$\mathcal{H}^{\rho,o}_{\mathbb{B}} = \{x \in \mathcal{C}_{\mathbb{B}} : \lim_{\delta \to 0} \omega_\rho(x,\delta) = 0\}.$$

Then $\mathcal{H}^{\rho,o}_{\mathbb{B}}$ is a closed separable subspace of $\mathcal{H}^{\rho}_{\mathbb{B}}$. We shall abbreviate $\mathcal{C}_{\mathbb{R}}$, $\mathcal{H}^{\rho}_{\mathbb{R}}$ and $\mathcal{H}^{\rho,o}_{\mathbb{R}}$ in $\mathcal{C}$, $\mathcal{H}^{\rho}$ and $\mathcal{H}^{\rho,o}$ correspondingly. Our main examples of Hölder spaces use as weight function $\rho = \rho_{a,b}$, $0 < a < 1$, $b \in \mathbb{R}$ defined by:

$$\rho_{a,b}(h) := h^a \ln^b(c/h), \quad 0 < h \le 1,$$

for a suitable constant c. For $\rho = \rho_{a,b}$, we shall write $\mathcal{H}^{a,b}_{\mathbb{B}}$ and $\mathcal{H}^{a,b,o}_{\mathbb{B}}$ for $\mathcal{H}^{\rho}_{\mathbb{B}}$ and $\mathcal{H}^{\rho,o}_{\mathbb{B}}$ respectively and we abbreviate $\mathcal{H}^{a,0,o}_{\mathbb{B}}$ in $\mathcal{H}^{a,o}_{\mathbb{B}}$. As above, the subscript $\mathbb{B}$ will be omitted when $\mathbb{B} = \mathbb{R}$.

In what follows, we assume that the weight function ρ satisfies the following technical conditions where c_1, c_2 and c_3 are positive constants:

$$\rho(0) = 0, \ \rho(\delta) > 0, \ 0 < \delta \le 1; \tag{9}$$

$$\rho \text{ is non decreasing on } [0,1]; \tag{10}$$

$$\rho(2\delta) \le c_1 \rho(\delta), \quad 0 \le \delta \le 1/2; \tag{11}$$

$$\int_0^\delta \frac{\rho(u)}{u}\, du \le c_2 \rho(\delta), \quad 0 < \delta \le 1; \tag{12}$$

$$\delta \int_\delta^1 \frac{\rho(u)}{u^2}\, du \le c_3 \rho(\delta), \quad 0 < \delta \le 1. \tag{13}$$

For instance, elementary computations show that the functions $\rho_{a,b}$ satisfy Conditions (9) to (13), for a suitable choice of the constant c, namely $c \ge \exp(b/a)$ if $b > 0$ and $c > \exp(-b/(1-a))$ if $b < 0$. For any ρ satisfying (9) to (13), we have the equivalence of norms :

$$\|x\|_\rho \sim \|x\|^{\mathrm{seq}}_\rho := \sup_{j \ge 0} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \|\lambda_r(x)\|,$$

where the $\mathbb{B}$-valued coefficients $\lambda_r(x)$ are still defined by (7) and (8).

The space $E = \mathcal{H}^{\rho,o}_{\mathbb{B}}$ may be used as the topological framework for limit theorems. Among various continuous functionals f for which the convergence (4) holds, let us mention the norms $f_1(x) = \|x\|_\rho$ and $f_2(x) = \|x\|^{\mathrm{seq}}_\rho$, which are closely connected to the test statistics proposed below for the detection of epidemic changes. Other examples of Hölder continuous functionals and operators like p-variation, fractional derivatives are given in Hamadouche [15].

4 Random elements in Hölder spaces

4.1 Processes with a version in Hölder space

Let $\mathbb{B}$ be a Banach space. We consider a given $\mathbb{B}$-valued stochastic process $\xi = \{\xi(t),\ t \in T\}$, continuous in probability and discuss the problem of existence of a version of ξ with almost all paths in $\mathcal{H}_{\mathbb{B}}^{\rho,o}$. For simplicity we restrict this presentation to the case $T = [0,1]$. The results presented here are proved in [34], in the more general case of $\mathbb{B}$-valued random fields. The main analytic tool in this problem is the following generalization of the Faber-Schauder decomposition.

Proposition 1. *For a $\mathbb{B}$-valued array $\nu = (\nu_r;\ j \geq 0, r \in D_j)$, consider the following conditions.*

(a) $\displaystyle\sum_{j=0}^{\infty} \max_{r\in D_j} \|\nu_r\| < \infty.$

(b) $\displaystyle\sup_{j\geq 0} \frac{1}{\rho(2^{-j})} \max_{r\in D_j} \|\nu_r\| < \infty.$

(c) $\displaystyle\lim_{J\to\infty} \sup_{j>J} \frac{1}{\rho(2^{-j})} \max_{r\in D_j} \|\nu_r\| = 0.$

Define the sequence $(y_J)_{J\geq 0}$ of continuous piecewise affine functions by

$$y_J := \sum_{j=0}^{J} \sum_{r\in D_j} \nu_r \Lambda_r.$$

Then (a) implies the convergence in $\mathcal{C}_{\mathbb{B}}$ of y_J to some function y. Condition (b) gives the same convergence plus the membership in $\mathcal{H}_{\mathbb{B}}^{\rho}$ for y. Condition (c) gives the convergence of y_J to y in $\mathcal{H}_{\mathbb{B}}^{\rho,o}$.

Corollary 1. *For any function $x : [0,1] \to \mathbb{B}$, define the $\mathbb{B}$-valued array $\nu = \nu(x) := (\lambda_r(x);\ j \geq 0, r \in D_j)$. Then x coincides at the dyadic points of $[0,1]$ with some function y which is in $\mathcal{C}_{\mathbb{B}}$ under (a), in $\mathcal{H}_{\mathbb{B}}^{\rho}$ under (b) and in $\mathcal{H}_{\mathbb{B}}^{\rho,o}$ under (c).*

From Corollary 1 and continuity in probability of ξ, it is easily seen that the problem of existence of $\mathcal{H}_{\mathbb{B}}^{\rho,o}$-versions of ξ reduces to the control of the $\lambda_r(\xi)$'s which are dyadic second differences of ξ. It is convenient here to define the second differences of ξ by

$$\Delta_h^2 \xi(t) := \xi(t+h) + \xi(t-h) - 2\xi(t), \quad t \in T,\ t \pm h \in T.$$

This leads to the general following result.

