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On the Hölderian Functional Central Limit Theorem for i.i.d. Random Elements in Banach Space[1]

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We investigate the weak Hölder convergence of polygonal partial sums processes of independent copies of a Banach valued random variable X. We establish a close connection between the relevant Hölder modulus of smoothness and the tail behaviour of X and of the normalized partial sums.

1. Introduction

Let B be a separable Banach space, ρ a modulus of smoothness and denote by $H_B^\rho = H\big(B,\rho(h)\big)$ the set of B-valued continuous functions $x : [0,1] \to B$ such that $w_\rho(x,1) < \infty$, where

$$w_\rho(x,\delta) := \sup_{0\le s,t\le 1,\ 0<|t-s|<\delta} \frac{\big\| x(t) - x(s)\big\|_B}{\rho\big(|s-t|\big)}.$$

As H_B^ρ is a non separable Banach space when endowed with the natural norm $\|x\|_\rho := \big\| x(0)\big\|_B + w_\rho(x,1)$, we mainly use its separable closed subspace

$$H_B^{\rho,o} = H\big(B,\rho(h),o\big) := \Big\{ x \in H^\rho;\ \lim_{\delta\to 0} w_\rho(x,\delta) = 0\Big\}.$$

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Let $X, X_1, \ldots, X_n, \ldots$ be i.i.d. random elements in B and $S_n := \sum_{k \leq n} X_k$. We say that X satisfies the central limit theorem in B, which we denote by $X \in \mathrm{CLT}\,(B)$, if $n^{-1/2} S_n$ converges in distribution in B. Define the polygonal partial sums process ζ_n by

$$\zeta_n(t) := n^{-1/2} S_{[nt]} + n^{-1/2} \big(nt - [nt]\big) X_{[nt]+1}, \quad t \in [0,1].$$

We say that X satisfies the functional central limit theorem in $H_B^{\rho,o}$, which we denote by $X \in \mathrm{FCLT}\,\big(B, \rho(h), o\big)$, if ζ_n converges in distribution in $H_B^{\rho,o}$. Our purpose is to study FCLT in $H_B^{\rho,o}$ in terms of the tail behavior of X and S_n. The present contribution is a (hopefully) self-contained part of a more general study on FCLT which should appear elsewhere. Our results hold under some technical conditions on ρ which are satisfied at least by the class of moduli

$$\rho(h) := h^\alpha \ln^\beta\left(\frac{c}{h}\right), \quad 0 < \alpha < 1, \quad \beta \in \mathbb{R}.$$

This class is well adapted to the known regularity of the limit Gaussian process. Kuelbs [5] proved that $X \in \mathrm{CLT}\,(B)$ if and only if X satisfies the FCLT in the space C_B of B valued continuous functions. Stronger is the topology on the space of paths, wider is the set of continuous functionals of paths and richer the corresponding FCLT. Of course replacing C_B by some $H_B^{\rho,o}$ in the statement of FCLT induces some extra cost on the conditions imposed on the distribution of X. Our main goal is to evaluate the right price of such a strenghtening of the FCLT. One interesting feature in Kuelbs result is that the geometry of the space B is involved in FCLT only through the (necessary) condition $X \in \mathrm{CLT}\,(B)$ (which does not reduces to the square integrability of X in general). We propose to extend this to the Hölderian setting.

As far as we know, the first Hölderian FCLT is due to Lamperti [6] who proved that if the real valued random variable X has a moment of order $p > 2$, then $X \in \mathrm{FCLT}\,(\mathbb{R}, h^\alpha, o)$, for $\alpha < 1/2 - 1/p$. Ciesielski [1] established convenient isomorphisms between $H(\mathbb{R}, h^\alpha, o)$ and some sequence spaces, using the Schauder basis of triangular functions. Kerkyacharian and Roynette [4] exploited these isomorphisms to propose a new and simple proof of Lamperti's result. Hamadouche [3] extended Lamperti's invariance principle to the case of weakly dependent random variables X_i and of convolution smoothing of partial sums. Erickson [2] established invariance principles for partial sums processes of i.i.d. real valued random variables indexed by $[0,1]^d$ with general moduli of smoothness ρ. Recently, Morel [9] study the FCLT for $[0,1]^d$ indexed partial sums processes of weakly dependent real valued random variables in the more general scale of Besov spaces, allowing an unified treatment of jump partial sums processes and smoothed ones.

