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The geometry of continued fractions and the topology of surface singularities

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Abstract.

We survey the use of continued fraction expansions in the algebraical and topological study of complex analytic singularities. We also prove new results, firstly concerning a geometric duality with respect to a lattice between plane supplementary cones and secondly concerning the existence of a canonical plumbing structure on the *abstract boundaries* (also called *links*) of normal surface singularities. The duality between supplementary cones gives in particular a geometric interpretation of a duality discovered by Hirzebruch between the continued fraction expansions of two numbers $\lambda > 1$ and $\lambda/(\lambda - 1)$.

Contents

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§1. Introduction

Continued fraction expansions appear naturally when one resolves germs of plane curves by sequences of plane blowing-ups, or Hirzebruch-Jung (that is, cyclic quotient) surface singularities by toric modifications.

They also appear when one passes from the natural plumbing decomposition of the abstract boundary of a normal surface singularity to its minimal JSJ decomposition. In this case it is very important to keep track of natural orientations. In general, as was shown by Neumann [57], if one changes the orientation of the boundary, the resulting 3-manifold is no more orientation-preserving diffeomorphic to the boundary of an isolated surface singularity. The only exceptions are *Hirzebruch-Jung singularities* and *cusp-singularities*. This last class of singularities got its name from its appearance in Hirzebruch's work [37] as germs at the compactified cusps of Hilbert modular surfaces. For both classes of singularities, one gets an involution on the set of analytical isomorphism types of the singularities in the class, by changing the orientation of the boundary. From the viewpoint of computations, Hirzebruch saw that both types of singularities have structures which can be encoded in continued fraction expansions of positive integers, and that the previous involution manifests itself in a duality between such expansions.

In the computations with continued fractions alluded to before, there appear in fact two kinds of continued fraction expansions. Some are constructed using only additions - we call them in the sequel *Euclidean continued fractions* - and the others using only subtractions - we call them *Hirzebruch-Jung continued fractions*. There is a simple formula, also attributed to Hirzebruch, which allows to pass from one type of continued fraction expansion of a number to the other one. Both types of expansions have geometric interpretations in terms of polygonal lines $P(\sigma)$. If (L, σ) is a pair consisting of a 2-dimensional lattice L and a strictly convex cone σ in the associated real vector space, $P(\sigma)$ denotes the boundary of the convex hull of the set of lattice points situated inside σ and different from the origin.

For Euclidean continued fractions this interpretation is attributed to Klein [45], while for the Hirzebruch-Jung ones it is attributed to Cohn [12].

It is natural to try to understand how both geometric interpretations fit together. By superimposing the corresponding drawings, we were led to consider *two supplementary cones* in a real plane, in the presence of a lattice. By *supplementary cones* we mean two closed strictly convex cones which have a common edge and whose union is a half-plane.

Playing with examples made us understand that the algebraic duality between continued fractions alluded to before has as geometric counterpart a duality between two supplementary cones in the plane with respect to a lattice. This duality is easiest to express in the case where the edges of the cones are irrational:

Suppose that the edges of the supplementary cones σ and σ' are irrational. Then the edges of each polygonal line $P(\sigma)$ and $P(\sigma')$ correspond bijectively in a natural way to the vertices of the other one.

When at least one of the edges is rational, the correspondence is slightly more complicated (there is a defect of bijectivity near the intersection points of the polygonal lines with the edges of the cones), as explained in Proposition 5.3. In this duality, points correspond to lines and conversely, as in the classical polarity relation between points and lines with respect to a conic. But the duality relation described in this paper is more elementary, in the sense that it uses only parallel transport in the plane. For this reason it can be explained very simply by drawing on a piece of cross-ruled paper.

The duality between supplementary cones gives a simple way to think about the relation between the pair (L, σ) and its dual pair $(\check{L}, \check{\sigma})$ and in particular about the relations between various invariants of toric surfaces (see Section 6). Indeed (see Proposition 5.11):

The supplementary cone of σ is canonically isomorphic over the integers with the dual cone $\check{\sigma}$, once an orientation of L is fixed.

