Computing the distribution of sequential Hölder norms of the Brownian motion

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Abstract

The distributions of Hölder norms of Brownian motion and of Brownian bridge have statistical applications for instance in the detection of some short "epidemic" changes in a sample. Unfortunately the exact distribution of these norms is not known. For practical reasons it is then convenient to replace these norms by sequential Hölder norms. The aim of this paper is to give explicit formulas and study practical computations for the distribution function of such norms.

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1. Introduction

Brownian motion with no doubt is the most computationally tractable stochastic process. Distributional properties of various its functionals receive tremendous attention from both theoreticians and practitioners in very broad area of research. Large number of facts and formulae associated to Brownian motion is given in the handbook by Borodin and Salminen (2002). Nothing seems to be known on the explicit distribution of the Hölder norms of the Brownian motion. These norms have some statistical interest since e.g. they are limit in distribution of some statistics used to detect short epidemics

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changes, see Račkauskas and Suquet (2004). Fortunately, there are equivalent sequential norms based on weighted dyadic increments. The aim of this paper is to study computational aspects of distribution of such sequential Hölder norms of Brownian motion. To be more precise let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j, i.e.

$$\mathrm{D}_0 = \{0,1\}, \qquad \mathrm{D}_j = \Big\{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\Big\}, \quad j \ge 1.$$

So the countable set D of dyadic numbers of $[0,1]$ may be written as

$$\mathrm{D} = \bigcup_{j=0}^{\infty} \mathrm{D}_j.$$

We use the enumeration $n \mapsto r_n$ of the elements of D provided by the lexicographic order on the couples (j,r) where j is the level of r. So $r_0 = 0$, $r_1 = 1$, $r_2 = 1/2$, $r_3 = 1/4$, $r_4 = 3/4$, $r_5 = 1/8, \ldots$ Hence a series like $\sum_{r\in \mathrm{D}} f(r)$ should be understood as $\sum_{n=0}^{\infty} f(r_n)$. For notational convenience, we put

$$\mathrm{D}^* := \mathrm{D} \setminus \{0\}.$$

Write for $r \in \mathrm{D}_j$, $j \ge 0$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

For a function $f : [0,1] \to \mathbb{R}$, we shall denote

$$\lambda_r(f) := \begin{cases} f(r) - \frac{1}{2}(f(r^+) + f(r^-)), & \text{if} \quad r \in \mathrm{D}_j, j \ge 1, \\ f(r) & \text{if} \quad r \in \mathrm{D}_0. \end{cases} \tag{1}$$

Dyadic increments functional $\mathrm{DI}(f,\rho)$ of a function $f : [0,1] \to \mathbb{R}$ depends on a weight function $\rho : [0,1] \to \mathbb{R}$, of the form $\rho(h) = h^\alpha L(1/h)$ with L slowly varying, and is defined by

$$\mathrm{DI}(f,\rho) := \max_{1\le j} \frac{1}{\rho(2^{-j})} \max_{r\in \mathrm{D}_j} |\lambda_r(f)|. \tag{2}$$

Let $W = \{W_t, t \in [0,1]\}$ be a standard Brownian motion process and $B = \{B_t, t \in [0,1]\}$ be the corresponding Brownian bridge $B_t = W_t - tW_1, t \in [0,1]$. Our aim is to provide practical computations for the distribution functions of $\mathrm{DI}(W, \rho)$ and $\mathrm{DI}(B, \rho)$ for a class of weight functions ρ. Such functionals appear as limits for dyadic increment statistics

$$\frac{1}{2} \max_{1 \leq 2^j \leq n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \Big| \sum_{nr^- < k \leq nr} X_k - \sum_{nr < k \leq nr^+} X_k \Big| \tag{3}$$

which are useful in so called epidemic change problems in a sample $X_1, \ldots, X_n$ (for more information we refer to Račkauskas and Suquet (2004, 2006) and the references therein.

The paper is organized as follows. Section 2 presents briefly some analytical background on Hölder spaces, sequential norms, Haar functions, triangular Faber-Schauder functions. This is used in Section 3 to generalize the classical Lévy-Kampé de Fériet expansion of Brownian motion in a series of triangular function to the Hölderian setting. In Section 4 this expansion leads to a representation of the distribution function of $\mathrm{DI}(W, \rho)$ and $\mathrm{DI}(B, \rho)$ as an infinite product of distributions functions. Section 5 discuss the rate of convergence of this product.

2. Haar and Faber-Schauder functions

In what follows $L^2[0,1]$ denote the classical Hilbert space of square integrable real valued functions on $[0,1]$ with respect to the Lebesgue measure and endowed with the scalar product $\langle f, g \rangle = \int_0^1 f(t) g(t) \, \mathrm{d}t$. We denote by $C[0,1]$ the space of real valued continuous functions on $[0,1]$ endowed with the supremum norm $\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$.

