Introduction to Jung's method of resolution of singularities

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> *Socrates*: Now, Glaucon, let's think about the ignorance of human beings and their education in the form of an allegory. Imagine them living underground in a kind of cave.... They see only the shadows the light from the fire throws on the wall of the cave in front of them.... So, it's obvious that for these prisoners the truth would be no more than the shadows of objects.... Now let's consider how they might be released and cured of their ignorance. Imagine that one man is set free and forced to turn around and walk toward the light. Looking at the light will be painful....
>
> (Plato: *The Republic.* Book Seven)

Contents

1. Introduction

The present notes originated in the introductory course given at the Trieste Summer School on Resolution of Singularities, in June 2006. They focus on the resolution of complex analytic curves and surfaces by Jung's method. They do not contain detailed proofs, but mainly explanations of the central concepts and of their interrelations, as well as heuristics.

If we have begun this text by a famous quotation from Plato, it is because we believe that the citation is related to the general philosophical idea of *resolution of singularities*. This idea corresponds to a *desire* which one can search to fulfil in various contexts, mathematics being one of them. And inside mathematics, one can search to fulfil it inside various of its branches, for various categories of objects. A formulation of the desire could be:

(1) $\qquad$ *Given a complicated object, represent it as the image of a less complicated one.*
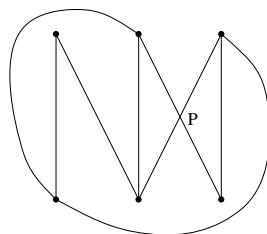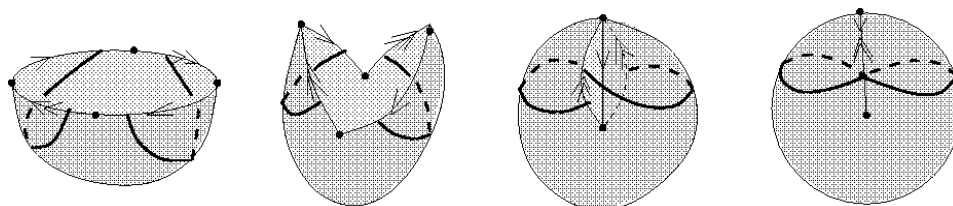
FIGURE 1. An undesired crossing

FIGURE 2. Construction of a cross-cap

As a very simple example, consider a well-known problem of elementary topology. It asks to prove that, given three villages and three wells, one cannot construct roads joining each village to each well, and such that the roads meet only at the villages or the wells. While dealing with the problem, one is naturally led to construct diagrams as in Figure 1. In it, there is an undesired crossing at the point P. One has to prove that such crossings are unavoidable. Or, reformulating the problem, that one cannot embed the abstract graph drawn in Figure 1 into the plane. In this drawing one has *to imagine that the crossing point P is not present.*

Now, a little introspection shows that we imagine very easily such an *elimination of crossing points*, that this is in fact part of our automatic toolkit for understanding images. For example, think of the perspective drawing of a cube, in which the 3-dimensional object "jumps to the eyes".

In the previous examples, we can isolate an elementary operation of local nature, in which we imagine two lines crossing on the plane as the projections of two uncrossing lines in space. This is the easiest example of *resolution of singularities*. The singularity which is being resolved is the germ of the union of the two lines at their meeting point P, their resolution is the union of the germs of the space lines at the preimages of the point P.

Let us give now a 2-dimensional example. We start again with a topological object, the *projective plane*. One can present it to beginners as the surface obtained from a disc by identifying opposite points of its boundary. A way to do this identification in 3-dimensional space is to divide first the boundary into four equal arcs; secondly, to deform the disc till one glues two opposite arcs along a segment; finally to glue the two remaining arcs along the same segment. One gets like this a so-called *cross-cap* (see Figure 2).

Arrived at this point, one has to eliminate by the imagination this segment of self-crossing of the surface in space, in order to get topologically the projective plane. This is an exercise analogous to the one performed before with the graph of
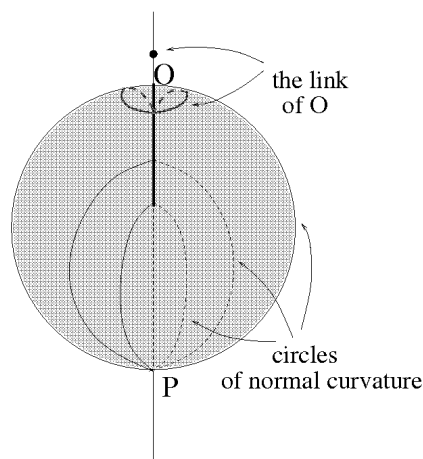
FIGURE 3. Why a cross-cap cannot be algebraic

Figure 1 in order to get rid of the crossing point P. But now it is more difficult to imagine it, as in everyday life we do not interpret self-crossing surfaces in space as projections of surfaces in a higher-dimensional space. In fact, the need to do such interpretations was historically one of the driving forces of the elaboration of a mathematical theory of higher-dimensional spaces.

For the moment we have constructed the cross-cap only as a topological space. But one can give an algebraic model of it. Let us cite Hilbert & Cohn-Vossen [**25**, VI.47], who explain a particularly nice way to get a defining equation:

> There is an algebraic surface of this form. Its equation is
>
> (2)
> $$(k_1 x^2 + k_2 y^2)(x^2 + y^2 + z^2) - 2z(x^2 + y^2) = 0.$$
>
> This surface is connected with a construction in differential geometry. On any surface F, we begin with a point P at which the curvature of F is positive. Then we construct all the circles of normal curvature at P. This family of circles sweeps out [a cross-cap], where the line of self-intersection is a segment of the normal to the surface F at P. The equation given above is referred to the rectangular coordinate system with P as origin and with the principal directions of F at the point P as x-axis and y-axis. k_1 and k_2 are the principal curvatures of F at the point P.

Something is subtly wrong in the previous statements: the locus of points satisfying equation (2) is not reduced to the cross-cap, but it contains also the entire axis of the variable z. Notice that the intersection of the cross-cap and of the z-axis is equal to the segment of self-intersection of the cross-cap (see Figure 3). This shows that the cross-cap described by Hilbert and Cohn-Vossen is in fact *a real semi-algebraic surface*, which means that it can be defined by a finite set of polynomial equations and *inequations*. At this point, we cannot resist the temptation of explaining why no (abstract) real algebraic surface can be homeomorphic to a cross-cap. This is a consequence of the following theorem of Sullivan [**45**]:

THEOREM 1.1. *Let X be a real algebraic set and O a point of X. Then the link of O in X has even Euler characteristic.*
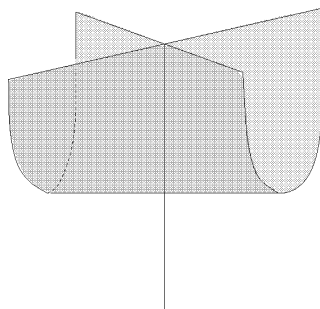
FIGURE 4. Whitney's umbrella

The *link* of O in X is, by definition, the boundary of a regular neighborhood of O in X, obtained by intersecting X with a sufficiently small ball centered at O, after having embedded X in an ambient euclidean space. In our case, consider as a point O the opposite of P on the cross-cap. The link of O is homeomorphic to an ∞-*shaped* curve, whose Euler characteristic is equal to -1. By Sullivan's theorem, this shows that a real algebraic surface containing the cross-cap must contain also other points in the neighborhood of O. In this way, one understands better the presence of the "stick" getting out of O.

There is a famous surface in the real 3-dimensional space, whose topology captures precisely the local topology of the surface (2) in the neighborhood of the point O. It is called *Whitney's umbrella*, and is defined by the equation:

$$(3) \qquad\qquad\qquad x^2 - zy^2 = 0.$$

Here again the axis of the variable z is contained in the algebraic set defined by the equation. The half-line where $z < 0$ appears separated from the points having neighborhoods with topological dimension 2, which are precisely those verifying the inequality $z \geq 0$ in addition of (3). That is why this half-line can be imagined as the stick of a (curious) umbrella (see Figure 4). We will come back later to this example, in Subsection 4.1.

Of course, the previous discussion deals with phenomena of real algebraic geometry. They do not occur in complex geometry. But we feel that it is important to develop intuitions from visual representations of objects, in particular from models of real algebraic surfaces, even if one is mainly interested in complex ones.

We introduced the cross-cap as a representation of the real projective plane. This representation is not faithful, as one identifies like this distinct points of the projective plane. It is a theorem of topology that the real projective plane cannot be represented faithfully in space, in the sense that it cannot be embedded in $\mathbb{R}^3$: indeed, it is non-orientable, and all properly embedded surfaces in $\mathbb{R}^3$ are orientable. Therefore, if one wants to represent it in space, some singular points are unavoidable, in the same way in which supplementary crossing points appear when one represents in the plane the abstract graph which was mentioned at the beginning of this section.

One faces here a general question, which can be asked in any mathematical category in which a convenient notion of *embedding* can be defined:

(4)
$$\text{Given two objects } X \text{ and } E,$$
$$\text{is it possible to represent faithfully } X \text{ inside } E?$$

When the answer is negative, another thing can be asked, which leads to singularity-theoretic questions:

(5)
$$\text{Given two objects } X \text{ and } E,$$
$$\text{how to represent } X \text{ inside } E \text{ with minimal distortion?}$$

For example, Figure 1 shows that the considered abstract graph can be mapped to the plane by introducing only one crossing point of the simplest type. Regarding the projective plane, the cross-cap is a more complicated object: it is an immersion nearly everywhere, with the exception of the extremities of the segment of self-intersection. A representation could be considered simpler if it is entirely an immersion. Boy showed in his thesis [**5**], done under the supervision of Hilbert, that the projective plane could be immersed in $\mathbb{R}^3$ (see the photographs at the end of [**25**, VI.48]).

Let us pass now to another category of geometry, namely *complex algebraic geometry*. A specialization of question (4) is: *given a smooth complex projective curve, is it possible to embed it algebraically in the complex projective plane?* We know that this is not always the case, as it is shown by the following classical theorem:

THEOREM 1.2. *Let C be a smooth algebraic curve inside $\mathbb{CP}^2$. Then $g = \frac{(d-1)(d-2)}{2}$, where g is the genus and d is the degree of C.*

This theorem shows that a smooth projective curve whose genus is not of the form $\frac{(d-1)(d-2)}{2}$ cannot be embedded in $\mathbb{CP}^2$. Theorem 3.5 below generalizes this statement to possibly singular curves.

Starting from a smooth curve C which cannot be embedded in $\mathbb{CP}^2$, one can specialize question (5). Here one gets the following classical theorem:

THEOREM 1.3. *Let C be a smooth projective curve. Then there exists an immersion $C \xrightarrow{\pi} \mathbb{CP}^2$ whose image has only normal crossings and which is an isomorphism over the complement of the singular points of $\pi(C)$.*

Here, $\pi(C)$ is said *to have normal crossings* if its germ at any singular point has only two irreducible components which are smooth and intersect transversely. There is a a higher-dimensional version of this notion (see Definition 2.21). One may prove the theorem by showing that, starting from any embedding of C in a projective space, a generic linear projection satisfies its conclusions.

What happens if instead of starting from a smooth curve, one starts from a curve which admits singular points? Then one has first to ask a specialization of the desire (1). An answer to this is the following:

THEOREM 1.4. *Let C be a projective curve. Then there exists a smooth projective curve $\tilde{C}$ and a morphism $\tilde{C} \to C$ which is an isomorphism over the complement of the set of singular points of C.*

This theorem is historically the first result of *resolution of singularities* in algebraic geometry. It goes back to the construction by Riemann of the surfaces bearing

nowadays his name, associated to any algebraic function of one variable (see the explanations which follow Proposition 2.2).

In the sequel, we shall explain different proofs of this result, as well as of the analogous result for surfaces. But instead of restricting to complex projective varieties, we shall work with the more general notion of *complex analytic spaces* (which we will call also shortly *analytic spaces*, or even *spaces*). General references about them are e.g. the encyclopaedia [**55**], as well as the books of Fischer [**13**] and Kaup & Kaup [**29**].

