Necessary and sufficient condition for the functional central limit theorem in Hölder spaces

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Revised version September 10, 2003

Abstract

Let $(X_i)_{i\geq 1}$ be an i.i.d. sequence of random elements in the Banach space B, $S_n := X_1+\cdots+X_n$ and ξ_n be the random polygonal line with vertices $(k/n, S_k)$, $k = 0, 1, \dots, n$. Put $\rho(h) = h^\alpha L(1/h)$, $0 \leq h \leq 1$ with $0 < \alpha \leq 1/2$ and L slowly varying at infinity. Let $\mathrm{H}^o_\rho(B)$ be the Hölder space of functions $x : [0, 1] \mapsto B$, such that $||x(t + h) - x(t)|| = o(\rho(h))$, uniformly in t. We characterize the weak convergence in $\mathrm{H}^o_\rho(B)$ of $n^{-1/2}\xi_n$ to a Brownian motion. In the special case where $B = \mathbb{R}$ and $\rho(h) = h^\alpha$, our necessary and sufficient conditions for such convergence are $\mathbf{E}X_1 = 0$ and $\mathbf{P}(|X_1| > t) = o(t^{-p(\alpha)})$ where $p(\alpha) = 1/(1/2 - \alpha)$. This completes Lamperti (1962) invariance principle.

MSC 2000 subject classifications. Primary-60F17; secondary-60B12.

Key words and phrases. Central limit theorem in Banach spaces, Hölder space, invariance principle, partial sums process.

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[3]Research supported by a cooperation agreement CNRS/LITHUANIA (4714).

[4]Partially supported by Vilnius Institute of Mathematics and Informatics.

1 Introduction

Let $(B, \| \, \|)$ be a separable Banach space and $X_1, \dots, X_n, \dots$ be i.i.d. random elements in B. Set $S_0 = 0$,

$$S_k = X_1 + \cdots + X_k, \quad \text{for} \quad k = 1, 2, \dots$$

and consider the partial sums processes

$$\xi_n(t) = S_{[nt]} + (nt - [nt]) X_{[nt]+1}, \quad t \in [0, 1]$$

and

$$\xi_n^{\mathrm{sr}} := n^{-1/2} \xi_n.$$

In the familiar case where B is the real line $\mathbb{R}$, classical Donsker-Prohorov invariance principle states, that if $\mathbf{E}\, X_1 = 0$ and $\mathbf{E}\, X_1^2 = 1$, then

$$\xi_n^{\mathrm{sr}} \xrightarrow{\mathcal{D}} W, \tag{1}$$

in $C[0, 1]$, where $(W(t), t \in \mathbb{R})$ is a standard Wiener process and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. The finiteness of the second moment of X_1 is clearly necessary here, since (1) yields that $\xi_n^{\mathrm{sr}}(1)$ satisfies the central limit theorem.

Replacing $C[0, 1]$ in (1) by a stronger topological framework provides more continuous functionals of paths. With this initial motivation, Lamperti [7] considered the convergence (1) with respect to some Hölderian topology. Let us recall his result.

For $0 < \alpha < 1$, let H_α^o be the vector space of continuous functions $x : [0, 1] \to \mathbb{R}$ such that $\lim_{\delta \to 0} \omega_\alpha(x, \delta) = 0$, where

$$\omega_\alpha(x, \delta) = \sup_{\substack{s, t \in [0,1], \\ 0 < t - s < \delta}} \frac{|x(t) - x(s)|}{|t - s|^\alpha}.$$

H_α^o is a separable Banach space when endowed with the norm

$$\|x\|_\alpha := |x(0)| + \omega_\alpha(x, 1).$$

Lamperti [7] proved that if $0 < \alpha < 1/2$ and $\mathbf{E}\, |X_1|^p < \infty$, where $p > p(\alpha) := 1/(1/2 - \alpha)$, then (1) takes place in H_α^o. This result was derived again by Kerkyacharian and Roynette [5] by another method based on Ciesielski [2] analysis of Hölder spaces by triangular functions. Further generalizations were given by Erickson [3] (partial sums processes indexed by $[0, 1]^d$), Hamadouche [4] (weakly dependent sequence (X_n)), Račkauskas and Suquet [10] (Banach space valued X_i's and Hölder spaces built on the moduli $\rho(h) = h^\alpha \ln^\beta(1/h)$).

Considering a symmetric random variable X_1 such that $\mathbf{P}\{X_1 \geq u\} = c u^{-p(\alpha)}$, $u \geq 1$, Lamperti [7] noticed that the sequence (ξ_n^{sr}) is not tight in H_α^o. This gives some hint that the cost of the extension of the invariance principle to the Hölderian setting is beyond the square integrability of X_1.

The simplest case of our general result provides a full answer to this question for the space H_α^o.

Theorem 1. *Let* $0 < \alpha < 1/2$ *and* $p(\alpha) = 1/(1/2 - \alpha)$. *Then*

$$\xi_n^{\mathrm{sr}} \xrightarrow[n \to \infty]{\mathcal{D}} W \quad \text{in the space} \quad \mathrm{H}_\alpha^o$$

if and only if $\mathbf{E}\, X_1 = 0$ *and*

$$\lim_{t \to \infty} t^{p(\alpha)} \mathbf{P}\{|X_1| \geq t\} = 0. \tag{2}$$

We would like to point here that Theorem 1 contrasts strongly with the Hölderian invariance principle for the *adaptive self-normalized* partial sums processes ζ_n^{se}. These are defined as random polygonal lines of interpolation between the vertices $(V_k^2/V_n^2, S_k/V_n)$, $k = 0, 1, \dots, n$, where $V_0^2 = 0$ and $V_k^2 = X_1^2 + \dots X_k^2$. It is shown in [11] that (ζ_n^{se}) converges in distribution to W in any H_α^o $(0 < \alpha < 1/2)$ provided that $\mathbf{E}\,|X_1|^{2+\varepsilon}$ is finite for some arbitrary small $\varepsilon > 0$. This condition can even be relaxed into "X_1 is in the domain of attraction of the normal distribution" in the case of symmetric X_i's (this last condition is also necessary).

To describe our general result, some notations are needed here. We write $\mathrm{C}(B)$ for the Banach space of continuous functions $x : [0,1] \to B$ endowed with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\|;\ t \in [0,1]\}$. Let ρ be a real valued non decreasing function on $[0,1]$, null and right continuous at 0, positive on $(0,1]$. Put

$$\omega_\rho(x,\delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{\|x(t)-x(s)\|}{\rho(t-s)}.$$

We associate to ρ the Hölder space

$$\mathrm{H}_\rho^o(B) := \{x \in \mathrm{C}(B);\ \lim_{\delta\to 0} \omega_\rho(x,\delta) = 0\},$$

equiped with the norm

$$\|x\|_\rho := \|x(0)\| + \omega_\rho(x,1).$$

We say that X_1 satisfies the central limit theorem in B, which we denote by $X_1 \in \mathrm{CLT}(B)$, if $n^{-1/2}S_n$ converges in distribution in B. This implies that $\mathbf{E}\,X_1 = 0$ and X_1 is *pregaussian*. It is well known (e.g. [8]), that the central limit theorem for X_1 cannot be characterized in general in terms of integrability of X_1 and involves the geometry of the Banach space B. Of course some integrability of X_1 and the partial sums is *necessary* for the CLT. More precisely, e.g. [8, Corollary 10.2], if $X_1 \in \mathrm{CLT}(B)$, then

$$\lim_{t\to\infty} t^2 \sup_{n\ge 1} \mathbf{P}\big\{\|S_n\| > t\sqrt{n}\big\} = 0.$$

The space $\mathrm{CLT}(B)$ may be endowed with the norm

$$\mathbf{clt}(X_1) := \sup_{n\ge 1} \mathbf{E}\,\|n^{-1/2}S_n\|. \tag{3}$$

Let us recall that a B valued Brownian motion W with parameter μ (μ being the distribution of a Gaussian random element Y on B) is a Gaussian process indexed by $[0,1]$, with independent increments such that $W(t) - W(s)$ has the same distribution as $|t-s|^{1/2}Y$.