A simple way to parametrize the collections of Haar and Faber-Schauder functions is to use directly the dyadic numbers. For $r \in \mathrm{D}_j$, $j \geq 1$, the $L^2[0,1]$ normalized Haar function H_r, is defined on $[0,1]$ by

$$H_r(t) = \begin{cases} +(r^+ - r^-)^{-1/2} = +2^{(j-1)/2} & \text{if } t \in (r^-, r]; \\ -(r^+ - r^-)^{-1/2} = -2^{(j-1)/2} & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

In the special case $j = 0$, as the support $[r^-, r^+]$ would extend beyond $[0,1]$ we have to modify the above formula to keep the $L^2[0,1]$ normalization. This leads us to define

$$H_0(t) := -\mathbf{1}_{[0,1]}(t), \quad H_1(t) := \mathbf{1}_{[0,1]}(t).$$

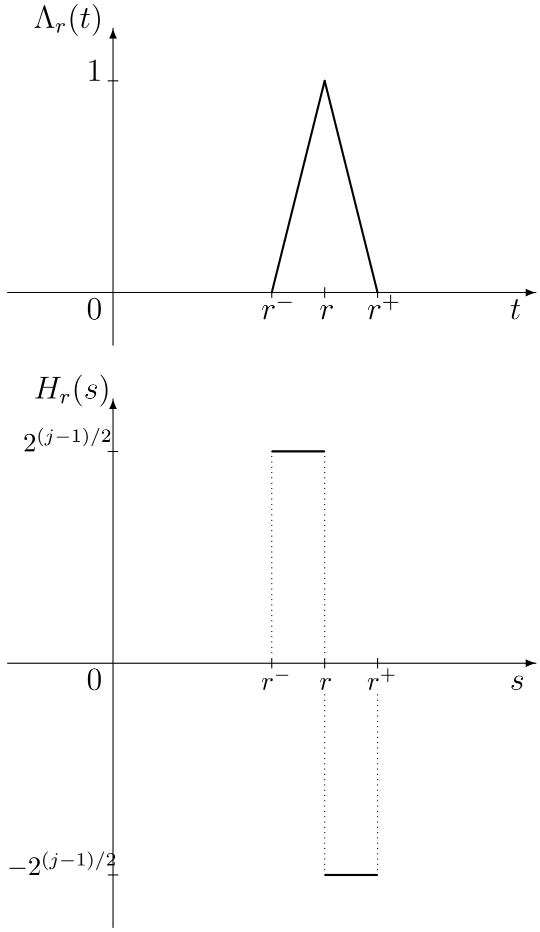

Figure 1: The Haar function H_r and the Faber-Schauder function Λ_r

The sequence $\{H_r; r \in \mathrm{D}, r \neq 0\}$ is an orthonormal basis of the Hilbert space $L^2[0,1]$.

For $r \in \mathrm{D}_j$, $j \geq 1$, the triangular Faber-Schauder functions Λ_r are continuous, piecewise affine with support $[r^-, r^+]$ and taking the value 1 at r:

$$\Lambda_r(t) = \begin{cases} (t - r^-)/(r - r^-) = 2^j(t - r^-) & \text{if } t \in (r^-, r]; \\ (r^+ - t)/(r^+ - r) = 2^j(r^+ - t) & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

In the special case $j = 0$, we just take the restriction to $[0,1]$ in the above formula, so

$$\Lambda_0(t) = 1 - t, \quad \Lambda_1(t) = t, \quad t \in [0,1].$$

The Λ_r's have the $C[0,1]$ normalization: $\|\Lambda_r\|_\infty = 1$. They are linked to the H_r's in the general case $r \in \mathrm{D}_j$, $j \geq 1$ by

$$\Lambda_r(t) = 2(r^+ - r^-)^{-1/2} \int_0^t H_r(s)\,\mathrm{d}s = 2^{(j+1)/2} \int_0^t H_r(s)\,\mathrm{d}s \tag{4}$$

and in the special case $j = 0$ by

$$\Lambda_0(t) = 1 + \int_0^t H_0(s)\,\mathrm{d}s, \quad \Lambda_1(t) = \int_0^t H_1(s)\,\mathrm{d}s. \tag{5}$$

The sequence $\{\Lambda_r; r \in \mathrm{D}\}$ is a Schauder basis of $C[0,1]$. Each $x \in C[0,1]$ has a unique expansion

$$x = \sum_{r \in \mathrm{D}} \lambda_r(x)\Lambda_r = \sum_{j=0}^{\infty} \sum_{r \in \mathrm{D}_j} \lambda_r(x)\Lambda_r,$$

with uniform convergence on $[0,1]$. The Schauder scalar coefficients $\lambda_r(x)$ are defined by (1).