All the spaces we consider will be assumed *reduced*. We will explain everything as intrinsically as possible, in order to emphasize the various morphisms used in the constructions.

If X is an analytic space, we denote by $\mathrm{Sing}(X)$ its singular locus.

Section 2 contains the general notions necessary to understand the proofs of the theorems concerning the existence of resolutions of curves and surfaces explained in sections 3 and 4. In Section 5 we state some open problems.

2. Generalities about finite morphisms and modifications

2.1. Finite morphisms.

When we draw on a piece of paper a real surface situated in 3-dimensional space, as we did before for the cross-cap and Whitney's umbrella, we trace some curves in the plane. Let us think for a moment about their relation with the surface. Suppose that the drawing is done by cylindrical projection to the plane. For the most economic drawings, as the one of Figure 4, one sees that the curves are of three types:

(1) projections of curves drawn on the surface in order to cut a part of it;
(2) projections of the curves contained in the singular locus of the surface;
(3) apparent contours of the surface with respect to the chosen projection.

The reader is encouraged to recognize each of these types in Figure 4.

Moreover, there are other important aspects of the drawings done before: each point of the plane was the image of *only a finite number of points of the surface* and *no point of the surface escapes to infinity*. This type of projection is of great importance in algebraic or analytic geometry:

DEFINITION 2.1. *A morphism $Y \xrightarrow{\psi} X$ of reduced complex analytic spaces is called **finite** if it is proper and with finite fibers.*

*Let $Y \xrightarrow{\psi} X$ be a finite morphism. Suppose moreover that Y is equidimensional and that ψ is surjective. The **degree** $\deg(\psi)$ of ψ is the maximal number of points in its fibers. The **critical locus** $C(\psi) \subset Y$ of ψ is the set of points $p \in Y$ such that ψ is not a local analytic isomorphism in the neighborhood of p. The **discriminant locus** $\Delta(\psi) \subset X$ is the image $\psi(C(\psi))$.*

The cardinal of the fibers of ψ is equal to $\deg(\psi)$ on the complement of a nowhere dense analytic subset of X. It is important to understand that this subset is contained in $\Delta(\psi)$, but that it is not necessarily equal to $\Delta(\psi)$. Think for example of the normalization morphism of an irreducible germ of curve, a notion explained in the next subsection.

Already for curves, the notion of discriminant is extremely rich, having a lot of avatars. We recommend Abhyankar's fascinating journey [**1**] among them. We mention also that a general program for studying discriminants in singularity theory was described by Teissier [**46**] and a general framework for studying discriminants in algebraic geometry was described by Gelfand, Kapranov & Zelevinsky in [**18**].

The name "discriminant locus" comes from the fact that for projections of hypersurfaces, it is defined by the discriminant of a polynomial :

PROPOSITION 2.2. *Let $f \in \mathbb{C}[t_1, ..., t_{n+1}]$. Denote by Y its vanishing locus in $\mathbb{C}^{n+1}$, by X the hyperplane of $\mathbb{C}^{n+1}$ defined by $t_{n+1} = 0$ and by ψ the restriction to Y of the canonical projection of $\mathbb{C}^{n+1}$ onto X. Then ψ is finite if and only if f is unitary with respect to the variable t_{n+1}, and if this is the case, then the discriminant locus of ψ is defined by the vanishing of the discriminant of the polynomial f with respect to the variable t_{n+1}.*

In the literature one also finds the names *ramification locus* instead of *critical locus* and *branch locus* instead of *discriminant locus.*

If $n = 1$ in the previous proposition, then from the equation $f(t_1, t_2) = 0$ one can express t_2 as a (*multivalued*) function of t_1. This kind of function was called *an algebraic function* in the XIX-th century. Riemann [**43**] associated to such a function a surface (called nowadays *the Riemann surface of the function*) over which the function $t_2(t_1)$ becomes *univalued.* This surface is smooth and projects canonically onto the t_1-axis. Riemann explained how one could construct it by cutting adequately the plane along curves connecting the various points of the discriminant locus, which in this case is a finite set of points on the t_1-axis, and by gluing adequately a finite number of copies of the trimmed surface. An important point to understand is that this Riemann surface does not project canonically only onto the affine line of the independent variable, but also on the affine curve of equation $f(t_1, t_2) = 0$, by a map which is a resolution of the curve. This is the reason why we stated in the introduction that Theorem 1.4 goes back to Riemann.

Returning to Definition 2.1, the discriminant locus of a finite surjective morphism is in fact a closed analytic subset of the target space. Moreover, it can be naturally endowed with a structure of (possibly non-reduced) complex space, whose formation commutes with base change (see Teissier [**46**]).

One can construct purely algebraically a finite morphism, starting from a convenient sheaf of $\mathcal{O}_X$-algebras. In order to explain why, recall one of the fundamental ideas of scheme theory: an affine algebraic variety is *completely determined* as a topological space by its algebra of regular functions. This gives a procedure to construct spaces by doing algebra: each time a new algebra (of finite type) is constructed, one gets automatically a new affine variety. More generally, this can be done over a base which is not an algebraic variety, for example over a complex analytic space X (see Peternell & Remmert [**55**, II.3]). In this case, one gets a new complex analytic space *over the initial one* X from a quasi-coherent sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of finite presentation. The new space is called *the analytic spectrum* of the sheaf $\mathcal{A}$, and is denoted Specan($\mathcal{A}$). Denote also by $\pi_{\mathcal{A}} : \text{Specan}(\mathcal{A}) \to X$ the canonical morphism associated with this construction. A particular case of it is:

PROPOSITION 2.3. *If $\mathcal{A}$ is coherent as an $\mathcal{O}_X$-module, then the morphism $\pi_{\mathcal{A}}$ is finite and $(\pi_{\mathcal{A}})_* \mathcal{O}_{\text{Specan}(\mathcal{A})} \simeq \mathcal{A}$.*

Let us state now the property of the discriminant loci which relates them with the discussion about the drawing of surfaces which took place at the beginning of this subsection:

PROPOSITION 2.4. *Suppose that* $Y \xrightarrow{\psi} X$ *is a finite surjective morphism between equidimensional reduced complex analytic spaces and that* X *is smooth. Then its discriminant locus is equal, set-theoretically, to the union of the image* $\psi(\mathrm{Sing}(Y))$ *of the singular locus of* Y *and of the closure of the apparent contour (that is, the set of critical values) of the restriction of* ψ *to the smooth locus of* Y.

2.2. The normalization morphism.

In the sequel, we shall examine desire (1) for reduced complex analytic spaces. We consider that such a space X is "complicated" if it is singular. Then we would like to represent it as the image of a non-singular one. But we have to decide first what we want to understand by "image". The most encompassing approach would be to consider a surjective morphism from any complex analytic space onto X. But as we consider that we are happy enough when a point is non-singular, it is natural to ask the morphism to be an isomorphism over the set of smooth points. Such morphisms are particular cases of those which are isomorphisms over dense open subsets (see Peternell [**55**, Chapter VII]):

DEFINITION 2.5. *Let* X *be a reduced complex space. A* **modification** *of* X *is a proper surjective morphism* $Y \xrightarrow{\rho} X$ *such that there exists a nowhere dense complex subspace* F *of* X *with the property:*

$$Y \setminus \rho^{-1}(F) \xrightarrow{\rho} X \setminus F \quad \text{is an isomorphism.}$$

The minimal subspace $\mathrm{Fund}(\rho)$ *with this property is called* **the fundamental locus** *of the modification* ρ. *The preimage* $\mathrm{Exc}(\rho) := \rho^{-1}(\mathrm{Fund}(\rho))$ *of the fundamental locus is called* **the exceptional locus** *of* ρ. *If* Z *is a closed irreducible subspace of* X, *not contained in the fundamental locus* $\mathrm{Fund}(\rho)$, *then its* **strict transform** Z'_ρ *by the modification* ρ *is the closure of* $\rho^{-1}(Z \setminus \mathrm{Fund}(\rho))$ *in* Y.

In the literature, strict transforms are also called *proper transforms.*

Informally, to *modify* X means to take out a nowhere dense analytic subset F and to replace it by another analytic set E. The important thing to remark is that the isomorphism between the "unmodified" parts of the two spaces must extend to an analytic morphism from the new space to the initial one.

If ρ is a modification, we can try to understand it by looking at its fibers, which are *compact* analytic spaces (remember that we asked ρ to be proper!). The simplest situation arises when all those fibers are finite from the set-theoretical viewpoint, that is, when the modification is a finite morphism (see Definition 2.1). Among such modifications, there is a unique one (up to *unique* isomorphism) which dominates all the other ones, the *normalization morphism.* Before stating precisely this result (see Theorem 2.10 below), we recall briefly the notion of *normal analytic space.*

This concept was first introduced in algebraic geometry by Zariski [**52**], inspired by the arithmetic notion of integral closure and by the notion of *normal projective variety* used by the Italian geometers (see Zariski [**52**, Footnote 26] and Teissier [**47**, Section 3.1]). It was extended to the complex analytic category in the years

1950. Here we prefer to give a "transcendental" (that is, function-theoretical, non-algebraic) definition, which has the advantage to allow us later on to construct very easily holomorphic functions on normal varieties. At the end of the subsection, we will briefly come back to Zariski's algebraic viewpoint.

The following theorem was proved by Riemann. It allows one to show that a function of one variable is holomorphic on a neighborhood of a point only by knowing its behaviour outside the point.

THEOREM 2.6. (**Riemann extension theorem**) *Let U be a neighborhood of 0 in $\mathbb{C}$ and f be a holomorphic and bounded function on $U \setminus 0$. Then f extends (in a unique way) to a function holomorphic on U.*

The previous theorem, also known as *Riemann's removable singularity theorem*, was extended to higher dimensions (see Kaup & Kaup [**29**, chapter 7]):

THEOREM 2.7. (**generalized Riemann extension theorem**) *Let U be a neighborhood of 0 in $\mathbb{C}^n$, $n \geq 1$ and f be a holomorphic and bounded function on $U \setminus Z$, where Z is a strict closed complex analytic subspace of U. Then f extends (in a unique way) to a function holomorphic on U.*

It is then natural to ask which complex analytic sets admit the same property as $\mathbb{C}^n$. In fact, at the beginning of the years 1950, some specialists of complex analytic geometry took this property as a definition of a *complex analytic set* (see Remmert [**55**, pages 30-31]). Later, as this name began to designate any set glued analytically from subsets of $\mathbb{C}^n$ which are defined locally by a finite number of analytic equations, sets with the Riemann extension property got a special name:

DEFINITION 2.8. *Let X be a reduced complex analytic space. If U is an open subspace of X, a **weakly holomorphic function** on U is a holomorphic and bounded function defined on $U \setminus Y$, where Y is a nowhere dense closed subspace of U. The space X is called **normal** if every weakly holomorphic function on U extends in a unique way to a holomorphic function on U, and this must occur for any open subset U of X.*

We have presented the normal spaces as those which have in common with the smooth ones, the truth of the generalized Riemann extension theorem. In the next theorem we state other similarities between them:

THEOREM 2.9. *A normal complex analytic space is locally irreducible and smooth in codimension 1 (that is, its singular set has codimension ≥ 2).*

Not any complex analytic space is normal. However, any complex analytic set can be canonically presented as the image of a normal one:

THEOREM 2.10. *Let X be a reduced complex space. Then there exists a modification $\overline{X} \overset{\nu}{\to} X$ such that: $\overline{X}$ is normal and ν is a finite morphism. Moreover, if ν is a fixed modification having these properties, then for any finite modification $X_1 \overset{\nu_1}{\to} X$, there exists a unique morphism $\overline{X} \overset{\chi}{\to} X_1$ making the following diagram commutative:*

$$\overline{X} \xrightarrow{\;\;\chi\;\;} X_1$$
$$\nu \searrow \quad \swarrow \nu_1$$
$$X$$

DEFINITION 2.11. *A morphism $\overline{X} \overset{\nu}{\to} X$ as in the previous theorem is called* **the normalization morphism** *of X.*

Theorem 2.10 explains why we have used the article "the" instead of "a" : it implies that a normalization morphism is unique up to unique isomorphism above X, which is the greatest type of uniqueness in a category. In this way, one characterizes the normalization morphism by a universal property.