Our approach of the Hölderian FCLT for Banach valued random variables follows essentially Kerkyacharian and Roynette. First we extend Ciesielski's isomorphisms. Because B (and a fortiori $H_B^{\rho,o}$) does not have necessarily a Schauder basis, this extension is non trivial. Next we use a sequential norm on $H_B^{\rho,o}$ to obtain conditions for existence of a $H_B^{\rho,o}$ version of a given B-valued stochastic process and for tightness of a sequence of random elements in $H_B^{\rho,o}$. For sake of brevity, we just state in Section 2 these preliminary results and refer to [12] for the proofs. Section 3 gives our sufficient conditions for Hölderian FCLT. These conditions are shown to be rather close to optimality in Section 4.

2. Tools

For any function $x : [0,1] \to B$, let us define the *second differences* by

$$\Delta_h^2 x(t) := x(t+h) + x(t-h) - 2x(t), \quad t, t+h \in [0,1].$$

Denote by $V_j := \left\{ (2k+1)2^{-j},\, 0 \le k < 2^{j-1} \right\}$ the set of dyadics of level j and define the B-valued coefficients $\lambda_{j,v}$ by $\lambda_{0,0}(x) = x(0)$, $\lambda_{0,1}(x) = x(1)$ and for $j \ge 1$,

$$\lambda_{j,v}(x) = -\frac{1}{2}\Delta_h^2 x(v) \quad \text{with} \quad h = 2^{-j}, \quad v \in V_j.$$

Introduce now the (possibly infinite) sequential seminorm

$$\|x\|_\rho^{\mathrm{seq}} := \sup_{j \ge 0} \frac{1}{\rho(2^{-j})} \max_{v \in V_j} \left\| \lambda_{j,v}(x) \right\|_B. \tag{1}$$

Observe that when x is continuous, $\|x\|_\rho^{\mathrm{seq}} = 0$ if and only if $x = 0$. In what follows we assume that the modulus of smoothness ρ satisfies the following conditions where c_1, c_2 and c_3 are positive constants:

$$\rho(0) = 0, \quad \rho(\delta) > 0, \quad 0 < \delta \le 1; \tag{2}$$

$$\rho \text{ is non decreasing on } [0,1]; \tag{3}$$

$$\rho(2\delta) \le c_1 \rho(\delta), \quad 0 \le \delta \le 1/2; \tag{4}$$

$$\int_0^\delta \frac{\rho(u)}{u}\,du \le c_2\rho(\delta), \quad 0 < \delta \le 1; \tag{5}$$

$$\delta \int_\delta^1 \frac{\rho(u)}{u^2}\,du \le c_3\rho(\delta), \quad 0 < \delta \le 1. \tag{6}$$

For instance, elementary computations show that the functions

$$\rho(\delta) := \delta^\alpha \ln^\beta\left(\frac{c}{\delta}\right), \quad 0 < \alpha < 1, \quad \beta \in \mathbb{R},$$

satisfy conditions (2) to (6), for a suitable choice of the constant c, namely $c \ge \exp(\beta/\alpha)$ if $\beta > 0$ and $c > \exp\big(-\beta/(1-\alpha)\big)$ if $\beta < 0$.

Proposition 1. *If ρ satisfies conditions (2) to (6), the norm $\|x\|_\rho$ is equivalent on C_B to the sequential norm, i.e. there are positive constants a, b such that for every $x \in C_B$,*

$$a\|x\|_\rho \le \|x\|_\rho^{\mathrm{seq}} \le b\|x\|_\rho,$$

with finite values if and only if $x \in H_B^\rho$.