The sequence $\{\Lambda_r; r \in \mathrm{D}\}$ is also a Schauder basis in a large class of Hölder spaces we are describing now. Let ρ be a real valued non decreasing function on $[0,1]$, null and right continuous at 0. Put

$$w_\rho(x,\delta) := \sup_{\substack{s,t \in [0,1],\\ 0<t-s<\delta}} \frac{|x(t)-x(s)|}{\rho(t-s)}.$$

We associate to ρ the separable Hölder space

$$\mathrm{H}_\rho^o[0,1] := \{x \in \mathrm{C}[0,1];\ \lim_{\delta \to 0} w_\rho(x,\delta) = 0\},$$

equiped with the norm

$$\|x\|_\rho := |x(0)| + w_\rho(x,1).$$

With the aim to use a sequential norm equivalent to $\|x\|_\rho$, we require, following Ciesielski (see e.g. (Semadeni, 1982, p.67)), that the modulus of smoothness ρ satisfies the conditions:

$$\rho(0) = 0,\ \rho(h) > 0,\ 0 < h \leq 1; \tag{6}$$

$$\rho \text{ is non decreasing on } [0, 1]; \tag{7}$$

$$\rho(2h) \leq c_1 \rho(h), \quad 0 \leq h \leq 1/2; \tag{8}$$

$$\int_0^h \frac{\rho(u)}{u} \,\mathrm{d}u \leq c_2 \rho(h), \quad 0 < h \leq 1; \tag{9}$$

$$h \int_h^1 \frac{\rho(u)}{u^2} \,\mathrm{d}u \leq c_3 \rho(h), \quad 0 < h \leq 1; \tag{10}$$

where c_1, c_2 and c_3 are positive constants. Let us observe in passing, that (6), (7) and (9) together imply the right continuity of ρ at 0. The class of functions ρ satisfying these requirements is rich enough according to the following.

Proposition 1. *For any* $0 < \alpha < 1$, *consider the function*

$$\rho(h) = h^\alpha L(1/h)$$

where L *is normalized slowly varying at infinity, continuous and positive on* $[1, \infty)$. *Then* ρ *fulfills conditions (6) to (10) up to a change of scale.*

The proof can be be found in (Račkauskas and Suquet, 2004, Prop.2).

Proposition 2. $\{\Lambda_r; r \in \mathrm{D}\}$ *is a Schauder basis of the space* $\mathrm{H}^o_\rho[0, 1]$ *and we have the equivalence of norms*

$$\|x\|_\rho \sim \|x\|_\rho^{\mathrm{seq}} := \sup_{j \geq 0} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} |\lambda_r(x)|.$$

For a proof see Semadeni (1982).

3. Expansion of the Brownian motion

The expansion of the standard Brownian motion as a series of triangular functions converging in $C[0, 1]$ is classical and goes back to Lévy and Kampé de Fériet (1957). It turns out that the same series converges also in the stronger topology of $\mathrm{H}^o_\rho[0, 1]$ for any ρ compatible with the smoothness of the Brownian motion. This provides a convenient expression of the ρ sequential norms of the Brownian motion and the Brownian bridge.

Theorem 3. *Assume that ρ fulfills Conditions (6) to (10) and that*

$$\sqrt{h|\ln h|} = o\Big(\rho(h)\Big), \quad h \to 0. \tag{11}$$

Let $\{X_r; r \in \mathrm{D}^*\}$ be a sequence of independent $\mathfrak{N}(0,1)$ random variables. Then the random series of functions

$$W := X_1\Lambda_1 + \sum_{j=1}^{\infty}\sum_{r\in\mathrm{D}_j} 2^{-(j+1)/2}X_r\Lambda_r, \tag{12}$$

converges a.s. in the space $\mathrm{H}^o_\rho[0,1]$ for any $\rho \in \mathcal{R}$. W is a Brownian motion started at 0. Removing the term $X_1\Lambda_1$ in (12) gives a Brownian bridge B. The ρ sequential norms of W and B may be written as

$$\|B\|^{\mathrm{seq}}_\rho = \sup_{j\geq 1}\frac{1}{\sqrt{2}\theta(2^j)}\max_{r\in D_j}|X_r|, \quad \|W\|^{\mathrm{seq}}_\rho = \max\Big\{\frac{|X_1|}{\rho(1)}, \|B\|^{\mathrm{seq}}_\rho\Big\}, \tag{13}$$

where

$$\theta(t) := t^{1/2}\rho(1/t), \quad t \geq 1. \tag{14}$$

We note that Condition (11) is optimal in view of the classical Lévy's result about the modulus of uniform continuity of the Brownian motion. Theorem 3 is a special case of Theorem 13 in Račkauskas and Suquet (2006) where the X_i's are random elements in some Banach space. We rewrite it here with its proof in the more usual setting of random variables for reader's convenience.

Proof. According to Prop. 3 c) in Račkauskas and Suquet (2001), the series (12) converges almost surely in the space $\mathrm{H}^o_\rho[0,1]$ if

$$\lim_{j\to\infty}\frac{1}{\rho(2^{-j})}\max_{r\in\mathrm{D}_j}|2^{-j/2}X_r| = 0, \text{ almost surely.} \tag{15}$$

A sufficient condition for the convergence (15) is that for every positive ε,

$$\sum_{j\geq 1}\mathbf{P}\Big\{\max_{r\in\mathrm{D}_j}|X_r| \geq \varepsilon\theta(2^j)\Big\} < \infty. \tag{16}$$

By identical distribution of the X_r's and the classical estimate $\mathbf{P}(|X_1| \geq t) \leq \exp(-t^2/2)$, (16) will follow in turn from

$$\sum_{j\geq 1} 2^j \exp\left(\frac{-\varepsilon^2\theta^2(2^j)}{2}\right) < \infty,$$

which is easily deduced from Condition (11). Therefore (15) is satisfied and the series (12) converges almost surely in the space $\mathrm{H}_\rho^o[0,1]$, defining a mean zero Gaussian random element $W = X_1\Lambda_1 + B$ in $\mathrm{H}_\rho^o[0,1]$.