As stated at the beginning of the introduction, computations with continued fractions appear also when one passes from the canonical plumbing structure on the boundary of a normal surface singularity to its minimal JSJ structure. Using this, Neumann [57] showed that the topological type of the minimal good resolution of the germ is determined by the topological type of the link. In fact all continued fractions appearing in Neumann's work are the algebraic counterpart of pairs (L, σ) canonically determined by the topology of the boundary. Using this remark, we prove the stronger statement (see Theorem 9.7):

The plumbing structure on the boundary of a normal surface singularity associated to the minimal normal crossings resolution is determined up to isotopy by the oriented ambient manifold. In particular, it is invariant up to isotopy under the group of orientation-preserving self-diffeomorphisms of the boundary.

In order to prove this theorem we have to treat separately the boundaries of Hirzebruch-Jung and cusp singularities. In both cases, we show that the oriented boundary determines naturally a pair (L, σ) as before. If one changes the orientation of the boundary, one gets a supplementary cone. In this way, the involution defined before on both sets of singularities is a manifestation of the geometric duality between supplementary cones (see Propositions 9.3 and 9.6).

For us, the moral of the story we tell in this paper is the following one:

If one meets computations with either Euclidean or Hirzebruch-Jung continued fractions in a geometrical problem, it means that somewhere behind is present a natural 2-dimensional lattice L and a couple of lines in the associated real vector space. One has first to choose one of the two pairs of opposite cones determined by the four lines and secondly an ordering of the edges of those cones. These choices may be dictated *by choices of orientations of the manifolds which led to the construction of the lattice and the cones. So, in order to think geometrically at the computations with continued fractions, recognize the lattice, the lines and the orientation choices.*

Let us outline now the content of the paper.

Someone who is interested only in the algebraic relations between the Euclidean and the Hirzebruch-Jung continued fraction expansions of a number can consult only Section 2. If one is also interested in their geometric interpretation, one can read Sections 3 and 4.

In Section 5 we prove geometrically the relations between the two kinds of continued fractions using the duality between supplementary cones described before. We introduce also a new kind of graphical representation which we call the *zigzag diagram*, allowing to visualize at the same time the algebra and the geometry of the continued fraction expansions of a number.

In Section 6 we give applications of zigzag diagrams to the algebraic description of special curve and surface singularities, defined using toric geometry.

Sections 8 and 9 are dedicated to the study of topological aspects of the links of normal surface singularities, after having recalled in Section 7 general facts about Seifert, graph, plumbing and JSJ-structures on 3-manifolds.

We think that the new results of the paper are Proposition 5.3, Theorem 9.1 and Theorem 9.7, as well as the very easy Proposition 5.11, which is nevertheless essential in order to understand the relation between dual cones in terms of parallelism, using Proposition 5.3.

We wrote this paper having in mind as a potential reader a graduate student who wants to be initiated either to the algebra of surface singularities or to their topology. That is why we tried to communicate basic intuitions, often referring to the references for complete proofs.

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§2. Algebraic comparison of Euclidean and Hirzebruch-Jung continued fractions

Definition 2.1. If $x_1, \ldots, x_n$ are variables, we consider two kinds of continued fractions associated to them:

$$[x_1, \dots, x_n]^+ := x_1 + \cfrac{1}{x_2 + \cfrac{1}{\cdots + \cfrac{1}{x_n}}}$$

$$[x_1, \dots, x_n]^- := x_1 - \cfrac{1}{x_2 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}}$$

We call $[x_1, \dots, x_n]^+$ a **Euclidean continued fraction** (abbreviated **E-continued fraction**) and $[x_1, \dots, x_n]^-$ a **Hirzebruch-Jung continued fraction** (abbreviated **HJ-continued fraction**).

The first name is motivated by the fact that E-continued fractions are tightly related to the Euclidean algorithm: if one applies this algorithm to a couple of positive integers (a, b) and the successive quotients are $q_1, \dots, q_n$, then $a/b = [q_1, \dots, q_n]^+$. See Hardy & Wright [32], Davenport [15] for an introduction to their arithmetics and Fowler [22] for the relation with the Greek theories of proportions. An extended bibliography on their applications can be found in Brezinski [7] and Shallit [72].

The second name is motivated by the fact that HJ-continued fractions appear naturally in the Hirzebruch-Jung method of resolution of singularities, originating in Jung [42] and Hirzebruch [35], as explained after Definition 6.4 below.