Theorem 3. *Let $\xi = \{\xi(t),\ t \in T\}$ be a $\mathbb{B}$-valued stochastic process, continuous in probability. Assume there exist a function $\sigma : [0,1] \to \mathbb{R}^+$, $\sigma(0) = 0$ and a function $\Psi : (0,\infty] \to \mathbb{R}^+$, $\Psi(\infty) = 0$ such that for all real numbers $z > 0$, $t \in T$, $t \pm h \in T$,*

$$P\Big(\big\|\Delta_h^2 \xi(t)\big\| > z\sigma(|h|)\Big) \le \Psi(z). \tag{14}$$

Put for $0 < u < \infty$,

$$R(u) = R(\Psi, \sigma, \rho, u) := \sum_{j=0}^{\infty} 2^{jd} \Psi\Big(u \frac{\rho}{\sigma}(2^{-j})\Big).$$

If $R(u_0)$ is finite for some $0 < u_0 < \infty$, then ξ has a version in $\mathcal{H}_B^{\rho}$. If $R(u)$ is finite for every $0 < u < \infty$, then ξ has a version in $\mathcal{H}_B^{\rho,o}$.

When Ψ is non increasing and σ non decreasing, the same conclusions hold replacing $R(u)$ by

$$I(u) := \int_0^1 \Psi\Big(u \frac{\rho}{\sigma}(s)\Big) \frac{\mathrm{d}s}{s^2}.$$

We only give here an application to the case of Gaussian processes. For other applications and examples we refer to [34].

Corollary 2. *Assume that the Gaussian $\mathbb{B}$-valued stochastic process $\xi = \{\xi(t), t \in T\}$ is continuous in probability and satisfies for each $t \in T$, $t \pm h \in T$,*

$$\mathbf{E}\, \big\|\Delta_h^2 \xi(t)\big\|^2 \le \sigma^2(|h|). \tag{15}$$

(i) If $\liminf_{j\to\infty} \dfrac{\rho(2^{-j})}{j^{1/2}\sigma(2^{-j})} > 0$, then ξ admits a version in $\mathcal{H}_B^{\rho}$.

(ii) If $\lim_{j\to\infty} \dfrac{\rho(2^{-j})}{j^{1/2}\sigma(2^{-j})} = \infty$, then ξ has a version in $\mathcal{H}_B^{\rho,o}$.

Example 1. Let $\mathbb{B}$ be a separable Banach space and Y a centered Gaussian random element in $\mathbb{B}$ with distribution μ. A $\mathbb{B}$-valued Brownian motion with parameter μ is a Gaussian process ξ indexed by $[0,1]$, with independent increments such that $\xi(t) - \xi(s)$ has the same distribution as $|t-s|^{1/2} Y$. Hence (15) holds with $\sigma(h) = h^{1/2} \mathbf{E}^{\,1/2} \|Y\|_B^2$ $(h \ge 0)$. Choosing the weight function $\rho(h) = \sqrt{h \ln(\mathrm{e}/h)}$, we see that

$$\lim_{j\to\infty} \frac{\rho(2^{-j})}{j^{1/2}\sigma(2^{-j})} = \frac{1}{\mathbf{E}^{\,1/2} \|Y\|^2} > 0.$$

Hence by Corollary 2 (ii), the $\mathbb{B}$-valued Brownian motion ξ has a version in $\mathcal{H}_{\mathbb{B}}^{\rho}$. This result is optimal because of Lévy's theorem on the modulus of uniform continuity of the standard Brownian motion.

4.2 Tightness

To deal with convergence in distribution of stochastic processes considered as random elements in $\mathcal{H}_{\mathbb{B}}^{\rho,o}$, a key tool is the following tightness criterion established in Račkauskas and Suquet [40].

Theorem 4. *Suppose the Banach space $\mathbb{B}$ is separable. Then the sequence $(\xi_n)_{n\geq 1}$ of random elements in $\mathcal{H}_{\mathbb{B}}^{\rho,o}$ is tight if and only if it satisfies the two following conditions.*

(i) For each dyadic *$t \in [0,1]$, the sequence of $\mathbb{B}$-valued random variables $(\xi_n(t))_{n\geq 1}$ is tight on $\mathbb{B}$.*

(ii) For each positive ε,

$$\lim_{J\to\infty} \sup_{n\geq 1} \mathbf{P}\left(\sup_{j>J} \frac{1}{\rho(2^{-j})} \max_{r\in \mathrm{D}_j} \|\lambda_r(\xi_n)\| > \varepsilon \right) = 0.$$

4.3 Partial sums processes

Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. random elements in the separable Banach space $\mathbb{B}$. Set $S_0 := 0$, $S_k := X_1 + \cdots + X_k$, for $k = 1, 2, \ldots$ and consider the partial sums processes

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0,1].$$

When $\mathbb{B} = \mathbb{R}$, Donsker-Prohorov invariance principle states, that if $\mathbf{E}\, X_1 = 0$ and $\mathbf{E}\, X_1^2 = \sigma^2 < \infty$, then

$$n^{-1/2}\sigma^{-1}\xi_n \xrightarrow{\mathcal{D}} W, \tag{16}$$

in $\mathcal{C}[0,1]$, where $\{W(t), t \in \mathbb{R}\}$ is a standard Wiener process. The necessity of $\mathbf{E}\, X_1^2 < \infty$ is clear here, since (16) implies the CLT for $n^{-1/2}S_n = n^{-1/2}\xi_n(1)$.