This convergence together with the obvious continuity of the λ_r's considered as linear functionals on $\mathrm{H}_\rho^o[0,1]$ legitimates the equality

$$\lambda_r(B) = \sum_{j=1}^{\infty} \sum_{r'\in \mathrm{D}_j} 2^{-(j+1)/2} \lambda_r\Big(X_{r'}\Lambda_{r'}\Big), \quad \text{a.s.}$$

By Lemma 1 in Račkauskas and Suquet (2001) we get

$$\lambda_r\Big(X_{r'}\Lambda_{r'}\Big) = X_{r'}\lambda_r\Big(\Lambda_{r'}\Big) = X_{r'}\mathbf{1}_{\{r=r'\}}, \quad r, r' \in \mathrm{D}^*.$$

Hence for $r \in \mathrm{D}_j$ $(j \geq 1)$, $\lambda_r(B) = 2^{-(j+1)/2}X_r$ and (13) follows.

To complete the proof, it remains to check that W is a standard Brownian motion. It is convenient here to recast (2) as $2^{-(j+1)/2}\Lambda_r(t) = \langle H_r, \mathbf{1}_{[0,t]}\rangle$, where $\langle\ ,\ \rangle$ denotes the scalar product in the space $L^2[0,1]$. Now (12) implies clearly for each $t \in [0,1]$ that

$$W(t) = \sum_{r\in \mathrm{D}^*} X_r \langle H_r, \mathbf{1}_{[0,t]}\rangle,$$

where the series converges almost surely for each $t \in [0,1]$. So for any $0 \leq s < t \leq 1$,

$$W(t) - W(s) = \sum_{r\in \mathrm{D}^*} X_r \langle H_r, \mathbf{1}_{(s,t]}\rangle$$

and this a.s. convergent series of independent mean zero Gaussian random variables converges also in quadratic mean, which legitimates the following covariance computation. For $0 \leq s < t \leq 1$, $0 \leq s' < t' \leq 1$, put

$$K(s,t,s',t') := \mathbf{E}\Big[\Big(W(t) - W(s)\Big)\Big(W(t') - W(s')\Big)\Big].$$

Noting that $\mathbf{E}\,(X_r X_{r'}) = \mathbf{1}_{\{r=r'\}}$ and using Parseval's identity for the Haar basis of $L^2[0,1]$ we obtain

$$K(s,t,s',t') = \sum_{r\in \mathrm{D}^*} \mathbf{E}\, X_r^2 \langle H_r, \mathbf{1}_{(s,t]}\rangle\langle H_r, \mathbf{1}_{(s',t']}\rangle = \langle \mathbf{1}_{(s,t]}, \mathbf{1}_{(s',t']}\rangle. \tag{17}$$

Whenever $(s,t] \cap (s',t'] = \emptyset$, (17) gives the independence of the Gaussian random variables $W(t) - W(s)$ and $W(t') - W(s')$, whence follows the independence of increments for the process W. Moreover (17) gives

$$K(s,t,s,t) = |t-s|.$$

As $W(0) = 0$, this achieves the identification of W as a version of the standard Brownian motion in the space $\mathrm{H}_\rho^o[0,1]$. □

4. Distributions of sequential norms

The distribution function of $\|B\|_\rho^{\mathrm{seq}}$ may be conveniently expressed in terms of the *error function*:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2)\,\mathrm{d}s = \mathbf{P}\Big(|X_1| \le x\sqrt{2}\Big), \quad x \ge 0,$$

where X_1 is $\mathfrak{N}(0,1)$ distributed. The following asymptotic expansion will be useful

$$\operatorname{erf} x = 1 - \frac{1}{x\sqrt{\pi}} \exp(-x^2)\Big(1 + O(x^{-2})\Big), \quad x \to \infty. \tag{18}$$