Define two sequences $(Z^{\pm}(x_1, \dots, x_n))_{n \geq 1}$ of polynomials with integer coefficients, by the initial data

$$Z^{\pm}(\emptyset) = 1, \ Z^{\pm}(x) = x$$

and the recurrence relations:

$$\begin{aligned} Z^{\pm}&(x_1, \dots, x_n) \\ &= x_1 Z^{\pm}(x_2, \dots, x_n) \pm Z^{\pm}(x_3, \dots, x_n), \quad \forall n \geq 2. \end{aligned} \tag{1}$$

Then one proves immediately by induction on n the following equality of rational fractions:

$$[x_1, \dots, x_n]^{\pm} = \frac{Z^{\pm}(x_1, \dots, x_n)}{Z^{\pm}(x_2, \dots, x_n)}, \quad \forall n \geq 1. \tag{2}$$

Also by induction on n, one proves the following twin of relation (1):

$$(3) \quad \begin{aligned} &Z^{\pm}(x_1, \dots, x_n) \\ &\quad = Z^{\pm}(x_1, \dots, x_{n-1})x_n \pm Z^{\pm}(x_1, \dots, x_{n-2}), \quad \forall n \geq 2. \end{aligned}$$

which, combined with (1), proves the following symmetry property:

$$(4) \qquad Z^{\pm}(x_1, \dots, x_n) = Z^{\pm}(x_n, \dots, x_1), \quad \forall n \geq 1.$$

If $(y_1, \dots, y_k)$ is a finite sequence of numbers or variables and $m \in \mathbf{N} \cup \{+\infty\}$, we denote by

$$(y_1, \dots, y_k)^m$$

the sequence obtained by repeating m times the sequence $(y_1, \dots, y_k)$. By convention, when $m = 0$, the result is the empty sequence.

Each number $\lambda \in \mathbf{R}$ can be expanded as (possibly infinite) Euclidean and Hirzebruch-Jung continued fractions:

$$\lambda = [a_1, a_2, \dots]^+ = [\alpha_1, \alpha_2, \dots]^-$$

with the conditions:

$$(5) \qquad a_1 \in \mathbf{Z},\ a_n \in \mathbf{N} - \{0\}, \quad \forall n \geq 1$$

$$(6) \qquad \alpha_1 \in \mathbf{Z},\ \alpha_n \in \mathbf{N} - \{0, 1\}, \quad \forall n \geq 1$$

Of course, we consider only indices n effectively present. For an infinite number of terms, these conditions ensure the existence of the limits $[a_1, a_2, \dots]^+ := \lim_{n \to +\infty}[a_1, \dots, a_n]^+$ and $[\alpha_1, \alpha_2, \dots]^- := \lim_{n \to +\infty}[\alpha_1, \dots, \alpha_n]^-$.

Any sequence $(a_n)_{n \geq 1}$ which verifies the restrictions (5) can appear and the only ambiguity in the expansion of a number as a E-continued fraction comes from the identity:

$$(7) \qquad [a_1, \dots, a_n, 1]^+ = [a_1, \dots, a_{n-1}, a_n + 1]^+$$

We deduce that any real number $\lambda \neq 1$ admits a unique expansion as a E-continued fraction such that condition (5) is satisfied and in the case that the sequence $(a_n)_n$ is finite, its last term is different from 1. When we speak in the sequel about *the E-continued fraction expansion* of a number $\lambda \neq 1$, it will be about this one. By analogy with the vocabulary of the Euclidean algorithm, we say that the numbers $(a_n)_{n \geq 1}$ are the *E-partial quotients* of λ.

Similarly, any sequence $(\alpha_n)_{n\geq 1}$ which verifies the restrictions (6) can appear and the only ambiguity in the expansion of a number as a HJ-continued fraction comes from the identity:

$$[\alpha_1, \ldots, \alpha_n, (2)^{\infty}]^- = [\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1]^- \tag{8}$$

We see that any real number λ admits a unique expansion as a HJ-continued fraction such that condition (6) is satisfied and the sequence $(\alpha_n)_n$ is not infinite and ultimately constant equal to 2. When we speak in the sequel about *the HJ-continued fraction expansion* of a number λ, it will be about this one. We call the numbers $(\alpha_n)_{n\geq 1}$ the *HJ-partial quotients* of λ.