The extension of the classical Donsker-Prohorov invariance principle to the case of B-valued partial sums is due to Kuelbs [6] who established that ξ_n^{sr} converges in distribution in $\mathrm{C}(B)$ to some Brownian motion W if and only if $X_1 \in \mathrm{CLT}(B)$. This convergence of ξ_n^{sr} will be referred to as the functional central limit theorem in $\mathrm{C}(B)$ and denoted by $X_1 \in \mathrm{FCLT}(B)$. Of course in Kuelbs FCLT, the parameter μ of W is the Gaussian distribution on B with same expectation and covariance structure as X_1. The stronger property of convergence in distribution of ξ_n^{sr} in $\mathrm{H}_\rho^o(B)$ will be denoted by $X_1 \in \mathrm{FCLT}(B,\rho)$.

An obvious preliminary requirement for the FCLT in $\mathrm{H}_\rho^o(B)$ is that the B-valued Brownian motion has a version in $\mathrm{H}_\rho^o(B)$. From this point of view, the critical ρ is $\rho_c(h) = \sqrt{h\ln(1/h)}$ due to Lévy's Theorem on the modulus of uniform continuity of the Brownian motion (see e.g. [12] and Proposition 4 below). So our interest will be restricted to functions ρ generating a weaker Hölder topology than ρ_c. More precisely, we consider the functions ρ of the form $\rho(h) = h^\alpha L(1/h)$ where $0 < \alpha \le 1/2$ and L is slowly varying at infinity. Moreover when $\alpha = 1/2$, we assume that $L(t)$ increases faster at infinity than $\ln^\beta t$ for some $\beta > 1/2$.

Throughout the paper we use the notation

$$\theta(t) = t^{1/2}\rho\Big(\frac{1}{t}\Big), \quad t \ge 1. \tag{4}$$

Our characterization of the FCLT in the Hölder space $\mathrm{H}_\rho^o(B)$ reads now simply: $X_1 \in \mathrm{FCLT}(B,\rho)$ *if and only if* $X_1 \in \mathrm{CLT}(B)$ and for every $A > 0$,

$$\lim_{t\to\infty} t\mathbf{P}\big\{\|X_1\| \ge A\theta(t)\big\} = 0.$$

Moreover when $\alpha < 1/2$, it is enough to take $A = 1$ in the above condition. Clearly in the special case $B = \mathbb{R}$ and $\rho(h) = h^\alpha$, this characterization is exactly Theorem 1. It is also worth noticing that like in Kuelbs FCLT, all the influence of the geometry of the Banach space B is absorbed by the condition $X_1 \in \mathrm{CLT}(B)$.

The paper is organized as follows. Section 2 presents some background on the sequential norm equivalent to the initial Hölder norm of $\mathrm{H}_\rho^o(B)$, the tightness in $\mathrm{H}_\rho^o(B)$ and the admissible Hölder topologies for the FCLT. Section 3 gives a general necessary condition which holds even for more general ρ. Section 4 contains the proof of the sufficient part in the characterization of Hölderian FCLT. Some technical auxilliary results are deferred in Section 5 to avoid overweighting of the exposition.

2 Preliminaries

2.1 Analytical background

With the aim to use a sequential norm equivalent to $\|x\|_\rho$, we require, following Ciesielski (see e.g. [13, p.67]), that the modulus of smoothness ρ satisfies the conditions:

$$\rho(0) = 0,\ \rho(h) > 0,\ 0 < h \le 1; \tag{5}$$

$$\rho \text{ is non decreasing on } [0,1]; \tag{6}$$

$$\rho(2h) \le c_1\rho(h), \quad 0 \le h \le 1/2; \tag{7}$$

$$\int_0^h \frac{\rho(u)}{u}\,\mathrm{d}u \le c_2\rho(h), \quad 0 < h \le 1; \tag{8}$$

$$h\int_h^1 \frac{\rho(u)}{u^2}\,\mathrm{d}u \le c_3\rho(h), \quad 0 < h \le 1; \tag{9}$$

where c_1, c_2 and c_3 are positive constants. Let us observe in passing, that (5), (6) and (8) together imply the right continuity of ρ at 0. The class of functions ρ satisfying these requirements is rich enough according to the following.

Proposition 2. *For any* $0 < \alpha < 1$, *consider the function*

$$\rho(h) = h^\alpha L(1/h)$$

where L is normalized slowly varying at infinity, continuous and positive on $[1,\infty)$. *Then ρ fulfills conditions (5) to (9) up to a change of scale.*

Proof. Let us recall that $L(t)$ is a positive continuous normalized slowly varying at infinity if it has a representation

$$L(t) = c \exp\left\{\int_b^t \varepsilon(u)\frac{\mathrm{d}u}{u}\right\}$$

with $0 < c < \infty$ constant and $\varepsilon(u) \to 0$ when $u \to \infty$. By a theorem of Bojanic and Karamata [1, Th. 1.5.5], the class of normalized slowly varying functions is exactly the Zygmund class *i.e.* the class of functions $f(t)$ such that for every $\delta > 0$, $t^\delta f(t)$ is ultimately increasing and $t^{-\delta} f(t)$ is ultimately decreasing. It follows that for some $0 < a \le 1$, ρ is non decreasing on $[0, a]$. Then (6) is satisfied by $\tilde\rho(h) := \rho(ah)$.

Due to the continuity and positivity of $\tilde\rho$ on $(0, 1]$, each inequality (7) to (9) will be fulfilled if its left hand side divided by $\tilde\rho(h)$ has a positive limit when h goes to 0. For (7), this limit is clearly 2^α.

For (8), we have by [1, Prop. 1.5.10],

$$\int_0^h \frac{\tilde\rho(u)}{u}\,\mathrm{d}u = a^\alpha \int_{1/h}^\infty v^{-1-\alpha} L(v/a)\,\mathrm{d}v \sim \frac{1}{\alpha}\tilde\rho(h).$$

Similarly for (9), we obtain by [1, Prop. 1.5.8],

$$h\int_h^1 \frac{\tilde\rho(u)}{u^2}\,\mathrm{d}u = a^\alpha h \int_1^{1/h} v^{-\alpha} L(v/a)\,\mathrm{d}v \sim \frac{\tilde\rho(h)}{1-\alpha}.$$

□

Write D_j for the set of dyadic numbers of level j in $[0, 1]$, *i.e.* $D_0 = \{0, 1\}$ and for $j \ge 1$,

$$D_j = \{\,(2k+1)2^{-j};\ 0 \le k < 2^{j-1}\,\}.$$

For any continuous function $x : [0, 1] \to B$, define

$$\lambda_{0,t}(x) := x(t), \quad t \in D_0$$

and for $j \ge 1$,

$$\lambda_{j,t}(x) := x(t) - \frac{1}{2}\big(x(t+2^{-j}) + x(t-2^{-j})\big), \quad t \in D_j.$$

The $\lambda_{j,t}(x)$ are the B-valued coefficients of the expansion of x in a series of triangular functions. The j-th partial sum $E_j x$ of this series is exactly the polygonal line interpolating x between the dyadic points $k2^{-j}(0 \le k \le 2^j)$. Under (5) to (9), the norm $\|x\|_\rho$ is equivalent (see e.g. [12]) to the sequence norm

$$\|x\|_\rho^{\mathrm{seq}} := \sup_{j\ge 0} \frac{1}{\rho(2^{-j})} \max_{t\in D_j} \|\lambda_{j,t}(x)\|.$$

It is easy to check that

$$\|x - E_j x\|_\rho^{\mathrm{seq}} = \sup_{i>j} \frac{1}{\rho(2^{-i})} \max_{t\in D_i} \|\lambda_{i,t}(x)\|. \tag{10}$$

2.2 Tightness

The dyadic affine interpolation which is behind the sequential norm is also useful to investigate the tightness in $\mathrm{H}_\rho^o(B)$. Indeed it is not difficult to check that $\mathrm{H}_\rho^o(B)$ can be expressed as a topological direct sum of closed subspaces (a Schauder decomposition) by

$$\mathrm{H}_\rho^o(B) = \bigoplus_{i=0}^{\infty} \mathbf{W}_i.$$

Here $\mathbf{W}_0$ is the space of B-valued functions defined and affine on $[0,1]$ and for $i \geq 1$, $\mathbf{W}_i$ is the space of B-valued polygonal lines with vertices at the dyadics of level at most i and vanishing at each dyadic of level less than i. It may be helpful to note here that each $\mathbf{W}_i$ has infinite dimension with B.