As another consequence of Theorem 2.10, notice that the process of normalization is of *local* nature, that is, the restriction of the normalization morphism of X to an open set $U \subset X$ is the normalization morphism of U.

The normalization morphism is a particular case of the construction of the analytic spectrum (see Proposition 2.3), in which $\mathcal{A} := \tilde{\mathcal{O}}_X$, the sheaf of weakly holomorphic function on X. This sheaf is coherent as an $\mathcal{O}_X$-module and can be defined algebraically, as was seen already by Riemann in the case of complex curves:

THEOREM 2.12. *Let X be a reduced complex space. The sheaf $\tilde{\mathcal{O}}_X$ of weakly holomorphic functions on X is coherent and equal to the sheaf of integral closures of the local rings of $\mathcal{O}_X$ in their total rings of fractions. The morphism $\pi_{\tilde{O}_X} :$ $\mathrm{Specan}(\tilde{O}_X) \to X$ is the normalization morphism of X.*

The *total ring of fractions* $\mathrm{Tot}(A)$ of a given ring A is by definition the ring of quotients in which all the elements of A which are not 0-divisors become invertible. If the initial ring is integral, that is, without 0-divisors, then its total ring of fractions is a field. If the ring is reduced but not integral, that is, the associated space is reduced but not irreducible, then $\mathrm{Tot}(A)$ is canonically the direct product of the fields of quotients of the rings associated to the irreducible components.

The previous theorem is the key for understanding why normalization has at the same time an algebraic and a transcendental aspect.

The concept of normalization is essential when one is thinking about resolution of singularities. Indeed, as shown by Theorem 2.9:

PROPOSITION 2.13. *The normalization morphism of X separates the local analytically irreducible components of X and resolves the singularities in codimension 1.*

The last statement means that $\mathrm{Sing}(\overline{X})$ has codimension at least 2 in $\overline{X}$.

Let us illustrate the proposition with a simple example, that of two smooth plane curves intersecting transversely, met in a topological context at the beginning of the introduction. Here we consider the union X of the two axes in the complex affine plane $\mathbb{C}^2$, with coordinates x, y. Thus, the associated algebra is $A := \mathbb{C}[x, y]/(xy)$. Consider the function $f = x/(x+y)$ restricted to X. It is weakly holomorphic, as it is holomorphic outside the origin and bounded in a neighborhood of it. Theorem 2.12 shows that f becomes a holomorphic function on the normalization of X. As f is constant outside the origin in restriction to both axes, taking the values 0 and 1 respectively, we see that there are two possible limits at the origin. Therefore f cannot be extended to a continuous function defined all over X. The abstract construction $\mathrm{Specan}(\tilde{O}_X)$ separates the lines, such that f becomes a function holomorphic all over the new curve, which is isomorphic to the disjoint union of the two axes.

To illustrate also Theorem 2.12, notice that f is indeed an element of the integral closure of A in its total ring of fractions $\mathrm{Tot}(A)$: $x + y$ is not a divisor of 0 and one has $f^2 - f = 0$, which is a relation of integral dependence of f over A.

For more details about normal varieties, one may consult Greco's book [**22**]. For details about the more general notion of *weakly normal complex spaces* (in which any *continuous* weakly holomorphic function is in fact holomorphic), one may consult Adkins, Andreotti and Leahy's book [**2**].

We conclude this subsection with a quotation from the introduction of Zariski's work [**52**]:

> Here we introduce the concept of a *normal* variety, both in the affine and in the projective space, and we are led to a geometric interpretation of the operation of integral closure. The importance of normal varieties is due to: ... *the singular manifold of a normal V_r is of dimension $\leq r - 2$* (in particular a normal curve (V_1) is free from singularities)... There is a definite class of normal varieties associated with and birationally equivalent to a given variety V_r. This class is obtained by a process of integral closure carried out in a suitable fashion for varieties in projective spaces....
>
> The special birational transformations effected by the operation of integral closure, and the properties of normal surfaces, play an essential rôle in our arithmetic proof for the reduction of singularities of an algebraic surface.

2.3. Blowing-up points and subschemes.

Let us begin by a theorem of elementary geometry (see Figure 5):

PROPOSITION 2.14. *Let ABC be a triangle in the euclidean plane. For each point P in the plane, consider the symmetric lines of PA, PB, PC with respect to the bisectors of the angles $\angle BAC, \angle CBA$ and $\angle ACB$ respectively. Then these three new lines intersect at another point $s(P)$ and the transformation $P \rightarrow s(P)$ is an involution.*

The proposition can be easily proved using the classical theorem of Ceva. It is also true that P and $s(P)$ are the two foci of a conic tangent to the edges of the triangle ABC (as an illustration of this fact, think at the inscribed circle, which is a conic tangent to the three edges, and whose center I verifies $I = s(I)$).

But what interests us here more is the fact that *the mapping s is not defined everywhere*. Indeed, it is not defined at the vertices of the triangle. By doing some drawings, one sees experimentally why: if one tends to a vertex by remaining on a line passing through it, then the limit of the transforms is well-defined, but *it depends on the chosen line*. Moreover, by varying the line, one gets as limits all the points situated on the line containing the opposite edge. Therefore, in a way:

$$s \text{ transforms each vertex into the opposite edge.}$$

As the dimension increases like this from 0 to 1, one assists to a kind of "blowing-up" of each vertex into a line. One has at the same time a phenomenon of "blowing-down" of each edge of the triangle into the opposite vertex. Indeed, all the points of the line containing an edge, with the only exception of the vertices, are sent by s into the opposite vertex of the triangle.

This kind of examples led Zariski to introduce a general notion of "blowing-up" and "blowing-down" in algebraic geometry. In order to explain it, let us first express algebraically the transformation s.
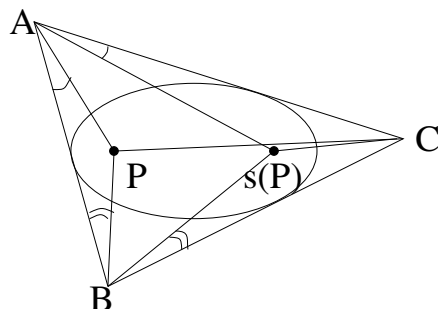
FIGURE 5. A birational involution of the plane

There are other points P for which $s(P)$ is not defined, those for which the three new lines are parallel. But in this case $s(P)$ can be interpreted as a point at infinity, which shows that it is better to think about s as a transformation of the *projective* plane into itself. There is then a choice of coordinates which makes the transformation particularly simple from the algebraical viewpoint: choose the unique system of projective coordinates $(X : Y : Z)$ such that the equations $X = 0, Y = 0, Z = 0$ define the edges of the triangle, and such that the center I of the incircle is $(1 : 1 : 1)$. Then the involution s can be written as:

$$(6) \qquad (X : Y : Z) \cdots \to (\frac{1}{X} : \frac{1}{Y} : \frac{1}{Z}).$$

Since the same map can be expressed as $(X : Y : Z) \cdots \to (YZ : ZX : XY)$, that is, as a map with quadratic polynomials as coordinates, one speaks about a *quadratic transformation of* $\mathbb{P}^2$. For a deeper understanding of this vocabulary, we refer the reader to the quotation from Zariski [**53**] at the end of this subsection.

We see that s can be expressed in projective coordinates using rational functions of the coordinates. That is why one says that s is a *rational map*. Because its inverse is also rational (as the map s is an involution), one says that the map $\mathbb{P}^2 \cdots\overset{s}{\to} \mathbb{P}^2$ is *birational*. Generally speaking:

DEFINITION 2.15. *Let X and Y be two reduced and irreducible algebraic varieties. A **rational map** $Y \cdots\overset{s}{\to} X$ is an algebraic morphism $U \to X$, where U is a dense Zariski open set of Y. **The indeterminacy locus** of a rational map is the complement of the maximal possible such U.*

*A **birational map** $Y \cdots\overset{s}{\to} X$ is a rational map which realizes an isomorphism between dense open subsets of Y and X. A **birational morphism** is a birational map which is defined everywhere.*

Birational geometry is the study of algebraic varieties up to birational isomorphism. It seems to have begun as a conscious domain of research with Riemann's definition [**44**, chapter XII] of the birational equivalence of plane algebraic curves, which we quote here:

> We shall consider now, as pertaining to a *same class, all the irreducible algebraic equations between two variable magnitudes, which can be transformed the ones into the others by rational substitutions.*

The notion of *modification* (see Definition 2.5) was introduced in complex analytic geometry in order to extend to it the notion of birational morphism, and to

create an analog of the birational geometry, the so-called *bimeromorphic geometry*. To understand this, notice that a proper birational morphism $Y \to X$ between complex algebraic varieties is a modification of the underlying complex analytic space of X.

By definition, the difference between the concepts of *rational map* and *rational morphism* is that for the first one we allow the presence of points of indeterminacy, while this is forbidden for the second notion. There is a general way to express a rational map in terms of rational morphisms. One simply considers *the closure of the graph of the rational map*. As this closure lives in the product space, it can be naturally projected onto the factor spaces, which are the source and the target of the initial map. But these two projections are now *morphisms*:

(7)
$$\overline{\text{Graph}(s)} \hookrightarrow Y \times X$$
$$Y \quad \overset{s}{\cdots\rightarrow} \quad X$$
with p_Y and p_X the projections.

The first one $\overline{\text{Graph}(s)} \overset{p_Y}{\longrightarrow} Y$ is a birational *morphism* and the second one is also a morphism, but not necessarily birational. The map s can be expressed as the composition $s = p_X \circ p_Y^{-1}$. If X is complete (that is, its underlying analytic space is compact), then p_Y is proper, and therefore p_Y is a *modification* of the underlying analytic space of Y (see Definition 2.5).

As a very important example, let us consider the canonical projection map from a vector space V of dimension $n \geq 2$ to its projectivization $\mathbb{P}(V)$:

(8)
$$B_0(V) \hookrightarrow V \times \mathbb{P}(V)$$
$$V \quad \overset{s}{\cdots\rightarrow} \quad \mathbb{P}(V)$$
with p_V and $p_{\mathbb{P}(V)}$ the projections.

One can study this diagram using a fixed coordinate system. Start from a basis of V, which determines an isomorphism between V and $\mathbb{C}^n$, and the associated canonical covering of $\mathbb{P}(V)$ with n affine charts isomorphic to $\mathbb{C}^{n-1}$. This gives a covering of $V \times \mathbb{P}(V)$ with n charts isomorphic to $\mathbb{C}^{2n-1}$. Being the roles of the different coordinates completely symmetric, one sees that it is enough to study the modification p_V inside one of these charts. One proves in this way:

PROPOSITION 2.16. *1)The algebraic variety $B_0(V)$ is smooth.*

2)The indeterminacy locus of the modification p_V is the point 0 and its exceptional locus is sent isomorphically to $\mathbb{P}(V)$ by the morphism $p_{\mathbb{P}(V)}$. Moreover, this second morphism is canonically isomorphic to the projection map of the total space of the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$.