Lamperti [22] was the first who considered the convergence (16) with respect to some Hölderian topology. He proved that if $0 < a < 1/2$ and $\mathbf{E}\, |X_1|^p < \infty$, where $p > p(a) := 1/(1/2 - a)$, then (16) takes place in $\mathcal{H}^{a,o}$. This result was derived again by Kerkyacharian and Roynette [19] by another method based on Ciesielski [9] analysis of Hölder spaces by triangular functions. Further generalizations were given by Erickson [13] (partial sums processes indexed by $[0,1]^d$), Hamadouche [15] (weakly dependent sequence (X_n)), Račkauskas and Suquet [37] (Banach space valued X_i's and Hölder spaces built on the weight $\rho(h) = h^a \ln^b(1/h)$). The following result is proved in Račkauskas and Suquet [38].

Theorem 5. *Let $0 < a < 1/2$ and $p(a) = 1/(1/2 - a)$. Then*

$$n^{-1/2}\sigma^{-1}\xi_n \xrightarrow[n\to\infty]{\mathcal{D}} W \quad \text{in the space } \mathcal{H}^{a,o}$$

if and only if $\mathbf{E}\, X_1 = 0$ and

$$\lim_{t\to\infty} t^{p(a)}\mathbf{P}(|X_1| \geq t) = 0. \tag{17}$$

Condition (17) yields the existence of moments $\mathbf{E}\,|X_1|^p$ for any $0 \le p < p(a)$. If a approaches $1/2$ then $p(a) \to \infty$. Hence, stronger invariance principle requires higher moments.

The description of more general results requires some background on Gaussian random elements and central limit theorem in Banach spaces. Let $\mathbb{B}'$ be the topological dual of $\mathbb{B}$. For a random element X in $\mathbb{B}$ such that for every $f \in \mathbb{B}'$, $\mathbf{E}\, f(X) = 0$ and $\mathbf{E}\, f^2(X) < \infty$, the covariance operator $Q = Q(X)$ is the linear bounded operator from $\mathbb{B}'$ to $\mathbb{B}$ defined by $Qf = \mathbf{E}\, f(X)X,\ f \in \mathbb{B}'$. A random element $X \in \mathbb{B}$ (or covariance operator Q) is said to be *pregaussian* if there exists a mean zero Gaussian random element $Y \in \mathbb{B}$ with the same covariance operator as X, i.e. for all $f, g \in \mathbb{B}'$, $\mathbf{E}\, f(X)g(X) = \mathbf{E}\, f(Y)g(Y)$. Since the distribution of a centered Gaussian random element is defined by its covariance structure, we denote by Y_Q a zero mean Gaussian random element with covariance operator Q.

For any pregaussian covariance Q there exists a $\mathbb{B}$-valued Brownian motion W_Q with parameter Q, a centered Gaussian process indexed by $[0, 1]$ with independent increments such that $W_Q(t) - W_Q(s)$ has the same distribution as $|t - s|^{1/2} Y_Q$.

We say that X_1 satisfies the *central limit theorem in* $\mathbb{B}$, which we denote by $X_1 \in \mathrm{CLT}(\mathbb{B})$, if $n^{-1/2} S_n$ converges in distribution in $\mathbb{B}$. This implies that $\mathbf{E}\, X_1 = 0$ and X_1 is pregaussian. It is well known, e.g. Ledoux and Talagrand [24] that the central limit theorem for X_1 cannot be characterized in general in terms of integrability of X_1 and involves the geometry of the Banach space $\mathbb{B}$.

We say that X_1 satisfies the *functional central limit theorem in* $\mathbb{B}$, which we denote by $X_1 \in \mathrm{FCLT}(\mathbb{B})$, if $n^{-1/2}\xi_n$ converges in distribution in $\mathcal{C}_{\mathbb{B}}$. Kuelbs [21] extended the classical Donsker-Prohorov invariance principle to the case of $\mathbb{B}$-valued partial sums by proving that $n^{-1/2}\xi_n$ converges in distribution in $\mathcal{C}_{\mathbb{B}}$ to some Brownian motion W if and only if $X_1 \in \mathrm{CLT}(\mathbb{B})$ (in short $X_1 \in \mathrm{CLT}(\mathbb{B})$ if and only if $X_1 \in \mathrm{FCLT}(\mathbb{B})$). Of course in Kuelbs theorem, the parameter Q of W is the covariance operator of X_1.

The convergence in distribution of $n^{-1/2}\xi_n$ in $\mathcal{H}^{\rho,o}_{\mathbb{B}}$, which we denote by $X_1 \in \mathrm{FCLT}(\mathbb{B}, \rho)$, is clearly stronger than $X_1 \in \mathrm{FCLT}(\mathbb{B})$.

An obvious preliminary requirement for the FCLT in $\mathcal{H}^{\rho,o}_{\mathbb{B}}$ is that the $\mathbb{B}$-valued Brownian motion has a version in $\mathcal{H}^{\rho,o}_{\mathbb{B}}$. From this point of view, the critical ρ is $\rho_c(h) = \sqrt{h \ln(\mathrm{e}/h)}$ due to Lévy's Theorem on the modulus of uniform continuity of the Brownian motion. So our interest will be restricted to functions ρ generating a weaker Hölder topology than ρ_c. More precisely, we consider the following class $\mathcal{R}$ of functions ρ.

Definition 2. We denote by $\mathcal{R}$ the class of functions ρ satisfying

i) for some $0 < a \le 1/2$, and some function L which is normalized slowly varying at infinity,

$$\rho(h) = h^a L(1/h), \quad 0 < h \le 1, \tag{18}$$

ii) $\theta(t) = t^{1/2}\rho(1/t)$ is C^1 on $[1, \infty)$,

iii) for some $b > 1/2$ and some $a > 0$, $\theta(t)\ln^{-b}(t)$ is non decreasing on $[a, \infty)$.

The following result is proved in Račkauskas and Suquet [36].