This Schauder decomposition of $\mathrm{H}_\rho^o(B)$ allows us to apply Theorem 3 in [14] and obtain the following tightness criterion.

Theorem 3. *The sequence (Y_n) of random elements in $\mathrm{H}_\rho^o(B)$ is tight if and only if the following two conditions are satisfied:*

i) For each dyadic $t \in [0,1]$, the sequence $(Y_n(t))_{n\geq 1}$ is tight on B.

ii) For each $\varepsilon > 0$,

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathbf{P}\{\|Y_n - E_j Y_n\|_\rho^{\mathrm{seq}} > \varepsilon\} = 0. \tag{11}$$

2.3 Admissible Hölder norms

Let us discuss now the choice of the functions ρ for wich it is reasonable to investigate a Hölderian FCLT. If $X_1 \in \mathrm{FCLT}(B,\rho)$ and ℓ is a linear continuous functional on B then clearly $\ell(X_1) \in \mathrm{FCLT}(\mathbb{R},\rho)$. So we may as well assume $B = \mathbb{R}$ in looking for a necessary condition on ρ. As polygonal lines, the paths of ξ_n^{sr} belong to H_ρ^o for any ρ such that $h/\rho(h) \to 0$, when $h \to 0$. The weaker smoothness of the limit process W and the necessity of its membership in H_ρ^o put a more restrictive condition on ρ.

Proposition 4. *Assume that for some X_1, the corresponding process ξ_n^{sr} converges weakly to W in H_ρ^o. Then*

$$\lim_{t\to\infty} \frac{\theta(t)}{\ln^{1/2} t} = \infty. \tag{12}$$

Proof. Let $\omega(W,\delta)$ denote the modulus of uniform continuity of W. Since W has necessarily a version in H_ρ^o, we see that $\omega(W,\delta)/\rho(\delta)$ goes a.s. to zero when $\delta \to 0$. This convergence may be recast as

$$\lim_{\delta\to 0} \frac{\omega(W,\delta)}{\sqrt{\delta \ln(1/\delta)}} \frac{\sqrt{\delta\ln(1/\delta)}}{\rho(\delta)} = 0 \quad \text{a.s.}$$

By Lévy's result [9, Th. 52,2] on the modulus of uniform continuity of W, we have with positive probability $\liminf_{\delta\to 0} \omega(W,\delta)/\sqrt{\delta\ln(1/\delta)} > 0$, so the above convergence implies

$$\lim_{\delta\to 0} \frac{\sqrt{\delta\ln(1/\delta)}}{\rho(\delta)} = 0,$$

which is the same as (12). □

3 A general requirement for the Hölderian FCLT

We prove now that a necessary condition for X_1 to satisfy the Hölderian FCLT in $\mathrm{H}_\rho^o(B)$ is that for every $A > 0$,

$$\lim_{t\to\infty} t\mathbf{P}\{\|X_1\| > A\theta(t)\} = 0.$$

In fact, the same tail condition must hold uniformly for the normalized partial sums, so the above convergence is a simple by-product of the following general result. We point out that Conditions (6) to (9) are not involved here. In this section the restriction on ρ comes from Proposition 4.

Theorem 5. *If the sequence $(\xi_n^{\mathrm{sr}})_{n\geq 1}$ is tight in $\mathrm{H}_\rho^o(B)$, then for every positive constant A,*

$$\lim_{t\to\infty} t \sup_{m\geq 1} \mathbf{P}\{\|S_m\| > m^{1/2}A\theta(t)\} = 0. \tag{13}$$

Proof. As a preliminary step, we claim and check that

$$\lim_{N\to\infty} \sup_{n\geq 1} \mathbf{P}\{\omega_\rho(\xi_n^{\mathrm{sr}}, 1/N) \geq A\} = 0. \tag{14}$$

From the tightness assumption, for every positive ε there is a compact subset K in $\mathrm{H}_\rho^o(B)$ such that

$$\mathbf{P}\{\omega_\rho(\xi_n^{\mathrm{sr}}, 1/N) \geq A\} \leq \mathbf{P}\{\omega_\rho(\xi_n^{\mathrm{sr}}, 1/N) \geq A \text{ and } \xi_n^{\mathrm{sr}} \in K\} + \varepsilon.$$

Define the functionals Φ_N on $\mathrm{H}_\rho^o(B)$ by $\Phi_N(f) := \omega_\rho(f; 1/N)$. By the definition of $\mathrm{H}_\rho^o(B)$, the sequence $(\Phi_N)_{N\geq 1}$ decreases to zero pointwise on $\mathrm{H}_\rho^o(B)$. Moreover each Φ_N is continuous in the strong topology of $\mathrm{H}_\rho^o(B)$. By Dini's theorem this gives the uniform convergence of $(\Phi_N)_{N\geq 1}$ to zero on the compact K. Then we have $\sup_{f\in K}\Phi_N(f) < A$ for every N bigger than some $N_0 = N_0(A, K)$. It follows that for $N > N_0$ and $n \geq 1$,

$$\mathbf{P}\{\omega_\rho(\xi_n^{\mathrm{sr}}, 1/N) \geq A \text{ and } \xi_n^{\mathrm{sr}} \in K\} = 0,$$

which leads to

$$\mathbf{P}\{\omega_\rho(\xi_n^{\mathrm{sr}}, 1/N) \geq A\} < \varepsilon, \quad N > N_0, n \geq 1,$$

completing the verification of (14). In particular we get

$$\lim_{N\to\infty} \sup_{m\geq 1} \mathbf{P}\{\omega_\rho(\xi_{mN}^{\mathrm{sr}}, 1/N) \geq A\} = 0. \tag{15}$$

Now we observe that

$$\max_{1\leq k\leq N} \frac{1}{\rho(1/N)} \|\xi_{mN}^{\mathrm{sr}}(k/N) - \xi_{mN}^{\mathrm{sr}}((k-1)/N)\| \leq \omega_\rho(\xi_{mN}^{\mathrm{sr}}, 1/N).$$

Writing for simplicity

$$Y_{k,m} = \|m^{-1/2}(S_{mk} - S_{m(k-1)})\|,$$

we have from (15) that

$$\lim_{N\to\infty} \sup_{m\geq 1} \mathbf{P}\Big\{\max_{1\leq k\leq N} Y_{k,m} > A\theta(N)\Big\} = 0, \tag{16}$$

recalling that $\theta(N) = N^{1/2}\rho(1/N)$. By independence and identical distribution of the X_i's,

$$\mathbf{P}\Big\{\max_{1\leq k\leq N} Y_{k,m} > A\theta(N)\Big\} = 1 - \big(1 - \mathbf{P}\{Y_{1,m} > A\theta(N)\}\big)^N. \tag{17}$$