3) If $y_i := \frac{x_i}{x_n}, \forall i \in \{1, ..., n-1\}$ are the coordinates of the canonical chart $U_n := \mathbb{P}(V) \backslash \{x_n = 0\}$ of $\mathbb{P}(V)$, then the canonical projection of the affine space $V \times U_n$ with coordinates $x_1, ..., x_n, y_1, ..., y_{n-1}$ onto the space with coordinates $y_1, ..., y_{n-1}, x_n$ is an isomorphism when restricted to $B_0(V)$.
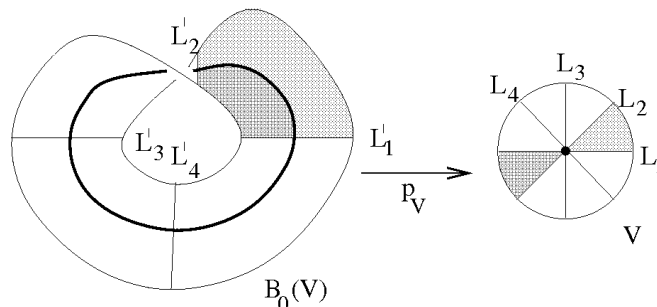
FIGURE 6. Blowing-up the origin in a real plane

4) *In terms of the coordinates $x_1, ..., x_n, y_1, ..., y_{n-1}$ of $B_0(V) \cap (V \times U_n)$ and $x_1, ..., x_n$ of V, the modification p_V is expressed as:*

(9) $x_1 = y_1 \cdot x_n \, , \, ... \, , \, x_{n-1} = y_n \cdot x_n \, , \, x_n = x_n.$

This proposition shows that one has modified V by replacing the origin with the projective space of all the directions of lines passing through the origin. Therefore, the origin has been "blown-up" into a higher dimensional space:

DEFINITION 2.17. *The birational morphism $B_0(V) \overset{p_V}{\to} V$ of diagram (8) is called* ***the blowing-up of the origin in*** V.

In Figure 6 we have represented the blowing-up of the origin in a real plane, by drawing its restriction over a disc centered at the origin. It is an excellent exercise to understand why one gets like this a Möbius band. We have represented also the strict transforms L_i' of four segments L_i passing through the origin. Please contemplate how they become disjoint on the blown-up disc!

The construction of the blowing-up of a point may be extended from an ambient vector space to an arbitrary complex manifold. One may blow-up a point of it by choosing a system of local coordinates and by identifying like this the point with the origin of the vector space defined by that coordinate system. Different coordinate systems lead to blown-up spaces which are canonically isomorphic over the initial manifold, which shows that the blow-up exists and is unique up to unique isomorphism.

Roughly speaking, one blows-up a point of a smooth surface by replacing it with a rational curve whose points correspond to the projectified algebraic tangent plane of the surface at that point. In the same way, one blows-up a point in a complex manifold by replacing it with the projectified tangent space at that point. More generally, one can blow-up a submanifold by replacing it with its projectified normal bundle. But one can still generalize this construction, and blow-up a non-necessarily smooth and even non-necessarily reduced subspace. The following theorem, characterizing blowing-ups by a universal property, was proved by Hironaka [26]:

THEOREM 2.18. *Let X be a (not necessarily reduced) complex analytic space. Let Y be a subspace of X, defined by the ideal sheaf $\mathcal{I}$. Then there exists a modification $B_Y(X) \overset{\beta_{X,Y}}{\to} X$ such that:*
   • *the preimage ideal sheaf $\beta_{X,Y}^{-1}\mathcal{I}$ is locally invertible;*

- *for any morphism $B \xrightarrow{\beta} X$ such that $\beta^{-1}\mathcal{I}$ is locally invertible, there exists a unique morphism γ such that the following diagram is commutative:*

$$
\begin{array}{ccc}
B & \xrightarrow{\ \ \gamma\ \ } & B_Y(X) \\
& {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\beta_{X,Y}} & \\
& X &
\end{array}
$$

DEFINITION 2.19. *A modification $B_Y(X) \xrightarrow{\beta_{X,Y}} X$ as in the previous theorem is called* **the blowing-up of** Y *(or* **with center** Y, *or* **of** $\mathcal{I}$) **in** X.

In algebraic geometry, blowing-ups are also known as *monoidal transforms* (see the quotation from Zariski at the end of this subsection) and in analytic geometry as σ-*processes*.

Not all the modifications can be obtained by blowing-up a subspace. Those which can are precisely the *projective modifications*. Moreover, a blowing-up does not determine the ideal sheaf $\mathcal{I}$ used to define it. In fact, a blowing-up in the sense of Hironaka [**26**] is the couple $(Y, \beta_{X,Y})$. Notice that the ideal sheaf $\mathcal{I}$ giving birth to it has been forgotten.

Let us consider again the example of the birational involution (6). We have seen that its indeterminacy locus in the projective plane is the set of vertices of the initial triangle. Moreover, we saw that the indeterminacy was caused by the fact that when one tends to a vertex along different lines passing through the vertex, one gets different limits of their images by the involution. This suggests that, by replacing each vertex with a curve parametrizing the lines passing through it, that is by blowing-up the three vertices, one modifies the projective plane in such a way that now the rational map is defined everywhere. This is indeed the case, as shown by the following quotations from the article [**53**] in which Zariski introduced the operation of blowing-up under the name of "monoidal transformation":

> With some non-essential modifications, and without their projective trimmings, the space Cremona transformations, known as monoidal transformations, are monoidal transformations in our sense.... A *quadratic transformation* is a special case of a monoidal transformation, the center is in that case a point.... A quadratic Cremona transformation is not at all a quadratic transformation in our sense. Our quadratic transformation has only one ordinary fundamental point and its inverse has no fundamental points at all, while a plane quadratic transformation and its inverse both have three fundamental points, which in special cases may be infinitely near points.... Of course, an ordinary quadratic transformation between two planes π and π' can be expressed as a product of quadratic transformations in our sense, or more precisely as the product of 3 successive quadratic transformations and of 3 inverses of quadratic transformations.

2.4. The definitions of resolution and embedded resolution.

Coming back to the desire (1), one could think that it is fulfilled for complex analytic spaces if one finds a modification whose initial space is smooth. Such modifications have a special name:

DEFINITION 2.20. *Let X be a reduced space. A **resolution** (or **resolution of the singularities**) of X is a morphism $\tilde{X} \xrightarrow{\pi} X$ such that:*
- π *is a modification of X;*
- $\tilde{X}$ *is smooth;*
- *the restriction $\tilde{X} \setminus \pi^{-1}(\mathrm{Sing}(X)) \xrightarrow{\pi} X \setminus \mathrm{Sing}(X)$ is an isomorphism.*

In some cases, people skip the last condition in the definition of a *resolution*. Other terms which were used in the literature are "reduction of the singularities" (see Walker [**49**] or the quotation at the end of Subsection 2.2) and "desingularization".

In the previous definition, if X is embedded in a smooth ambient space, one needs sometimes to get a resolution as a restriction of a modification of the ambient space. It is also important to have a modification in which the subspaces of interest are as simple as possible from a local viewpoint. The subspaces we are speaking about are the exceptional locus of the modification and the strict transform of the space X (see Definition 2.5). As the ambient space is supposed to be smooth, it can be shown that the exceptional locus is necessarily of codimension 1. The usual condition of local simplicity for a hypersurface is that of being a *divisor with normal crossings*, and for the union of a hypersurface and a subvariety of possibly lower dimension, of *crossing normally*:

DEFINITION 2.21. *Let M be a complex manifold and D a divisor of M. Denote by $|D|$ the underlying reduced hypersurface of D.*

*One says that D is a **divisor with normal crossings** if for each point $p \in |D|$, there is a system of local coordinates centered at p such that in some neighborhood of p the hypersurface $|D|$ is the union of some hyperplanes of coordinates.*

*If V is a reduced subvariety of M of a possibly lower dimension, one says that D **crosses** V **normally** if for each point $p \in |D| \cap V$, there is a system of local coordinates centered at p such that in some neighborhood of p the hypersurface $|D|$ is the union of some hyperplanes of coordinates and V is the intersection of some of the remaining hyperplanes of coordinates.*

In order to understand this definition, remark that if one takes as the hypersurface $|D|$ the union of two of the coordinate hyperplanes in $\mathbb{C}^3$, then no line intersecting them only at the origin crosses them normally. But after blowing-up the intersection of the two planes, the total transform of their union and the strict transform of the line cross normally.

In the previous definition, we do not suppose that each irreducible component of D is smooth. If this is moreover the case, one says usually that D is a *strict normal crossings divisor*.

DEFINITION 2.22. *Let X be a closed reduced subspace of a complex manifold M. An **embedded resolution** of X in M is a morphism $\tilde{M} \xrightarrow{\pi} M$ such that:*
- π *is a modification of M;*
- $\tilde{M}$ *is smooth;*
- *the restriction $\tilde{M} \setminus \pi^{-1}(\mathrm{Sing}(X)) \xrightarrow{\pi} M \setminus \mathrm{Sing}(X)$ is an isomorphism;*
- *the strict transform X'_π is smooth;*
- *the exceptional locus of π has normal crossings and also crosses normally the strict transform X'_π of X by the modification π.*

2.5. The special case of surfaces.

From now on, we will restrict our considerations to surfaces. Inside them, it will be important to consider also various curves and their intersection numbers. That is why we recall first basic facts about intersection theory of curves on smooth surfaces. For more details, one may consult Hartshorne's book [**23**, Chapter V.1].

Let C_1 and C_2 be two (not necessarily reduced) properly embedded curves (that is, effective divisors) contained on a (possibly non-compact) smooth surface M. Let P be a common point of C_1 and C_2. Suppose that their analytic germs at P have no common components. Then their *intersection number at P* may be defined by the formula:
$$(C_1 \cdot C_2)_P := \dim_{\mathbb{C}} \mathcal{O}_{M,P}/(f_1, f_2) > 0,$$
where $f_i \in \mathcal{O}_{M,P}$ denotes a holomorphic function defining the curve C_i in a neighborhood of P on M. This definition is independent of the choice of the functions f_1, f_2.

If C_1 is compact (but not necessarily C_2), then one may define the *global intersection number*:
$$(C_1 \cdot C_2) := \deg \mathcal{O}_M(C_2)|_{C_1},$$
where $\mathcal{O}_M(C_2)$ denotes the line bundle generated by the divisor C_2 on M and $\mathcal{O}_M(C_2)|_{C_1}$ denotes its restriction to C_1. This number depends only on the germ of C_2 along C_1. If *both* C_1 and C_2 are compact, then one has : $(C_1 \cdot C_2) = (C_2 \cdot C_1)$.

By bilinearity, one may extend the definition of intersection number to the case where C_1 and C_2 are possibly non-effective compact divisors on the surface M.

If C_1 and C_2 have no common components, then they have only a finite set of common points and:
$$(C_1 \cdot C_2) = \sum_{P \in C_1 \cap C_2} (C_1 \cdot C_2)_P.$$
This shows that, with the previous hypothesis, $(C_1 \cdot C_2) > 0$ whenever $C_1 \cap C_2$ is non-empty.

This positivity result is no longer necessarily true if C_1 and C_2 have common components, for example if $C_1 = C_2$. The simplest example is provided by the self-intersection number of the exceptional curve created by blowing-up a point on a smooth surface, which is equal to -1 (see below).

If M is a smooth complex surface and D is a compact reduced curve in M, one associates to it an abstract unoriented *weighted dual graph* $\Gamma(D)$. Its vertices v_i correspond bijectively to the irreducible components D_i of D and its edges correspond bijectively to the unordered pairs of distinct vertices $\{v_i, v_j\}$ whose corresponding irreducible components D_i, D_j intersect. Each vertex v_i is weighted by the self-intersection number $e_i := D_i^2$ of the corresponding component and each edge by the intersection number of the components associated to its vertices. Denote by $e_{i,j} = e_{j,i} := D_i \cdot D_j$ the weight of the edge joining v_i and v_j. Therefore, if there is no edge between v_i and v_j, one has $e_{i,j} = 0$.

One may associate to the curve D *the intersection form* on the free abelian group of the divisors supported by D, given by the intersection number. This intersection form depends only on the associated dual graph $\Gamma(D)$. Indeed, if $\sum_i x_i D_i$ is a divisor supported by D, then its self-intersection number is :
$$\sum_i e_i x_i^2 + 2 \sum_{\{i,j\mid i \neq j\}} e_{i,j} x_i x_j.$$

One can particularize the previous constructions to the case where D is the exceptional divisor of a resolution of a normal surface singularity. The divisors appearing like this are very special, as is shown by the following theorem. Point 1) was proved by Du Val [**48**] and Mumford [**36**], and point 2) was proved by Grauert [**21**].

THEOREM 2.23. *1) Let D be the exceptional locus of a resolution of a complex analytic normal surface singularity. Then D is a connected curve whose intersection form is negative definite.*

2) Let D be a reduced divisor with compact support in a smooth complex analytic surface. If the intersection form of D is negative definite, then there exists a neighborhood of D which is the resolution of a normal surface having only one singular point, such that D is the exceptional divisor of this resolution.