Theorem 2. *Let $\xi = \big\{\xi(t),\ t \in [0,1]\big\}$ be a B-valued stochastic process, continuous in probability. Assume there exist a function $\sigma : [0,1] \to \mathbb{R}^+$, $\sigma(0) = 0$ and a function $\Psi : (0,\infty] \to \mathbb{R}^+$, $\Psi(\infty) = 0$ such that for all $r > 0$, $t, t+h \in [0,1]$,*

$$P\Big(\big\|\Delta_h^2\xi(t)\big\|_B > r\sigma\big(|h|\big)\Big) \le \Psi(r). \tag{7}$$

Put for $0 < u < \infty$,

$$R(u) = R(\Psi,\sigma,\rho,u) := \sum_{j=0}^\infty 2^j\,\Psi\left(u\frac{\rho}{\sigma}(2^{-j})\right). \tag{8}$$

If $R(u_0)$ is finite for some $0 < u_0 < \infty$, then ξ has a version in H_B^ρ. If $R(u)$ is finite for every $0 < u < \infty$, then ξ has a version in $H_B^{\rho,o}$.

Corollary 3. *Assume that the Gaussian B-valued stochastic process $\xi = (\xi_t,\ t \in [0,1])$ is continuous in probability and satisfies for each $t, t+h \in [0,1]$,*

$$\mathbf{E}\big\| \Delta_h^2 \xi(t) \big\|_B^2 \le \sigma\big(|h|\big)\,. \tag{9}$$

If for some $u > 0$,

$$\sum_{j=0}^{\infty} 2^j \exp\left(-u^2 \frac{\rho^2}{\sigma^2}(2^{-j}) \right) < \infty, \tag{10}$$

then ξ admits a version in H_B^ρ. If (10) holds for each $u > 0$ then ξ has a version in $H_B^{\rho,o}$.

Example. Let Y be a centered Gaussian random element in B with distribution μ. A B-valued Brownian motion W with parameter μ is a Gaussian process ξ indexed by $[0,1]$, with independent increments such that $\xi(t) - \xi(s)$ has the same distribution as $|t-s|^{1/2} Y$. Hence (9) holds with $\sigma(h) = h^{1/2}\mathbf{E}^{1/2}\|Y\|_B^2$ $(h \ge 0)$. With the modulus of smoothness $\rho(h) = \sqrt{h \ln(e/h)}$ and some tail estimate for Gaussian distribution (see e.g. inequality (3.5) p. 59 in [8]) the left hand side of (10) converges if $u > 2\sqrt{2}\mathbf{E}^{1/2}\|Y\|_B^2$. So the B valued Brownian motion ξ has a version in $H\big(B, \sqrt{h \ln(e/h)}\,\big)$. On the other hand, should ξ have a version in $H\big(B, \sqrt{h \ln(e/h)}, o\big)$, then for any continuous linear functional f on $H_B^{\rho,o}$, $f(\xi)$ would be a real valued Brownian motion with a version in $H\big(\mathbb{R}, \sqrt{h \ln(e/h)}, o\big)$. This is impossible due to Lévy's theorem on the modulus of uniform continuity of the standard Brownian motion.

Suppose now that $X \in \mathrm{FCLT}\big(B, \rho(h), o\big)$, then $X \in \mathrm{FCLT}(C_B)$ and X is pregaussian in B. Denote by μ the Gaussian measure on B with the same covariance structure as X. Then by Kuelbs FCLT, the limit of the polygonal partial sums process ζ_n is a B-valued Brownian motion with parameter μ. So the above example tells us precisely that the stronger FCLT (from the topological point of view) we can expect in the scale of Hölder spaces $H\big(B, h^\alpha \ln^\beta(c/h), o\big)$ corresponds to an open condition $\alpha = 1/2$, $\beta > 1/2$.

The use of sequential norm provides convenient tightness conditions. Recall that throughout the paper B is assumed separable.

Theorem 4. *The sequence $(\xi_n)_{n\ge1}$ of random elements in $H_B^{\rho,o}$ is tight if and only if it satisfies the two following conditions.*

(i) *For each* $t \in [0,1]$, *the sequence of* B*-valued random variables* $\big(\xi_n(t)\big)_{n\geq 1}$ *is tight in* B.

(ii) *For each positive* ε,

$$\lim_{J\to\infty}\sup_{n\geq 1} P\Big(\sup_{j\geq J}\frac{1}{\rho(2^{-j})}\max_{v\in V_j}\big\|\lambda_{j,v}(\xi_n)\big\|_B>\varepsilon\Big)=0.$$

Corollary 5. *Let* $(\xi_n)_{n\geq 1}$ *be a the sequence of random elements in* $H_B^{\rho,o}$ *satisfying* (i) *of Theorem* 4. *Suppose moreover that* (7) *holds for each* ξ_n *with the same* σ *and* Ψ *independent of* n. *If* $R(u)$ *defined by* (8) *is finite for each* $u>0$, *then* $(\xi_n)_{n\geq 1}$ *is tight in* $H_B^{\rho,o}$.