Consider the function $g_N(u) := 1 - (1-u)^N$, $0 \le u \le 1$. As g_N is increasing on $[0,1]$, we have

$$g_N(u) \ge g_N(1/N) = 1 - (1-1/N)^N > 1 - \mathrm{e}^{-1}, \quad 1/N \le u \le 1. \tag{18}$$

By concavity of g_N, we also have

$$g_N(u) \ge N g_N(1/N) u \ge N(1-\mathrm{e}^{-1})u, \quad 0 \le u \le 1/N. \tag{19}$$

Write $u_{m,N} := \mathbf{P}\bigl\{Y_{1,m} > A\theta(N)\bigr\}$ and $u_N := \sup_{m\ge 1} u_{m,N}$. By increasingness and continuity of g_N, $\sup_{m\ge1} g_N(u_{m,N}) = g_N(u_N)$. This together with (16) and (17) shows that $\lim_{N\to\infty} g_N(u_N) = 0$. By (18), it follows that $0 \le u_N \le 1/N$, for N large enough. In view of (19), we have then $\lim_{N\to\infty} N u_N = 0$. This last convergence can be recast more explicitly as

$$\lim_{N\to\infty} N \sup_{m\ge1} \mathbf{P}\bigl\{\lVert S_m\rVert > m^{1/2} A\theta(N)\bigr\} = 0,$$

which is clearly equivalent to (13). □

4 Characterizing the Hölderian FCLT

Before proving our main result let us set assumptions for $\rho(h)$.

Definition 6. We denote by $\mathcal{R}$ the class of non decreasing functions ρ satisfying

i) for some $0 < \alpha \le 1/2$, and some positive function L which is normalized slowly varying at infinity,

$$\rho(h) = h^\alpha L(1/h), \quad 0 < h \le 1; \tag{20}$$

ii) $\theta(t) = t^{1/2}\rho(1/t)$ is C^1 on $[1,\infty)$;

iii) there is a $\beta > 1/2$ and some $a > 1$, such that $\theta(t)\ln^{-\beta}(t)$ is non decreasing on $[a,\infty)$.

Remark 7. Clearly $L(t)\ln^{-\beta}(t)$ is normalized slowly varying for any $\beta > 0$, so when $\alpha < 1/2$, $t^{1/2-\alpha}L(t)\ln^{-\beta}(t)$ is ultimately non decreasing and iii) is automatically satisfied.

The assumption ii) of C^1 regularity for θ is not a real restriction, since the function $\rho(1/t)$ being α-regularly varying at infinity is asymptoticaly equivalent to a C^∞ α-regularly varying function $\tilde{\rho}(1/t)$ (see [1]). Then the corresponding Hölderian norms are equivalent.

Put $b := \inf_{t\ge1}\theta(t)$. Since by iii), $\theta(t)$ is ultimately increasing and $\lim_{t\to\infty}\theta(t) = \infty$, we can define its generalized inverse φ on $[b,\infty)$ by

$$\varphi(u) := \sup\{t \ge 1;\ \theta(t) \le u\}. \tag{21}$$

With this definition, we have $\theta(\varphi(u)) = u$ for $u \ge b$ and $\varphi(\theta(t)) = t$ for $t \ge a$.

Theorem 8. *Let $\rho \in \mathcal{R}$. Then $X_1 \in \mathrm{FCLT}(B,\rho)$ if and only if $X_1 \in \mathrm{CLT}(B)$ and for every $A > 0$,*

$$\lim_{t\to\infty} t\mathbf{P}\bigl\{\lVert X_1\rVert \ge A\theta(t)\bigr\} = 0. \tag{22}$$

Corollary 9. *Let $\rho \in \mathcal{R}$ with $\alpha < 1/2$ in (20). Then $X_1 \in \mathrm{FCLT}(B,\rho)$ if and only if $X_1 \in \mathrm{CLT}(B)$ and*

$$\lim_{t\to\infty} t\mathbf{P}\bigl\{\lVert X_1\rVert \ge \theta(t)\bigr\} = 0. \tag{23}$$

Corollary 10. *Let $\rho(h) = h^{1/2}\ln^{\beta}(c/h)$ with $\beta > 1/2$. Then $X_1 \in \mathrm{FCLT}(B,\rho)$ if and only if $X_1 \in \mathrm{CLT}(B)$ and*

$$\mathbf{E}\exp\Big(d\|X_1\|^{1/\beta}\Big) < \infty, \quad \text{for each } d > 0. \tag{24}$$

Corollary 11. *Let $\rho \in \mathcal{R}$ and $B = \mathbb{R}$. Then $X_1 \in \mathrm{FCLT}(\mathbb{R},\rho)$ if and only if $\mathbf{E}\,X_1 = 0$ and either (22) or (23) holds according to the case $\alpha = 1/2$ or $\alpha < 1/2$.*

Remark 12. The requirement "for every $A > 0$" in (22) cannot be avoided in general. For instance let us choose $B = \mathbb{R}$, X_1 symmetric such that $\mathbf{P}\{|X_1| \ge u\} = \exp(-u/c)$, $(c > 0)$ and $\rho(h) = h^{1/2}\ln(1/h)$, so $\theta(t) = \ln t$. Clearly (22) is satisfied only for $A > c$, so $X_1 \notin \mathrm{FCLT}(\mathbb{R},\rho)$.

Proof of Theorem 8. The necessity of "$X_1 \in \mathrm{CLT}(B)$" is obvious while that of (22) is contained in Theorem 5. For the converse part, Kuelbs FCLT gives us for any $m \ge 1$ and $0 \le s_1 < \cdots < s_m \le 1$

$$\big(\xi_n^{\mathrm{sr}}(s_1),\dots,\xi_n^{\mathrm{sr}}(s_m)\big) \xrightarrow{\mathcal{D}} \big(W(s_1),\dots,W(s_m)\big)$$

in the space B^m. In particular, Condition i) of Theorem 3 is automatically fulfilled. So the remaining work is to check Condition (11).

Write for simplicity $t_k = t_{kj} = k2^{-j}$, $k = 0,1,\dots,2^j$, $j = 1,2,\dots$ In view of (10), it is sufficient to prove that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big\{\sup_{J\le j}\frac{1}{\rho(2^{-j})}n^{-1/2}\max_{1\le k<2^j}\|\xi_n(t_{k+1})-\xi_n(t_k)\| \ge \varepsilon\Big\} = 0. \tag{25}$$

To this end, we bound the probability in the left hand side of (25) by $P_1 + P_2$ where

$$P_1 := \mathbf{P}\Big\{\sup_{J\le j\le \log n}\frac{1}{\rho(2^{-j})}n^{-1/2}\max_{1\le k\le 2^j}\|\xi_n(t_{k+1})-\xi_n(t_k)\| \ge \varepsilon\Big\}$$

and

$$P_2 := \mathbf{P}\Big\{\sup_{j>\log n}\frac{1}{\rho(2^{-j})}n^{-1/2}\max_{1\le k\le 2^j}\|\xi_n(t_{k+1})-\xi_n(t_k)\| \ge \varepsilon\Big\}.$$

Here and throughout the paper, log denotes the logarithm *with basis* 2, while ln denotes the natural logarithm ($\log(2^x) = x = \ln(\mathrm{e}^x)$).

Estimation of P_2. If $j > \log n$, then $t_{k+1} - t_k = 2^{-j} < 1/n$ and therefore with $t_k \in [i/n,(i+1)/n)$, either t_{k+1} is in $(i/n,(i+1)/n]$ or belongs to $\big((i+1)/n,(i+2)/n\big]$, where $1 \le i \le n-2$ depends on k and j.

