Weak convergence in $\mathrm{L}^p(0,1)$ of the uniform empirical process under dependence

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Abstract: The weak convergence of the empirical process of strong mixing or associated random variables is studied in $L^p(0,1)$. We find minimal rates of convergence to zero of the mixing coefficients or the covariances, in either case, supposing stationarity of the underlying variables. The rates obtained improve, for p not too large, the corresponding results in the classical $D(0,1)$ framework.

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1 Introduction

Let $(U_n, n \geq 1)$ be a stationary sequence of uniform random variables on $[0,1]$ and define, as usual, the uniform empirical process ξ_n by:

$$\xi_n(t) = n^{-1/2} \sum_{i=1}^{n} \big(\mathbb{I}_{[0,t]}(U_i) - t \big), \qquad t \in [0,1]. \tag{1}$$

When the U_i are independent, $(\xi_n,\ n \geq 1)$ is well known to converge weakly in the Skorokhod space $D(0,1)$ to the Brownian bridge. This functional central limit theorem has been widely extended to dependent U_i's, giving a limiting process ξ, which is Gaussian centered with covariance

$$\Gamma(s,t) = s \wedge t - st + \sum_{k=2}^{\infty} \Big(\mathrm{P}(U_1 \leq s, U_k \leq t) - st \Big) + \sum_{k=2}^{\infty} \Big(\mathrm{P}(U_k \leq s, U_1 \leq t) - st \Big).$$

In each particular dependence context, results are stated using some "dependence rate", which can be a mixing coefficient or the covariance $\mathrm{Cov}(U_1, U_k)$ in case of associated U_i's. A natural bound for this dependence rate is clearly the best known rate to obtain the finite dimensional CLT for the random variables $\mathbb{I}_{(s,t]}(U_i)$. Those bounds were obtained under ρ-mixing (see Shao and Yu [12]) and under β-mixing up to a logarithmic factor (see Doukhan, Massart and Rio [6]). As for the strong mixing case or the associated one, this achievement remains an open question. Yoshihara [16] proved the weak $D(0,1)$ convergence of $(\xi_n,\ n \geq 1)$ under the strong mixing rate $\alpha(n) = O(n^{-a})$ with $a > 3$, which was later improved by Shao [11] requiring only $a > 2$. For associated U_i's, Yu obtained the weak $D(0,1)$ convergence of $(\xi_n,\ n \geq 1)$ under the assumption $\mathrm{Cov}(U_1, U_n) = O(n^{-a})$ with $a > 7.5$. Shao and Yu [12] improved this result proving that it is enough to have $a > (3 + \sqrt{33})/2 \simeq 4.373$.

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The classical framework to study the weak convergence of $(\xi_n,\ n \geq 1)$ to a Gaussian process is the Skorokhod topology. This approach provides useful results about the asymptotic behaviour of a large class of functionals. ¿From a statistical point of view, one can be interested in weakening the dependence rate assumptions for a class of functionals of paths continuous in a weaker topology than Skorokhod's one. For instance, the Cramér-von Mises ω^2 test statistic, the Watson statistic [19], the Shepp statistic [13], the Anderson Darling statistics [1], some von Mises functionals, some functionals of the type $\int G\big(t, \xi_n(t)\big)\, \mu(dt)$ require only the L^2 or L^p topology on the space of paths [5]. In [10], the authors obtained the $\mathrm{L}^2(0,1)$ weak convergence of (ξ_n) to the same process ξ as above, under the hypotheses:

$$\sum_{n\geq 1} \alpha(n) < \infty \qquad \text{in the } \alpha\text{-mixing stationary case} \tag{2}$$

$$\sum_{n\geq 1} \mathrm{Cov}^{1/3}(U_1, U_n) < \infty \qquad \text{in the associated stationary case.} \tag{3}$$

Let us observe that (2) is simply the condition giving the Ibragimov's unidimensional CLT for bounded strong mixing random variables [7]. In this case the sequence $(\mathbb{1}_{(s,t]}(U_n),\ n \geq 1)$ is strong mixing with mixing coefficients dominated by those of $(U_n,\ n \geq 1)$. In the associated case, the best known dependence rate to obtain the unidimensional CLT for the U_i's is the summability of the sequence $\mathrm{Cov}(U_1, U_n)$. Unfortunately the sequence $(\mathbb{1}_{(s,t]}(U_n),\ n \geq 1)$ does not inherit the association property from $(U_n,\ n \geq 1)$ and the control of covariances of indicators explains the extra exponent $1/3$.