The following lemma (see Hirzebruch [37, page 257]) can be easily proved by induction on the integer $b \geq 1$.

Lemma 2.2. *If $a \in \mathbf{Z}$, $b \in \mathbf{N} - \{0\}$ and x is a variable, then*:

$$[a, b, x]^+ = [a+1, (2)^{b-1}, x+1]^-$$

Using this lemma one sees how to pass from the E-continued fraction expansion of a real number λ to its HJ-continued fraction expansion:

Proposition 2.3. *If $(a_n)_{n\geq 1}$ is a (finite or infinite) sequence of positive integers, then*:

$$\begin{aligned}
&[a_1, \ldots, a_{2n}]^+ \\
&\qquad = [a_1+1, (2)^{a_2-1}, a_3+2, (2)^{a_4-1}, \ldots, (2)^{a_{2n}-1}]^- \\
&[a_1, \ldots, a_{2n+1}]^+ \\
&\qquad = [a_1+1, (2)^{a_2-1}, a_3+2, (2)^{a_4-1}, \ldots, (2)^{a_{2n}-1}, a_{2n+1}+1]^- \\
&[a_1, a_2, a_3, a_4, \ldots]^+ \\
&\qquad = [a_1+1, (2)^{a_2-1}, a_3+2, (2)^{a_4-1}, a_5+2, (2)^{a_6-1}, \ldots]^-
\end{aligned}$$

(*recall that, by convention, $(2)^0$ denotes the empty sequence*).

Example 2.4. $11/7 = [(1)^3, 3]^+ = [2, 3, (2)^2]^-$.

Notice that this procedure can be inverted. In particular, an immediate consequence of the previous proposition is that a number has bounded E-partial quotients if and only if it has bounded HJ-partial quotients. Similarly, it has ultimately periodic E-continued fraction (which happens if and only if it is a quadratic number, see Davenport [15]) if and only if it has ultimately periodic HJ-continued fraction. In this case, Proposition 2.3 explains how to pass from its E-period to its HJ-period.

The continued fraction expansions of two numbers which differ by an integer are related in an evident and simple way. For this reason, from now on we restrict our attention to real numbers $\lambda > 1$. The map

$$\lambda \longrightarrow \frac{\lambda}{\lambda - 1} \tag{9}$$

is an involution of the interval $(1, +\infty)$ on itself. The E-continued fraction expansions of the numbers in the same orbit of this involution are related in a very simple way:

Lemma 2.5. *If $\lambda \in (1, +\infty)$ and $\lambda = [a_1, a_2, \dots]^+$ is its expansion as a (finite or infinite) continued fraction, then*:

$$\frac{\lambda}{\lambda - 1} = \begin{cases} [1 + a_2, a_3, a_4, \dots]^+, & \textit{if } a_1 = 1, \\ [1, a_1 - 1, a_2, a_3, \dots]^+, & \textit{if } a_1 \geq 2 \end{cases}$$

The proof is immediate, once one notices that $\lambda/(\lambda-1) = [1, \lambda-1]^+$. Notice also that the involutivity of the map (9) shows that the first equality in the previous lemma is equivalent to the second one.

Example 2.6. If $\lambda = 11/7 = [(1)^3, 3]^+$, then $11/4 = \lambda/(\lambda - 1) = [2, 1, 3]^+$.

By combining Proposition 2.3 and Lemma 2.5, we get the following relation between the HJ-continued fraction expansions of the numbers in the same orbit of the involution (9):

Proposition 2.7. *If $\lambda \in \mathbf{R}$ is greater than 1 and*

$$\lambda = [(2)^{m_1}, n_1 + 3, (2)^{m_2}, n_2 + 3, \dots]^-$$

is its expression as a (finite or infinite) continued fraction, with $m_i, n_i \in \mathbf{N}$, $\forall i \geq 1$, then:

$$\frac{\lambda}{\lambda - 1} = [m_1 + 2, (2)^{n_1}, m_2 + 3, (2)^{n_2}, m_3 + 3, \dots]^-$$

For λ rational, this was proved in a different way by Neumann [57, Lemma 7.2]. It reads then:

$$\lambda = [(2)^{m_1}, n_1 + 3, (2)^{m_2}, \dots, n_s + 3, (2)^{m_{s+1}}]^- \Longrightarrow$$
$$\frac{\lambda}{\lambda - 1} = [m_1 + 2, (2)^{n_1}, m_2 + 3, \dots, (2)^{n_s}, m_{s+1} + 2]^-$$

The important point here is that even a value $m_{s+1} = 0$ contributes to the number of partial quotients in the HJ-continued fraction expansion of $\lambda/(\lambda - 1)$.