Theorem 6. *Let $\rho \in \mathcal{R}$. Then $X_1 \in \mathrm{FCLT}(\mathbb{B}, \rho)$ if and only if $X_1 \in \mathrm{CLT}(\mathbb{B})$ and for every $A > 0$,*

$$\lim_{t\to\infty} t\mathbf{P}\big(\|X_1\| \geq At^{1/2}\rho(1/t)\big) = 0. \tag{19}$$

If $\rho \in \mathcal{R}$ with $a < 1/2$ in (18) then it suffices to check (19) for $A = 1$ only. Of course the special case $\mathbb{B} = \mathbb{R}$ and $\rho(h) = h^a$ gives back Theorem 5.

In the case where $\rho(h) = h^{1/2}\ln^b(c/h)$ with $b > 1/2$, Condition (19) is equivalent to $\mathbf{E}\exp\Big(\gamma\,\|X_1\|^{1/b}\Big) < \infty$, for each $\gamma > 0$. Let us note, that for the spaces $\mathbb{B} = L_p(0,1)$, $2 \leq p < \infty$, as well as for each finite dimensional space, Condition (19) yields $X_1 \in \mathrm{CLT}(\mathbb{B})$. On the other hand it is well known that for some Banach spaces existence of moments of any order does not guarantee central limit theorem. It is also worth noticing that like in Kuelbs FCLT, all the influence of the geometry of the Banach space $\mathbb{B}$ is absorbed by the condition $X_1 \in \mathrm{CLT}(\mathbb{B})$.

It would be useful to extend the Hölderian FCLT to the case of dependent X_i's. A first step was done by Hamadouche [15] in the special case where $\mathbb{B} = \mathbb{R}$ and under weak dependence (association and α-mixing). The result presented in Račkauskas and Suquet [37] provides a very general approach for $\mathbb{B}$-valued X_i's and any dependence structure, subject to obtaining a good estimate of the partial sums. Laukaitis and Račkauskas [23] obtained Hölderian FCLT for a polygonal line process based on residual partial sums of a stationary Hilbert space valued autoregression (ARH(1)) and applied it to the problem of testing stability of ARH(1) model under different types of alternatives.

4.4 Adaptive self-normalized partial sums processes

In order to relax moment assumptions like (19) in the FCLT for i.i.d. mean zero random variables X_i, Račkauskas and Suquet [33] consider the so called *adaptive self-normalized* partial sums processes. *Self-normalized* means that the classical normalization by $\sqrt{n}$ is replaced by

$$V_n = (X_1^2 + \cdots + X_n^2)^{1/2}.$$

Adaptive means that the vertices of the corresponding random polygonal line have their abscissas at the random points V_k^2/V_n^2 $(0 \leq k \leq n)$ instead of the deterministic equispaced points k/n. By this construction the slope of each line adapts itself to the value of the corresponding random variable.

By ζ_n (respect. ξ_n) we denote the random polygonal partial sums process defined on $[0,1]$ by linear interpolation between the vertices $(V_k^2/V_n^2, S_k)$, $k = 0, 1, \ldots, n$ (respect. $(k/n, S_k)$, $k = 0, 1, \ldots, n$). For the special case $k = 0$, we put $S_0 = 0$, $V_0 = 0$. By convention the random functions $V_n^{-1}\xi_n$ and $V_n^{-1}\zeta_n$ are defined to be the null function on the event $\{V_n = 0\}$. Figure 2 displays the polygonal lines $n^{-1/2}\xi_n$ and $V_n^{-1}\zeta_n$ built on a simulated sample of size $n = 800$ of the symmetric distribution given by $\mathbf{P}(|X_1| > t) = 0.5\,t^{-2.2}\mathbf{1}_{[1,\infty)}(t)$. For these simulated paths we have $\|n^{-1/2}\xi_n\|_{0.49} \simeq 24.75$, while $\|V_n^{-1}\zeta_n\|_{0.49} \simeq 3.05$. This picture shows how adaptive partition of the time interval improves slopes of polygonal line process.

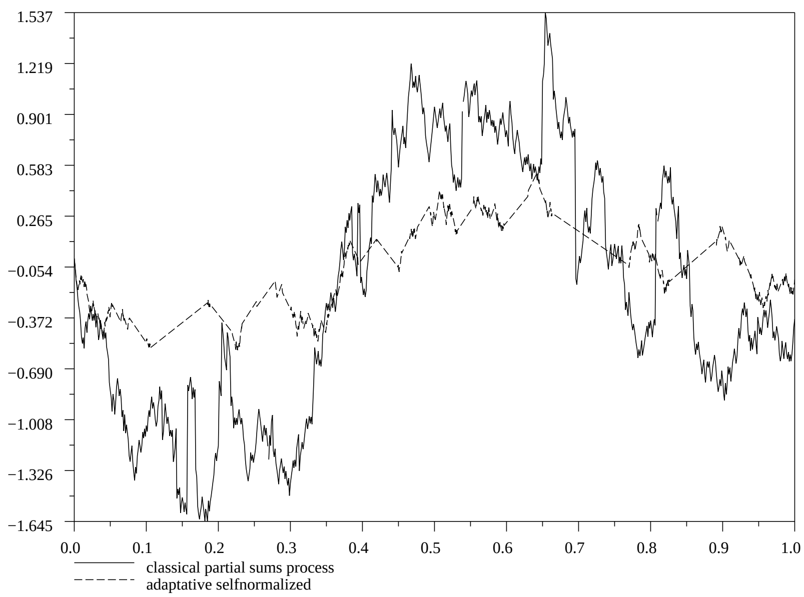

Figure 2: Partial sums processes $n^{-1/2}\xi_n$ and $V_n^{-1}\zeta_n$

Membership of X_1 in the domain of attraction of the normal distribution (DAN) means that there exists a sequence $b_n \uparrow \infty$ such that

$$b_n^{-1} S_n \xrightarrow{\mathcal{D}} N(0,1).$$

The following result is proved in Račkauskas and Suquet [33].