3. Sufficient Conditions for Hölderian FCLT

Theorem 6. *Let* X *be a random element in the Banach space* B *such that*

a) $X\in \mathrm{CLT}\,(B)$;

b) *There is an increasing positive function* φ *such that for each integer* $m\geq 1$ *and each* $u>0$,

$$P\big(\|S_m\|_B>m^{1/2}u\big)\leq\frac{1}{\varphi(u)};$$

c) *For each* $\delta>0$,

$$\sum_{j\geq 1}\frac{2^j}{\varphi\big(\delta 2^{j/2}\rho(2^{-j})\big)}<\infty.$$

Then $X\in\mathrm{FCLT}\,(B,\rho,o)$.

Proof. From a) and Corollary 1 p. 166 in [5], $X\in\mathrm{FCLT}\,(C_B)$, which gives the tightness in B of $\big(\zeta_n(t)\big)_{n\geq 1}$ and the convergence of finite dimensional distributions of ζ_n. Hence we only have to check the condition on $R(u)$ in Corollary 5.

In this problem, the use of second differences brings nothing more than first ones, essentially due to the independence of increments. Hence it suffices to prove the uniform convergence (with respect to n) of the series

$$\Sigma_{\varepsilon,n} := \sum_{j=1}^{\infty} 2^j \max_{v\in V_j} P\Big(\big\|\,\zeta_n(v+2^{-j}) - \zeta_n(v)\big\|_B > \varepsilon\rho(2^{-j})\Big).$$

What we need now, is an estimate of

$$\Delta_n(t,h,r) := P\Big(\big\|\,\zeta_n(t+h) - \zeta_n(t)\big\|_B > r\Big),$$

uniform in n. Consider first the following configuration:

$$0 \le \frac{k-1}{n} \le t < \frac{k}{n} < \frac{l}{n} \le t+h < \frac{l+1}{n} \le 1.$$

Then we have

$$\begin{aligned}\Delta_n(t,h,r) \le P\Big(\big\|\,\zeta_n(t+h) - \zeta_n(l/n)\big\|_B > r/3\Big) &+ \\ + P\Big(\big\|\,\zeta_n(l/n) - \zeta_n(k/n)\big\|_B > r/3\Big) &+ \\ + P\Big(\big\|\,\zeta_n(k/n) - \zeta_n(t)\big\|_B > r/3\Big)&.\end{aligned}$$

To bound the central increment, putting $m = l-k$ and noting that $m/n \le h$, we get by b),

$$\begin{aligned}P\Big(\big\|\,\zeta_n(l/n) - \zeta_n(k/n)\big\|_B > r/3\Big) &= P\left(\left(\frac{m}{n}\right)^{1/2}\left\|\frac{S_{k+m}-S_k}{m^{1/2}}\right\|_B > r/3\right) \le \\ &\le P\big(h^{1/2}\|m^{-1/2}S_m\|_B > r/3\big) \le \\ &\le \frac{1}{\varphi\big(\frac{1}{3}rh^{-1/2}\big)}.\end{aligned}$$

To handle the left increment, observing that $n^{1/2}(k/n - t) \le n^{-1/2} \le h^{1/2}$ and using b) with $m = 1$, we obtain

$$P\Big(\big\|\,\zeta_n(k/n) - \zeta_n(t)\big\|_B > r/3\Big) \le \frac{1}{\varphi\big(\frac{1}{3}rh^{-1/2}\big)}.$$

Clearly the same estimate holds for the right increment and is also useful when t and $t+h$ are in contiguous intervals (i.e. $k = l$) or in the same interval ($k = l+1$). Finally we have in all the possible configurations

$$P\Big(\big\|\,\zeta_n(t+h) - \zeta_n(t)\big\|_B > r\Big) \le \frac{3}{\varphi\big(\frac{1}{3}rh^{-1/2}\big)},$$

whence

$$\Sigma_{\varepsilon,n} \le \sum_{j\ge 1} \frac{3\cdot 2^j}{\varphi\big(\frac{\varepsilon}{3}2^{-j/2}\rho(2^{-j})\big)}.$$

So by c) the condition on $R(u)$ in Corollary 5 is satisfied and the proof is complete.