As one can see, there is an hiatus for intermediate results allowing applications to a larger set of functionals than $\mathrm{L}^2(0,1)$ continuous ones, at a lower cost than $D(0,1)$'s conditions. For α-mixing stationary underlying variables U_i, Suquet [15] proves the $\mathrm{L}^p(0,1)$-weak convergence of ξ_n under the conditions $\alpha(n) = O(n^{-a})$ with $a > p/2$. The main tool used in [10] and [15] is some special wavelets multiresolution analysis of L^p spaces. In this paper we propose a more classical approach of $\mathrm{L}^p(0,1)$ weak convergence allowing a unified treatment of strong mixing and associated cases obtaining a general condition which gives us back the results of [15].

Our results are stated in Section 2 together with some tightness conditions which can have their own interest. Proofs are given in Section 3 and some applications to functionals of paths are presented in Section 4.

2 Results

Theorem 1 *Let $(\xi_n,\ n \geq 1)$ be a sequence of random elements in $\mathrm{L}^p(0,1)$, $p \geq 1$, verifying:*

(i) For some $\gamma > 1$, $\sup_{n\geq 1} \mathbb{E}\, \|\xi_n\|_1^\gamma < \infty$,

(ii) $\lim_{h\to 0} \sup_{n\geq 1} \mathbb{E}\, \|\xi_n(\cdot + h) - \xi_n(\cdot)\|_p^p = 0$.

Then $(\xi_n,\ n \geq 1)$ is tight in $\mathrm{L}^p(0,1)$.

Corollary 2 *A sufficient condition for the sequence of random functions $(\xi_n,\ n \geq 1)$ to be tight in $\mathrm{L}^p(0,1)$, $p \geq 1$, is the existence of positive exponents $q \leq p < r$ such that:*

(a) For some constant c,

$$\mathbb{E}\, |\xi_n(t)|^r \leq c, \quad t \in [0,1], \quad n \geq 1;$$

(b) For some function $\varepsilon(h) = o(1)$ as $h \to 0$:

$$\mathbb{E}\, |\xi_n(t+h) - \xi_n(t)|^q \leq \varepsilon(h), \quad 0 \leq h < 1, \quad 0 \leq t \leq 1-h, \quad n \geq 1.$$

In the next theorem, we suppose that $p \geq 2$, $\{U_n,\, n \geq 1\}$ is a stationary sequence of $[0,1]$-uniformly distributed random variables and ξ_n denotes the uniform empirical process defined by (1).

Theorem 3 *Assume the sequence $\{U_n,\, n \geq 1\}$ is either strong mixing with rate*

$$\begin{aligned} \sum_{n\geq 1} \alpha(n) &< \infty \qquad \text{if } p = 2, && (4)\\ \alpha(n) &= O(n^{-a}) \quad \text{for some } a > p/2, \text{ if } p > 2; && (5) \end{aligned}$$

either associated with

$$\begin{aligned} \sum_{n\geq 1} \mathrm{Cov}^{1/3}(U_1, U_n) &< \infty \qquad \text{if } p = 2, && (6)\\ \mathrm{Cov}(U_1, U_n) &= O(n^{-a}) \quad \text{for some } a > 3p/2, \text{ if } p > 2. && (7) \end{aligned}$$

Then the uniform empirical process ξ_n weakly converges in $\mathrm{L}^p(0,1)$ to a centered Gaussian process ξ with a.s. continuous paths and covariance Γ:

$$\Gamma(s,t) = s \wedge t - st + \sum_{k=2}^{\infty} \Big(\mathrm{P}(U_1 \leq s, U_k \leq t) - st \Big) + \sum_{k=2}^{\infty} \Big(\mathrm{P}(U_k \leq s, U_1 \leq t) - st \Big).$$