In the first case we have

$$\|\xi_n(t_{k+1})-\xi_n(t_k)\| = \|X_{i+1}\|2^{-j}n \le 2^{-j}n\max_{1\le i\le n}\|X_i\|.$$

If t_k and t_{k+1} are in consecutive intervals, then

$$\begin{aligned}\|\xi_n(t_{k+1})-\xi_n(t_k)\| &\le \|\xi_n(t_k)-\xi_n((i+1)/n)\| + \|\xi_n((i+1)/n)-\xi_n(t_{k+1})\|\\ &\le 2^{-j+1}n\max_{1\le i\le n}\|X_i\|.\end{aligned}$$

With both cases taken into account we obtain

$$\begin{aligned} P_2 &\leq \mathbf{P}\Big\{ \sup_{j>\log n} \frac{1}{\rho(2^{-j})} n^{-1/2} n 2^{-j+1} \max_{1\leq i\leq n} \|X_i\| \geq \varepsilon \Big\} \\ &\leq \mathbf{P}\Big\{ \sup_{j>\log n} \frac{1}{\theta(2^j)} \max_{1\leq i\leq n} \|X_i\| \geq \frac{\varepsilon}{2} \Big\} \\ &\leq \mathbf{P}\Big\{ \max_{1\leq i\leq n} \|X_i\| \geq \frac{\varepsilon}{2} \min_{j>\log n} \theta(2^j) \Big\} \\ &\leq n\mathbf{P}\Big\{ \|X_1\| \geq \frac{\varepsilon}{2}\theta(n) \Big\}, \end{aligned}$$

for $n \geq a$ (see Definition 6.iii)). Hence, due to (22), for each $\varepsilon > 0$, $\lim_{n\to\infty} P_2 = 0$.

Estimation of P_1. Let $u_k = [nt_k]$. Then $u_k \leq nt_k \leq 1 + u_k$ and $1 + u_k \leq u_{k+1} \leq nt_{k+1} \leq 1 + u_{k+1}$. Therefore

$$\|\xi_n(t_{k+1}) - \xi_n(t_k)\| \leq \|\xi_n(t_{k+1}) - S_{u_{k+1}}\| + \|S_{u_{k+1}} - S_{u_k}\| + \|S_{u_k} - \xi_n(t_k)\|.$$

Since $\|S_{u_k} - \xi_n(t_k)\| \leq \|X_{1+u_k}\|$ and $\|\xi_n(t_{k+1}) - S_{u_{k+1}}\| \leq \|X_{1+u_{k+1}}\|$ we obtain $P_1 \leq P_{1,1} + P_{1,2}$, where

$$\begin{aligned} P_{1,1} &:= \mathbf{P}\Big\{ \sup_{J\leq j\leq \log n} \frac{1}{\rho(2^{-j})} n^{-1/2} \max_{1\leq k\leq 2^j} \|S_{u_{k+1}} - S_{u_k}\| \geq \frac{\varepsilon}{2} \Big\} \\ P_{1,2} &:= \mathbf{P}\Big\{ \max_{J\leq j\leq \log n} \frac{1}{\rho(2^{-j})} n^{-1/2} \max_{1\leq i\leq n} \|X_i\| \geq \frac{\varepsilon}{4} \Big\}. \end{aligned}$$

In $P_{1,2}$, the maximum over j is realized for $j = \log n$, so

$$P_{1,2} = \mathbf{P}\Big\{ \frac{1}{\theta(n)} \max_{1\leq i\leq n} \|X_i\| \geq \frac{\varepsilon}{4} \Big\} \leq n\mathbf{P}\Big\{ \|X_1\| \geq \frac{\varepsilon}{4}\theta(n) \Big\},$$

which goes to zero by (22).

To estimate $P_{1,1}$, we use truncation arguments. For a positive δ, that will be precised later, define

$$\widetilde{X}_i := X_i \mathbf{1}\big(\|X_i\| \leq \delta\theta(n)\big), \qquad X'_i := \widetilde{X}_i - \mathbf{E}\,\widetilde{X}_i,$$

where $\mathbf{1}(E)$ denotes the indicator function of the event E. Let $\widetilde{S}_{u_k}$ and $\widetilde{P}_{1,1}$ be the expressions obtained by replacing X_i with $\widetilde{X}_i$ in S_{u_k} and $P_{1,1}$. Similarly we define S'_{u_k} and $P'_{1,1}$ by replacing X_i with X'_i and ε with $\varepsilon/2$. Due to (22), the control of $P_{1,1}$ reduces to that of $\widetilde{P}_{1,1}$ because

$$P_{1,1} \leq \widetilde{P}_{1,1} + \mathbf{P}\Big\{ \max_{1\leq i\leq n} \|X_i\| > \delta\theta(n) \Big\} \leq \widetilde{P}_{1,1} + n\mathbf{P}\big\{ \|X_1\| > \delta\theta(n) \big\}.$$

Now to deal with centered random variables, we shall prove that $\widetilde{P}_{1,1} \leq P'_{1,1}$. It suffices to prove that for n and J large enough, the following holds

$$\sup_{J\leq j\leq \log n} \frac{1}{\rho(2^{-j})} n^{-1/2} \max_{1\leq k\leq 2^j} \sum_{i=1+u_k}^{u_{k+1}} \|\mathbf{E}\,\widetilde{X}_i\| < \frac{\varepsilon}{4}. \tag{26}$$

As $j \leq \log n$,

$$1 \leq u_{k+1} - u_k \leq n2^{-j} + 1 \leq 2n2^{-j}, \quad 0 \leq k < 2^j, \tag{27}$$

so it suffices to have

$$2n^{1/2}\|\mathbf{E}\,\widetilde{X}_1\|\max_{J\le j\le\log n}\frac{2^{-j}}{\rho(2^{-j})}<\frac{\varepsilon}{4}. \tag{28}$$

Writing $2^{-j}/\rho(2^{-j})=2^{-j/2}/\theta(2^j)$ and recalling that θ is non decreasing on $[a,\infty)$, we see that for $J\ge\log a$, (28) reduces to

$$\frac{2n^{1/2}}{2^{J/2}\theta(2^J)}\|\mathbf{E}\,\widetilde{X}_1\|<\frac{\varepsilon}{4}. \tag{29}$$

Now, as $\mathbf{E}\,X_1=0$, we get

$$\|\mathbf{E}\,\widetilde{X}_1\|\le\int_{\delta\theta(n)}^{\infty}\mathbf{P}\big\{\|X_1\|>t)\big\}\,\mathrm{d}t=\int_n^{\infty}\mathbf{P}\big\{\|X_1\|>\delta\theta(u)\big\}\delta\theta'(u)\,\mathrm{d}u.$$

By (22), there is an $u_0(\delta)\ge 1$ such that for $u\ge u_0(\delta)$, $u\mathbf{P}\big\{\|X_1\|>\delta\theta(u)\big\}\le 1$, whence

$$\|\mathbf{E}\,\widetilde{X}_1\|\le-\delta\frac{\theta(n)}{n}+\delta\int_n^{\infty}\frac{\theta(u)}{u^2}\,\mathrm{d}u,\quad n\ge u_0(\delta).$$

As $\theta(u)u^{-1/2}=\rho(1/u)$ is non increasing, this last integral is dominated by $n^{-1/2}\theta(n)\int_n^{\infty}u^{-3/2}\,\mathrm{d}u=2n^{-1}\theta(n)$. Now plugging the estimate

$$\|\mathbf{E}\,\widetilde{X}_1\|\le\delta n^{-1}\theta(n) \tag{30}$$

into the left hand side of (29) gives

$$\frac{2n^{1/2}}{2^{J/2}\theta(2^J)}\|\mathbf{E}\,\widetilde{X}_1\|\le\frac{2\delta}{2^J}\frac{2^{J/2}}{\theta(2^J)}\frac{\theta(n)}{n^{1/2}}\le\frac{2\delta}{2^J}.$$

This concludes (26) provided $\delta/2^J<\varepsilon/8$.

