Optimal Hölderian functional central limit theorems for uniform empirical and quantile processes

Djamel Hamadouche
Laboratoire de Mathématiques
Faculté des Sciences
Université M. Mammeri
Tizi-Ouzou Algérie

Charles Suquet
Laboratoire P. Painlevé (UMR 8524 CNRS)
U.F.R. de Mathématiques Bâtiment M2
Université des Sciences et Technologies de Lille
F-59655 Villeneuve d'Ascq Cedex France

Abstract

Let H_ρ^o be the Hölder space of functions $x : [0,1] \to \mathbb{R}$ such that $|x(t+h) - x(t)| = o(\rho(h))$ uniformly in $t \in [0,1]$, where $\rho(h) = h^\alpha L(1/h)$ with $0 < \alpha < 1$ and L is normalized slowly varying. Denote by ξ_n^{pg} the polygonal smoothing of the uniform empirical process. We prove that ξ_n^{pg} converges weakly in H_ρ^o to the Brownian bridge B if and only if $h^{1/4} = o(\rho(h))$. We also prove that the polygonal smoothing χ_n^{pg} of the uniform quantile process converges weakly in H_ρ^o to B if and only if $h^{1/2}\ln(1/h) = o(\rho(h))$.

Keywords : Brownian bridge, empirical process, Hölder space, quantile process random polygonal line, spacings.

MSC 2000 : 60B12, 60F17, 62G30.

1 Introduction

Let $U_1, \dots, U_n, \dots$ be independent and $[0,1]$ uniformly distributed random variables. Let $F_n(t)$ be the empirical distribution function based on $U_1, \dots, U_n$ and $G_n(t)$ be the empirical quantile function. It is well known that the empirical and quantile processes

$$\xi_n(t) := \sqrt{n}\big(F_n(t) - t\big), \qquad \chi_n(t) := \sqrt{n}\big(G_n(t) - t\big), \qquad t \in [0,1],$$

both converge in law to the Brownian bridge B. The usual topological setting for this convergence is the Skorohod's space $D(0,1)$. Changing the topological framework changes the set of continuous functionals of the paths and hence modifies the scope of the convergence in law of ξ_n and χ_n. For instance, Dudley [8] proves that the convergence of ξ_n to B holds with respect to the p-variation norm for $p \in (0,2)$. In another direction, Morel and Suquet [14] give a necessary and sufficient condition for the convergence in law in $L^2(0,1)$ of ξ_n to a Gaussian process when the U_i's are associated. Under the same positive dependence assumption, they also investigate the convergence in law of ξ_n in some Besov spaces.

Smoothing ξ_n and χ_n allows us to look for stronger topologies. Denote by F_n^{pg} the polygonal cumulative empirical distribution function (which interpolates linearly F_n between its consecutive jumps). Denote by G_n^{pg} the polygonal uniform sample quantile function (precise definitions are given in subsection 2.1). Then define the processes

$$\xi_n^{\mathrm{pg}}(t) := \sqrt{n}\big(F_n^{\mathrm{pg}}(t) - t\big), \qquad \chi_n^{\mathrm{pg}}(t) := \sqrt{n}\big(G_n^{\mathrm{pg}}(t) - t\big), \qquad t \in [0,1].$$

It is well known that ξ_n^{pg} and χ_n^{pg} converge in law to B in the space $\mathrm{C}[0,1]$ of continuous functions endowed with the supremum norm. Because the Hölder spaces are topologically embedded in $\mathrm{C}[0,1]$, they support more continuous functionals than $\mathrm{C}[0,1]$. From this point of view, the alternative framework of Hölder spaces gives functional limit theorems of a wider scope than $\mathrm{C}[0,1]$. This choice may be relevant when the paths of ξ_n^{pg} and χ_n^{pg} and of the limit process B share some α-Hölder regularity. Due to the well known regularity of Brownian paths, this requirement is clearly satisfied with any $\alpha < 1/2$ and it seems rather natural to ask for some Hölderian convergence in law of ξ_n^{pg} and χ_n^{pg}.

Considering the Hölder spaces H_α^o of functions $x : [0,1] \to \mathbb{R}$ such that $|x(t+h) - x(t)| = o(h^\alpha)$ uniformly in $t \in [0,1]$, Hamadouche [9] established that the sequence $(\xi_n^{\mathrm{pg}})_{n\geq 1}$ converges weakly to B in H_α^o for every $\alpha < 1/4$ but is not tight in H_α^o as soon as $\alpha \geq 1/4$. In some sense this result means that polygonal smoothing is too violent. With convolution smoothing of ξ_n, it is possible to reach the weak Hölder convergence for any $\alpha < 1/2$, see [10].

This paper investigates the convergence in law of ξ_n^{pg} and of χ_n^{pg} with respect to the more general class of Hölder spaces H_ρ^o of functions x satifying $|x(t+h) - x(t)| = o(\rho(h))$ uniformly in $t \in [0,1]$, where $\rho(h) = h^\alpha L(1/h)$ with $0 < \alpha < 1$ and L is slowly varying at infinity and ultimately monotonic. In particular, this covers the case of spaces H_ρ^o built on weight functions $\rho(h) = h^\alpha \ell_1^{\beta_1}(c_1/h) \dots \ell_k^{\beta_k}(c_k/h)$, where the ℓ_j's are j-iterated logarithms. The critical weight function for the Hölder regularity of B being $\rho_0(h) = h^{1/2} \ln^{1/2}(c/h)$, no stronger topological framework than $\mathrm{H}_{\rho_0}^o$ can be expected for the convergence in law to B of a sequence of polygonal processes. Recent limit theorems in the spaces H_ρ^o may be found in Račkauskas and Suquet [16, 18, 19]. Some statistical applications of weak Hölder convergence are proposed by the same authors [17, 20, 21].

We prove in the present contribution that ξ_n^{pg} converges in law to B in H_ρ^o

if and only if $h^{1/4} = o(\rho(h))$. Hence the weight function $\rho(h) = h^{1/4}$ *is really the right critical one* in this problem. The polygonal quantile process behaves better with respect to Hölder topologies. Indeed we prove that χ_n^{pg} converges in law to B in H_ρ^o if and only if $h^{1/2}\ln(1/h) = o(\rho(h))$.

The paper is organized as follows. The relevant background on Hölder spaces and weak convergence therein is presented in Section 2. Section 3 contains the limit theorem for ξ_n^{pg} and its proof. Section 4 does the same for χ_n^{pg}.

2 Preliminaries

2.1 Polygonal processes

Let $U_1, \dots, U_n$ be a sample of i.i.d. random variables uniformly distributed on $[0, 1]$. We denote by $U_{n:i}$ the order statistics of the sample

$$0 = U_{n:0} \le U_{n:1} \le \cdots \le U_{n:n} \le U_{n:n+1} = 1,$$

which are distinct with probability one. For notational convenience, put

$$u_{n:i} = \mathbf{E}\, U_{n:i} = \frac{i}{n+1}, \quad i = 0, 1, \dots, n+1.$$

We recall the distributional equality (see e.g. [24])

$$(U_{n:1}, \dots, U_{n:n}) \overset{d}{=} \Big(\frac{S_1}{S_{n+1}}, \dots, \frac{S_n}{S_{n+1}}\Big), \tag{1}$$

where $S_k = X_1 + \cdots + X_k$ and the X_k's are i.i.d 1-exponential random variables.