Theorem 4. *Let* $c = \limsup_{j\to\infty} j^{1/2}/\theta(2^j)$, *where* θ *is defined by* (14)*.*

i) If $c = \infty$ *then* $\|B\|_\rho^{\mathrm{seq}} = \|W\|_\rho^{\mathrm{seq}} = \infty$ *almost surely.*

ii) If $0 \le c < \infty$*, then* $\|B\|_\rho^{\mathrm{seq}}$ *and* $\|W\|_\rho^{\mathrm{seq}}$ *are almost surely finite and their distribution functions are given by*

$$\mathbf{P}\Big(\|B\|_\rho^{\mathrm{seq}} \le x\Big) = \prod_{j=1}^{\infty} \Big\{\operatorname{erf}\Big(\theta(2^j)x\Big)\Big\}^{2^{j-1}}, \quad x > 0 \tag{19}$$

and

$$\mathbf{P}\Big(\|W\|_\rho^{\mathrm{seq}} \le x\Big) = \operatorname{erf}\Big(2^{-1/2}\theta(1)x\Big) \prod_{j=1}^{\infty} \Big\{\operatorname{erf}\Big(\theta(2^j)x\Big)\Big\}^{2^{j-1}}, \quad x > 0. \tag{20}$$

The support of these distributions is $[c\sqrt{\ln 2}, \infty)$*.*

Proof. Recalling (13), let us consider the non decreasing sequence of random variables $(M_J)_{J\ge1}$ where

$$M_J := \max_{1\le j\le J} \frac{1}{\sqrt{2}\theta(2^j)} \max_{r\in D_j} |X_r|.$$

Considering $\|B\|_\rho^{\mathrm{seq}}$ as a random element in $[0,\infty]$, we have for each $x \in \mathbb{R}$,

$$\mathbf{P}(M_J \le x) \downarrow \mathbf{P}(\|B\|_\rho^{\mathrm{seq}} \le x) =: F_{B,\rho}(x), \quad (J \to \infty). \tag{21}$$

By independence and equal distribution of the X_r's,

$$\mathbf{P}(M_J \le x) = \prod_{j=1}^{J} \Big\{ \operatorname{erf}\Big(x\theta(2^j) \Big) \Big\}^{2^{j-1}} . \tag{22}$$

Then (19) and (20) follow from (21) and (22) and this holds true without any requirement on the value of c (including the case where $c = \infty$).

As $\|B\|_\rho^{\mathrm{seq}}$ may take infinite value, (21) does not guarantee that $F_{B,\rho}$ is a probability distribution function on $[0, \infty)$. Now, we are left with two questions.

a) Does there exists $x_0 < \infty$ such that $F_{B,\rho}(x_0) > 0$? In this case $F_{B,\rho}$ will be positive on $[x_0, \infty)$.

b) $\lim_{x\to\infty} F_{B,\rho}(x) = 1$? In this case $P(\|B\|_\rho^{\mathrm{seq}} = \infty) = 0$.

Let us get rid first of the rough subcase of i) where $\theta(2^j)$ does not go to ∞ with j. Then there is a constant A and an infinite subset $\mathbb{J}_1$ of $\mathbb{N}$ such that $\theta(2^j) \le A$ for each $j \in \mathbb{J}_1$. Then noting that

$$0 \le M_J \le \min_{j\in\mathbb{J}_1,\, j\le J} \operatorname{erf}\Big(Ax\Big)^{2^{j-1}}$$

we directly see that $\lim_{J\to\infty} M_J = 0$ for every $0 \le x < \infty$. So $F_{B,\rho}(x) = 0$ for every $x \in \mathbb{R}$, which means exactly that $\|B\|_\rho^{\mathrm{seq}} = \infty$ with probability one. The same holds for W since $\|W\|_\rho^{\mathrm{seq}} \ge \|B\|_\rho^{\mathrm{seq}}$.

From now on, we assume that $\lim_{j\to\infty} \theta(2^j) = \infty$. Taking logarithms in (22) and using (18), it is easily seen that a positive answer to question a) is equivalent to the convergence of the series $\sum_{j=1}^\infty u_j(x_0)$ where

$$u_j(x) := \frac{2^j}{x\theta(2^j)} \exp\Big(-x^2\theta(2^j)^2\Big) = \frac{1}{x}\exp\Big(-x^2\theta(2^j)^2 + j\ln 2 - \ln\theta(2^j)\Big).$$

If $c = \infty$, there is an infinite subset $\mathbb{J}_2$ of $\mathbb{N}$ such that $\lim_{\mathbb{J}_2\ni j\to\infty} j\theta(2^j)^{-2} = \infty$, so that $\lim_{\mathbb{J}_2\ni j\to\infty} u_j(x) = \infty$ and the series $\sum_{j=1}^\infty u_j(x)$ diverges for every $x \ge 0$. It follows that $F_{B,\rho}(x) = 0$ for every $x \in \mathbb{R}$.