Corollary 7. *Let X be a random element in B such that $X \in \mathrm{CLT}\,(B)$ and $\mathbf{E}\,\|X\|_B^p < \infty$ for some $p > 2$. Then $X \in \mathrm{FCLT}\,\big(B, h^{1/2-1/p}\ln^\beta(a/h), o\big)$ for each $\beta > 1/p$.*

Proof. By Theorem 6.20 p. 171 in [8]), there is a constant K depending only on p such that

$$\mathbf{E}^{1/p}\|S_m\|_B^p \le K\bigg(\mathbf{E}\,\|S_m\|_B + \mathbf{E}^{1/p}\big(\max_{1\le i\le m}\|X_i\|_B\big)^p\bigg).$$

Since $X \in \mathrm{CLT}\,(B)$, we have $\sup_{m\ge 1} m^{-1/2}\mathbf{E}\,\|S_m\|_B < \infty$. To bound the second term in the above estimate, observe that

$$\mathbf{E}^{1/p}\Big(\max_{1\le i\le m}\|X_i\|_B\Big)^p = \mathbf{E}^{1/p}\max_{1\le i\le m}\|X_i\|_B^p \le m^{1/p}\mathbf{E}^{1/p}\|X\|_B^p.$$

Since $p > 2$, there is a constant K' depending on p and the distribution of X only, such that $\mathbf{E}^{1/p}\|S_m\|_B^p \le K' m^{1/2}$, so condition b) of Theorem 6 is satisfied with $\varphi(u) = K'^{-p}u^p$. Now with our choice of $\rho(h)$, condition c) reduces to the convergence of $\sum_{j\ge 1} j^{-\beta p}$.

The moment assumption in Corollary 7 can be slightly relaxed if B has the Λ-Rosenthal property for which the reader is referred to Ledoux [7].

Corollary 8. *Suppose the separable Banach space B has the property Λ-$\mathrm{Ros}\,(p)$ for some $p > 2$. If X is a pregaussian random element in B such that $\|X\|_{p,\infty} := \sup^{1/p} t^p P\big(\|X\|_B > t\big) < \infty$, then*

$$X \in \mathrm{FCLT}\,\big(B, h^{1/2-1/p}\ln^\beta(a/h), o\big)$$

for each $\beta > 1/p$.

Proof. By Theorem 2.4 (ii) in [7], B has also the property Ros (p). Hence by Theorem 10.10 p. 285 in [8], $X \in \mathrm{CLT}\,(B)$.

Denoting by $G(X_i)$, Gaussian independent random elements with the same covariance as X, the property Λ-Ros (p) writes here

$$\|S_m\|_{p,\infty} \le C\left(\sup_{t>0}{}^{1/p} t^p \sum_{i=1}^m P\bigl(\|X_i\|_B > t\bigr) + \left\|\sum_{i=1}^m G(X_i)\right\|_{p,\infty}\right) \le$$

$$\le C\bigl(m^{1/p}\|X\|_{p,\infty} + c_p m^{1/2}\bigr).$$

So we have $\|S_m\|_{p,\infty} = O(m^{1/2})$ and the proof is completed as in Corollary 7.

Remarks. The space $\mathbb{R}$ satisfies the property Λ-Ros (p) for each $p \ge 2$. This follows from the classical Rosenthal inequality and Theorem 2.4(ii) in [7]. So in the special case $B = \mathbb{R}$, Corollary 7 improves on Lamperti's invariance principle, alltogether by relaxing the hypothesis and strenghtening the conclusion. As already pointed out by Lamperti, it is impossible to prove $X \in \mathrm{FCLT}\,(\mathbb{R}, h^{1/2-1/p})$ under the assumption $\|X\|_{p,\infty} < \infty$ (it suffices to consider the symetric random variable X_1 whose distribution is given by $P(X_1 \ge u) = 1/(2u^p)$, $u \ge 1$).