Theorem 7. *Assume that ρ satisfies Conditions (9) to (13) and*

$$\lim_{j\to\infty} \frac{2^j \rho^2(2^{-j})}{j} = \infty. \tag{20}$$

If X_1 is symmetric then

$$V_n^{-1}\zeta_n \xrightarrow{\mathcal{D}} W, \quad in \quad \mathcal{H}^{\rho,o}$$

if and only if $X_1 \in DAN$.

When tested with $\rho(h) = h^{1/2}\ln^b(c/h)$, (20) reduces to $j^{2b-1} \to \infty$. Due to the inclusions of Hölder spaces, this shows that Theorem 7 gives the best result possible in the scale of the separable Hölder spaces $\mathcal{H}^{a,b,o}$. Moreover, no high order moments are needed except the condition $X_1 \in DAN$ which due to well known O'Briens result is equivalent to

$$V_n^{-1} \max_{1\le k\le n} |X_k| \xrightarrow{P} 0.$$

It seems worth noticing here, that without adaptive construction of the polygonal process, the existence of moments of order bigger than 2 is necessary for Hölder weak convergence. Indeed, if the process $V_n^{-1}\xi_n$ converges weakly to W in $\mathcal{H}^{a,o}$ for some $a > 0$, then its maximal slope $n^{-1/2}V_n^{-1}\max_{1\le k\le n}|X_k|$ converges to zero in probability. This on its turn yields $V_n^{-1}\max_{1\le k\le n}|X_k| \to 0$ almost surely, and according to Maller and Resnick (1984), $\mathbf{E}\, X_1^2 < \infty$. Hence $n^{-1}V_n^2$ converges almost surely to $\mathbf{E}\, X_1^2$ by the strong law of large numbers. Therefore $n^{-1/2}\xi_n$ converges weakly to W in $\mathcal{H}^{a,o}$ and by Theorem 5 the moment restriction (17) is necessary.

Naturally it is very desirable to remove the symmetry assumption in Theorem 7. Although the problem remains open, we can propose the following partial result in this direction (for more on this problem see Račkauskas and Suquet [33]).

Theorem 8. *If for some $\varepsilon > 0$, $\mathbf{E}\,|X_1|^{2+\varepsilon} < \infty$, then for any $b > 1/2$, $V_n^{-1}\zeta_n$ converges weakly to W in the space $\mathcal{H}^{1/2,b,o}$.*

Some extensions of this result for the non i.i.d. case are given in Račkauskas and Suquet [39].

4.5 Empirical processes

In asymptotic statistics, the empirical distribution function F_n of an i.i.d. sample $X_1,\dots,X_n$ plays a central rôle. When the distribution function F of the X_i's is continuous, the transformation $U_i := F(X_i)$ reduces the study of the asymptotical behavior of F_n to the case of uniform $[0,1]$ distributed random variables U_i. The corresponding uniform empirical process is defined by

$$\xi_n(t) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(\mathbf{1}_{\{U_i\le t\}} - t\big), \quad t \in [0,1].$$

It is well known that ξ_n converges weakly in the Skorokhod space to the Brownian bridge B. From this convergence follows the weak $\mathcal{C}[0,1]$ convergence to B of ξ_n^{pg}, the polygonal smoothing of ξ_n. This polygonal smoothing is simply the empirical process associated to the polygonal cumulative distribution function. More precisely, let us denote by $U_{n:i}$ the order statistics of the sample $U_1,\dots,U_n$

$$0 = U_{n:0} \le U_{n:1} \le \dots \le U_{n:n} \le U_{n:n+1} = 1.$$

We define $\xi_n^{\mathrm{pg}} = (\xi_n^{\mathrm{pg}}(t),\ t \in [0,1])$ as the random polygonal line with vertices $(U_{n:k}, \xi_n(U_{n:k}))$, $k = 0, 1, \dots, n+1$.

Investigating the weak Hölder convergence of ξ_n^{pg}, Hamadouche [14] proved the following result.

Theorem 9. *The sequence $(\xi_n^{\mathrm{pg}})_{n\geq 1}$ converges weakly to the Brownian bridge B in every $\mathcal{H}^{a,o}$ for $0 < a < 1/4$. Moreover $(\xi_n^{\mathrm{pg}})_{n\geq 1}$ is not tight in $\mathcal{H}^{a,o}$ for $a \geq 1/4$.*

At first sight this result looks somewhat surprising because the limiting process B has a version in every $\mathcal{H}^{a,o}$ for $a < 1/2$ and the paths of ξ_n^{pg} are Lipschitz functions. It illustrates the fact that polygonal smoothing of the empirical distribution function is in some sense too violent. With a convolution smoothing it is possible to achieve the convergence in any $\mathcal{H}^{a,o}$ for $a < 1/2$ (see [16] and the references therein).

Another important stochastic process connected to the empirical distribution function is the so-called uniform quantile process. Put for notational convenience

$$u_{n:i} = \mathbf{E}\, U_{n:i} = \frac{i}{n+1}, \quad i = 0, 1, \dots, n+1.$$

The (discontinuous) uniform quantile process χ_n is given by

$$\chi_n(t) := \sqrt{n}\Big(\sum_{i=1}^{n+1} U_{n:i} \mathbf{1}_{]u_{n:i-1}, u_{n:i}]}(t) - t \Big), \quad t \in [0,1]. \tag{21}$$

We associate to χ_n the polygonal uniform quantile process χ_n^{pg} which is affine on each $[u_{n:i-1}, u_{n:i}]$, $i = 1, \dots, n+1$ and such that

$$\chi_n^{\mathrm{pg}}(u_{n:i}) = \sqrt{n}(U_{n:i} - u_{n:i}), \quad i = 0, 1, \dots, n+1. \tag{22}$$

Using the Hölderian FCLT (Theorem 6), Hamadouche and Suquet [17] obtained the following optimal result.