If $0 \le c < \infty$, we have for any positive ε some $J_0 = J_0(\varepsilon)$ such that

$$\theta(2^j) \ge 1 \quad \text{and} \quad \frac{j}{\theta(2^j)^2} \le (c+\varepsilon)^2, \quad j \ge J_0.$$

It follows that

$$u_j(x) \le \frac{1}{x} \exp\left\{ -j \left(\frac{x^2}{(c+\varepsilon)^2} - \ln 2 \right) \right\}, \quad j \ge J_0,$$

so the series $\sum_{j=1}^{\infty} u_j(x)$ converges at geometric rate for $x > (c+\varepsilon)\sqrt{\ln 2}$. Now the answer to question b) is also positive noting that for $x > c\sqrt{\ln 2}$

$$F_{B,\rho}(x) = \prod_{j=1}^{\infty} \left\{ \operatorname{erf}\left(\theta(2^j)x\right) \right\}^{2^{j-1}}$$

and applying the bounded convergence theorem (with $x \to \infty$) to the series

$$\ln F_{B,\rho}(x) = \sum_{j=1}^{\infty} 2^{j-1} \ln \operatorname{erf}\left(\theta(2^j)x\right).$$

To complete the proof, it remains to show that the support of the distribution of $\|B\|_{\rho}^{\mathrm{seq}}$ is $[c\sqrt{\ln 2}, \infty)$, for which it is enough to check that if $x < c\sqrt{\ln 2}$, $F_{B,\rho}(x) = 0$. The case $c = 0$ being obvious, let us assume $0 < c < \infty$. Having fixed $0 < x < c\sqrt{\ln 2}$, let us choose $0 < p < q < 1$, such that $x \le pc\sqrt{\ln 2}$. From the definition of c, $\theta(2^j)j^{-1/2}$ converges to $1/c$ along some subsequence. Hence we can find an *infinite* subset $\mathbb{J} \subset \mathbb{N}$ (depending on θ, x and q) such that

$$\theta(2^j) \le \frac{j^{1/2}}{qc}, \quad j \in \mathbb{J}.$$

Puting $s = p^2 q^{-2}$, we obtain the lower bound

$$u_j(x) = \frac{2^j}{x\theta(2^j)} \exp\left(-x^2\theta(2^j)^2\right) \ge \frac{1}{c\sqrt{s \ln 2}} j^{-1/2} 2^j 2^{-sj}, \quad j \in \mathbb{J}.$$

As $\mathbb{J}$ is infinite and $s < 1$, this is enough to entail $\sum_{j=1}^{\infty} u_j(x) = \infty$ and hence $F_{B,\rho}(x) = 0$. □

5. Practical computations

For the numerical computation of $F_{B,\rho}(x)$, we need to estimate the convergence rate of the corresponding infinite product. First we observe that for every positive x

$$\prod_{j=1}^{\infty} \left\{ \operatorname{erf}\left(\theta(2^j)x\right) \right\}^{2^{j-1}} < \prod_{j=1}^{J} \left\{ \operatorname{erf}\left(\theta(2^j)x\right) \right\}^{2^{j-1}} \quad J \ge 1.$$

So to control the relative error committed when approximating the infinite product by its J-th partial product, we simply need a practically computable lower bound $m(J)$ for

$$R_J := \prod_{j=J+1}^{\infty} \left\{\operatorname{erf}\left(\theta(2^j)x\right)\right\}^{2^{j-1}}.$$

Then we will have

$$m(J) \prod_{j=1}^{J} \left\{\operatorname{erf}\left(\theta(2^j)x\right)\right\}^{2^{j-1}} < F_{B,\rho}(x) < \prod_{j=1}^{J} \left\{\operatorname{erf}\left(\theta(2^j)x\right)\right\}^{2^{j-1}}. \tag{23}$$

First we note the elementary estimate

$$\operatorname{erf} x \geq 1 - \exp(-x^2), \quad x \geq 0, \tag{24}$$

which is easily checked by the change of variable $s = x + u$ in the integral $1 - \operatorname{erf} x = 2\pi^{-1/2} \int_x^\infty \exp(-s^2)\, \mathrm{d}s$ and the majorization of $\exp(-2xu)$ by 1 inside the new integral. The lower bound (24) is better than (18) for small values of x. Combining (24) with the inequality

$$1 - y \geq \exp(-2y), \quad 0 \leq y \leq 0.7968,$$

we obtain provided that $\exp(-\theta(2^{J+1})^2 x^2) \leq 0.7968$,

$$\left\{\operatorname{erf}\left(\theta(2^j)x\right)\right\}^{2^{j-1}} \geq \left\{1 - \exp\left(-\theta(2^j)^2 x^2\right)\right\}^{2^{j-1}} \geq \exp\left(-2^j \exp\left(-\theta(2^j)^2 x^2\right)\right).$$

Hence we get for $x\theta(2^{J+1}) \geq 0.4767 > \sqrt{-\ln(0.7968)}$,

$$R_J \geq \exp\left(-\sum_{j=J+1}^{\infty} 2^j \exp\left(-\theta(2^j)^2 x^2\right)\right).$$

Assuming that θ is non decreasing and comparing series and integral leads now to

$$R_J \geq \exp\left(-2 \int_{2^J}^{\infty} \exp\left(-x^2\theta(t)^2\right) \mathrm{d}t\right), \quad \text{if} \quad x\theta(2^{J+1}) \geq 0.4767. \tag{25}$$

We are mainly interested in the two cases $\rho(h) = h^\alpha$ $(0 < \alpha < 1/2)$ and $\rho(h) = h^{1/2} \ln^\beta(\mathrm{e}^{2\beta}/h)$ $(\beta > 1/2)$. The constant $b = \mathrm{e}^{2\beta}$ is the smallest for which $\rho(h) = h^{1/2} \ln^\beta(b/h)$ fulfils Conditions (6) to (10). In both cases, θ is increasing on $[1, \infty)$.