The next proposition is equivalent to the previous one, as an easy inspection shows. Its advantage is that it gives a graphical way to pass from the HJ-continued fraction expansion of a number $\lambda > 1$ to the analogous expansion of $\lambda/(\lambda - 1) > 1$.

Proposition 2.8. *Consider a number $\lambda \in \mathbf{R}$ greater than 1 and let*

$$\lambda = [\alpha_1, \alpha_2, \ldots]^-, \quad \frac{\lambda}{\lambda - 1} = [\beta_1, \beta_2, \ldots]^-$$

be the expressions of λ and $\lambda/(\lambda-1)$ as (finite or infinite) HJ-continued fractions. Construct a diagram made of points organized in lines and columns in the following way:

- *its lines are numbered by the positive integers;*
- *the line numbered $k \geq 1$ contains $\alpha_k - 1$ points;*
- *the first point in the line numbered $k+1$ is placed under the last point of the line numbered k.*

Then the k-th column contains $\beta_k - 1$ points.

This graphical construction seems to have been first noticed by Riemenschneider in [66] when $\lambda \in \mathbf{Q}_+$. Nowadays one usually speaks about *Riemenschneider's point diagram* or *staircase diagram*.

Example 2.9. If $\lambda = 11/7 = [2, 3, (2)^2]^-$, the associated point diagram is:

$$\begin{array}{cc} \bullet & \\ \bullet & \bullet \\ & \bullet \\ & \bullet \end{array}$$

One deduces from it that $\lambda/(\lambda - 1) = [3, 4]^-$.

§3. Klein's geometric interpretation of Euclidean continued fractions

We let Klein [46] himself speak about his interpretation, in order to emphasize his poetical style:

> Let us now enliven these considerations with geometric pictures. Confining our attention to positive numbers, let us *mark all those points* in the positive quadrant of the xy plane *which have integral coordinates*, forming thus a so-called *point lattice*. Let us examine this lattice, I am tempted to say this "firmament" of points, with our point of view at the origin. [...]

> Looking from 0, then, one sees points of the lattice *in all rational directions and only in such directions*. The field of view is everywhere "densely" but not completely and continuously filled with "stars". One might be inclined to compare this view with that of the milky way. With the exception of 0 itself there is *not a single integral point lying upon an irrational ray $x/y = \omega$, where ω is irrational*, which is very remarkable. If we recall Dedekind's definition of irrational number, it becomes obvious that such a ray makes a *cut in the field of integral points by separating the points into two point sets*, one lying to the right of the ray and one to the left. If we inquire how these point sets converge toward our ray $x/y = \omega$, we shall find a very simple relation to the continued fraction for ω. By marking each point ($x = p_\nu$, $y = q_\nu$), corresponding to the convergent p_ν/q_ν, we see that the rays to these points approximate to the ray $x/y = \omega$ better and better, alternately from the left and from the right, just as the numbers p_ν/q_ν approximate to the number ω. Moreover, if one makes use of the known number-theoretic properties of p_ν, q_ν, one finds the following theorem: *Imagine pegs or needles affixed at all the integral points, and wrap a tightly drawn string about the sets of pegs to the right and to the left of the ω-ray, then the vertices of the two convex string-polygons which bound our two point sets will be precisely the points (p_ν, q_ν) whose coordinates are the numerators and denominators of the successive convergents to ω, the left polygon having the even convergents, the right one the odd.* This gives a new and, one may well say, an extremely graphical definition of a continued fraction.

In the original article [45], one finds moreover the following interpretation of the E-partial quotients:

> Each edge of the polygons [...] may contain integral points. The number of parts in which the edge is decomposed by such points is exactly equal to a partial quotient.