Theorem 10. *Let $\rho(h) = h^{1/2} L(1/h)$ be a weight function in the class $\mathcal{R}$. Then χ_n^{pg} converges weakly in $\mathcal{H}^{\rho,o}$ to the Brownian bridge if and only if*

$$\lim_{t\to\infty} \frac{L(t)}{\ln t} = \infty. \tag{23}$$

A third process related to empirical process is the empirical characteristic function $\mathfrak{c}_n$. Functional limit theorems for $\mathfrak{c}_n$ in Hölderian framework are investigated in [35] in the multivariate case. For simplicity we shall describe the results in the univariate case only. Let X be a real valued random variable and $(X_k)_{k\geq 1}$ a sequence of independent copies of X. Define respectively the empirical characteristic function $\mathfrak{c}_n$ and the characteristic function $\mathfrak{c}$ by

$$\mathfrak{c}_n(t) := n^{-1} \sum_{k=1}^{n} \exp(itX_k), \quad \mathfrak{c}(t) := \mathbf{E}\, \exp(itX), \quad t \in \mathbb{R}.$$

Here the paths of $\mathfrak{c}_n$ are smooth enough to allow membership in any $\mathcal{H}^{\rho,o}$, so we do not need any smoothing. Clearly $\mathfrak{c}_n$ appears as the sum of i.i.d. random elements in $\mathcal{H}^{\rho,o}$, so that the almost sure convergence in $\mathcal{H}^{\rho,o}$ of $\mathfrak{c}_n$ reduces to some strong law of large numbers in $\mathcal{H}^{\rho,o}$, while the weak $\mathcal{H}^{\rho,o}$ convergence of $n^{1/2}(\mathfrak{c}_n - \mathfrak{c})$ is just a central limit theorem for the random element $\xi :\ t \mapsto \exp(itX)$. The Hölder functions considered here (as elements of the spaces $\mathcal{H}^{\rho,o}$) can be defined on any compact interval of $\mathbb{R}$, but we shall keep $T = [0, 1]$ for simplicity.

Theorem 11. *Assume that the weight function ρ belongs to the class $\mathcal{R}$. Then the convergence*

$$\sup_{\substack{t,s\in T,\\ s\neq t}} \frac{|\mathfrak{c}_n(t) - \mathfrak{c}(t)|}{\rho(|t-s|)} \xrightarrow[n\to\infty]{\text{a.s.}} 0 \tag{24}$$

holds if and only if

$$\mathbf{E}\,\rho^*(|X|) < \infty, \tag{25}$$

where

$$\rho^*(h) := \frac{1}{\rho\big(\min(1; 1/h)\big)}, \quad 0 < h < \infty. \tag{26}$$

In the special case where $\rho(h) = h^a$ for some $0 < a < 1$, Condition (25) writes $\mathbf{E}\,|X|^a < \infty$.

We refer to [35] for a discussion of the rate of convergence in (24), based on some Marcinkiewicz-Zygmund strong law of large numbers in $\mathcal{H}^{\rho,o}$.

Now consider the empirical characteristic process

$$Y_n(t) = \sqrt{n}\big(\mathfrak{c}_n(t) - \mathfrak{c}(t)\big), \quad t \in T.$$

By the multidimensional CLT, the finite dimensional distributions of (Y_n) converge to those of a complex Gaussian process Y with zero mean and covariance $\mathbf{E}\,Y(t)\overline{Y(s)} = \mathfrak{c}(t-s) - \mathfrak{c}(t)\mathfrak{c}(-s)$, $s, t \in T$.

Theorem 12. *If the distribution of X satisfies*

$$\sum_{j=1}^{\infty} \frac{\sqrt{j}}{\rho(2^{-j})} \mathbf{E}^{\,1/2} |\sin(2^{-j}X)|^4 < \infty, \tag{27}$$

then (Y_n) converges in distribution to Y in the space $\mathcal{H}^{\rho,o}$.

Roughly speaking, Condition (27) may be interpreted as the square integrability of the random element $\xi : t \mapsto \exp(itX)$ in a norm a bit stronger than $\|\xi\|_\rho$, see [35]. This is not surprising because the bad geometric properties of $\mathcal{H}^{\rho,o}$ do not allow to deduce the CLT for ξ from the square integrability of $\|\xi\|_\rho$.

4.6 Detection of epidemic changes

Hölderian invariance principles like Theorems 6 and 8 have statistical applications to detection of epidemic change. This question is investigated in [41] for the case of real valued observations and in [42] for the case of Banach space valued (in fact functional) observations. We just sketch here the method.

The epidemic model may be described as follows. Having observed a sample $X_1, X_2, \dots, X_n$ of random variables, we want to test the standard null hypothesis of constant mean

(H_0): $X_1, \dots, X_n$ *all have the same mean denoted by* μ_0,

against the epidemic alternative

(H_A): *there are integers* $1 < k^* < m^* < n$ *and a constant* $\mu_1 \neq \mu_0$ *such that* $\mathbf{E}\, X_i = \mu_0 + (\mu_1 - \mu_0)\mathbf{1}_{\{k^* < i \leq m^*\}}$, $i = 1, 2, \dots, n$.

To simplify notation put

$$\varrho(h) := \rho\big(h(1-h)\big), \quad 0 \leq h \leq 1.$$

For $\rho \in \mathcal{R}$, define with $t_k := k/n$, $0 \leq k \leq n$, $S(t) := \sum_{1 \leq k \leq t} X_k$,

$$\begin{aligned} \mathrm{UI}(n,\rho) &= \max_{1 \leq i < j \leq n} \frac{\big|S(j) - S(i) - S(n)(t_j - t_i)\big|}{\varrho(t_j - t_i)} \\ \mathrm{DI}(n,\rho) &= \max_{1 \leq j \leq \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \Big| S(nr) - \frac{1}{2}\big(S(nr^+) + S(nr^-)\big)\Big|. \end{aligned}$$

These test statistics may be viewed as some discrete Hölder norms of the partial sums process built on the X_k's. Their relevance will be clear from the next result. In what follows, we naturally assume that the numbers of observations k^*, $m^* - k^*$, $n - m^*$ before, during and after the epidemic go to infinity with n. Write $l^* := m^* - k^*$ for the length of the epidemic.