Case $\rho(h) = h^\alpha$, *(*$0 < \alpha < 1/2$*).* We can express the integral

$$I(J) := \int_{2^J}^\infty \exp\Big(-x^2\theta(t)^2\Big)\,\mathrm{d}t$$

in terms of the Gamma distribution. Recall that the Gamma distribution with scale parameter λ and shape parameter q has density

$$g_{\lambda,q}(y) = \frac{\lambda^q}{\Gamma(q)} \mathrm{e}^{-\lambda y} y^{q-1}, \quad y \geq 0.$$

We denote by $G_{\lambda,q}$ the corresponding distribution function:

$$G_{\lambda,q}(x) = \int_0^x g_{\lambda,q}(y)\,\mathrm{d}y, \quad x \geq 0.$$

$G_{\lambda,q}(x)$ is computable by a software routine. In the current case,

$$\theta(t) = t^{1/p} \quad \text{where} \quad \frac{1}{p} = \frac{1}{2} - \alpha.$$

The change of variable $u = t^{2/p}$ gives now

$$I(J) = \int_{2^{2J/p}}^\infty \exp\Big(-x^2 u\Big) \frac{p}{2} u^{p/2-1}\,\mathrm{d}u = \frac{p\Gamma(p/2)}{2x^p} \int_{2^{2J/p}}^\infty g_{x^2,p/2}(u)\,\mathrm{d}u.$$

Case $\rho(h) = h^{1/2} \ln^\beta(\mathrm{e}^{2\beta}/h)$ *(*$\beta \geq 1/2$*).* Then

$$\theta(t)^2 = (2\beta + \ln t)^{2\beta}$$

and since $2\beta \geq 1$, we get for $t \geq 2^J$:

$$\theta(t)^2 \geq (2\beta)^{2\beta} + (\ln t)^{2\beta} \geq (2\beta)^{2\beta} + (J \ln 2)^{2\beta-1} \ln t.$$

Pluging this in $I(J)$ gives

$$I(J) \leq \exp\Big(-x^2(2\beta)^{2\beta}\Big) \int_{2^J}^\infty t^{-x^2(J\ln 2)^{2\beta-1}}\,\mathrm{d}t.$$

When $\beta > 1/2$, the support of the distribution of $\|B\|_\rho^{\rm seq}$ is $[0,\infty)$ and for any $x > 0$, the integral above converges as soon as $x^2(J\ln 2)^{2\beta-1} > 1$, that is $J > x^{-2/(2\beta-1)}(\ln 2)^{-1}$. Clearly then

$$I(J) \le \exp\Big(-x^2(2\beta)^{2\beta}\Big)\frac{2^{J(1-x^2(J\ln 2)^{2\beta-1})}}{x^2(J\ln 2)^{2\beta-1}-1} \quad \text{if} \quad J > x^{-2/(2\beta-1)}(\ln 2)^{-1}.$$

In the critical case $\beta = 1/2$, we have $\limsup_{j\to\infty} j^{1/2}/\theta(2^j) = (\ln 2)^{-1/2}$, so the support of the distribution is $[1,\infty)$. Then for $x > 1$, we can use the same estimate as for the case $\beta > 1/2$. Here it takes the simpler form

$$I(J) \le \frac{\exp(-x^2)}{x^2-1}2^{J(1-x^2)}, \quad x > 1,\ J \ge 1.$$

Let us observe moreover that when $x > 1$, the preliminary condition $x\theta(2^{J+1}) \ge 0.4767$ writes here $x\Big(1 + (J+1)\ln 2\Big)^{1/2} \ge 0.4767$ and is automatically satisfied for $J \ge 1$.

Let us summarize now this discussion on the relative error.

Proposition 5. *The distribution function $F_{B,\rho}(x)$ of $\|B\|_\rho^{\rm seq}$ is always overestimated by the finite products $\prod_{j=1}^J\Big\{\mathrm{erf}\Big(\theta(2^j)x\Big)\Big\}^{2^{j-1}}$. Moreover, when θ is non decreasing, the relative error of approximation is given by*

$$m(J)\prod_{j=1}^J\Big\{\mathrm{erf}\Big(\theta(2^j)x\Big)\Big\}^{2^{j-1}} < F_{B,\rho}(x) < \prod_{j=1}^J\Big\{\mathrm{erf}\Big(\theta(2^j)x\Big)\Big\}^{2^{j-1}}$$

where

$$m(J) = \exp\left(-2\int_{2^J}^\infty \exp\Big(-x^2\theta(t)^2\Big)\,\mathrm{d}t\right), \quad \textit{for } x\theta(2^{J+1}) \ge 0.4767.$$

- *In the case where $\rho(h) = h^\alpha$, $0 < \alpha < 1/2$, putting $1/p = 1/2 - \alpha$,*

  $$m(J) = \exp\left(-p\Gamma(p/2)\frac{1 - G_{x^2,p/2}\Big(2^{2J/p}\Big)}{x^p}\right), \quad \textit{for } x2^{(J+1)/p} \ge 0.4767,$$

  where $G_{\lambda,q}$ is the Gamma d.f. with scale parameter λ and shape parameter q.