When the hypothesis of point 2) are satisfied, one says that D can be *contracted*.

For more details about dual graphs and intersection matrices, one can consult Laufer [**31**], Némethi [**37**] or Popescu-Pampu [**42**].

Let us consider again the blowing-up of a point on a smooth surface, which is illustrated in Figure 6 for the case of real surfaces. As a particular case of point 2) of Proposition 2.16, one shows that its exceptional locus E is a smooth rational curve of self-intersection number $E^2 = -1$. Such a curve passing only through smooth points of a surface is called classically an *exceptional curve of the first kind*. More generally, an *exceptional curve* is a reduced divisor which can be contracted. It can be shown that an exceptional curve of the first kind must contract to a smooth point of the new normal surface.

Moreover, it is a classical theorem of Castelnuovo that if one starts from a (not necessarily smooth) projective surface, then the surface obtained after having contracted an exceptional curve of the first kind is again projective. This is to be contrasted with the general case, when the contraction of an exceptional curve cannot be always done in the projective category, or even in the category of schemes (see an example of Nagata in Bădescu [**4**, chapter 3]).

Let (S, s) be a normal surface singularity. In Section 4 we explain a proof of Theorem 4.1, which says in particular that a resolution of (S, s) always exists. It is then natural to try to compare all possible resolutions. We have the following theorem concerning them:

THEOREM 2.24. *Let (S, s) be a germ of normal surface. There exists a minimal resolution $S_{min} \xrightarrow{\pi_{min}} S$ of (S, s), in the sense that any other resolution $S' \xrightarrow{\pi'} S$ can be factored through a composition γ of blowing-ups of points:*

$$S' \xrightarrow{\gamma} S_{min}$$
$$\pi' \searrow \quad \swarrow \pi_{min}$$
$$S$$

The minimal resolution can be characterized by the property that no irreducible component of E_{min} is exceptional of the first kind.

The previous theorem is specific to surfaces: it is no longer true in higher dimensions.

3. Resolutions of curve singularities

3.1. Abstract resolution.

Using theorems 2.9 and 2.10, we get immediately:

THEOREM 3.1. *If C is a reduced analytic curve, then its normalization morphism is a resolution of C.*

Analytically, a normalization of a germ of curve (C, c) is given by a set of parametrizations $(\mathbb{C}, 0) \xrightarrow{\nu_i} C_i$ of the irreducible components of C, with the condition that each parametrization realizes a homeomorphism onto its image. If an irreducible germ of curve is embedded in some space $(\mathbb{C}^n, 0)$, such a parametrization is given by n convergent power series $x_1(t), ..., x_n(t)$ in a variable t, with the restriction that one cannot write them as convergent power series of a new variable w, with w a convergent power series of t of order ≥ 2.

Let us explain how one can deduce the existence of a normalization of an analytically irreducible germ of curve with topological arguments, in the spirit of Riemann. Consider an embedding of the germ in a smooth space, and choose local coordinates $(x_1, ..., x_n)$ in this space such that the canonical projection onto the axis of the first coordinate x_1 is finite (which means that the curve is not contained in the hyperplane of the other coordinates). Denote by $(C, c) \xrightarrow{\alpha} (\mathbb{C}, 0)$ the restriction of this projection to the curve. Look at the induced morphism of fundamental groups $\pi_1(U \setminus c) \xrightarrow{\alpha_*} \pi_1(V \setminus 0)$, where U and V are neighborhoods of c in a representative of C and of 0 in $\mathbb{C}$, which are homeomorphic to discs. Since the covering is finite and has a connected total space, the image group $\alpha_*(\pi_1(U \setminus c)) \subset \pi_1(V \setminus 0)$ is infinite cyclic and has a finite index $m \geq 1$. Notice that m can also be interpreted as the degree of the finite morphism α.

Consider then another copy of $\mathbb{C}$, with parameter t, and the morphism $\mathbb{C}_t \xrightarrow{\tau} \mathbb{C}_{x_1}$ defined by the equation $x_1 = t^m$. By construction, $\alpha_*(\pi_1(U \setminus c)) = \tau_*(\pi_1(\mathbb{C} \setminus 0))$, which shows that the map τ can be lifted to a homeomorphism ν from a pointed neighborhood of the origin in $\mathbb{C}_t$ to $U \setminus c$. Compose this morphism ν with the ambient coordinate functions $(x_1, ..., x_n)$ at the target. By construction, all the functions $x_i \circ \nu$ are holomorphic and bounded on a pointed neighborhood of 0. By Riemann's extension theorem 2.7, all of them can be extended to functions holomorphic also at the origin. This shows that ν extends to a map holomorphic all over the chosen neighborhood of 0 in $\mathbb{C}_t$:

(10)
$$(\mathbb{C}_t, 0) \xrightarrow{\nu} (C, c)$$
$$\tau \searrow \qquad \downarrow \alpha$$
$$(\mathbb{C}_{x_1}, 0)$$

The map ν constructed in this way is the normalization morphism of (C, c).

The problem with this resolution process by the normalization morphism is that, given a fixed embedding of the curve, it does not extend naturally to a modification of the ambient space. But in many applications, and in particular for Jung's method of resolution of surfaces presented in Section 4, it is important to resolve the curve by a morphism which is the restriction of an ambient one.

Let us consider this second problem in the case of plane curves. At first, Max Noether *simplified* the singularities of plane curves by doing sequences of quadratic transforms of the type (6), with respect to conveniently chosen triangles (see [**38**] and the obituary [**10**] by Castelnuovo, Enriques & Severi). If we used the term "simplified" and not "resolved" in the previous sentence, it is because he did not really resolve them with the modern definition 2.20. He proved instead the theorem:

THEOREM 3.2. *Let C be a plane curve. Then one can transform the curve C into another curve C' which has only ordinary multiple points, by a sequence of involutions isomorphic to the involution (6).*

An *ordinary multiple point* designates a point of the curve at which its analytically irreducible components are smooth and pairwise transverse. The strategy to prove the previous theorem was to iterate the following steps, given the curve $C \subset \mathbb{P}^2$ we want to simplify:

1) Choose a singular point c of C which is not an ordinary multiple point.

2) Choose a triangle having a vertex at c, and whose edges are transverse to C (that is, they cross it normally) outside the set of its vertices.

3) Choose a quadratic transformation of the plane whose reference triangle is the fixed one, then take the transform of the curve C under this map.

The theorem is deduced from the fact that one can arrive, after a finite number of iterations, at a curve having only ordinary multiple points as singularities. The fact that one cannot obtain only ordinary double points as singularities comes from step 2).

Once the elementary operation of blowing-ups of points was isolated, it was possible to deduce immediately the following theorem from the previous one:

THEOREM 3.3. *Let C be a reduced curve embedded in a smooth surface S. Then C can be resolved by a finite sequence of blowing-ups of points. At each step of the algorithm, one simply blows-up the singular points of the strict transform of C.*

Why is Theorem 3.3 a consequence of Theorem 3.2? To understand this, look at step 3) of the previous iteration. In it, the singular point c of the curve is blown up. At the same time other things happen to the plane, well described in Zariski's quotation at the end of subsection 2.3. But if one concentrates one's attention in the neighborhood of the point c, the effect on the germ (C, c) is the same as if one had only blown up that point. As it can be shown that any germ of reduced analytic curve embedded in a smooth surface can be embedded in the projective plane, one sees that the study done during the proof of Theorem 3.2 allows one to prove Theorem 3.3.

Let us be more explicit. The explanations which led to diagram (10) show that if $(C, c) \subset S$ is an irreducible germ of curve in a smooth complex surface, then there are local coordinates (x, y) on S centered at c, such that (C, c) is given by a parametrization of the form:

$$(11) \qquad \begin{cases} x = t^m \\ y = \sum_{k \geq n} a_k t^k \end{cases} .$$

where $a_k \in \mathbb{C}$, $\forall\, k \geq n, a_n \neq 0$ and $\min(m, n)$ is equal to the multiplicity of C at c. A parametrization of the form (11) is called a *Puiseux parametrization* or a *Newton-Puiseux parametrization*. Such parametrizations are of the utmost importance in

the detailed study of singularities of plane curves (see e.g. Brieskorn & Knörrer [**7**], Teissier [**47**], Wall [**50**]).

It is possible to show that if c is a singular point of C, that is, if $m \geq 2$, then the choice of local coordinates can be done such that $n > m$ and n is not divisible by m. Then, as a consequence of Proposition 2.16, one can show that the strict transform of C by the blowing-up of c on S can be parametrized in suitable local coordinates by:

$$\begin{cases} x_1 = t^m \\ y_1 = \sum_{k \geq n} a_k t^{k-m} \end{cases} .$$

Continuing like this, we see that after exactly $[\frac{n}{m}]$ blowing-ups on the strict transform of C, one arrives at a strict transform with multiplicity strictly less than m. Therefore, *multiplicity can be dropped by doing blowing-ups.* In the same way, one can show that *the intersection number of two germs of curves embedded in a surface diminishes strictly after one blowing-up of their intersection point.* The theorem is a direct consequence of these two facts.

In the previous theorem it is not essential to suppose that the curve can be embedded, even locally, in a smooth surface. One has in general:

THEOREM 3.4. *Let (C, c) be a germ of reduced curve. Then it can be resolved by a finite sequence of blowing-ups of points. At each step of the algorithm, one simply blows-up the singular points of its strict transform.*

Let us sketch a more intrinsic proof (that is, which does not work with local coordinates) for the case when the germ is irreducible. Consider the normalization morphism $(\overline{C}, \overline{c}) \xrightarrow{\nu} (C, c)$. One has the inclusion of the corresponding local rings: $\mathcal{O}_{C,c} \subset \mathcal{O}_{\overline{C},\overline{c}}$. Denote by F their common field of fractions. Denote by $(C_k, c_k) \xrightarrow{\pi_k} (C, c)$, $k \geq 1$ the composition of the first k blowing-ups of the germ (C, c) or of its strict transforms. Denote by $\mathcal{O}_k$ the local ring of the germ (C_k, c_k). By Theorem 2.10, the normalization morphism ν can be factored through the morphism π_k, which shows that one has the sequence of inclusions:

$$\mathcal{O}_{C,c} \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_{\overline{C},\overline{c}} \subset F.$$

As ν is a finite morphism, one deduces that $\mathcal{O}_{\overline{C},\overline{c}}/\mathcal{O}_{C,c}$ is a finite dimensional $\mathbb{C}$-vector space, which shows that one has to arrive at an index $p \geq 1$ such that $\mathcal{O}_p = \mathcal{O}_{p+1}$. By Proposition 2.16, point 3), one sees that $\mathcal{O}_{p+1} = \mathcal{O}_p[\frac{y_2}{y_1}, ..., \frac{y_r}{y_1}]$, where $y_1, ..., y_r$ are generators of the maximal ideal of the local ring $\mathcal{O}_p$, chosen such that y_1 has the smallest multiplicity when we look at the generators as functions on the germ $(\overline{C}, \overline{c})$. The equality $\mathcal{O}_p = \mathcal{O}_{p+1}$ implies then that

$$\frac{y_2}{y_1}, ..., \frac{y_r}{y_1} \in \mathcal{O}_p$$

which shows that the maximal ideal $(y_1, ..., y_r)\mathcal{O}_p$ is principal, and generated by y_1. But this shows that the local ring $\mathcal{O}_p$ is regular, that is, the germ (C_p, c_p) is smooth. Again by Theorem 2.10, we deduce that π_p is the normalization morphism of (C, c). As a consequence, one has desingularized the germ (C, c) after p blowing-ups.

When (C, c) is not irreducible, the total ring of fractions of $\mathcal{O}_{C,c}$ is no more a field, but a direct product of fields. At some steps of the blowing-ups irreducible components may be separated, but the overall analysis remains the same.

For a careful proof written in the language of commutative algebra and for many details on abstract singularities of not necessarily plane curves, one can consult Castellanos & Campillo's book [**8**].