The empirical distribution function F_n of the sample is

$$F_n(t) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{U_i \le t\}}.$$

It is also the piecewise constant function which jumps at the U_i's and satisfies

$$F_n(U_{n:i}) = \frac{i}{n}, \quad 0 \le i \le n.$$

Note that $F_n(U_{n:n+1}) = 1$. We define the polygonal empirical function F_n^{pg} as the polygonal random line with vertices $\big(U_{n:i}, F_n(U_{n:i})\big)$, $i = 0, 1, \dots, n+1$. The uniform empirical process ξ_n and the polygonal uniform empirical process ξ_n^{pg} are defined respectively by

$$\xi_n(t) := \sqrt{n}\big(F_n(t) - t\big), \quad \xi_n^{\mathrm{pg}}(t) := \sqrt{n}\big(F_n^{\mathrm{pg}}(t) - t\big), \qquad t \in [0, 1].$$

The polygonal smoothing being non linear, F_n^{pg} does not inherit of the unbiasedness of F_n. Nevertheless the obvious estimate

$$\|\xi_n - \xi_n^{\mathrm{pg}}\|_\infty \le \frac{1}{\sqrt{n}}, \tag{2}$$

implies that ξ_n^{pg}, like ξ_n, converges in the sense of the finite dimensional distributions to the Brownian bridge B.

We define the (discontinuous) uniform quantile process χ_n by

$$\chi_n(t) = \sqrt{n}\Big(\sum_{i=1}^{n+1} U_{n:i}\mathbf{1}_{(u_{n:i-1},u_{n:i}]}(t) - t\Big), \quad t \in [0,1]. \tag{3}$$

Definition (3) differs slightly of the most usual one for the uniform quantile process, see e.g. [5]. This later, denoted here $\tilde{\chi}_n$ is given by

$$\tilde{\chi}_n(t) := \sqrt{n}\big(F_n^{-1}(t) - t\big), \qquad t \in [0,1],$$

where $F_n^{-1}(t) := \inf\{u;\ F_n(u) \geq t\}$. The advantage of (3) is that at each jump of χ_n, the process has null expectation.

We associate to χ_n the polygonal uniform quantile process χ_n^{pg} which is affine on each $[u_{n:i-1}, u_{n:i}]$, $i = 1, \dots, n+1$ and satisfies

$$\chi_n^{\mathrm{pg}}(u_{n:i}) = \sqrt{n}(U_{n:i} - u_{n:i}), \quad i = 0, 1, \dots, n+1. \tag{4}$$

We shall also consider the polygonal smoothing $\tilde{\chi}_n^{\mathrm{pg}}$ of $\tilde{\chi}_n$ defined as the polygonal line which is affine on each $[(i-1)/n, i/n]$, $i = 1, \dots, n$ and satisfies

$$\tilde{\chi}_n^{\mathrm{pg}}(i/n) = \sqrt{n}(U_{n:i} - i/n), \quad i = 0, 1, \dots, n. \tag{5}$$

2.2 Hölder spaces

Throughout this paper we deal with Hölder spaces H_ρ or H_ρ^o built on some weight function ρ satisfying the following condition.

(r1) The function $\rho : [0,1] \to \mathbb{R}$ is non decreasing continuous and such that

$$\rho(h) = h^\alpha L(1/h), \quad 0 < h \leq 1, \tag{6}$$

where $0 < \alpha < 1$, and L is some positive function which is normalized slowly varying at infinity.

Let us recall that $L(t)$ is positive continuous normalized slowly varying at infinity if it has a representation

$$L(t) = c \exp\left\{\int_b^t \varepsilon(u)\frac{\mathrm{d}u}{u}\right\},$$

with $0 < c < \infty$ constant and $\varepsilon(u) \to 0$ when $u \to \infty$. By a theorem of Bojanic and Karamata [1, Th.1.5.5], the class of normalized slowly varying functions is exactly the Zygmund class i.e. the class of functions $f(t)$ such that for every $\delta > 0$, $t^\delta f(t)$ is ultimately increasing and $t^{-\delta} f(t)$ is ultimately decreasing. Here and below "ultimately" means "on some interval $[b, \infty)$".

We shall also use one of the following extra assumptions.

(r2) The function L in (6) is ultimately monotonic.

(r3) The function $\theta(t) := t^{1/2}\rho(1/t)$ is C^1 on $[1,\infty)$ and there is a $\beta > 1/2$, such that $\theta(t)\ln^{-\beta}(t)$ is ultimately non decreasing.

Note that such a β always (resp. never) exists when $\alpha < 1/2$ (resp. $\alpha > 1/2$). In particular (r3) is satisfied by $\rho(h) = h^{1/2}\ln^{\gamma}(c/h)$ for $\gamma > 1/2$. Here c denotes any constant compatible with the requirement of increasingness of ρ on $[0,1]$.

Denote as usual by $\mathrm{C}[0,1]$ the space of continuous functions $x : [0,1] \to \mathbb{R}$ endowed with the supremum norm $\|x\|_\infty$. For ρ satisfying (r1), put

$$\omega_\rho(x,\delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{|x(t)-x(s)|}{\rho(t-s)}, \qquad 0 < \delta \le 1.$$

We associate to ρ the Hölder space

$$\mathrm{H}_\rho := \{x \in \mathrm{C}[0,1];\ \omega_\rho(x,1) < \infty\},$$

endowed with the norm

$$\|x\|_\rho := |x(0)| + \omega_\rho(x,1).$$

As H_ρ is a non separable Banach space, it is more convenient to work with its closed separable subspace

$$\mathrm{H}^o_\rho := \{x \in \mathrm{H}_\rho;\ \lim_{\delta\to 0}\omega_\rho(x,\delta) = 0\}.$$

When $\rho(h) = h^\alpha$, the corresponding Hölder spaces H_ρ and H^o_ρ will be denoted by H_α and H^o_α respectively.

As polygonal lines, the paths of ξ_n^{pg} and χ_n^{pg} belong clearly to H^o_ρ for any ρ satisfying (r1).