For $0 < \gamma \le 2$, consider the Young functions $\psi_\gamma(t) := \exp(t^\gamma) - 1$ (with classical modification to handle the problem of convexity near the origin when $\gamma < 1$) and denote by $\|X\|_{\psi_\gamma}$ the corresponding Orlicz norms.

Corollary 9. *Let X be a random element in B such that $X \in \mathrm{CLT}\,(B)$ and $\|X\|_{\psi_\gamma} < \infty$ for some $\gamma \in (0, 2]$. Then $X \in \mathrm{FCLT}\,\bigl(B, h^{1/2}\ln^\beta(a/h), o\bigr)$ for each $\beta > 1/\gamma$.*

Proof. Suppose for the moment that $C := \sup_{m\ge 1}\|S_m\|_{\psi_\gamma} < \infty$. Then the uniform estimate of condition b) is given by

$$P\bigl(\|S_m\|_B > m^{1/2}u\bigr) \le \left\{\psi_\gamma\left(\frac{u}{\|S_m\|_{\psi_\gamma}}\right)\right\}^{-1} \le \frac{1}{\psi_\gamma(u/C)}.$$

Now condition c) reduces to $\sum_{j\ge 1} 2^j \exp(-\delta j^{\beta\gamma}) < \infty$. Clearly this convergence holds for each $\delta > 0$ if and only if $\beta\gamma > 1$.

So it remains to check that $\|S_m\|_{\psi_\gamma} = O(m^{1/2})$. To this end we invoke Theorem 6.21 p. 172 in [8] which gives a constant K_γ such that if $0 < \gamma \le 1$,

$$\|S_m\|_{\psi_\gamma} \le K_\gamma\left(\mathbf{E}\,\|S_m\|_B + \left\|\max_{1\le i\le m}\|X_i\|_B\right\|_{\psi_\gamma}\right) \tag{11}$$

and if $1 < \gamma \le 2$, with $\gamma' = \gamma/(\gamma-1)$,

$$\|S_m\|_{\psi_\gamma} \le K_\gamma \Bigl(\mathbf{E}\,\|S_m\|_B + \Bigl(\sum_{i=1}^m \|X_i\|_{\psi_\gamma}^{\gamma'} \Bigr)^{1/\gamma'} \Bigr). \tag{12}$$

In both cases, $\mathbf{E}\,\|S_m\|_B$ is $O(m^{1/2})$ since $X \in \mathrm{CLT}\,(B)$. In (12), the second term in the right hand side is $O(m^{1/2})$ since $\gamma' \ge 2$. It remains to check that $\bigl\| \max_{1\le i\le m} \|X_i\|_B \bigr\|_{\psi_\gamma} = O(m^{1/2})$. In fact mimicking the subgaussian case ($\gamma = 2$), it is easy to obtain the much better estimate $O(\ln^{1/\gamma} n)$.

4. Necessary Conditions for Hölderian FCLT

We now consider the converse problem: knowing that X satisfies the FCLT in $H_B^{\rho,o}$, what can be said about the tail behavior of X? In fact we shall obtain more: the tail behavior of the normalized partial sums $m^{-1/2}S_m$, uniformly in m. The results of this section seem completely new, even in the case $B = \mathbb{R}$.

To study the converse of FCLT, it is convenient to introduce the properties of bounded CLT and FCLT. We say that X satisfies the bounded CLT in B, if the sequence $(n^{-1/2}S_n)_{n\ge1}$ is bounded in probability, that is for each $\varepsilon > 0$, there is a positive A such that $\sup_{n\ge1} P\bigl(\|n^{-1/2}S_n\|_B > A\bigr) < \varepsilon$. This will be denoted by $X \in \mathrm{bCLT}\,(B)$. The bounded FCLT is defined similarly with $\|\zeta_n\|_\rho$ instead of $\|n^{-1/2}S_n\|_B$. Clearly bCLT and bFCLT are weaker than CLT and FCLT respectively. Observe also that if $X \in \mathrm{FCLT}\,(B,\rho,o)$ (resp. $\mathrm{bFCLT}\,(B,\rho)$), then $X \in \mathrm{CLT}\,(B)$ (resp. $\mathrm{bCLT}\,(B)$).

In the general results of this section, we do not require that the modulus of smoothness ρ satisfies conditions (4) to (6). Our first step is to discard some moduli giving too strong a topology.