The next theorem is a generalization of Theorem 1.2. It shows how the finite dimension of the quotient $\overline{\mathcal{O}}_{C,c}/\mathcal{O}_{C,c}$ used in the previous proof appears in the computation of the genus of the normalization. By $\overline{\mathcal{O}}_{C,c}$ we denote the integral closure of $\mathcal{O}_{C,c}$ in its total ring of fractions, that is (see Theorem 2.12), the direct sum of the local rings of the normalization $\overline{C}$ at the preimages of the point c.

THEOREM 3.5. *Let C be a reduced algebraic curve inside $\mathbb{CP}^2$. Then the genus of its normalization $\overline{C}$ is equal to $\frac{(d-1)(d-2)}{2} - \sum \delta(C,c)$, where the sum is done over the singular points of C, and at such points $\delta(C,c) := \dim_{\mathbb{C}} \overline{\mathcal{O}}_{C,c}/\mathcal{O}_{C,c}$.*

For many more details about possibly singular plane curves, we recommend the leisurely introduction done in Brieskorn & Knörrer [**7**].

3.2. Embedded resolution of plane curves.

Let us consider again Theorem 3.3. If π denotes the composition of the blowing-ups which resolves the curve C, one knows by the definition of resolution that the strict transform C'_π is smooth. But the total transform $\pi^{-1}(C)$ has not necessarily only normal crossings. Nevertheless, by blowing-up more, one can arrive at an embedded resolution:

THEOREM 3.6. *Let $C \hookrightarrow S$ be a reduced curve embedded in a smooth complex analytic surface. Start from the identity morphism $S_0 = S \overset{\pi_0}{\to} S$. Then the following algorithm stops after a finite number of steps:*
   • *If $S_k \overset{\pi_k}{\to} S$ is given and the total transform $\pi_k^{-1}(C)$ has more complicated singularities than normal crossings inside the surface S_k, then blow-up each point of $\pi_k^{-1}(C)$ at which its irreducible components do not cross normally. The composition of π_k and of these blowing-ups is by definition $S_{k+1} \overset{\pi_{k+1}}{\to} S$.*
   • *If $\pi_k^{-1}(C)$ has normal crossings inside the surface S_k, then STOP.*

Moreover, the embedded resolution obtained in this way can be distinguished among all embedded resolutions by a *minimality property*, to be compared with the one stated in Theorem 2.24:

THEOREM 3.7. *Let $S_{min} \overset{\pi_{min}}{\longrightarrow} S$ be the embedded resolution of the curve C obtained by the algorithm 3.6. Then π_{min} is minimal among all the embedded resolutions of C, in the sense that any other resolution $S' \overset{\pi'}{\to} S$ factorizes as $S' \overset{\psi}{\to} S_{min} \overset{\pi_{min}}{\longrightarrow} S$, where ψ is a composition of blowing-ups of points.*

One can use the previous theorem as a way to analyze the structure of a singular point of a curve embedded in a smooth surface. More precisely, one can look at various aspects of the resolution π_{min} and of the sequence of blowing-ups leading to it:
   (a) The sequence of multiplicities of the strict transforms of the germ.
   (b) The dual graph of the total transform of (C,c), each vertex being decorated by the order of vanishing of the preimage of the maximal ideal $\mathcal{O}_{S,c}$ on the corresponding component.

(c) The dual graph of the total transform of (C, c), each vertex being decorated by the self-intersection number of the corresponding component.

(d) A graph which represents the strict transforms of the components of the curve (C, c) at each step of the process of blowing-ups, and whose edges are drawn in such a way as to remember if the strict transform passes or not through a smooth point of the exceptional locus.

It can be shown that all these encodings are equivalent. They are also equivalent with information readable on Newton-Puiseux parametrizations of the germ (C, c) (that is, parametrizations of the type (11)). Furthermore, all the encodings describe completely the *embedded topological type* of the germ. For the state of the art around 2004 on the relations with the embedded topology of germs, see Wall [**50**].

The comparison between information readable on Newton-Puiseux expansions and on sequences of blowing-ups (in fact quadratic transformations, as we explained before) seems to have been started by Max Noether in [**39**]. It was carefully explored by Enriques & Chisini [**12**], who introduced the viewpoint (d) of the previous list. A recent textbook emphasizing the usefulness of such graphs (called nowadays *Enriques diagrams*) in the study of plane curve singularities is Casas-Alvero [**9**].

A detailed comparison between the viewpoint (b) and information readable on Newton-Puiseux expansions was done in García Barroso [**15**].

The four viewpoints (a)-(d) are also compared in Campillo & Castellanos [**8**], where some of them are extended to arbitrary reduced germs, not necessarily embeddable in smooth surfaces.

A common aspect of all the comparisons is the use of expansions of rational numbers into continued fractions. In [**42**] we made a detailed study of the convex geometry lying behind the use of continued fractions, and of its applications via toric geometry to the study of singularities of curves and surfaces.

4. Resolution of surface singularities by Jung's method

4.1. Strategy.

Our aim in this section is to prove the following:

THEOREM 4.1. *Any reduced complex surface admits a resolution.*

Unlike for the case of plane curves, the normalization morphism is no longer always a resolution. Indeed, the world of *normal* surface singularities is huge and fascinating. For example, all isolated surface singularities of complete intersections are normal. But as the normalization morphism exists and it is an isomorphism over the smooth locus, one can reduce the proof of Theorem 4.1 to the task of proving it for normal surfaces, although this is not much of a simplification.

Many methods to prove Theorem 4.1 have been proposed, sometimes for special types of surfaces (projective surfaces in $\mathbb{P}^3$, arbitrary projective surfaces, algebraic surfaces, analytic surfaces, arithmetic surfaces, etc.) For the first approaches on the problem, one can consult Gario's papers [**16**], [**17**]. For the state of the art around 1935, see Zariski's book [**51**]. For the progresses made up to the year 2000, one can consult Hauser's index [**24**]. We recommend also Lipman's survey [**33**], Cutkosky's book [**11**] and Kollár's book [**30**]. In these last two references, one may find detailed proofs of the existence of resolution of singularities of complex algebraic varieties in arbitrary dimension.

We will now explain one of the methods, usually known as *Jung's method*. It is probably the most amenable one to computations by hand on examples defined by explicit equations. For special types of singularities, other methods could be more suitable. For example, if the surface admits a $\mathbb{C}^*$-action, then one knows how to describe equivariantly a resolution (see Orlik & Wagreich [**40**] or Müller [**35**]).

From a very general viewpoint, one could express the fundamental idea of this method by the injunction:

(12)
In order to represent an object as the image of a simpler one
first choose an image of the object
then simplify this image.

Given a reduced complex analytic surface, Jung's method consists in analysing its structure in a neighborhood of one of its singular points by projecting it to a plane and by considering an embedded resolution of the discriminant curve. It was introduced by Jung [**28**] as a way to uniformize locally a surface, and extended by Walker [**49**] in order to prove resolution of singularities of algebraic surfaces. This second paper was considered by Zariski [**51**, Chapter 1] to be the first complete proof of the resolution of singularities for surfaces. Hirzebruch [**27**] used again the method in order to prove the resolution of singularities for complex analytic surfaces (for an excellent summary of Hirzebruch's work on singularities we refer to Brieskorn [**6**]). Here we will explain Hirzebruch's proof (see also Laufer [**31**, Chapter II] and Lipman [**33**]).

Before starting our explanation, we would like to emphasize that Hirzebruch's motivation was to extend to the case of 2 complex variables Riemann's construction of a smooth (real) surface associated to a multivalued analytic function of one variable. We quote from the introduction of [**27**]:

> The «algebroid» function elements of a multivalued function $f(z_1, z_2)$ defined in a complex manifold (with two complex dimensions) can be easily associated to the points of a Hausdorff space of dimension 4, which covers part of M, and which we will call *the Riemann domain* of the function f. But this Riemann domain is not in general a topological manifold....

The main steps of Jung's method of resolution of reduced complex analytic surfaces are:

(A) Take a germ of the given surface and consider a finite morphism to a germ of smooth surface.

(B) Consider an embedded resolution of the discriminant curve of this morphism, and pull-back the initial germ by this resolution morphism.

(C) Normalize the surface obtained by this pull-back.

(D) Resolve explicitly the singularities of the new normal surface, by using the fact that they admit a finite morphism to a smooth surface whose discriminant curve has normal crossings.

(E) Glue together all the previous constructions to get a global resolution of the initial surface.

Let us explain the previous steps with more details. We emphasize that the surface S is not supposed to be normal.

(A) Let S be the given reduced surface. Consider one of its points $s \in S$, and the germ (S, s). Denote by $(S, s) \overset{\alpha}{\to} (R, r)$ a finite morphism, where (R, r) is a germ of smooth surface. Consider its discriminant locus $\Delta(\alpha) \subset R$. Three different situations may occur: either it is empty, either it is the point r, either it is a germ of curve.

(B) When $\Delta(\alpha)$ is empty, α is a local isomorphism, which shows that S is smooth at s.

When $\Delta(\alpha) = r$, the normalization morphism $\coprod (\overline{S}_i, \overline{s}_i) \overset{\chi}{\to} (S, s)$, whose total space is a multigerm, is also a resolution. Indeed, each restriction $(\overline{S}_i, \overline{s}_i) \overset{\alpha \circ \chi}{\longrightarrow} (R, r)$ is unramified outside r, and as $\pi_1(V \setminus r) = 0$ for each polydisc representative of R, one sees that a finite representative of $\alpha \circ \chi$ is a trivial covering over $V \setminus r$. The generalized Riemann extension theorem 2.7 implies that $\alpha \circ \chi$ is an isomorphism, which shows that each $(\overline{S}_i, \overline{s}_i)$ is smooth.

Let us suppose finally that $(\Delta(\alpha), r) \subset (R, r)$ is a germ of curve. Take an embedded resolution $(\tilde{R}, E) \overset{\psi}{\to} (R, r)$ of the germ, then the pull-back $\psi^*(\alpha)$ of the morphism α. One gets like this the following diagram of analytic morphisms:

(13)
$$
\begin{array}{ccc}
(\tilde{S}, F) & \overset{\alpha^*(\psi)}{\longrightarrow} & (S, s) \\
{\scriptstyle \psi^*(\alpha)} \downarrow & & \downarrow {\scriptstyle \alpha} \\
(\tilde{R}, E) & \overset{\psi}{\longrightarrow} & (R, r)
\end{array}
$$

As the mapping ψ is a proper modification, one deduces that $\alpha^*(\psi)$ is also a proper modification.

By construction, $\psi^{-1}(\Delta(\alpha))$ is a curve with normal crossings inside $\tilde{R}$ and $\psi^{-1}(\Delta(\alpha)) = E \cup \Delta(\alpha)'_\psi$, where $\Delta(\alpha)'_\psi$ denotes the strict transform of $\Delta(\alpha)$ by the modification ψ.

Note that the discriminant locus $\Delta(\psi^*(\alpha))$ of the morphism $\psi^*(\alpha)$ is contained in $\psi^{-1}(\Delta(\alpha))$. Therefore the germs of the surface $\tilde{S}$ at the points of $F = (\alpha^*(\psi))^{-1}(s)$ have a special property: they can be projected by finite morphisms (the localization of $\psi^*(\alpha)$) to a smooth surface, such that the discriminant locus has normal crossings. In many ways such germs are much more tractable than arbitrary germs of surfaces, that is why they received a special name:

DEFINITION 4.2. *Let $(\mathcal{X}, x)$ be a germ of reduced equidimensional complex surface. The germ $(\mathcal{X}, x)$ is called **quasi-ordinary** if there exists a finite morphism ϕ from $(\mathcal{X}, x)$ to a germ of smooth surface, whose discriminant locus is contained in a curve with normal crossings. Such a morphism ϕ is also called **quasi-ordinary**.*

An example of quasi-ordinary germ is the germ at the origin of Whitney's umbrella defined by equation (3). A quasi-ordinary morphism associated to it is the restriction of the canonical projection of the ambient space to the plane of coordinates (y, z). Please contemplate how this is visible in Figure 4, where the discriminant curve is the union of two lines, one being the projection of the singular locus and the other one being the apparent contour. At this point, recall also the comments about the drawing of surfaces from the beginning of Subsection 2.1.