3 Proofs

Proof of Theorem 1: We shall use a convolution approximation. To avoid notational complications, we define $\xi_n(t) = 0$ for any t outside $[0,1]$. Let κ be a probability density with support $[-1,1]$. We suppose moreover that κ is Lipschitz, that is:

$$\|\kappa\|_{\mathrm{Lip}} := \sup_{s \neq t} \frac{|\kappa(t) - \kappa(s)|}{|t-s|} < \infty.$$

For any positive integer j, define $\kappa_j(t) = j\kappa(jt)$. The sequence $(\kappa_j,\, j \geq 1)$ is an approximate identity.

Using the classical trick:

$$\kappa_j * \xi_n(x) - \xi_n(x) = \int_{|t|\leq 1/j} \{\xi_n(x-t) - \xi_n(x)\} \kappa_j(t)\, dt$$

and the Jensen inequality with respect to the probability measure $\kappa_j(t)\, dt$, we easily obtain:

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}} |\kappa_j * \xi_n(x) - \xi_n(x)|^p\, dx &\leq \mathbb{E} \int_{|t|\leq 1/j} \kappa_j(t)\, dt \int_{\mathbb{R}} |\xi_n(x-t) - \xi_n(x)|^p\, dx \\ &\leq \sup_{|t|\leq 1/j} \mathbb{E}\, \|\xi_n(\cdot) - \xi_n(\cdot + t)\|_p^p. \end{aligned}$$

Hence by (ii),

$$\lim_{j\to\infty} \sup_{n\geq 1} \mathbb{E}\, \|\kappa_j * \xi_n - \xi_n\|_p^p = 0. \qquad (8)$$

Now it is easily checked that

$$\mathbb{E}\, \|\kappa_j * \xi_n\|_\infty \leq \|\kappa_j\|_\infty \sup_{i\geq 1} \mathbb{E}\, \|\xi_i\|_1.$$

and for any $0 \leq s < t \leq 1$:

$$\mathbb{E}\,|\kappa_j * \xi_n(t) - \kappa_j * \xi_n(s)|^\gamma \leq \|\kappa_j\|_{\mathrm{Lip}}^\gamma \sup_{i\geq 1} \mathbb{E}\,\|\xi_i\|_1^\gamma\,|t-s|^\gamma.$$

It follows (see for instance Theorem 12.3, p. 95 in [3]) that for each fixed j the sequence $(\kappa_j * \xi_n,\, n \geq 1)$ is tight in $C(0,1)$ and hence also in $\mathrm{L}^p(0,1)$.

For fixed $\eta > 0$, define $\eta_i = 2^{-i}\eta$ $(i \geq 1)$ and choose a sequence of positive ε_i decreasing to 0. By (8) and the Markov inequality, we can extract a subsequence of indexes j_i such that:

$$P(\|\xi_n - \kappa_{j_i} * \xi_n\|_p > \varepsilon_i) < \eta_i, \quad n \geq 1,\ i \geq 1. \tag{9}$$

By the $\mathrm{L}^p(0,1)$-tightness of the sequence $(\kappa_{j_i} * \xi_n,\, n \geq 1)$, there is some compact C_i in $\mathrm{L}^p(0,1)$ such that

$$P(\kappa_{j_i} * \xi_n \notin C_i) < \eta_i, \quad n \geq 1,\ i \geq 1. \tag{10}$$

¿From (9) and (10) we deduce:

$$P(\xi_n \in A) > 1 - 2\eta, \quad n \geq 1,$$

where

$$A = \{f \in \mathrm{L}^p(0,1),\ \kappa_{j_i} * f \in C_i \text{ and } \|\kappa_{j_i} * f - f\|_p \leq \varepsilon_i,\ i \geq 1\}.$$

Clearly for any fixed $\delta > 0$, A can be recovered by a finite number of balls with radius 2δ. As $\mathrm{L}^p(0,1)$ is complete, the compacity of A follows and $(\xi_n,\, n \geq 1)$ is tight in $\mathrm{L}^p(0,1)$. ■