Lemma 10. *Let ρ be a modulus of smoothness and put*

$$\theta(t) := t^{1/2}\rho(1/t), \quad t > 0. \tag{13}$$

Suppose there exists some Banach space B and some random element X in B such that $X \in \mathrm{bFCLT}\,(B,\rho)$. Then $\lim_{t\to\infty} \theta(t) = \infty$.

Proof. If the conclusion is false, there is a sequence t_k increasing to infinity such that $\theta(t_k)$ converges to some $a \in [0, \infty)$. Putting $h_k = 1/t_k$, we have $\rho(h_k) = O(h_k^{1/2})$. It then follows from $X \in \mathrm{bFCLT}\,(B, \rho)$ that $\sup_{k \geq 1} h_k^{-1/2} \|\, \zeta_n(h_k) \|_B$ is bounded in probability (uniformly in n). The same holds for $\sup_{k \geq 1} h_k^{-1/2} f\big(\zeta_n(h_k)\big)$ where f is any element of B'. So there is a constant A such that

$$P\Big(\forall k \geq 1,\ \big|f\big(\zeta_n(h_k)\big)\big| \leq A h_k^{1/2}\Big) > \frac{1}{2}.$$

By linearity, $f\big(\zeta_n(t)\big)$ is the polygonal line built on the normalized partial sums of the i.i.d. real valued random variables $f(X_i)$. Recalling that on the real line, bounded CLT and CLT are equivalent, we deduce from the boundedness in probability of $f\big(\zeta_n(1)\big)$ that $f(X) \in \mathrm{CLT}\,(\mathbb{R})$ and hence $\mathbf{E}\, f(X)^2 < \infty$. Now by the classical Donsker–Prokhorov's invariance principle, $f(\zeta_n)$ converges in distribution in $C_{\mathbb{R}}[0,1]$ to a real valued Brownian motion $\tilde{W}$ with variance $\mathbf{E}\, \tilde{W}(t)^2 = t \mathbf{E}\, f(X)^2$. It follows that for each closed set F in $C_{\mathbb{R}}[0,1]$, $\limsup_{n \to \infty} P\big(f(\zeta_n) \in F\big) \leq P(\tilde{W} \in F)$. Applying this with the closed $F := \big\{x \in C_{\mathbb{R}}[0,1];\ \forall k \geq 1,\ |x(h_k)| \leq A h_k^{1/2}\big\}$, we obtain

$$P\Big(\forall k \geq 1,\ \big|\tilde{W}(h_k)\big| \leq A h_k^{1/2}\Big) \geq \frac{1}{2}.$$

This contradicts the well known fact that $t^{-1/2}\big|\tilde{W}(t)\big|$ goes almost surely to infinity when t goes to 0.

Theorem 11. *Let $X \in \mathrm{bFCLT}\,(B, \rho)$. Then there are two positive constants A and C such that for each $m \geq 1$, $u \geq A\rho(1)$,*

$$P\big(\|S_m\|_B > m^{1/2} u\big) \leq \frac{C}{\phi(u/A)}, \tag{14}$$

where ϕ is the continuous function defined by

$$\phi(u) := \sup\big\{t \geq 1;\ \theta(t) \leq u\big\}, \quad A\rho(1) \leq u < \infty. \tag{15}$$

Furthermore, if $X \in \mathrm{FCLT}\,(B, \rho, o)$, then for any positive A,

$$\lim_{u \to \infty} \phi(u/A) \sup_{m \geq 1} P\big(\|S_m\|_B > m^{1/2} u\big) = 0. \tag{16}$$

Proof. Remark that, due to Lemma 10, ϕ is well defined. To prove the claim on the tail of $m^{-1/2} S_m$, observe first that for $n \geq m$

$$\max_{1 \leq l \leq n/m} \frac{1}{\rho(m/n)} \Big\|\zeta_n(ml/n) - \zeta_n\big(m(l-1)/n\big)\Big\|_B \leq \omega_\rho(\zeta_n, m/n) \leq \omega_\rho(\zeta_n, 1).$$

From $X \in \mathrm{bFCLT}\,(B,\rho)$, we have for each $0<\varepsilon<1$ a constant A, *not depending on* m such that

$$P\Big(\max_{1\le l\le n/m}\big\|\zeta_n(ml/n)-\zeta_n\big(m(l-1)/n\big)\big\|_B > A\rho(m/n)\Big)<\varepsilon.$$