The name "quasi-ordinary" was probably introduced in reference to the previously named "ordinary" singularities. In [**51**, page 18], Zariski says that a surface in the projective space has an ordinary singularity at a point if either it is locally

isomorphic at this point to a singular normal crossings divisor, or to an "ordinary cuspidal point", defined geometrically. In fact these last germs are isomorphic to the germ at the origin of Whitney's umbrella.

In many respects, quasi-ordinary germs are more amenable to study than arbitrary singularities, because one can extend to them by analogy many constructions done first for curves. For example, in what concerns resolution of singularities, González Pérez [20] gave two methods for finding embedded resolutions of quasi-ordinary germs of hypersufaces in arbitrary dimensions, by developing a method analogous to the one proposed for the case of curves by Goldin & Teissier [19]. As an introduction to quasi-ordinary singularities in arbitrary dimensions, we recommend Lipman's foundational work [34].

(C) Coming back to the diagram (13), let us normalize the surface $\tilde{S}$. Denote by $\overline{\tilde{S}} \xrightarrow{\nu} \tilde{S}$ the normalization morphism. One gets the diagram:

(14)
$$
\begin{array}{ccc}
(\overline{\tilde{S}}, G) & & \\
 & \searrow^{\overline{\psi}} & \\
\downarrow^{\nu} & & \\
\overline{\alpha} \quad (\tilde{S}, F) & \xrightarrow{\alpha^*(\psi)} & (S, s) \\
\downarrow^{\psi^*(\alpha)} & & \downarrow^{\alpha} \\
(\tilde{R}, E) & \xrightarrow{\psi} & (R, r)
\end{array}
$$

By definition, $\overline{\alpha} := \psi^*(\alpha) \circ \nu$ and $\overline{\psi} := \alpha^*(\psi) \circ \nu$. As normalization morphisms are finite modifications, we see that $\overline{\alpha}$ is finite and $\overline{\psi}$ is a modification. Moreover, the discriminant locus $\Delta(\overline{\alpha})$ of $\overline{\alpha}$ is contained in the discriminant locus of $\psi^*(\alpha)$, which shows that $\Delta(\overline{\alpha})$ has again normal crossings. Therefore, the singularities of $\overline{\tilde{S}}$ are still more special than those of $\tilde{S}$. They are the so-called *Hirzebruch-Jung singularities*:

DEFINITION 4.3. *A **Hirzebruch-Jung** germ (or singularity) of complex surface is a* normal *quasi-ordinary germ of surface.*

(D) We explain in the next subsection how one can use Definition 4.3 directly in order to give an explicit resolution of any Hirzebruch-Jung singularity. For the moment, please accept the fact that Hirzebruch-Jung singularities admit resolutions.

As the surface $\overline{\tilde{S}}$ has only this special kind of singularities, one sees that it can be resolved. Denote by $T \xrightarrow{\rho} \overline{\tilde{S}}$ a resolution of $\overline{\tilde{S}}$. Then $\alpha^*(\psi) \circ \nu \circ \rho$ is a modification of (S, s), being a composition of three modifications. Moreover, its source is smooth and it is an isomorphism over the smooth locus of S, which shows that it is a resolution of the germ (S, s).

(E) Once the germ (S, s) was fixed, the finite morphism α used to project it on a smooth surface was arbitrary. One can choose then a representative $U \xrightarrow{\alpha} V$ of the germ of morphism α which is a finite morphism of analytic surfaces, and such that the germ $(\Delta(\alpha), r)$ admits a closed representative in V which is smooth outside r. Then the composed morphism $\alpha^*(\psi) \circ \nu \circ \rho$ is by construction a normalization

of $U \setminus s$ (recall that s is not necessarily an isolated singular point of S). Indeed, its source is smooth, it is proper and bimeromorphic as a composition of proper and bimeromorphic morphisms, and its only possible fiber of positive dimension lies over the point r. We use then Theorem 2.10 to complete the argument.

This shows that the set of points of the surface (S, s) which do not have neighborhoods resolved by the normalization of S is *discrete*. If $\Sigma(S)$ is this set, choose neighborhoods $(U_s)_{s \in \Sigma(S)}$ which are pairwise disjoint and which are at the same time sources of finite morphisms as explained in the previous paragraph. Apply then the previous process for each one of them, getting like this resolution morphisms $T_s \to U_s$ for any $s \in \Sigma(S)$. If one considers also the normalization morphism of $S \setminus \Sigma(S)$, all these modifications of open sets which form a covering of the surface S *agree on overlaps*. This implies that they can be *glued* into a resolution of the entire surface S.

Examples of applications of this method are given in Laufer [**31**], Lê & Weber [**32**] and Némethi [**37**]. In this last reference are described the dual graphs of the resolutions obtained by this method for the germs of surfaces defined by equations of the form $z^n + f(x, y) = 0$, where f is reduced. The description is done in terms of the embedded resolution of the curve $f(x, y) = 0$ and the integer $n \geq 1$.

We see that the application of the injunction (12) depends heavily on the possibility to do an embedded resolution of curves contained in smooth surfaces. Nevertheless, this does not lead to an embedded resolution of a surface, only to an abstract one. If the method could be adapted to give an embedded resolution, then the same strategy would prove the resolution of 3-folds, and then one could try to get on these lines an inductive resolution in all dimensions. Untill now, this strategy was not succesfull (see Problem (1) in Section 5).

4.2. Resolution of Hirzebruch-Jung singularities.

The aim of this section is to explain a proof of the following theorem:

THEOREM 4.4. *Let (S, s) be a Hirzebruch-Jung singularity. Then it can be resolved. Moreover, the exceptional divisor of its minimal resolution has normal crossings, its components are smooth rational curves and its dual graph is a segment.*

The method we will present is a generalization of the one used to construct topologically a normalization morphism for an irreducible germ of curve (see the explanations which precede diagram (10)). We saw there that one could work in convenient local coordinates with a morphism defined by $x_1 = t^m$. Similarly, one can resolve Hirzebruch-Jung surface singularities using only morphisms defined by *monomials* in two variables. There is a branch of algebraic geometry which studies intrinsically such morphisms, called *toric geometry*. In what follows we will explain how it appears in our context. For an introduction to toric geometry, one may consult Fulton [**14**].

Let (S, s) be a Hirzebruch-Jung singularity and $(S, s) \overset{\alpha}{\to} (R, r)$ be a finite morphism whose discriminant locus has normal crossings. Choose local coordinates in the neighborhood of the point $r \in R$ such that the discriminant curve is contained in the union of the coordinate axis. Therefore, from now on we suppose that $R = \mathbb{C}^2$ and that α is unramified over $(\mathbb{C}^*)^2$.

Choose a finite representative $U \overset{\alpha}{\to} V$ where V is a polycylinder in the coordinates (x, y). Denote by V^* the complement of the axis of coordinates and by U^*

the preimage $\alpha^{-1}(V^*)$. Then the restricted morphism $U^* \xrightarrow{\alpha} V^*$ is a finite (unramified) covering with connected source (because (S, s) was supposed to be normal). Consider the associated morphism of fundamental groups $\pi_1(U^*) \xrightarrow{\alpha_*} \pi_1(V^*)$. As V was chosen to be a polycylinder, V^* is the product of two pointed discs, which shows that $N := \pi_1(V^*)$ is a free abelian group of rank 2. Therefore all its finite index subgroups are also free abelian of rank 2. In particular this holds for $N(\alpha) := \alpha_*(\pi_1(U^*))$. Notice that the abelianity of those groups implies that we do not have to worry about base points.

Let us look at the multiplicative semigroup of Laurent monomials $x^a y^b$ which are regular in a neighborhood of the origin of $\mathbb{C}^2$, which means that $a \geq 0, b \geq 0$. One can think about them as functions $(\mathbb{C}^*)^2 \to \mathbb{C}^*$ and restrict them to the loops which represent the elements of $N(\alpha)$. One associates like this to each pair (a, b) a map from an oriented circle to $\mathbb{C}^*$, map whose degree (its linking coefficient with the origin) is a well-defined integer. Consider the elements of $N(\alpha)$ whose associated degree is *non-negative*. They form a sub-semigroup of $(N(\alpha), +)$. Denote by σ the closed convex cone generated by its elements inside the associated real vector space $N(\alpha)_{\mathbb{R}} := N(\alpha) \otimes_{\mathbb{Z}} \mathbb{R}$. As $N(\alpha)$ has finite index in N, one has a canonical identification $N(\alpha)_{\mathbb{R}} = N_{\mathbb{R}}$. Therefore, σ may also be seen as the cone in $N_{\mathbb{R}}$. It is precisely the first quadrant !

The pair $(N(\alpha), \sigma)$ determines a two-dimensional normal affine toric surface $\mathcal{X}(N(\alpha), \sigma)$ and the inclusion $N(\alpha) \hookrightarrow N$ induces a canonical toric morphism:

$$\mathcal{X}(N(\alpha), \sigma) \xrightarrow{\gamma_{N:N(\alpha)}} \mathbb{C}^2 = \mathcal{X}(N, \sigma).$$

One sees then that the morphism of fundamental groups induced over $(\mathbb{C}^*)^2$ has $N(\alpha)$ as its image. From this, one concludes that $\gamma_{N:N(\alpha)}$ can be lifted over $(\mathbb{C}^*)^2$ to a morphism with target space (S, s). By construction, this lift is bounded in a neighborhood of the special point $\underline{0}$ of $\mathcal{X}(N(\alpha), \sigma)$ (its unique closed orbit under the torus action). As this last variety is normal, one deduces that it can be extended to a morphism μ defined on a neighborhood of $\underline{0} \in \mathcal{X}(N(\alpha), \sigma)$. Thus, one gets a diagram:

(15)
$$(\mathcal{X}(N(\alpha), \sigma), \underline{0}) \xrightarrow{\mu} (S, s)$$
$$\gamma_{N:N(\alpha)} \searrow \qquad \downarrow \alpha$$
$$(\mathbb{C}^2, 0)$$

By construction, μ is a finite morphism which is an isomorphism over $(\mathbb{C}^*)^2$. As both its source and target are normal, one deduces that μ is an isomorphism. This shows that a Hirzebruch-Jung singularity is analytically isomorphic to a germ of toric surface.

But the singularities of normal toric surfaces admit explicit minimal resolutions, which can be deduced from the geometry of the pair $(N(\alpha), \sigma)$. They verify the properties listed in Theorem 4.4.

When Hirzebruch did his work [**27**], toric geometry did not exist. Nevertheless, he gave an explicit resolution of Hirzebruch-Jung singularities in a way which nowadays can be recognized to be toric, by gluing affine planes through monomial maps. We recommend Brieskorn's article [**6**] for comments on this approach, as well as on other contributions by Hirzebruch to singularity theory.

$$\{x_1=0\}' \quad -b_1 \quad -b_2 \quad \cdots\cdots \quad -b_r \quad \{x_2=0\}'$$

FIGURE 7. Total transform of $x_1 x_2 = 0$

It was one of our contributions to the study of Hirzebruch-Jung singularities to construct the affine toric surface $\mathcal{X}(N(\alpha), \sigma)$. Our motivation was to be able to compute the normalizations of explicit quasi-ordinary singularities. We showed that in arbitrary dimensions the normal quasi-ordinary singularities could be characterized as the germs of normal affine toric varieties defined by a simplicial cone. Moreover, we gave an algorithm of normalization for hypersurface quasi-ordinary germs in arbitrary dimensions (see [**41**]).