One interesting feature of the spaces H^o_α is the existence of a basis of triangular functions, see [3]. We write this basis as a triangular array of functions, indexed by the dyadic numbers. Let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j, i.e.

$$D_0 = \{0,1\}, \qquad D_j = \{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\}, \quad j \ge 1.$$

Write for $r \in D_j$, $j \ge 0$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

For $r \in D_j$, $j \ge 1$, define the triangular Faber-Schauder functions Λ_r by:

$$\Lambda_r(t) := \begin{cases} 2^j(t-r^-) & \text{if } t \in (r^-, r]; \\ 2^j(r^+-t) & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

When $j = 0$, we just take the restriction to $[0, 1]$ in the above formula, so

$$\Lambda_0(t) = 1 - t, \quad \Lambda_1(t) = t, \quad t \in [0, 1].$$

The sequence $\{\Lambda_r; r \in D_j,\ j \geq 0\}$ is a Schauder basis of $\mathrm{C}[0, 1]$. Each $x \in \mathrm{C}[0, 1]$ has a unique expansion

$$x = \sum_{j=0}^{\infty} \sum_{r \in D_j} \lambda_r(x) \Lambda_r, \tag{7}$$

with uniform convergence on $[0, 1]$. The Schauder scalar coefficients $\lambda_r(x)$ are given by

$$\lambda_r(x) = x(r) - \frac{x(r^+) + x(r^-)}{2}, \quad r \in D_j,\ j \geq 1,$$

and in the special case $j = 0$ by

$$\lambda_0(x) = x(0), \quad \lambda_1(x) = x(1).$$

The partial sum

$$E_J x := \sum_{j=0}^{J} \sum_{r \in D_j} \lambda_r(x) \Lambda_r \tag{8}$$

in the series (7) gives the linear interpolation of x by a polygonal line between the dyadic points of level at most J.

Ciesielski [3] proved that $\{\Lambda_r; r \in D_j,\ j \geq 0\}$ is also a Schauder basis of each space H_α^o (hence the convergence (7) holds in the H_α topology when $x \in \mathrm{H}_\alpha^o$) and that the norm $\|x\|_\alpha$ is equivalent to the following sequence norm :

$$\|x\|_\alpha^{\mathrm{seq}} := \sup_{j \geq 0} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(x)|.$$

This equivalence of norms provides a very convenient discretization procedure to deal with Hölder spaces and is extended in [18] to the spaces H_ρ^o with ρ satisfying (r1). The sequence norm $\|x\|_\rho^{\mathrm{seq}}$ equivalent to $\|x\|_\rho$ is then defined by

$$\|x\|_\rho^{\mathrm{seq}} := \sup_{j \geq 0} \frac{1}{\rho(2^j)} \max_{r \in D_j} |\lambda_r(x)|. \tag{9}$$

It is worth noticing that

$$\|x - E_J x\|_\rho^{\mathrm{seq}} = \sup_{j > J} \frac{1}{\rho(2^j)} \max_{r \in D_j} |\lambda_r(x)|. \tag{10}$$

2.3 Tightness in Hölder spaces

We write

$$Y_n \xrightarrow[n \to \infty]{\mathrm{H}_\rho^o} Y,$$

for the convergence in law in the separable Banach space H_ρ^o of a sequence $(Y_n)_{n \geq 1}$ of random elements in H_ρ^o (also called here convergence in distribution in

H^o_ρ or weak convergence in H^o_ρ). Such a convergence is equivalent to the tightness of $(Y_n)_{n\geq 1}$ on H^o_ρ together with convergence of finite dimensional distributions.

The following characterization of tightness in H^o_ρ looks very similar to the classical one for the space $\mathrm{C}[0,1]$, obtained by combining Ascoli's and Prohorov's theorems.

Theorem 1. *The sequence $(Y_n)_{n\geq 1}$ of random elements in H^o_ρ is tight if and only if the following two conditions are satisfied:*

i) For each $t\in[0,1]$, the sequence $(Y_n(t))_{n\geq 1}$ is tight on $\mathbb{R}$.

ii) For each $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}(\omega_\rho(Y_n,\delta)>\varepsilon)=0. \tag{11}$$

Proof sketched. For the sufficiency, we refer to Theorem 3 in [18] noting that with the notations used therein,

$$\|Y_n-E_jY_n\|^{\mathrm{seq}}_\rho\leq\omega_\rho(Y_n,2^{-j}).$$

For the necessity, introduce the functionals Φ_N defined on H^o_ρ by $\Phi_N(x):=\omega_\rho\big(x,1/N\big)$. By the definition of H^o_ρ, the sequence $(\Phi_N)_{N\geq 1}$ decreases to zero pointwise on H^o_ρ. Moreover each Φ_N is continuous in the strong topology of H^o_ρ. By Dini's theorem this gives the uniform convergence of $(\Phi_N)_{N\geq 1}$ to zero on any compact K of H^o_ρ. This remark combined with the assumption of tightness of $(Y_n)_{n\geq 1}$ leads easily to

$$\lim_{N\to\infty}\sup_{n\geq 1}\mathbf{P}(\omega_\rho(Y_n,1/N)>\varepsilon)=0, \tag{12}$$

from which we obtain (11). □

The special following tightness result extends Theorem 1 in [9]. It may be relevant in the case of processes Y_n whose "not too small increments" behave smoothly, while their smaller increments can be controlled differently. This typically happens with the empirical processes ξ_n^{pg}.

Theorem 2. *Assume that the sequence $(Y_n)_{n\geq 1}$ of random elements in H^o_ρ fulfils the following conditions.*

a) For each $r\in D$, $\big(Y_n(r)\big)_{n\geq 1}$ is tight in $\mathbb{R}$.

b) For some non increasing sequence $(a_n)_{n\geq 1}$ in $(0,1)$, converging to zero,

$$\mathbf{P}\big(|Y_n(t)-Y_n(s)|\geq u\big)\leq|t-s|Q(|t-s|,u),\quad u>0,\ |t-s|\geq a_n, \tag{13}$$

where the function $Q:\mathbb{R}_+^*\times\mathbb{R}_+^*\to\mathbb{R}_+$ satisfies for every positive ε,

$$\sum_{j=1}^{+\infty}Q\big(2^{-j},\varepsilon\rho(2^{-j})\big)<\infty. \tag{14}$$

c) $\omega_\rho(Y_n, a_n)$ *converges to* 0 *in probability* $(n \to \infty)$.

Then $(Y_n)_{n\geq 1}$ *is tight in* H_ρ^o.

When using Theorem 2 to prove a weak Hölder convergence, Condition a) is automatically satisfied as soon as the convergence of finite dimensional distributions of Y_n holds true. Note also that if b) is satisfied without the restriction $|t-s| \geq a_n$, then c) follows. Of course our interest in this theorem focuses on the case where there is no possibility to obtain b) without the restriction $|t-s| \geq a_n$. Condition c) may be tractable when some information is available on the local smoothness of Y_n. The following corollary is well adapted to the case of the polygonal uniform empirical process. Recall that $\rho(h) = h^\alpha L(1/h)$ with $0 < \alpha < 1$ and L slowly varying.

Corollary 3. *Assume that the sequence* $(Y_n)_{n\geq 1}$ *of random elements in* H_ρ^o *satisfies Conditions a) and c) of Theorem 2 and that for some real numbers* $\gamma > 0$, $p > 2$ *and some non increasing sequence* $(a_n)_{n\geq 1}$ *in* $(0,1)$, *converging to zero,*

$$\mathbf{E}\,|Y_n(t) - Y_n(s)|^p \leq C_p |t-s|^{1+\gamma}, \quad |t-s| \geq a_n, \tag{15}$$

for some positive constant C_p. *Then* $(Y_n)_{n\geq 1}$ *is tight in* H_ρ^o *if either* $\alpha < \gamma/p$ *or* $\alpha = \gamma/p$ *with* $\sum_{j\geq 1} L(2^j)^{-p} < \infty$.