Proof of Corollary 2: If $q = p$ it is evident that (b) implies (ii). Suppose now that $q < p$ and denote by $1/u$ and $1/v$ the barycentric coordinates of p in the segment (q, r), that is:

$$p = \frac{1}{u}q + \frac{1}{v}r, \quad \frac{1}{u} + \frac{1}{v} = 1, \quad u, v > 0.$$

The Hölder inequality applied to any non negative random variable Y writes:

$$\mathbb{E}\,Y^p \leq \mathbb{E}^{1/u}\,Y^q\,\mathbb{E}^{1/v}\,Y^r.$$

Choosing $Y = |\xi_n(t+h) - \xi_n(t)|$ and integrating with respect to t, we get:

$$\begin{aligned}
\mathbb{E}\,\|\xi_n(\cdot + h) - \xi_n(\cdot)\|_p^p &\leq \int \mathbb{E}^{1/u}\,|\xi_n(t+h) - \xi_n(t)|^q\,\mathbb{E}^{1/v}\,|\xi_n(t+h) - \xi_n(t)|^r\,dt \\
&\leq (2^{r-1}c)^{1/v}\varepsilon(h)^{1/u},
\end{aligned}$$

using (a) and (b). Hence $\mathbb{E}\,\|\xi_n(\cdot + h) - \xi_n(\cdot)\|_p^p$ converges to zero, uniformly in n, as h goes to zero and the hypotheses of Theorem 1 are satisfied. ■

Proof of Theorem 3: We first check the tightness of $(\xi_n,\, n \geq 1)$.
Case $p = 2$: By stationarity of $(U_i,\, i \geq 1)$, we have

$$\mathbb{E}\,|\xi_n(t) - \xi_n(s)|^2 = g_1(s,t) + 2\sum_{k=2}^{n}\Big(1 - \frac{k}{n}\Big)g_k(s,t),$$

where

$$g_k(s,t) = \mathrm{Cov}\big(\mathbb{I}_{(s,t]}(U_1), \mathbb{I}_{(s,t]}(U_k)\big), \quad s, t \in [0,1], \quad k \geq 1.$$

In the strong mixing case, the bound $|g_k(s,t)| \leq \alpha(k-1)$ follows obviously from the definition of the mixing coefficients. In the associated case, Lemma 4.5 of Yu [18] provides the uniform bound:

$$|g_k(s,t)| \leq 16\,\mathrm{Cov}^{1/3}(U_1, U_k), \quad s, t \in [0,1]. \tag{11}$$

So under (4) or (6),

$$\mathbb{E}\,|\xi_n(t)-\xi_n(s)|^2 \le g_1(s,t) + 2\sum_{k=2}^{\infty} |g_k(s,t)| \tag{12}$$

where the right hand side is a normally convergent series of continuous functions. Its sum, say $g(s,t)$, is uniformly continuous on the square $[0,1]^2$ and vanishes on the diagonal. This gives the uniform estimate

$$\mathbb{E}\,|\xi_n(t+h)-\xi_n(t)|^2 \le \sup_{0\le s\le 1} g(s,s+h) = \varepsilon(h), \quad t\in[0,1],\ n\ge 1. \tag{13}$$

So we have condition (b) of Corollary 2 and hence (ii) of Theorem 1 is verified. Condition (i) with $\gamma = 2$ is easily obtained choosing $s=0$ in (12).

Case $p>2$: We use Corollary 2. From (13), condition (b) is satisfied with $q=2$.

By Theorem 2 of Yokoyama [17], condition (a) holds in the strong mixing case if the sequence $(n^{r/2-1}\alpha(n),\ n\ge 1)$ is summable. Choosing $p<r<2a$, this follows from our assumption (5).

In the associated case, the random variables $\mathbb{I}_{[0,t]}(U_i)$ are associated as non increasing functions of associated variables. So combining the moment inequality of Birkel [4] with (11), we obtain condition (a) of Corollary 2 by choosing $p<r<2a/3$.