Theorem 13. *Let* $\rho \in \mathcal{R}$. *Assume under* (H_A) *that the* X_i*'s are independent and* $\sigma_0^2 := \sup_{k \geq 1} \operatorname{Var} X_k$ *is finite. If*

$$\lim_{n \to \infty} n^{1/2} \frac{h_n}{\rho(h_n)} = \infty, \quad \textit{where} \quad h_n := \frac{l^*}{n}\Big(1 - \frac{l^*}{n}\Big), \tag{28}$$

then

$$n^{-1/2}\mathrm{UI}(n,\rho) \xrightarrow[n \to \infty]{\mathrm{P}} \infty, \quad \textit{and} \quad n^{-1/2}\mathrm{DI}(n,\rho) \xrightarrow[n \to \infty]{\mathrm{P}} \infty.$$

When $\rho(h) = h^a$, (28) leads to detect *short epidemics* such that $l^* = o(n)$ and $l^* n^{-\delta} \to \infty$, where $\delta = (1-2a)(2-2a)^{-1}$. Symmetrically one can detect *long epidemics* such that $n - l^* = o(n)$ and $(n - l^*)n^{-\delta} \to \infty$. When $\rho(h) = h^{1/2} \ln^b(c/h)$ with $b > 1/2$, (28) is satisfied provided that $h_n = n^{-1} \ln^\gamma n$, with $\gamma > 2b$. This leads to detect short epidemics such that $l^* = o(n)$ and $l^* \ln^{-\gamma} n \to \infty$ as well as of long ones verifying $n - l^* = o(n)$ and $(n - l^*) \ln^{-\gamma} n \to \infty$.

Let $W = \{W(t), t \in [0,1]\}$ be a standard Wiener process and $B = \{B(t), t \in [0,1]\}$ the corresponding Brownian bridge $B(t) = W(t) - tW(1)$, $t \in [0,1]$. Consider for ρ in $\mathcal{R}$, the following random variables

$$\mathrm{UI}(\rho) := \sup_{0<t-s<1} \frac{|B(t) - B(s)|}{\varrho(t-s)} \tag{29}$$

and

$$\mathrm{DI}(\rho) = \sup_{j\geq 1} \frac{1}{\rho(2^{-j})} \max_{r\in \mathrm{D}_j} \Big| W(r) - \frac{1}{2}W(r^+) - \frac{1}{2}W(r^-)\Big| = \|B\|_\rho^{\mathrm{seq}}. \tag{30}$$

These variables serve as limiting for uniform increment (UI) and dyadic increment (DI) statistics respectively. No analytical form seems to be known for the distribution function of $\mathrm{UI}(\rho)$, whereas the distribution of $\mathrm{DI}(\rho)$ is completely specified in terms of the *error function* $\operatorname{erf} x = 2\pi^{-1/2}\int_0^x \exp(-s^2)\,\mathrm{d}s$.

Theorem 14. *Let* $c = \limsup_{j\to\infty} j^{1/2}/\theta(2^j)$, *where* $\theta(t) = t^{1/2}\rho(1/t)$.

i) If $c = \infty$ *then* $\mathrm{DI}(\rho) = \infty$ *almost surely.*

ii) If $0 \leq c < \infty$, *then* $\mathrm{DI}(\rho)$ *is almost surely finite and its distribution function is given by*

$$\mathbf{P}\big(\mathrm{DI}(\rho) \leq x\big) = \prod_{j=1}^{\infty} \big\{\operatorname{erf}\big(\theta(2^j)x\big)\big\}^{2^{j-1}}, \quad x > 0. \tag{31}$$

The distribution function of $\mathrm{DI}(\rho)$ *is continuous with support* $[c\sqrt{\ln 2}, \infty)$.

The infinite product in (31) converges very fast and in practice one need to compute only four or five factors.

Theorem 15. *Under* (H_0) *with i.i.d* X_k*'s, assume that* $\rho \in \mathcal{R}$ *and for every* $A > 0$,

$$\lim_{t\to\infty} t\,\mathbf{P}\big(|X_1| > At^{1/2}\rho(1/t)\big) = 0.$$

Then

$$\sigma^{-1}n^{-1/2}\mathrm{UI}(n,\rho) \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{UI}(\rho) \quad \text{and} \quad \sigma^{-1}n^{-1/2}\mathrm{DI}(n,\rho) \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{DI}(\rho),$$

where $\sigma^2 = \operatorname{Var} X_1$ *and* $\mathrm{UI}(\rho)$, $\mathrm{DI}(\rho)$ *are defined by (29) and (30).*

Of course when the variance σ^2 is unknown the results remain valid if σ^2 is substituted by its standard estimator $\widehat{\sigma}^2$.

References

[1] Ayache A (2001) Du mouvement brownien fractionnaire au mouvement brownien multifractionnaire. Technique et Science Informatiques. 20 9:1133–1152

[2] Ayache A, Lévy Véhel J (1999) Generalized multifractional Brownian motion: definition and preliminary results. In Dekking M, Lévy Véhel J, Lutton E, Tricot C (eds) Fractals: Theory and Applications in Engineering. Springer 17–32

[3] Ayache A, Lévy Véhel J (2000) The Generalized multifractional Brownian motion. Statistical Inference for Stochastic Processes 3:7–18

[4] Ayache A, Lévy Véhel J (2004) Identification of the pointwise Hölder exponent of Generalized Multifractional Brownian Motion. Stochastic Processes and their Applications 111:119–156

[5] Ayache A, Léger S (2000) The multifractional Brownian sheet. To appear in Ann Mat Blaise Pascal

[6] Ayache A, Taqqu MS (2003) Multifractional Processes with Random Exponent. Preprint to appear in Publicacions Matemàtiques

[7] Benassi A, Jaffard S, Roux D (1997) Elliptic Gaussian random processes. Rev Mat Iberoamericana 13 1:19–89