Proof of Theorem 2. By Theorem 2 and Remark 1 in [19], it suffices to prove that for every positive ε,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \mathbf{P}\big(\|Y_n - E_J Y_n\|_\rho^{\mathrm{seq}} > \varepsilon\big) = 0, \tag{16}$$

with the projectors E_J defined by (8). Define the integer J_n by the condition $2^{-J_n - 1} < a_n \leq 2^{-J_n}$. Then accounting (10), we have for each $J \geq 1$

$$\mathbf{P}\big(\|Y_n - E_J Y_n\|_\rho^{\mathrm{seq}} > \varepsilon\big) \leq P'_{J,n} + P''_n, \tag{17}$$

where

$$\begin{aligned} P'_{J,n} &:= \mathbf{P}\Big(\max_{J\leq j\leq J_n} \frac{1}{\rho(2^{-j})} \max_{r\in D_j} |\lambda_r(Y_n)| > \varepsilon\Big), \\ P''_n &:= \mathbf{P}\Big(\sup_{j>J_n} \frac{1}{\rho(2^{-j})} \max_{r\in D_j} |\lambda_r(Y_n)| > \varepsilon\Big). \end{aligned}$$

The estimate (17) is clear when $J \leq J_n$. When $J_n < J$, it remains true with the usual convention "$\sup\emptyset = -\infty$", which gives $P'_{J,n} = 0$.

To control P''_n, let us simply note that

$$P''_n \leq \mathbf{P}\big(\omega_\rho(Y_n, a_n) > \varepsilon\big),$$

so Condition c) in Theorem 2 gives

$$\limsup_{n\to\infty} P''_n = 0. \tag{18}$$

Now to control $P'_{J,n}$, having in mind the usual convention that a sum indexed by the emptyset is defined as null, we get by Condition b)

$$\begin{aligned} P'_{J,n} &\leq \sum_{J \leq j \leq J_n} \sum_{r \in D_j} \mathbf{P}\big(|\lambda_r(Y_n)| > \varepsilon\rho(2^{-j})\big) \\ &\leq \sum_{J \leq j \leq J_n} 2^j 2^{-j} Q\big(2^{-j}, \varepsilon\rho(2^{-j})\big) \\ &\leq \sum_{j=J}^{\infty} Q\big(2^{-j}, \varepsilon\rho(2^{-j})\big). \end{aligned}$$

In view of (14), this leads to

$$\lim_{J\to\infty} \limsup_{n\to\infty} P''_n = 0. \tag{19}$$

Then (16) and hence the tigthness of (Y_n) follow from (18) and (19). □

Proof of Corollary 3. By Markov's inequality, (15) leads to (13) with $Q(v,u) := C_p u^\gamma v^{-p}$. Hence Condition (14) reduces to

$$\sum_{j=1}^{\infty} \frac{2^{j(p\alpha-\gamma)}}{L(2^j)^p} < \infty.$$

Since L is slowly varying in the neighbourhood of infinity, $z^\delta L(z)^p$ goes to infinity with z for any positive δ. It follows that the above series converges for any $\alpha < \gamma/p$ whatever the choice of the slowly varying function L may be. □

2.4 An Hölderian FCLT

We shall need the following invariance principle for partial sums processes, which is proved in [18] in a more general setting. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables in $\mathbb{R}$ with null expectation and $\mathbf{E}\, X_1^2 = 1$. Set $S_0 = 0$,

$$S_k = X_1 + \cdots + X_k, \quad \text{for} \quad k = 1, 2, \ldots$$

and consider the polygonal partial sums processes

$$\Xi_n(t) = n^{-1/2} S_{[nt]} + n^{-1/2}(nt - [nt]) X_{[nt]+1}, \quad t \in [0,1]. \tag{20}$$

Theorem 4 (Račkauskas, Suquet [18]). *Let ρ satisfying (r1). If Ξ_n converges weakly in H^o_ρ to the standard Brownian motion W, then for every $A > 0$,*

$$\lim_{t\to\infty} t\mathbf{P}\big(|X_1| \geq A\theta(t)\big) = 0. \tag{21}$$

If ρ satisfies (r1), (r3) and (21), then Ξ_n converges weakly in H^o_ρ to W.

Recall that Condition (r3) implies that $\alpha \le 1/2$ and that θ is defined by

$$\theta(t) = t^{1/2}\rho(1/t) = t^{1/2-\alpha}L(t).$$

If $a < 1/2$ then it suffices to check (21) for $A = 1$ only. In the classical case where $\rho(h) = h^\alpha$, $0 < \alpha < 1/2$, (21) is equivalent to $\mathbf{P}(|X_1| \ge t) = o(t^{-p(\alpha)})$ with $p(\alpha) := (1/2-\alpha)^{-1}$. This improves on Lamperti's Theorem [11] which obtained the weak H^o_α convergence of Ξ_n under the finiteness of $\mathbf{E}\,|X_1|^p$ for some $p > p(\alpha)$. In the case where $\rho(h) = h^{1/2}\ln^b(c/h)$ with $b > 1/2$, (21) is equivalent to $\mathbf{E}\exp\bigl(\gamma|X_1|^{1/b}\bigr) < \infty$, for each $\gamma > 0$.

2.5 Spacings

The study of the asymptotical behavior in H^o_ρ of the polygonal lines ξ_n^{pg} and χ_n^{pg} involves clearly their ρ-weighted increments between vertices. This requires some probabilistic control on the increments $U_{n:i+k} - U_{n:i}$. The quantities of interest are more precisely the minimal and maximal spacing $\delta_{n:1}$ and $\delta_{n:n+1}$ defined by

$$\delta_{n:1} := \min_{0\le i\le n}(U_{n:i+1} - U_{n:i}), \qquad \delta_{n:n+1} := \max_{0\le i\le n}(U_{n:i+1} - U_{n:i}) \tag{22}$$

and the minimal k-spacing $m_n(k)$ defined for $k = 1, \dots, n$ by

$$m_n(k) := \min_{0\le i\le n-k+1}(U_{n:i+k} - U_{n:i}). \tag{23}$$

Let us recall the classical Lévy's results about $\delta_{n:1}$ and $\delta_{n:n+1}$.

Lemma 5 (Lévy [13], [24]). *The sequence of random variables $\bigl(n^2\delta_{n:1}\bigr)_{n\ge1}$ converges in distribution to the exponential distribution with parameter 1.*

Lemma 6 (Lévy [24], p.726). *The limiting distribution of the maximal spacing $\delta_{n:n+1}$ is given by*

$$\mathbf{P}\bigl((n+1)\delta_{n:n+1} - \log(n+1) \le t\bigr) \xrightarrow[n\to\infty]{} \exp(-\mathrm{e}^{-t}), \quad t > 0.$$

To control $m_n(k)$, we use the following lemma by Deheuvels which extends an earlier result of Devroye [7] concerning $m_n(1) = \delta_{n:1}$.