As a particular case of this normalization algorithm, one gets the following lemma, which is needed when one applies steps (C) and (D) described in Section 4.1 to concrete examples (as the ones presented in Laufer [**31**], Lê & Weber [**32**] and Némethi [**37**]):

LEMMA 4.5. *Let* $(S, s) \hookrightarrow \mathbb{C}^3_{x_1, x_2, y}$ *be the quasi-ordinary irreducible singularity corresponding to the algebraic function with two variables* $y := x_1^{\frac{p_1}{q_1}} x_2^{\frac{p_2}{q_2}}$, *where* $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$ *and* $p_1, q_1, p_2, q_2 \in \mathbb{N}^*$. *Denote* $d := \gcd(q_1, q_2)$, $j_1 = \frac{q_1}{d}$ *and let* $k_1 \in \{0, 1, ..., q_1 - 1\}$ *be the unique number in this set which satisfies the congruence equation* $k_1 p_1 + j_1 p_2 \equiv 0 \pmod{q_1}$. *Denote also:*

$$q_1' := \frac{q_1}{\gcd(q_1, k_1)}, \ k_1' := \frac{k_1}{\gcd(q_1, k_1)}.$$

Consider $\frac{q_1'}{k_1'} = b_1 - 1/(b_2 - 1/(\cdots - b_r))$, *the decomposition of* $\frac{q_1'}{k_1'}$ *as a Hirzebruch-Jung continued fraction (that is,* $b_i \geq 2$, $\forall i \in \{1, ..., r\}$*). Then:*

(1) (S, s) *has the same normalization over* $\mathbb{C}^2_{x_1, x_2}$ *as the surface* (S', s) *defined by the algebraic function*

$$x_1^{\frac{1}{q_1'}} x_2^{\frac{q_1' - k_1'}{q_1'}}.$$

(2) *The total transform of the function* $(x_1 x_2)|_{(S,s)}$ *by the minimal resolution of* (S, s) *has a dual graph as drawn in Figure 7, where* $\{x_i = 0\}'$ *denotes the strict transform of* $x_i = 0$.

The important point to notice is the way the strict transforms of the germs of curves defined by $x_1 = 0$ and $x_2 = 0$ intersect the exceptional divisor of the minimal resolution of (S, s). Be careful not to permute x_1 and x_2!

We conclude this section with a theorem which characterizes Hirzebruch-Jung surface singularities from many different viewpoints. References can be found in [**3**], [**41**] and [**42**].

THEOREM 4.6. *Hirzebruch-Jung surface singularities can be characterized* among normal singularities *by the following equivalent properties:*

(1) *They are quasi-ordinary.*
(2) *They are singularities of toric surfaces.*
(3) *They are cyclic quotient singularities.*

(4) *The exceptional divisor of their minimal resolution has normal crossings,
    its components are smooth rational curves and its dual graph is a segment.*
(5) *Their link is a lens space.*
(6) *The fundamental group of their link is abelian.*

5. Open problems

(1) Adapt Jung's method to get *embedded* resolution of germs of surfaces in
    $\mathbb{C}^3$.
(2) Use Jung's method to get obstructions on the topology of germs of surfaces
    with isolated singularities in $\mathbb{C}^3$.
(3) If (S, s) is a germ of a normal surface and $(S, s) \xrightarrow{\alpha} (\mathbb{C}^2, 0)$ is a finite
    morphism, one gets by Jung's method an associated resolution of (S, s).
    In general, one gets more components of the exceptional divisor than
    in the minimal resolution. Denote by $\mathrm{md}(S, s)$ the minimum number of
    supplementary components when one varies α, the germ (S, s) being fixed
    ('*md*' being the initials of '*minimal difference*'). Is $\mathrm{md}(S, s)$ bounded
    from above when one varies (S, s) among the normal germs with fixed
    topology ?

    By construction, $\mathrm{md}(S, s)$ attains a minimal value when one varies
    (S, s) like this. Compute it in terms of the weighted dual graph of the
    minimal good resolution of (S, s) (which encodes the topology of (S, s),
    as ensured by a theorem of Neumann).

References

[1] Abhyankar, S.S. *Historical ramblings in algebraic geometry and related algebra.* Amer. Math.
    Monthly **83** (1976), no. 6, 409–448.
[2] Adkins, W.A., Andreotti, A., Leahy, J.V. *Weakly normal complex spaces.* Accademia
    Nazionale dei Lincei, 1981.
[3] Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A. *Compact complex surfaces.* Second
    enlarged edition, Springer, 2004.
[4] Bădescu, L. *Algebraic Surfaces.* Springer, 2001.
[5] Boy, W. *Über die curvatura integra und die Topologie geschlossener Flächen.* Math. Ann. **57**
    (1903) , 151-184.
[6] Brieskorn, E. *Singularities in the work of Friedrich Hirzebruch.* Surveys in Differential Ge-
    ometry **VII** (2000), 17-60.
[7] Brieskorn, E., Knörrer, H. *Plane algebraic curves.* Birkhäuser, 1986.
[8] Campillo, A., Castellanos, J. *Curve singularities. An algebraic and geometric approach.* Her-
    mann, 2005.
[9] Casas-Alvero, E. *Singularities of plane curves.* London Mathematical Society Lecture Note
    Series **276**, Cambridge U.P., 2000.
[10] Castelnuovo, G., Enriques, F., Severi, F. *Max Noether.* Math.Annalen **93** (1925), 161-181.
[11] Cutkosky, D. *Resolution of singularities.* American Mathematical Society, 2004.
[12] Enriques, F., Chisini, O. *Lezioni sulla teoria geometrica delle equazioni e delle funzioni
     algebriche.* vol. II, N. Zanichelli, Bologna, 1918.
[13] Fischer, G. *Complex analytic geometry.* LNM **538**, Springer-Verlag, 1976.
[14] Fulton, W. *Introduction to toric varieties.* Annals of Math. Studies **131**, Princeton Univ.
     Press, 1993.

[15] García Barroso, E. R. *Invariants des singularités de courbes planes et courbure des fibres de Milnor.* Tesis doctoral, Univ. de La Laguna, Tenerife (Spain), 1996.

[16] Gario, P. *Resolution of singularities of surfaces by P. Del Pezzo. A mathematical controversy with C. Segre.* Archive for History of Exact Sciences **40** (1989), 247-274.

[17] Gario, P. *Singolarità e Geometria sopra una Superficie nella Corrispondenza di C. Segre a G. Castelnuovo.* Archive for History of Exact Sciences **43** (1991), 145-188.

[18] Gelfand, I. M., Kapranov, M. M., Zelevinsky, A. V. *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Birkhäuser, 1994.

[19] Goldin, R., Teissier, B. *Resolving plane branch singularities with one toric morphism.* In *Resolution of singularities, a research textbook in tribute to Oscar Zariski.* Birkhäuser, Progress in Math. **181**, 2000, 315-340.

[20] González Pérez, P.D. *Toric embedded resolutions of quasi-ordinary hypersurface singularities.* Ann. Inst. Fourier, Grenoble **53**, 6 (2003), 1819-1881.

[21] Grauert, H. *Über Modifikationen und exzeptionnelle analytische Mengen.* Math. Ann. **146** (1962), 331-368.

[22] Greco, S. *Normal varieties.* Academic Press, 1978.

[23] Hartshorne, R. *Algebraic geometry.* Graduate Texts in Mathematics, No. **52**. Springer-Verlag, 1977.

[24] Hauser, H. *Resolution of singularities 1860-1999.* Progress in Maths. **181** (2000), 5-36.

[25] Hilbert, D., Cohn-Vossen, S. *Geometry and the imagination.* Chelsea, 1990. Translation of the first German edition, 1932.

[26] Hironaka, H. *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II.* Annals of Maths. **79**, 1 (1964), 109-326.

[27] Hirzebruch, F. *Über vierdimensionale Riemannsche Flächen Mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen.* Math. Ann. **126**, 1-22 (1953).

[28] Jung, H.W.E. *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen x,y in der Umgebung einer Stelle x=a, y=b.* J. Reine Angew. Math. **133** (1908), 289-314.

[29] Kaup, L., Kaup, B. *Holomorphic functions of several variables.* Walter de Gruyter, 1983.

[30] Kollár, J. *Lectures on resolution of singularities.* Annals of Mathematics Studies, 166. Princeton University Press, 2007.

[31] Laufer, H.B. *Normal two-dimensional singularities.* Princeton Univ. Press, 1971.

[32] Lê, D.T., Weber, C. *Résoudre est un jeu d'enfants.* Revista del seminario Iberoamericano de Matemáticas, **3** I (2000), 3-23.

[33] Lipman, J. *Introduction to resolution of singularities.* Proceedings of Symposia in Pure Mathematics **29** (1975), 187- 230.

[34] Lipman, J. *Topological Invariants of Quasi-ordinary Singularities.* Memoirs Amer. Math. Soc. **388**, 1988.

[35] Müller, G. *Resolution of weighted homogeneous surface singularities.* Progress in Mathematics **181**, 507-517, Birkhäuser, 2000.

[36] Mumford, D. *The topology of normal singularities of an algebraic surface and a criterion for simplicity.* Publ. Math. IHES **9** (1961), 5-22.

[37] Némethi, A. *Five lectures on normal surface singularities.* Bolyai Soc. Mathematical Studies **8**, Low Dimensional Topology, 269-351, 1999.

[38] Noether, M. *Ueber die singulären Werthsysteme einer algebraischen Function und die singulären Punkte einer algebraischen Curve.* Mathematische Annalen **9** (1875), 166-182.

[39] Noether, M. *Les combinaisons caractéristiques dans la transformation d'un point singulier.* Rend. Circ. Mat. Palermo **IV** (1890), 89-108, 300-301.

[40] Orlik, P., Wagreich, P. *Isolated singularities of algebraic surfaces with $\mathbb{C}^*$-action.* Ann. of Math. (2) **93** (1971), 205-228.

[41] Popescu-Pampu, P. *On higher dimensional Hirzebruch-Jung singularities.* Rev. Mat. Complut. **18** (2005), no.1, 209-232.

[42] Popescu-Pampu, P. *The geometry of continued fractions and the topology of surface singularities.* Advanced Studies in Pure Mathematics, Kinokuniya, Tokyo, 2007, 119-195.

[43] Riemann, B. *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse.* Inauguraldissertation, Göttingen (1851), 1-48.

[44] Riemann, B. *Theorie der Abelschen Funktionen.* Journal für reine und angew. Math. **54**, 14, 115-155 (1857).

[45] Sullivan, D. *Combinatorial invariants of analytic spaces.* In *Proceedings Liverpool Singularities Symposium I*, 165-168, LNM **192**, Springer, 1971.

[46] Teissier, B. *The hunting of invariants in the geometry of discriminants.* In *Real and complex singularities.* Per Holm ed., 565-678, Sijthoff and Noordhoff (1977).

[47] Teissier, B. *Introduction to curve singularities.* In *Singularity theory.* D.T. Lê, K. Saito, B.Teissier eds. World Scientific, 1995.

[48] P. Du Val. *On absolute and non-absolute singularities of algebraic surfaces.* Revue de la Faculté des Sciences de l'Univ. d'Istanbul (A) **91** (1944), 159-215.

[49] Walker, R.J. *Reduction of the Singularities of an Algebraic Surface.* Annals of Maths. **36**, 2 (1935), 336-365.

[50] Wall, C.T.C. *Singular points of plane curves.* Cambridge UP, 2004.

[51] Zariski, O. *Algebraic Surfaces.* Springer-Verlag, 1971. Reedition of the first one appeared in 1935. With appendices by S.S. Abhyankar, J. Lipman, D. Mumford.

[52] Zariski, O. *Some results in the arithmetic theory of algebraic varieties.* Amer. J. Math. **61** (1939), 249-294. Also in *Collected Papers I*, H. Hironaka and D. Mumford eds., The MIT Press, 73-118, 1971.

[53] Zariski, O. *Normal varieties and birational correspondences.* Bull. Amer. Math. Soc. **48** (1942), 402-413. Also in *Collected Papers I*, H. Hironaka and D. Mumford eds., The MIT Press, 1971.

[54] Zariski, O. *Foundations of a general theory of birational correspondences.* Trans. Amer. Math. Soc. **53** (1943), 490-542. Also in *Collected Papers I*, H. Hironaka and D. Mumford eds., The MIT Press, 1971.

[55] *** *Several complex variables VII.* H. Grauert, Th. Peternelll, R. Remmert eds., Encyclopaedia of Mathematical Sciences **74**, Springer-Verlag, 1994.

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