Estimation of $P'_{1,1}$. Recalling (27), we have

$$\begin{aligned}P'_{1,1}&\le\sum_{j=J}^{\log n}\mathbf{P}\Big\{n^{-1/2}\max_{1\le k\le 2^j}\|S'_{u_{k+1}}-S'_{u_k}\|\ge\frac{\varepsilon}{4}\rho(2^{-j})\Big\}\\&\le\sum_{j=J}^{\log n}\mathbf{P}\Big\{\max_{1\le k\le 2^j}\frac{\|S'_{u_{k+1}}-S'_{u_k}\|}{(u_{k+1}-u_k)^{1/2}}\ge\frac{\varepsilon}{4\sqrt{2}}\theta(2^j)\Big\}\\&\le\sum_{j=J}^{\log n}\sum_{k=1}^{2^j}\mathbf{P}\Big\{\frac{\|S'_{u_{k+1}}-S'_{u_k}\|}{(u_{k+1}-u_k)^{1/2}}\ge\frac{\varepsilon}{4\sqrt{2}}\theta(2^j)\Big\}\end{aligned} \tag{31}$$

At this stage we use tail estimates related to ψ_γ-Orlicz norms (see (38) and (39) in Section 5). By Talagrand's inequality (Theorem 15 below), (27), Lemmas 16 and 17, we get for $1<\gamma\le 2$,

$$\begin{aligned}\Big\|\frac{S'_{u_{k+1}}-S'_{u_k}}{(u_{k+1}-u_k)^{1/2}}\Big\|_{\psi_\gamma}&\le K_\gamma\Big(2\mathbf{clt}(X_1)+(u_{k+1}-u_k)^{1/2-1/\gamma}\|X'_1\|_{\psi_\gamma}\Big)\\&\le K'\left(1+\big(n2^{-j}\big)^{1/2-1/\gamma}\frac{\delta\theta(n)}{\ln^{1/\gamma}\varphi\big(\delta\theta(n)\big)}\right),\end{aligned}$$

with a constant K' depending only on γ and of the distribution of X_1.

Now we choose γ such that

$$\frac{1}{2} < \frac{1}{\gamma} < \beta, \tag{32}$$

so $\theta(t)\ln^{-1/\gamma} t \to \infty$, as $t \to \infty$. With $t = \varphi\bigl(\delta\theta(n)\bigr)$, this gives

$$\lim_{n\to\infty} \frac{\delta\theta(n)}{\ln^{1/\gamma}\varphi\bigl(\delta\theta(n)\bigr)} = \infty. \tag{33}$$

Put for notational convenience

$$w_{n,j} := \bigl(n2^{-j}\bigr)^{1/2-1/\gamma}\frac{\delta\theta(n)}{\ln^{1/\gamma}\varphi\bigl(\delta\theta(n)\bigr)}, \quad 0 \le j \le \log n.$$

By (33), for $n \ge n_0$,

$$\frac{\delta\theta(n)}{\ln^{1/\gamma}\varphi\bigl(\delta\theta(n)\bigr)} > 1,$$

which gives in particular $w_{n,\log n} > 1$. As $\gamma < 2$, $w_{n,j}$ is increasing in j, so with $J'_n := \min\{j \le \log n;\ w_{n,j} \ge 1\}$ we have for $n \ge n_0$,

$$\left\| \frac{S'_{u_{k+1}} - S'_{u_k}}{(u_{k+1}-u_k)^{1/2}} \right\|_{\psi_\gamma} \le \begin{cases} 2K' & \text{if} \quad 0 \le j < J'_n \\ 2K'w_{n,j} & \text{if} \quad J'_n \le j \le \log n. \end{cases} \tag{34}$$

Put $J_n := \max(J, J'_n)$. With the usual convention of nullity of a sum indexed by the empty set, we can split the upper bound (31) in two sums Q_1 and Q_2 indexed respectively by $J \le j < J_n$ and $J_n \le j \le \log n$.

Estimation of Q_1. Due to (34) and (39), we have with some constant $c = c(K', \varepsilon)$,

$$\begin{aligned} Q_1 \quad &:= \quad \sum_{J \le j < J_n} \sum_{k=1}^{2^j} \mathbf{P}\Bigl\{ \frac{\|S'_{u_{k+1}} - S'_{u_k}\|}{(u_{k+1}-u_k)^{1/2}} \ge \frac{\varepsilon}{4\sqrt{2}}\theta(2^j) \Bigr\} \\ &\le \quad \sum_{J \le j < J_n} 2^{j+1}\exp\bigl(-c\theta(2^j)^\gamma\bigr). \end{aligned}$$

Since $\theta(2^j)^\gamma/j$ goes to infinity, we have $c\theta(2^j)^\gamma/j \ge 1 + \ln 2$ for J large enough, and then

$$Q_1 \le \frac{2\mathrm{e}^{-J}}{1-\mathrm{e}^{-1}}.$$

Estimation of Q_2. Using again (34) and (39), we get

$$\begin{aligned} Q_2 \quad &:= \quad \sum_{J_n \le j \le \log n} \sum_{k=1}^{2^j} \mathbf{P}\Bigl\{ \frac{\|S'_{u_{k+1}} - S'_{u_k}\|}{(u_{k+1}-u_k)^{1/2}} \ge \frac{\varepsilon}{4\sqrt{2}}\theta(2^j) \Bigr\} \\ &\le \quad \sum_{J_n \le j \le \log n} 2^{j+1}\exp\bigl(-c\theta(2^j)^\gamma w_{n,j}^{-\gamma}\bigr). \end{aligned}$$

Puting $z_j := 2^{j+1}\exp\bigl(-c\theta(2^j)^\gamma w_{n,j}^{-\gamma}\bigr)$, we now observe that for j large enough $z_{j+1}/z_j \ge 2$. Indeed

$$\frac{z_{j+1}}{z_j} = 2\exp\Biggl\{ cn^{1-\gamma/2}\frac{\ln\varphi\bigl(\delta\theta(n)\bigr)}{\bigl(\delta\theta(n)\bigr)^\gamma} 2^{j(\gamma/2-1)}\bigl[\theta(2^j)^\gamma - 2^{\gamma/2-1}\theta(2^{j+1})^\gamma\bigr] \Biggr\}.$$

As $\rho(h) = h^\alpha L(1/h)$, $\theta(t) = t^{1/2-\alpha}L(t)$ with L slowly varying at infinity, we have

$$\lim_{j\to\infty} \frac{\theta(2^{j+1})^\gamma}{\theta(2^j)^\gamma} = 2^{\gamma/2-\alpha\gamma}.$$

Hence there is a j_0 *independent of* n such that for $j \ge j_0$, $z_{j+1}/z_j \ge 2$, provided that $2^{\gamma/2-1}2^{\gamma/2-\alpha\gamma} < 1$, i.e.

$$\gamma < \frac{1}{1-\alpha}. \tag{35}$$

Note that for $\alpha = 1/2$, the inequality (35) does not impose any additional restriction on the choice of γ. For $0 < \alpha < 1/2$, we have $1 < 1/(1-\alpha) < 2$ which is compatible with the condition $1 < \gamma \le 2$ used in the above Talagrand's inequality. Moreover, the compatibility between (32) and (35) requires $\beta > 1 - \alpha$, which is not a problem since for $\alpha < 1/2$, Condition iii) in the definition of $\mathcal{R}$ is satisfied with any $\beta > 0$.