[8] Barral J, Lévy Véhel J (2004) Multifractal analysis of a class of additive processes with correlated nonstationary increments. Electronic Journal of Probability 9:508–543

[9] Ciesielski Z (1960) On the isomorphisms of the spaces H_α and m. Bull Acad Pol Sci Ser Sci Math Phys 8:217–222

[10] Davies S, Hall P (1999) Fractal analysis of surface roughness by using spatial data. J R Statist B 61 1:3–37

[11] Doukhan P, Oppenheim G, Taqqu MS, (eds) (2002) Theory and Applications of Long-range Dependence. Birkhäuser, Boston

[12] Falconer K (1990) Fractal Geometry: Mathematical Fondations and Applications. John Wiley and Sons, New York

[13] Erickson RV (1981) Lipschitz smoothness and convergence with applications to the central limit theorem for summation processes. Annals of Probability 9:831–851

[14] Hamadouche D (1998) Weak convergence of smoothed empirical process in Hölder spaces. Stat Probab Letters 36:393–400

[15] Hamadouche D (2000) Invariance principles in Hölder spaces. Portugaliae Mathematica 57:127–151

[16] Hamadouche D, Suquet Ch (1999) Weak Hölder convergence of stochastic processes with application to the perturbed empirical process. Applicationes Math 26:63–83

[17] Hamadouche D, Suquet Ch (2004) Smoothed quantile processes in Hölder spaces. Pub IRMA Lille (Preprint) 62 IV

[18] Jaffard S, Meyer Y, Ryan RD (2001) Wavelets Tools for Science & Technology. SIAM Philadelphia.

[19] Kerkyacharian G, Roynette B (1991) Une démonstration simple des théorèmes de Kolmogorov, Donsker et Ito-Nisio. Comptes Rendus de l'Académie des Sciences Paris, Série I 312:877–882

[20] Kolmogorov AN (1940) Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. Comptes Rendus (Doklady) de l'Académie des Sciences de l' URSS (NS) 26:115–118

[21] Kuelbs J (1973) The invariance principle for Banach space valued random variables. Journal of Multivariate Analysis 3:161–172

[22] Lamperti J (1962) On convergence of stochastic processes. Transactions of the American Mathematical Society 104:430–435

[23] Laukaitis A, Račkauskas A (2002) Testing changes in Hilbert space autoregressive models. Lietuvos Matematikos Rinkinys 42:434–447

[24] Ledoux M, Talagrand M (1991) Probability in Banach Spaces. Springer-Verlag, Berlin Heidelberg.

[25] Lévy P (1937) Théorie de l'addition des variables aléatoires. Gauthier-Villars, Paris, Second Edition (1954)

[26] Lévy Véhel J (1998) Introduction to the Multifractal Analysis of Images. In Fisher Y (ed) Fractal Image Encoding and Analysis. Springer

[27] Lévy Véhel J (2002) Signal enhancement based on Hölder regularity analysis. IMA Volumes in Mathematics and its Applications 132:197–209

[28] Peltier RF, Lévy Véhel J (1995) Multifractional Brownian Motion: definition and preliminary results. Rapport de recherche de l'INRIA, No 2645

[29] Lifshits MA (1995) Gaussian Random Functions. Kluwer Academic Publishers, Dordrecht Boston London

[30] Mandelbrot BB, Van Ness JW (1968) Fractional Brownian motions, fractional noises and applications. SIAM Review 10:422–437

[31] Račkauskas A, Suquet Ch (1999) Central limit theorem in Hölder spaces. Probability and Mathematical Statistics 19:155–174

[32] Račkauskas A, Suquet Ch (1999) Random fields and central limit theorem in some generalized Hölder spaces. In: Grigelionis B et al (eds) Prob Theory and Math Statist. Proceedings of the 7th Vilnius Conference (1998) TEV Vilnius and VSP Utrecht 599–616

[33] Račkauskas A, Suquet Ch (2001) Invariance principles for adaptive self-normalized partial sums processes. Stochastic Processes and their Applications 95:63–81

[34] Račkauskas A, Suquet Ch (2001) Hölder versions of Banach spaces valued random fields. Georgian Mathematical Journal 8 2:347–362

[35] Račkauskas A, Suquet Ch (2002) Hölder convergences of multivariate empirical characteristic functions. Mathematical Methods of Statistics vol. 11:3:

[36] Račkauskas A, Suquet Ch (2004) Necessary and sufficient condition for the Hölderian functional central limit theorem. Journal of Theoretical Probability 17 1:221–243

[37] Račkauskas A, Suquet Ch (2002) On the Hölderian functional central limit theorem for i.i.d. random elements in Banach space. In: Berkes I, Csáki E, Csörgő M (eds) Limit Theorems in Probability and Statistics. Balatonlelle (1999) (János Bolyai Mathematical Society, Budapest 2:485–498

[38] Račkauskas A, Suquet Ch (2003) Necessary and sufficient condition for the Lamperti invariance principle. Theory of Probability and Mathematical Statistics 68:115–124

[39] Račkauskas A, Suquet Ch (2003) Invariance principle under self-normalization for nonidentically distributed random variables. Acta Applicandae Mathematicae 79 1-2:83–103

[40] Račkauskas A, Suquet Ch (2004) Central limit theorems in Hölder topologies for Banach space valued random fields. Theory of Probability and its Applications 49 1:109–125

[41] Račkauskas A, Suquet Ch (2004) Hölder norm test statistics for epidemic change. Journal of Statistical Planning and Inference 126 2:495–520

[42] Račkauskas A, Suquet Ch (2003) Testing epidemic change of infinite dimensional parameters. Preprint to appear in Statistical Inference for Stochastic Processes

[43] Rosinski J, Samorodnitsky G (1996) Symmetrization and concentration inequalities for multilinear forms with applications to zero-one laws for Lévy chaos. Ann Probab 24 1:422–437

[44] Samorodnitsky G, Taqqu MS (1994) Stable non-Gaussian random processes. Chapman & Hall.

[45] Tricot C (1993) Curves and Fractal dimensions. Springer-Verlag.