Lemma 7 (Deheuvels [6]). *Let $k \ge 1$ be a fixed integer. Then, whenever a non increasing sequence $(a_n)_{n\ge1}$ of positive constants satisfies the condition*

$$\sum_{n=1}^\infty (na_n)^k < \infty, \tag{24}$$

we have $\mathbf{P}(m_n(k) \le a_n \text{ i.o.}) = 0$.

3 Polygonal uniform empirical process

Theorem 8. *Let $\rho(h) = h^{1/4}L(1/h)$ be a weight function satisfying (r1) and (r2). Then ξ_n^{pg} converges weakly in H_ρ^o to the Brownian bridge if and only if*

$$\lim_{t\to\infty} L(t) = \infty. \tag{25}$$

In view of the embeddings of Hölder spaces and of the result in [9] for the weight functions $\rho(h) = h^\alpha$, $\alpha \geq 1/4$, Theorem 8 leads to the following characterization of the weak H_ρ^o convergence of ξ_n^{pg} when ρ satisfies (r1) and (r2) :

$$\xi_n^{\mathrm{pg}} \xrightarrow[n\to\infty]{\mathrm{H}_\rho^o} B \quad \text{if and only if} \quad \lim_{h\to 0} h^{-1/4}\rho(h) = \infty. \tag{26}$$

It is worth noticing that Condition (r2) is needed here only to prove the necessity of (25).

The following elementary lemma plays a key rôle in the proof of Theorem 8.

Lemma 9. *Let the weight function ρ satisfy (r1) with $0 < \alpha < 1$. Then, there is a $\eta \in (0,1]$, such that if $0 \leq t < t' \leq 1$ with $t' - t \leq \eta$ and if f is any real valued function whose restriction to $[t, t']$ is affine, we have*

$$\sup_{t\leq s<s'\leq t'} \frac{|f(s') - f(s)|}{\rho(s'-s)} = \frac{|f(t') - f(t)|}{\rho(t'-t)}, \tag{27}$$

where η depends only on ρ.

Proof of Lemma 9. Because the function L in (6) is normalized slowly varying, $t^\varepsilon L(t)$ is ultimately non decreasing for any positive ε, i.e. is non decreasing on some interval $[b, \infty)$, where $b \geq 1$ depends on ε and L. Choosing $\varepsilon = 1 - \alpha$, it follows that the function $h/\rho(h)$ is non decreasing on $(0, \eta]$, where $\eta = 1/b$ depends only on ρ. This together with the fact that f is affine between t and t' leads for $t \leq s < s' \leq t'$ to the estimate

$$\frac{|f(s') - f(s)|}{\rho(s'-s)} = \frac{s'-s}{\rho(s'-s)} \frac{|f(t') - f(t)|}{t'-t} \leq \frac{t'-t}{\rho(t'-t)} \frac{|f(t') - f(t)|}{t'-t},$$

whence (27) follows. □

Proof of Theorem 8. As we have already noted that ξ_n^{pg} converge to B in the sense of the finite dimensional distributions, we only have to prove that Condition (25) is necessary and sufficient for the tightness in H_ρ^o of the sequence $(\xi_n^{\mathrm{pg}})_{n\geq 1}$.

The necessity of (25) follows essentially of the fact that with $\rho_0(h) := h^{1/4}$, $(\xi_n^{\mathrm{pg}})_{n\geq 1}$ is not tight in $\mathrm{H}_{\rho_0}^o$, see [9]. Now if (25) fails, then $\rho(h)/\rho_0(h)$ is bounded near 0 by (r2) and hence H_ρ^o is topologically embedded in $\mathrm{H}_{\rho_0}^o$. This forbids the tightness of $(\xi_n^{\mathrm{pg}})_{n\geq 1}$ in H_ρ^o.

To prove the sufficiency of (25) for the tightness of $(\xi_n^{\mathrm{pg}})_{n\geq 1}$, we shall use Corollary 3, recalling that its Condition a) is obviously satisfied, due to the convergence of the one dimensional distributions of $(\xi_n^{\mathrm{pg}})_{n\geq 1}$.

To check Conditions b) and c), we choose $a_n := n^{-c}$ with $3/2 < c < 2$. In order to obtain the estimate (15) for the increments of ξ_n^{pg}, let us observe first that the corresponding increments of the discontinuous empirical process ξ_n may be expressed as

$$\xi_n(t) - \xi_n(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i(s,t), \quad 0 \le s \le t \le 1, \tag{28}$$

where the $Z_i(s,t) := \mathbf{1}_{(s,t]}(U_i) - (t-s)$ are i.i.d. bounded random variables. Note moreover that for any real $p \ge 2$,

$$\mathbf{E}\,|Z_i(s,t)|^p \le |t-s|. \tag{29}$$

Now in view of (2), (28) and (29), Rosenthal's inequality, see [22] or [12], gives us for any $p > 2$,

$$\begin{aligned} \mathbf{E}\,|\xi_n^{\mathrm{pg}}(t) - \xi_n^{\mathrm{pg}}(s)|^p &\le \mathbf{E}\left(|\xi_n(t) - \xi_n(s)| + 2n^{-1/2}\right)^p \\ &\le 2^{p-1}\mathbf{E}\,|\xi_n(t) - \xi_n(s)|^p + 2^{2p-1}n^{-p/2} \\ &\le A_p\left(n^{1-p/2}|t-s| + |t-s|^{p/2}\right) + 2^{2p-1}n^{-p/2}, \end{aligned} \tag{30}$$

with a constant A_p depending only on p. The estimate (30) is valid for any $s, t \in [0,1]$. When $|t-s| \ge n^{-c}$, we have moreover

$$n^{-p/2} \le |t-s|^{\frac{p}{2c}}, \quad n^{1-p/2} \le |t-s|^{\frac{p-2}{2c}},$$

so (30) leads to the inequality

$$\mathbf{E}\,|\xi_n^{\mathrm{pg}}(t) - \xi_n^{\mathrm{pg}}(s)|^p \le C_p|t-s|^{\frac{p}{2c}}, \quad |t-s| \ge a_n = n^{-c},$$

where the constant C_p depends only on p. Hence (15) is satisfied for any $p > 2c$ with an exponent $\gamma = \frac{p}{2c} - 1$. Finally recalling that $c < 2$, we choose p large enough to have

$$\frac{\gamma}{p} = \frac{1}{2c} - \frac{1}{p} > \frac{1}{4}.$$

To check Condition c), it is convenient to introduce the random variable N with values in $\mathbb{N}^* \cup \{\infty\}$ defined by

$$N := \inf\{l \in \mathbb{N}^*;\ \forall n \ge l,\ m_n(2) \ge a_n\}, \tag{31}$$

with the usual convention $\inf \emptyset := \infty$. Because $a_n = n^{-c}$ with $c > 3/2$, Lemma 7 implies that N is almost surely finite. Hence for any positive ε there is some integer $n_1 = n_1(\varepsilon)$ such that

$$\mathbf{P}(N > n_1) < \varepsilon. \tag{32}$$

It follows that for any positive ε_0,

$$\mathbf{P}\big(\omega_\rho(\xi_n^{\mathrm{pg}}, a_n) > \varepsilon_0\big) \le \mathbf{P}\big(\omega_\rho(\xi_n^{\mathrm{pg}}, a_n) > \varepsilon_0,\ N \le n_1\big) + \varepsilon. \tag{33}$$