Writing $Z_l^{(m)} := \big\|m^{-1/2}\big(S_{ml}-S_{m(l-1)}\big)\big\|_B$ and restricting ourselves for simplicity to n multiple of m ($n=Nm$ and from now on, we fix m), this can be recasted as

$$P\Big(\max_{1\le l\le N} Z_l^{(m)} > A\theta(N)\Big)<\varepsilon,\quad N\ge 1.$$

By independence and identical distribution of the $Z_l^{(m)}$ ($l\ge 1$), it follows

$$\big[1-P\big(Z_1^{(m)}>A\theta(N)\big)\big]^N > 1-\varepsilon,\quad N\ge 1.$$

By convexity of $t\mapsto (1-\varepsilon)^t$, we have $(1-\varepsilon)^{1/N}\ge 1+N^{-1}\ln(1-\varepsilon)$, from which we obtain

$$P\big(Z_1^{(m)}>A\theta(N)\big)\le -\frac{1}{N}\ln(1-\varepsilon),\quad N\ge 1.$$

Now let $u\ge A\rho(1)$ and using Lemma 10, denote by N_u the greatest integer N such that $A\theta(N)\le u$. Then we have $1\le N_u\le \phi(u/A)<1+N_u$, whence

$$P\big(Z_1^{(m)}>u\big)\le \frac{-\ln(1-\varepsilon)}{N_u}\le\frac{-2\ln(1-\varepsilon)}{\phi(u/A)}.$$

The proof of (14) is complete.

To prove (16), observe that the tightness of $(\zeta_n)_{n\ge 1}$ in $H_B^{\rho,o}$ (note the o here, which was not needed above) gives the *uniform* convergence in probability of $\omega_\rho(\zeta_n,1/N)$ (with respect to n) when N goes to infinity (see e.g. [10], Lemma 1.4). Using the same trick as above we obtain that for any positive A,

$$\lim_{N\to\infty}\sup_{m\ge 1}\Big\{1-\big[1-P\big(Z_1^{(m)}>A\theta(N)\big)\big]^N\Big\}=0.$$

Replacing ε by $\varepsilon_N\downarrow 0$ in the above estimates, it is easily seen that

$$P\big(Z_1^{(m)}>A\theta(N)\big)\le\frac{-\ln(1-\varepsilon_N)}{N},$$

from which we obtain similarly

$$P\big(Z_1^{(m)}>u\big)\le\frac{-2\ln\big(1-\varepsilon(u)\big)}{\phi(u/A)},$$

with $\lim_{u\to\infty}\varepsilon(u)=0$.

Corollary 12. *Suppose that* $X \in \mathrm{bFCLT}\left(B, h^{\alpha} \ln^{\beta}(c/h)\right)$, $0 < \alpha < 1/2$, $\beta \in \mathbb{R}$. *Put* $1/p = 1/2 - \alpha$, *then*

$$\sup_{m \geq 1} P\big(\|m^{-1/2} S_m\|_B > u\big) \leq K \frac{\ln^{p\beta} u}{u^p}, \quad u > 0.$$

Proof. The result follows by elementary estimates of ϕ. Here the constant A is irrelevant.

Remark. When $\beta p = 1 + \varepsilon$, the gap between the sufficient condition of Corollary 8 and the necessary one of Corollary 12 is only the logarithmic factor $\ln^{1+\varepsilon} u$ in the estimate of the tail of X.

Corollary 13. *Suppose that* $X \in \mathrm{bFCLT}\left(B, h^{1/2} \ln^{\beta}(c/h)\right)$ *where* $\beta \geq 1/2$. *Define* $\gamma = 1/\beta$. *Then*

$$\sup_{m \geq 1} \|m^{-1/2} S_m\|_{\psi_\gamma} < \infty.$$

Proof. Here we have an explicit formula for the inversion of θ, that is $\phi(u) = c^{-1} \exp(v^{\gamma})$. We conclude by noting that, due to the properties of exponential functions, the existence of ψ_γ Orlicz norm is equivalent to a bound of the form $1/\psi_\gamma(\delta u)$ for the tail.

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