Once the tightness of (ξ_n) established, to prove the weak $\mathrm{L}^p(0,1)$ convergence of ξ_n to ξ, it suffices to verify the convergence in distribution of $\int_0^1 f(t)\xi_n(t)\,dt$ to $\int_0^1 f(t)\xi(t)\,dt$ for each f in $\mathrm{L}^{p'}(0,1)$ the dual space of $\mathrm{L}^p(0,1)$. Observe now that:

$$\int_0^1 f(t)\xi_n(t)\,dt = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\int_{U_i}^1 f(t)\,dt - \mathbb{E}\int_{U_i}^1 f(t)\,dt\Big).$$

As $\mathrm{L}^{p'}(0,1)\subset \mathrm{L}^1(0,1)$, the function $F(x)=\int_x^1 f(t)\,dt$ is absolutely continuous. In the strong mixing case, Ibragimov's central limit theorem [7] for strong mixing bounded random variables applies to the $F(U_i)$'s and gives the desired convergence and the covariance function of ξ. In the associated case, the same conclusion is obtained from the following theorem. ■

Theorem 4 (Newman, [9]) *Let $(Y_n)_{n\ge1}$ be strictly stationary associated random variables and $X_n = F(Y_n)$, where F is an absolutely continuous function. Put $\bar{X}_n = \bar{F}(Y_n)$ where $\bar{F}(t) = \int_{[0,t]} |F'(u)|\,du$. If*

$$\int_{[0,1]^2} F'(x)F'(y)\Gamma(x,y)\,dx\,dy < \infty, \tag{14}$$

Γ being defined by:

$$\Gamma(x,y) = \Gamma_1(x,y) + \sum_{k\ge 2}\big(\Gamma_k(x,y)+\Gamma_k(y,x)\big),$$

where

$$\Gamma_k(x,y) = \mathrm{P}(Y_1>x, Y_k>y) - \mathrm{P}(Y_1>x)\,\mathrm{P}(Y_k>y),$$

then $\bar{X}_1$ is square integrable and $n^{-1/2}\sum_{j=1}^{n}(X_j-\mathbb{E}\,X_j)$ converges weakly to a centered Gaussian random variable with variance

$$\sigma^2 = \int_{[0,1]^2} F'(x)F'(y)\Gamma(x,y)\,dx\,dy.$$

4 Some functionals of paths

As an application, we consider now some examples of functionals of paths $T(\xi_n)$ whose convergence in distribution follows from the $\mathrm{L}^p(0,1)$ weak convergence of the empirical process ξ_n.

Example 1: Watson statistic.

Watson [19] proposed the following statistic for testing the "goodness of fit" on a circle in the i.i.d. case:

$$W_n^2 = \frac{1}{2}\int_0^1\int_0^1 \big[\xi_n(t)-\xi_n(s)\big]^2 ds\,dt = \int_0^1 \xi_n(t)^2\,dt - \Big(\int_0^1 \xi_n(t)\,dt\Big)^2.$$

Clearly this functional of paths is continuous in the $\mathrm{L}^2(0,1)$ topology.

Example 2: Weighted Mallows statistics.

The convergence of the Mallows statistic $M_n = \int_0^1 |\xi_n(t)|\,dt$ needs only the $\mathrm{L}^1(0,1)$ weak convergence of ξ_n. More generally, for any non negative weight function ψ in $\mathrm{L}^{p'}(0,1)$, the dual space of $\mathrm{L}^p(0,1)$, we have the convergence in distribution of

$$M_{n,\psi} = \int_0^1 \psi(t)|\xi_n(t)|\,dt,$$

under $\mathrm{L}^p(0,1)$ weak convergence of ξ_n.

Example 3: Some Anderson Darling statistics.