Now it remains to control $\omega_\rho(\xi_n^{\mathrm{pg}}, a_n)$ on the event

$$E_\varepsilon := \{N \le n_1\}.$$

To do that, let us consider an arbitrary interval $[t, t + a_n]$ and estimate on E_ε the ratios

$$q_n(s, s') := \frac{|\xi_n^{\mathrm{pg}}(s') - \xi_n^{\mathrm{pg}}(s)|}{\rho(s' - s)}, \quad t \le s < s' \le t + a_n.$$

Due to (23) and (31) it is clear that on E_ε and for every $n \ge n_1$, the open interval $(t, t + a_n)$ contains at most two order statistics $U_{n:i}$. We observe also that the increment of ξ_n^{pg} between two consecutive vertices of its restriction to $[t, t+a_n]$ (including artificial vertices at t and $t+a_n$) is bounded by $n^{-1/2}$ because $n^{1/2}a_n = n^{1/2-c} < n^{-1}$. Then considering the three possible configurations and using Lemma 9 it is easily seen that on E_ε,

$$q_n(s, s') \le \frac{3}{n^{1/2}\rho(\delta_{n:1})}, \quad t \le s < s' \le t + a_n,\ n \ge n_1.$$

This estimate being uniform in $t \in [0, 1 - a_n]$, it follows that

$$\omega_\rho(\xi_n^{\mathrm{pg}}, a_n) \le \frac{3}{n^{1/2}\rho(\delta_{n:1})}, \quad n \ge n_1, \quad \text{on } E_\varepsilon. \tag{34}$$

Next we note that if Z is a random variable having the exponential distribution with parameter 1, then with $b := -\ln(1 - \varepsilon)$, $\mathbf{P}(Z \le b) = \varepsilon$. In view of Lemma 5 we can find an integer $n_2 = n_2(\varepsilon)$ such that

$$\mathbf{P}\Big(\delta_{n:1} \le \frac{b}{n^2}\Big) < 2\varepsilon, \quad n \ge n_2. \tag{35}$$

Finally, by Condition (25) we can find an integer $n_3 = n_3(\varepsilon, \varepsilon_0)$ such that

$$\frac{3}{n^{1/2}\rho(bn^{-2})} = \frac{3}{b^{1/4}L(n^2/b)} < \varepsilon_0, \quad n \ge n_3. \tag{36}$$

Gathering (33), (34), (35) and (36), we see that with $n_4 := \max(n_1, n_2, n_3)$,

$$\mathbf{P}\big(\omega_\rho(\xi_n^{\mathrm{pg}}, a_n) > \varepsilon_0\big) \le 3\varepsilon, \quad n \ge n_4.$$

Since ε and ε_0 were arbitrary, Condition c) of Corollary 3 is satisfied and the proof of Theorem 8 is complete. □

4 Polygonal uniform quantile processes

Theorem 10. *Let $\rho(h) = h^{1/2}L(1/h)$ be a weight function satisfying (r1) and (r3). Then χ_n^{pg} converges weakly in H_ρ^o to the Brownian bridge, if and only if*

$$\lim_{t\to\infty} \frac{L(t)}{\ln t} = \infty. \tag{37}$$

Due to the embeddings of Hölder spaces, note that if ρ satisfies (r1) with $\alpha < 1/2$, then χ_n^{pg} converges weakly in H_ρ^o to B. Note also that for the special class of weight functions of the form $\rho(h) = h^{1/2} \ln^\gamma(c/h)$, (37) implies (r3).

Proof. Let us establish first the sufficiency of (37). With the i.i.d. 1-exponential random variables X_i and their partial sums S_k introduced in (1), let ζ_n be the polygonal process which is affine on each interval $[u_{n:i-1}, u_{n:i}]$, $i = 1, \dots, n+1$ and such that

$$\zeta_n(u_{n:i}) = \sqrt{n}\Big(\frac{S_i}{S_{n+1}} - u_{n:i}\Big), \quad i = 0, 1, \dots, n+1. \tag{38}$$

Put for notational convenience

$$X_i' = X_i - \mathbf{E}\, X_i, \quad S_k' = S_k - \mathbf{E}\, S_k$$

and let Ξ_n be the partial sums process (20) built on the S_k''s instead of the S_k's. Note also that $\mathbf{E}\, {X_1'}^2 = 1$. Then we have for $i = 0, 1, \dots, n+1$,

$$\begin{aligned}
\zeta_n(u_{n:i}) &= \frac{\sqrt{n}}{S_{n+1}}\Big(S_i - \frac{iS_{n+1}}{n+1}\Big) \\
&= \frac{\sqrt{n}}{S_{n+1}}\Big(S_i' - \frac{i}{n+1}S_{n+1}'\Big) \\
&= \frac{\sqrt{n(n+1)}}{S_{n+1}}\Big(\frac{S_i'}{\sqrt{n+1}} - \frac{i}{n+1}\frac{S_{n+1}'}{\sqrt{n+1}}\Big) \\
&= \frac{\sqrt{n(n+1)}}{S_{n+1}}\big(\Xi_{n+1}(u_{n:i}) - u_{n:i}\,\Xi_{n+1}(1)\big).
\end{aligned}$$

It follows that

$$\zeta_n(t) = \frac{\sqrt{n(n+1)}}{S_{n+1}}\big(\Xi_{n+1}(t) - t\Xi_{n+1}(1)\big), \quad t \in [0,1], \tag{39}$$

because both sides of (39) are polygonal lines with the same vertices. By the strong law of large numbers,

$$\frac{\sqrt{n(n+1)}}{S_{n+1}} \xrightarrow[n\to\infty]{\text{a.s.}} 1. \tag{40}$$

By Theorem 4, Ξ_{n+1} converges weakly in H_ρ^o to the Brownian motion if X_1' satisfies (21) for every positive A. Noting that $\theta(t) = L(t)$, and $|X_1'| \le X_1 + 1$, this condition follows from (37) since

$$t\mathbf{P}\big(|X_1'| \ge A\theta(t)\big) \le t\mathbf{P}\big(X_1 + 1 \ge A\theta(t)\big) = t\exp(-AL(t) + 1).$$