Now $\sum_{j_0 \le j \le m} z_j \le 2z_m$, so for $J \ge j_0$,

$$Q_2 \le 2z_{\log n} = 4n \exp\Bigl(-c\frac{\ln\varphi\bigl(\delta\theta(n)\bigr)}{\delta^\gamma}\Bigr).$$

To finish the proof it suffices to check that

$$\lim_{\delta\downarrow 0}\liminf_{n\to\infty} \frac{\ln\varphi\bigl(\delta\theta(n)\bigr)}{\delta^\gamma \ln n} = \infty. \tag{36}$$

The condition iii) in the definition of the class $\mathcal{R}$ provides the representation $\theta(t) = f(t)\ln^\beta t$, $t > 1$, with f ultimately non decreasing. This gives in turn $\varphi(u) = \exp\bigl(u^{1/\beta}g(u)\bigr)$ with g ultimately non increasing. Indeed, puting $u = \theta(t)$ and taking the logarithms in this last formula yields $g\bigl(\theta(t)\bigr) = f(t)^{-1/\beta}$ where θ is continuous and ultimately non decreasing.

Now for $\delta < 1$ and n large enough,

$$\frac{\ln\varphi\bigl(\delta\theta(n)\bigr)}{\delta^\gamma \ln n} = \frac{\delta^{1/\beta}\theta(n)^{1/\beta}g\bigl(\delta\theta(n)\bigr)}{\delta^\gamma \ln\varphi\bigl(\theta(n)\bigr)} \ge \frac{\delta^{1/\beta}\theta(n)^{1/\beta}g\bigl(\theta(n)\bigr)}{\delta^\gamma\theta(n)^{1/\beta}g\bigl(\theta(n)\bigr)} = \delta^{1/\beta-\gamma}.$$

As $\gamma > 1/\beta$, (36) follows. □

Proof of Corollary 9. The only thing to check is that we can drop the constant A in (22). As $\alpha < 1/2$, we can write $\alpha = 1/2 - 1/p$ ($p > 2$), so $\theta(t) = t^{1/p}L(1/t)$, with L slowly varying at 0. It follows that $A\theta(t)$ is asymptoticaly equivalent to $\theta(A^p t)$. So for some function ε, vanishing at infinity and with $v = A^p t$,

$$\begin{aligned} t\mathbf{P}\bigl\{\|X_1\| \ge A\theta(t)\bigr\} &= A^{-p}v\mathbf{P}\bigl\{\|X_1\| \ge \theta(v)(1+\varepsilon(v))\bigr\} \\ &\le 2A^{-p}\frac{v}{2}\mathbf{P}\bigl\{\|X_1\| \ge \theta(v/2)\bigr\}, \end{aligned}$$

for v large enough, using the fact that

$$\lim_{v\to\infty} \frac{\theta(v)}{\theta(v/2)} = 2^{1/p} > 1.$$

Now (22) follows clearly from (23). □

Proof of Corollary 10. When $\rho(h) = h^{1/2}\ln^\beta(c/h)$, then putting $u := A\theta(t) = A\ln^\beta(ct)$ and $\gamma := 1/\beta$, (22) is clearly equivalent to

$$\mathbf{P}\big(\|X_1\| \geq u\big) = o\Big(\exp\big(-(u/A)^\gamma\big)\Big). \tag{37}$$

for each $A > 0$. As (37) gives the finiteness of $\mathbf{E}\exp\Big(d\|X_1\|^{1/\beta}\Big)$ for any $d < 1/A$, (24) follows. Conversely from (24), Markov inequality leads directly to (37) and then to (22). □

Proof of Corollary 11. As "$X_1 \in \mathrm{CLT}(\mathbb{R})$" is equivalent to $\mathbf{E}\,X_1 = 0$ and $\mathbf{E}\,X_1^2 < \infty$, we just have to check that the finiteness of $\mathbf{E}\,X_1^2$ follows from (22). Using (22), we get for any $a > 0$,

$$\begin{aligned}
\mathbf{E}\,X_1^2 &= \int_0^\infty 2x\mathbf{P}\big\{|X_1| > x\big\}\,\mathrm{d}x \\
&\leq a^2 + \int_a^\infty 2x\mathbf{P}\big\{|X_1| > x\big\}\,\mathrm{d}x \\
&= a^2 + \int_{\varphi(a)}^\infty 2\theta(t)\mathbf{P}\big\{|X_1| > \theta(t)\big\}\theta'(t)\,\mathrm{d}t \\
&\leq a^2 + C\int_{\varphi(a)}^\infty \frac{2\theta(t)\theta'(t)}{t}\,\mathrm{d}t.
\end{aligned}$$

Noting that $\theta(t)^2/t = \rho(1/t)^2$ vanishes at infinity, integration by parts gives

$$\mathbf{E}\,X_1^2 \leq a^2 + \frac{a^2}{\varphi(a)} + C\int_{\varphi(a)}^\infty \frac{\theta(t)^2}{t^2}\,\mathrm{d}t.$$

So everything reduces to the convergence of the integral

$$I = \int_1^\infty \frac{\theta(t)^2}{t^2}\,\mathrm{d}t = \int_1^\infty \frac{\rho(1/t)^2}{t}\,\mathrm{d}t.$$

The convergence follows easily from our general Assumption (8) and Proposition 2. The monotonicity of ρ and the substitution $u = 1/t$ gives $\int_1^\infty \rho(1/t)^2/t\,\mathrm{d}t \leq c_2\rho(1)^2 < \infty$. □

The following corollary of Theorems 5 and 8, which might be of independent interest concludes this section.

Corollary 13. *Let $X_1, \dots, X_n, \dots$ be i.i.d. random elements in the separable Banach space B such that $X_1 \in \mathrm{CLT}(B)$. Let $\theta(t) = t^{1/2-\alpha}L(t)$ with $0 < \alpha \leq 1/2$ and L normalized slowly varying at infinity. Assume moreover that when $\alpha = 1/2$, $L(t)\ln^{-\beta} t$ is ultimately non decreasing for some $\beta > 1/2$. Then the condition*

$$\lim_{t\to\infty} t\sup_{m\geq 1}\mathbf{P}\big\{\|S_m\| > m^{1/2}A\theta(t)\big\} = 0, \quad \textit{for every } A > 0$$

is equivalent to

$$\lim_{t\to\infty} t\mathbf{P}\big\{\|X_1\| > A\theta(t)\big\} = 0, \quad \textit{for every } A > 0.$$

When $\alpha < 1/2$, it is enough to take $A = 1$ in the second condition.

5 Auxiliary technical results

Set for $\gamma > 0$, and X a random variable

$$\|X\|_{\psi_\gamma} := \inf\{c > 0;\ \mathbf{E}\exp(\|X/c\|^\gamma) \le 2\}. \tag{38}$$

For $\gamma \ge 1$, $\|X\|_{\psi_\gamma}$ is a norm and is equivalent to a norm for $0 < \gamma < 1$. From this definition, Beppo Levi theorem (monotone convergence) and Markov inequality, we have immediately

$$\mathbf{P}\{\|X\| \ge x\} \le 2\exp\Big(-\frac{x^\gamma}{\|X\|_{\psi_\gamma}^\gamma}\Big), \quad x > 0. \tag{39}$$

Lemma 14. *If for some constants K and λ, a random variable Y satisfies*

$$\mathbf{P}\{\|Y\| \ge t\} \le K\exp\Big\{-\Big(\frac{t}{\lambda}\Big)^\gamma\Big\}, \quad t > 0,$$

then

$$\|Y\|_{\psi_\gamma} \le \Big(1 + \frac{K}{2}\Big)^{1/\gamma}\lambda.$$

Proof. For $c > \lambda$ we have

$$\begin{aligned}
\mathbf{E}\exp(\|Y/c\|^\gamma) &= \int_0^\infty \frac{\gamma}{c}\Big(\frac{t}{c}\Big)^{\gamma-1}\exp\Big\{\Big(\frac{t}{c}\Big)^\gamma\Big\}\mathbf{P}\{\|Y\| > t\}\,\mathrm{d}t \\
&\le \frac{K}{c^\gamma}\int_0^\infty \gamma t^{\gamma-1}\exp\{t^\gamma(c^{-\gamma} - \lambda^{-\gamma}\}\,\mathrm{d}t \\
&= \frac{K}{c^\gamma}\frac{1}{\lambda^{-\gamma} - c^{-\gamma}}\int_0^\infty \gamma u^{\gamma-1}\exp(-u^\gamma)\,\mathrm{d}u = \frac{K}{(c/\lambda)^\gamma - 1}.
\end{aligned}$$