Anderson and Darling [1] studied the asymptotic behaviour of the statistics:

$$A_{n,\psi}^2 = \int_0^1 \xi_n(t)^2\psi(t)\,dt$$

for non negative weight functions ψ. The choice of the suitable $\mathrm{L}^p(0,1)$ space depends on ψ. Indeed to check the continuity of the functional $T:\ f\mapsto \int_0^1 f^2\psi\,dt$, Hölder inequality provides the bound:

$$|Tf-Tg| \le \|\psi\|_r\cdot\|f-g\|_{2s}\cdot\|f+g\|_{2s}, \qquad \frac{1}{r}+\frac{1}{s}=1.$$

So if ψ belongs to $\mathrm{L}^r(0,1)$ we can take $p=2s$. For instance, choosing the weight $\psi(t)=t^{-a}(1-t)^{-a}$, with $a<1$, we obtain the convergence in distribution of $A_{n,\psi}^2$ under the $\mathrm{L}^p(0,1)$ weak convergence of ξ_n for $p>2/(1-a)$. The case $a=1$ seems out of reach of the $\mathrm{L}^p(0,1)$ method. Its study would require more knowledge about the regularity of the limiting process ξ.

Example 4: Hölder functions and local time.

Define the local time of ξ_n ([14] p. 398) by:

$$L_n(x) = n^{-1/2}\{\text{number of times } \xi_n(t)=x,\ 0\le t\le 1\}$$

For any suitable Borel function f we have:

$$\int_0^1 f\big(\xi_n(t)\big)\,dt = \int_{-\infty}^{+\infty} f(x)L_n(x)\,dx.$$

Suppose now $f\in \dot{C}^p(\mathbb{R})$, $p\notin\mathbb{N}$, the homogeneous Hölder space of order p ([8] p. 177), so we have the representation:

$$f(x) = \sum_{k=0}^{[p]} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R(x_0,x), \qquad x_0, x\in\mathbb{R},$$

where $|R(x_0,x)|\le C|x-x_0|^p$, together with the estimate:

$$\big|f^{([p])}(x_0)\big| \le C'\big(f^{([p])}(0)+|x_0|^{p-[p]}\big).$$

It follows that $|f^{(k)}(x_0)|\le C''\big(1+|x_0|^{p-k}\big)$ for $0\le k\le [p]$, so the existence and continuity on $\mathrm{L}^p(0,1)$ of the functional $T: h\mapsto \int_0^1 f\big(h(t)\big)\,dt$ is easily checked.

Consider now the occupation measure μ of ξ defined by

$$\mu(B) = \lambda\{t \in [0,1] : \ \xi(t) \in B\}$$

for any Borel subset B of $\mathbb{R}$ (λ denotes the Lebesgue measure). Following Berman ([2], Ex. 3.2), a sufficient condition for μ to be absolutely continuous with respect to λ is

$$\int_{[0,1]^2} Q(s,t)^{-1/2}\, ds\, dt < \infty, \quad \text{where} \quad Q(s,t) = \Gamma(s,s) + \Gamma(t,t) - 2\Gamma(s,t).$$

Define analogously $Q_1(s,t)$ with $\Gamma_1(s,t) = s \wedge t - st$ and $\tilde{Q}(s,t)$ with $\tilde{\Gamma}(s,t) = \Gamma(s,t) - \Gamma_1(s,t)$. Then $\tilde{Q}$ is non negative on $[0,1]^2$, so the integrability of $Q^{-1/2}$ follows from integrability of $Q_1^{-1/2}$ that is from:

$$\int_{[0,1]^2} |t-s|^{-1/2}\big(1 - |t-s|\big)^{-1/2}\, ds\, dt < \infty,$$

which is easily checked. So the occupation measure μ has a density L (the local time of ξ) and for each $f \in \dot{C}^p(\mathbb{R})$

$$\int_0^1 f\big(\xi(t)\big)\, dt = \int_0^1 f(t)\, \mu(dt) = \int_{\mathbb{R}} f(x) L(x)\, dx.$$

Finally by $\mathrm{L}^p(0,1)$ continuity of the functional T for any $f \in \dot{C}^p(\mathbb{R})$, the moments $\int_{\mathbb{R}} f(x) L_n(x)\, dx$ of the local time L_n converge in distribution to the corresponding moments of the local time L of ξ.

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