Denote by I the identity $s \mapsto s$ on $[0,1]$. The map $\varphi : \mathrm{H}_\rho^o \to \mathrm{H}_\rho^o$, $x \mapsto x - x(1)I$ is obviously continuous for any ρ. Hence by preservation of the weak convergence by continuous maping, see e.g. [2, p.29],

$$\varphi(\Xi_{n+1}) \xrightarrow[n\to\infty]{\mathrm{H}_\rho^o} B. \tag{41}$$

From (40) and (41) it follows that

$$\zeta_n \xrightarrow[n\to\infty]{\mathrm{H}^o_\rho} B. \tag{42}$$

By (1), χ_n^{pg} and ζ_n have the same finite dimensional distributions. This implies that the distributions of χ_n^{pg} and ζ_n coincide as probability measures on the Borel σ-field of H^o_ρ. A simple way to see it is to use the decomposition of any element $x \in \mathrm{H}^o_\rho$ on the Faber Schauder basis of triangular functions, noting that the coefficients are dyadic second differences of x, see [15] or [23]. Hence (42) gives

$$\chi_n^{\mathrm{pg}} \xrightarrow[n\to\infty]{\mathrm{H}^o_\rho} B. \tag{43}$$

To prove now the necessity of (37), assume that the convergence (43) holds for some ρ satisfying (r1), with $\alpha = 1/2$. This implies the tightness of $\big(\varphi(\Xi_{n+1})\big)_{n\geq 1}$ in H^o_ρ. The sequence of degenerated processes $(Z_n)_{n\geq 1}$ defined by $Z_n = \Xi_{n+1}(1)I$ is tight in H^o_ρ in view of the CLT in $\mathbb{R}$. As $\Xi_{n+1} = \varphi(\Xi_{n+1}) + Z_n$, this gives the tightness in H^o_ρ of $(\Xi_{n+1})_{n\geq 1}$. By Theorem 5 in [18] X_1' must satisfy (21) for every positive A, which implies (37) because

$$\mathbf{P}(|X_1'| \geq A\theta(t)) \geq \mathbf{P}(X_1 \geq 1 + AL(t)) = \exp(-1 - AL(t)).$$

The proof is complete. □

Remark 11. From Theorem 10 we see that the critical ρ for the weak H^o_ρ convergence of χ_n^{pg} to B is $\rho_1(h) = h^{1/2}\ln(c/h)$. This fact is somewhat connected to the following observation. Denote by $i^\star$ the random index where the maximal spacing of the sample is realized, i.e.

$$U_{n:i^\star} - U_{n:i^\star-1} = \delta_{n:n+1} = \max_{1\leq i\leq n+1}(U_{n:i} - U_{n:i-1}).$$

By Lemma 6, we have for n large enough,

$$\mathbf{P}\Big(\delta_{n:n+1} \geq \frac{\ln(n+1)}{n+1}\Big) > 1 - 2\mathrm{e}^{-1} > 0.$$

Now the following lower bound holds at least with probability $1 - 2\mathrm{e}^{-1}$.

$$\frac{|\chi_n^{\mathrm{pg}}(u_{n:i^\star}) - \chi_n^{\mathrm{pg}}(u_{n:i^\star-1})|}{\rho_1(1/(n+1))} \geq \frac{n^{1/2}\big(\frac{\ln(n+1)}{n+1} - \frac{1}{n+1}\big)}{(n+1)^{-1/2}\ln(c(n+1))} \sim 1.$$

Hence $\omega_{\rho_1}(\chi_n^{\mathrm{pg}}, 1/n)$ cannot converge to zero in probability, which forbids the tightness in $\mathrm{H}^o_{\rho_1}$ of $(\chi_n^{\mathrm{pg}})_{n\geq 1}$ in view of (12).

Since we already noted that our definition of the quantile process is not the most classical, it is natural to ask if Theorem 10 still holds true with the classical polygonal quantile process $\tilde{\chi}_n^{\mathrm{pg}}$, based on the left continuous inverse of the empirical distribution function F_n. This process is the random polygonal line with vertices $\big(i/n, n^{1/2}(U_{n:i} - i/n)\big)$, $0 \leq i \leq n$.

Corollary 12. *Let $\rho(h) = h^{1/2}L(1/h)$ be a weight function satisfying (r1) and (r3). Then $\tilde{\chi}_n^{\mathrm{pg}}$ converges weakly in H_ρ^o to the Brownian bridge, if and only if L satisfies (37).*

Proof. To show that (37) implies the weak-H_ρ^o convergence of $\tilde{\chi}_n^{\mathrm{pg}}$ to B, it suffices clearly to check the tightness. To this end, we estimate $\omega_\rho(\tilde{\chi}_n^{\mathrm{pg}}, \delta)$ in terms of $\omega_\rho(\chi_n^{\mathrm{pg}}, \delta)$ in order to apply Theorem 10. To control the increments of $\tilde{\chi}_n^{\mathrm{pg}}$ between s and t we discuss according to the positions of s and t relatively to the grid $\{i/n, i = 0, \dots, n\}$.
Case 1, $\frac{i-1}{n} < s < t \le \frac{i}{n}$. Interpolating linearly between $(i-1)/n$ and i/n and using the relationship between $U_{n:i} - U_{n:i-1}$ and $\chi_n^{\mathrm{pg}}(u_{n:i}) - \chi_n^{\mathrm{pg}}(u_{n:i-1})$, we get

$$\tilde{\chi}_n^{\mathrm{pg}}(t) - \tilde{\chi}_n^{\mathrm{pg}}(s) = n(t-s)\big(\chi_n^{\mathrm{pg}}(u_{n:i}) - \chi_n^{\mathrm{pg}}(u_{n:i-1})\big) - \frac{n^{1/2}(t-s)}{n+1}.$$

Due to (r1), $h/\rho(h)$ increases on $(0, 1/n]$ for n large enough. Hence

$$\begin{aligned}\frac{|\tilde{\chi}_n^{\mathrm{pg}}(t) - \tilde{\chi}_n^{\mathrm{pg}}(s)|}{\rho(t-s)} &\le n\frac{t-s}{\rho(t-s)}\omega_\rho\Big(\chi_n^{\mathrm{pg}}, \frac{1}{n+1}\Big)\rho\Big(\frac{1}{n+1}\Big) + \frac{n^{1/2}}{n+1}\frac{t-s}{\rho(t-s)} \\ &\le \omega_\rho\Big(\chi_n^{\mathrm{pg}}, \frac{1}{n}\Big) + \frac{1}{n^{3/2}\rho(1/n)}. \end{aligned} \tag{44}$$

Case 2, $\frac{i-1}{n} < s \le \frac{i}{n} \le (j-1)/n < t \le j/n$. By chaining we get

$$\frac{|\tilde{\chi}_n^{\mathrm{pg}}(t) - \tilde{\chi}_n^{\mathrm{pg}}(s)|}{\rho(t-s)} \le T_1 + T_2 + T_3,$$

where

$$T_1 := \frac{|\tilde{\chi}_n^{\mathrm{pg}}(i/n) - \tilde{\chi}_n^{\mathrm{pg}}(s)|}{\rho(i/n - s)}, \quad T_2 := \frac{|\tilde{\chi}_n^{\mathrm{pg}}(t) - \tilde{\chi}_n^{\mathrm{pg}}((j-1)/n)|}{\rho(t - (j-1)/n)},$$

$$T_3 := \frac{|\tilde{\chi}_n^{\mathrm{pg}}((j-1)/n) - \tilde{\chi}_n^{\mathrm{pg}}(i/n)|}{\rho((j-1)/n - i/n)},$$

with the convention that $T_k := 0$ when its denominator vanishes. From Case 1, T_1 and T_2 are bounded by (44). To bound T_3 when $i < j-1$, we note that