- *In the case where* $\rho(h) = h^{1/2}\ln^{\beta}(\mathrm{e}^{2\beta}/h)$, $\beta > 1/2$,

$$m(J) \geq \exp\left(-2\exp\left(-x^2(2\beta)^{2\beta}\right)\frac{2^{J(1-x^2(J\ln 2)^{2\beta-1})}}{x^2(J\ln 2)^{2\beta-1}-1}\right),$$

for

$$J > \max\left(\frac{1}{x^{2/(2\beta-1)}\ln 2}; \frac{0.4767^{1/\beta}}{x^{1/\beta}\ln 2} - \frac{2\beta}{\ln 2} - 1\right), \quad x > 0.$$

- *In the case where* $\rho(h) = h^{1/2}\ln^{1/2}(\mathrm{e}/h)$,

$$m(J) = \exp\left(-\frac{2\exp(-x^2)}{x^2-1}2^{J(1-x^2)}\right), \quad x > 1, \ J \geq 1.$$

As an illustration, we give in table 1 the values of the distribution function of $\|B\|_{\rho}^{\mathrm{seq}}$ when $\rho(h) = h^{0.45}$, computed with the software Scilab. Although the support of the distribution is $[0, \infty)$, one can observe that the probability that $\|B\|_{\rho}^{\mathrm{seq}}$ belongs to $[1.7, 3.18]$ is bigger than $1 - 2 \times 10^{-5}$.

	$\alpha = 0.45$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.7	.00001	.00004	.00013	.00034	.00082	.00177	.00350	.00642	.01101	.01778
1.8	.02722	.03976	.05572	.07526	.09839	.12496	.15468	.18715	.22189	.25838
1.9	.29608	.33447	.37307	.41142	.44916	.48594	.52152	.55569	.58830	.61925
2.0	.64847	.67595	.70168	.72570	.74805	.76880	.78801	.80575	.82212	.83719
2.1	.85105	.86378	.87545	.88616	.89596	.90493	.91314	.92064	.92750	.93376
2.2	.93948	.94471	.94948	.95383	.95781	.96144	.96475	.96777	.97053	.97305
2.3	.97535	.97745	.97937	.98112	.98272	.98419	.98552	.98674	.98786	.98888
2.4	.98981	.99066	.99144	.99216	.99281	.99341	.99395	.99445	.99491	.99533
2.5	.99572	.99607	.99639	.99669	.99696	.99721	.99744	.99765	.99784	.99801
2.6	.99817	.99832	.99846	.99858	.99870	.99880	.99890	.99899	.99907	.99915
2.7	.99921	.99928	.99934	.99939	.99944	.99948	.99952	.99956	.99960	.99963
2.8	.99966	.99969	.99971	.99973	.99976	.99978	.99979	.99981	.99983	.99984
2.9	.99985	.99986	.99987	.99988	.99989	.99990	.99991	.99992	.99992	.99993
3.0	.99994	.99994	.99995	.99995	.99995	.99996	.99996	.99996	.99997	.99997
3.1	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999

Table 1: Table of the values of $F_{B,\rho}(x)$ for $\rho(h) = h^{0.45}$

Figure 2, produced also with Scilab, shows the influence of α on the localisation of the mass of the distribution of $\|B\|_{\rho}^{\mathrm{seq}}$ when $\rho(h) = h^{\alpha}$. The six diplayed curves from left to right correspond to the values of α in increasing order. The fact that when α approaches the critical value 1/2, the mass is shifted to infinity is of course not surprising.

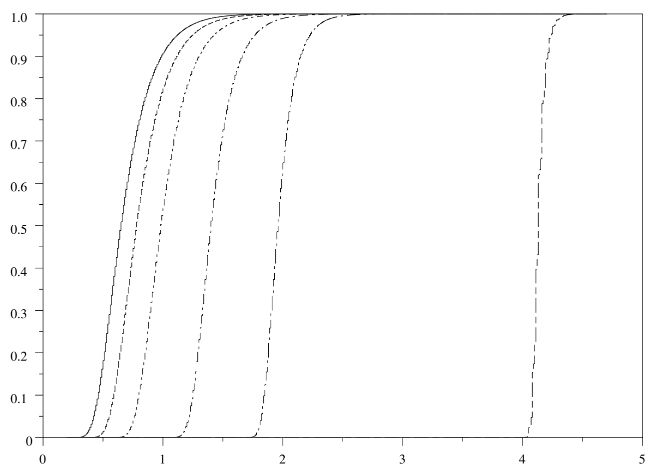

Figure 2: $F_{B,\rho}(x)$ for $\rho(h) = h^{\alpha}$, $\alpha = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.45,\ 0.49$

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