Now the choice $c = \lambda\big(1 + K/2\big)^{1/\gamma}$, gives $\mathbf{E}\exp(\|Y/c\|^\gamma) \le 2$. The result follows since $\|Y\|_{\psi_\gamma} = \inf\{c > 0;\ \mathbf{E}\exp(\|Y/c\|^\gamma) \le 2\}$. □

Theorem 15 (Talagrand [15, Th. 4]). *Let $(Y_i)_{i\ge1}$ be a sequence of independent mean zero Banach-space valued random variables. Then for $1 < \gamma \le 2$, we have*

$$\Big\|\sum_{i\le N} Y_i\Big\|_{\psi_\gamma} \le K_\gamma\Big\{\mathbf{E}\Big\|\sum_{i\le N} Y_i\Big\| + \Big(\sum_{i\le N}\|Y_i\|_{\psi_\gamma}^{\gamma'}\Big)^{1/\gamma'}\Big\},$$

where $1/\gamma + 1/\gamma' = 1$ and K_γ depends on γ only.

Lemma 16. *Let $\theta(t)$ be positive and C^1 on $[1,\infty)$ and $\lim_{t\to\infty}\theta(t) = \infty$. Assume that*

i) $M := \sup_{u>1} u\mathbf{P}\big\{\|X_1\| > \theta(u)\big\} < \infty$;

ii) there is some $a > 1$ such that on $[a,\infty)$, $\theta(t)$ is non decreasing and $\theta(t)^{-\gamma}\ln t$ is non increasing.

Then the truncated random variable $\widetilde{X}_1 = (X_1; \|X_1\| \le \delta\theta(n))$, satisfies, for n large enough

$$\|\widetilde{X}_1\|_{\psi_\gamma} \le K\frac{\delta\theta(n)}{\ln^{1/\gamma}\{\varphi(\delta\theta(n))\}}, \tag{40}$$

where φ is the generalized inverse of θ defined by (21). The same upper bound remains valid for $\widetilde{X}_1 - \mathbf{E}\widetilde{X}_1$, with a different K.

Condition ii) is fulfilled particularly by $\theta(t) = t^{1/2}\rho(1/t)$ when $\rho(h) = h^\alpha L(h)$ with $0 < \alpha < 1/2$ and when $\rho(h) = h^{1/2}\ln^\beta(c/h)$ with $\beta > 1/\gamma$.

Proof. Let us consider for any $s > 0$, the exponential moments

$$I_{n,s} := \mathbf{E}\exp(\|s\widetilde{X}_1\|^\gamma) = \int_0^{\delta\theta(n)} \gamma s^\gamma x^{\gamma-1}\exp\bigl((sx)^\gamma\bigr)\mathbf{P}\{\|X_1\| > x\}\,\mathrm{d}x.$$

Put $m := \max\{\theta(t);\ 1 \le t \le a\}$. From i) and change of variable, we obtain for $\delta\theta(n) \ge \theta(a)$,

$$\begin{aligned} I_{n,s} &\le \exp\bigl(s^\gamma m^\gamma\bigr) + \int_{\theta(a)}^{\delta\theta(n)} \gamma s^\gamma x^{\gamma-1}\exp\bigl((sx)^\gamma\bigr)\mathbf{P}\{\|X_1\| > x\}\,\mathrm{d}x \\ &\le \exp\bigl(s^\gamma m^\gamma\bigr) + M\int_a^{\varphi(\delta\theta(n))} \frac{\exp\bigl(s^\gamma\theta(v)^\gamma\bigr)}{v}\gamma s^\gamma\theta(v)^{\gamma-1}\theta'(v)\,\mathrm{d}v \end{aligned}$$

Integrating by parts and observing that

$$\frac{\exp\bigl(s^\gamma\theta(v)^\gamma\bigr)}{v^2} = \exp\Bigl\{\theta(v)^\gamma\Bigl(s^\gamma - \frac{2\ln v}{\theta(v)^\gamma}\Bigr)\Bigr\}$$

is non decreasing on $[a,\infty)$ *for any* $s > 0$ thanks to ii), we obtain

$$I_{n,s} \le \exp\bigl(s^\gamma m^\gamma\bigr) + 2M\frac{\exp\bigl(s^\gamma\delta^\gamma\theta(n)^\gamma\bigr)}{\varphi\bigl(\delta\theta(n)\bigr)},$$

for each $s > 0$ and each n such that $\delta\theta(n) \ge \theta(a)$. Now the choice

$$s = s(n) = \bigl(\delta\theta(n)\bigr)^{-1}\ln^{1/\gamma}\varphi\bigl(\delta\theta(n)\bigr)$$

gives $I_{n,s(n)} \le \exp\bigl(m^\gamma s(n)^\gamma\bigr) + 2M$. By ii) and the change of variable $u = \theta(t)$, $u^{-\gamma}\ln\varphi(u)$ is non increasing on $[\theta(a),\infty)$, whence $s(n)^\gamma \le \theta(a)^{-\gamma}\ln a$ and

$$\mathbf{E}\exp(|s(n)\widetilde{X}_1|^\gamma) \le \exp\Bigl\{\Bigl(\frac{m}{\theta(a)}\Bigr)^\gamma\ln a\Bigr\} + 2M.$$

This together with Markov inequality and Lemma 14, provides a constant K depending only on θ, M and γ such that for $\delta\theta(n) \ge \theta(a)$,

$$\|\widetilde{X}_1\|_{\psi_\gamma} \le Ks(n)^{-1} = K\frac{\delta\theta(n)}{\ln^{1/\gamma}\{\varphi\bigl(\delta\theta(n)\bigr)\}}.$$

□

With the notation as in the proof of Theorem 8, the following result holds.

Lemma 17. *Assume that θ is C^1, ultimately non decreasing and that X_1 satisfies (22). Then when $j \le \log n$,*

$$\frac{\mathbf{E}\,\|S'_{u_{k+1}} - S'_{u_k}\|}{(u_{k+1} - u_k)^{1/2}} \le 2\mathbf{clt}(X_1), \quad n \ge n_0,$$

with n_0 depending on δ and the distribution of X_1.

Proof. Recall (3) and note that $X_i' = X_i - (X_i - \widetilde{X}_i) + \mathbf{E}\,(X_i - \widetilde{X}_i)$. As $j \leq \log n$, we have $u_{k+1} - u_k \leq 2n2^{-j}$. Consequently

$$\begin{aligned} \frac{\mathbf{E}\,\|S'_{u_{k+1}} - S'_{u_k}\|}{(u_{k+1} - u_k)^{1/2}} &\leq \mathbf{clt}(X_1) + 2(u_{k+1} - u_k)^{1/2}\mathbf{E}\,\|X_1 - \widetilde{X}_1\| \\ &\leq \mathbf{clt}(X_1) + 2\sqrt{2}\delta\rho(1/n)2^{-j/2}, \end{aligned}$$

where the estimate for $\mathbf{E}\,\|X_1 - \widetilde{X}_1\|$ is the same as for $\|\mathbf{E}\,\widetilde{X}_1\|$, (see the proof of (30)). The result follows. □

Acknowledgment. The authors are grateful to the anonymous Referee for careful reading of the manuscript and many suggestions helping to improve the readability of the text.

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