$$\tilde{\chi}_n^{\mathrm{pg}}\Big(\frac{j-1}{n}\Big) - \tilde{\chi}_n^{\mathrm{pg}}\Big(\frac{i}{n}\Big) = \chi_n^{\mathrm{pg}}(u_{n:j-1}) - \chi_n^{\mathrm{pg}}(u_{n:i}) - \frac{j-1-i}{n^{1/2}(n+1)},$$

whence

$$\begin{aligned} T_3 &\le \frac{\omega_\rho\big(\chi_n^{\mathrm{pg}}, \frac{j-1-i}{n+1}\big)\rho\big(\frac{j-1-i}{n+1}\big)}{\rho\big(\frac{j-1-i}{n}\big)} + \frac{1}{n^{1/2}}\frac{\frac{j-1-i}{(n+1)}}{\rho\big(\frac{j-1-i}{n}\big)} \\ &\le \omega_\rho(\chi_n^{\mathrm{pg}}, t-s) + Cn^{-1/2}, \end{aligned} \tag{45}$$

where $C := \sup_{0<h\leq 1} h/\rho(h)$. Accounting (44), we obtain finally in Case 2 and for n large enough,

$$\frac{|\tilde{\chi}_n^{\mathrm{pg}}(t) - \tilde{\chi}_n^{\mathrm{pg}}(s)|}{\rho(t-s)} \leq 2\omega_\rho\Big(\chi_n^{\mathrm{pg}}, \frac{1}{n}\Big) + \omega_\rho(\chi_n^{\mathrm{pg}}, t-s) + (C+1)n^{-1/2}.$$

To conclude this discussion, let us simply retain that for any $0 < \delta < 1$ and $n > 1/\delta$,

$$\omega_\rho(\tilde{\chi}_n^{\mathrm{pg}}, \delta) \leq 3\omega_\rho(\chi_n^{\mathrm{pg}}, \delta) + (C+1)n^{-1/2}.$$

From this it follows that

$$\limsup_{n\to\infty} \mathbf{P}\big(\omega_\rho(\tilde{\chi}_n^{\mathrm{pg}}, \delta) > \varepsilon\big) \leq \limsup_{n\to\infty} \mathbf{P}\big(\omega_\rho(\chi_n^{\mathrm{pg}}, \delta) > \varepsilon/4\big). \tag{46}$$

From (37), Theorem 10 and Theorem 1 applied to $(\chi_n^{\mathrm{pg}})_{n\geq 1}$ and (46) we deduce that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbf{P}\big(\omega_\rho(\tilde{\chi}_n^{\mathrm{pg}}, \delta) > \varepsilon\big) = 0.$$

Now Theorem 1 applied to the sequence $(\tilde{\chi}_n^{\mathrm{pg}})_{n\geq 1}$ gives its tightness in H_ρ^o.

The necessity of (37) for the weak H_ρ^o convergence of $(\tilde{\chi}_n^{\mathrm{pg}})_{n\geq 1}$ is easily checked by the argument given in Remark 11. □

References

[1] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular variation,* Encyclopaedia of Mathematics and its Applications, Cambridge University Press, 1987.

[2] P. Billingsley, *Convergence of probability measures,* Wiley (1968).

[3] Z. Ciesielski, *On the isomorphisms of the spaces H_α and m,* Bull. Acad. Pol. Sci. Ser. Sci. Math. Phys., 8 (1960), pp. 217–222.

[4] Z. Ciesielski, *Hölder conditions for realizations of Gaussian processes*, Trans. Amer. Math. Soc., 99 (1961), pp. 403–413.

[5] M. Csörgő, *Quantile Processes with Statistical Applications,* CBMS-NSF Regional Conference Series in Applied Mathematics No 42, SIAM Philadelphia, 1983.

[6] P. Deheuvels, *Functional laws of the iterated logarithm for small increments of empirical processes,* Statistica Neerlandica, 50 no 2 (1996), pp. 261–280.

[7] L. Devroye, *Upper and lower class sequences for minimal uniform spacings,* Zeitschrift für Wahrscheinlichkeit und verwandte Gebiete 61 (1982), pp. 237–254.

[8] R.M. Dudley, *Fréchet differentiability, p-variation and uniform Donsker classes,* Ann. Probab. 20 (1992), No. 4, pp. 1968–1982.

[9] D. Hamadouche, *Weak convergence of smoothed empirical process in Hölder spaces,* Statist. Probab. Letters 36 (1998), pp. 393–400.

[10] D. Hamadouche, *Hölder convergence of smoothed empirical process,* Statist. Probab. Letters, 39 (1998), pp. 385–386.

[11] J. Lamperti, *On convergence of stochastic processes*, Trans. Amer. Math. Soc., 104 (1962), pp. 430–435.

[12] M. Ledoux and M. Talagrand, *Probability in Banach Spaces.* Springer-Verlag, Berlin, Heidelberg (1991).

[13] P. Lévy, *Théorie de l'addition des variables aléatoires.* Gauthier-Villars, Paris, Second Edition, 1954.

[14] B. Morel and Ch. Suquet, *Hilbertian invariance principles for the empirical process under association,* Mathematical Methods of Statistics, vol. 11, No 2, (2002), pp. 203–220.

[15] A. Račkauskas and Ch. Suquet, *Random fields and central limit theorem in some generalized Hölder spaces,* In: B. Grigelionis *et al.* (Eds), Prob. Theory and Math. Statist., Proceedings of the 7th Vilnius Conference (1998), TEV Vilnius and VSP Utrecht, pp. 599–616.

[16] A. Račkauskas and Ch. Suquet, *Invariance principles for adaptive self-normalized partial sums processes,* Stochastic Process. Appl., 95 (2001), pp. 63–81.

[17] A. Račkauskas and Ch. Suquet, *Hölder convergences of multivariate empirical characteristic functions,* Mathematical Methods of Statistics vol. 11, no. 3, (2002).

[18] A. Račkauskas and Ch. Suquet, *Necessary and sufficient condition for the Hölderian functional central limit theorem,* Journal of Theoretical Probability, vol. 17, No 1, January 2004, pp. 221–243.

[19] A. Račkauskas and Ch. Suquet, *Central limit theorems in Hölder topologies for Banach space valued random fields,* Theory of Probability and its Applications, vol. 49, No 1, 2004, pp. 109–125.

[20] A. Račkauskas and Ch. Suquet, *Hölder norm test statistics for epidemic change,* Journal of Statistical Planning and Inference 2004, vol. 126, Issue 2, pp. 495–520.

[21] A. Račkauskas and Ch. Suquet, *Testing epidemic changes of infinite dimensional parameters,* to appear in Statistical Inference for Stochastic Processes.

[22] H.P. Rosenthal, *On the subspaces of L_p $(p > 2)$ spanned by sequences of independent random variables,* Israel J. of Math. 1970, vol. 8, pp. 273–303.

[23] Z. Semadeni, *Schauder bases in Banach spaces of continuous functions.* Lecture Notes Math. 918, Springer Verlag, Berlin, Heidelberg, New York, 1982.

[24] G.R. Shorack and J.A. Wellner, *Empirical Processes with applications to statistics,* Wiley 1986.