Before Klein, Smith expressed a related idea in [73]:

> If with a pair of rectangular axes in a plane we construct a system of unit points (*i.e.* a system of points of which the coordinates are integral numbers), and draw the line $y = \theta x$, we learn from that theorem that if (x, y) be a unit point lying nearer to that line than any other unit point having a less abscissa (or, which comes to the same thing, lying at a less distance from the origin), y/x is a convergent to θ; and, *vice*

> *versa*, if y/x is a convergent, $(x,\, y)$ is one of the 'nearest points'. Thus the 'nearest points' lie alternately on opposite sides of the line, and the double area of the triangle, formed by the origin and any two consecutive 'nearest points', is unity.

Proofs of the preceding properties can be found in Stark [75]. Here we only sketch the reason of Klein's interpretation. For explanations about our vocabulary, read next section.

Let $\lambda > 1$ be a real number. In the first quadrant σ_0, consider the half-line L_λ of slope λ (see Fig. 1). It is defined by the equation $y = \lambda x$, which shows that $\lambda = \omega^{-1} = \theta$, where ω is Klein's notation and θ is Smith's. It subdivides the quadrant σ_0 into two closed cones with vertex the origin, $\sigma_x(\lambda)$ adjacent to the axis of the variable x and $\sigma_y(\lambda)$ adjacent to the axis of the variable y.

Lemma 3.1. *The segment which joins the lattice points of coordinates* $(1,\, 0)$ *and* $(1,\, a_1)$ *is a compact edge of the convex hull of the set of lattice points different from the origin contained in the cone* $\sigma_x(\lambda)$*, where* $\lambda = [a_1,\, a_2,\, \dots]^+$ *is the E-continued fraction expansion of* λ*.*

Proof. Indeed, the half-line starting from $(1,\, 0)$ and directed towards $(1,\, a_1)$ cuts the half-line L_λ inside the segment $[(1,\, [\lambda]),\, (1,\, [\lambda] + 1))$, where $[\lambda]$ is the integral part of λ. But $[\lambda] = a_1$, which finishes the proof. Q.E.D.

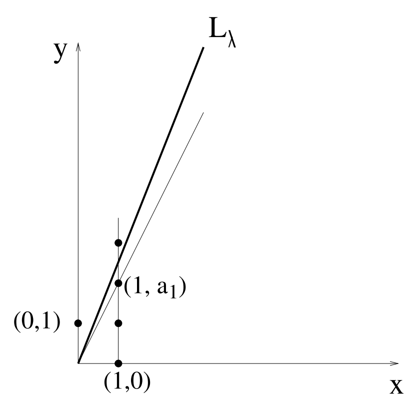

Fig. 1. Figure illustrating the proof of Lemma 3.1

Replace now the initial basis of the lattice by $(0,\, 1)$, $(1,\, a_1)$. With respect to this new basis, the slope of the half-line L_λ is $(\lambda - a_1)^{-1} =$

$[a_2, a_3, \dots]^+$. This allows one to prove Klein's interpretation by induction.

If one considers all lattice points on the compact edges of the boundaries of the two previous convex hulls instead of only the vertices, and then one looks at the slopes of the lines which join them to the origin, one obtains the so-called *slow approximating sequence* of λ. This kind of sequence appears naturally when one desingularizes germs of complex analytic plane curves by successively blowing up points (see Enriques & Chisini [19], Michel & Weber [53] and Lê, Michel & Weber [51]). We leave as an exercise for the interested reader to interpret this geometrically (first, read Section 6.3).

As explained by Klein himself in [45], his interpretation suggests to generalize the notion of continued fraction to higher dimensions by taking the boundaries of convex hulls of lattice points situated inside convex cones. For references about recent research in this area, see Arnold [1] and Moussafir [54].

§4. Cohn's geometric interpretation of Hirzebruch-Jung continued fractions

A geometric interpretation of HJ-continued fractions analogous to Klein's interpretation of Euclidean ones was given by Cohn [12] (see the comment on his work in Hirzebruch [37, 2.3]). It seems to have soon become folklore among people doing toric geometry (see Section 6). Before describing this interpretation, let us introduce some vocabulary in order to speak with more precision about convex hulls of lattice points in the plane.

Let L be a lattice of rank 2, that is, a free abelian group of rank 2. It embeds canonically into the associated real vector space $L_{\mathbf{R}} = L \otimes_{\mathbf{Z}} \mathbf{R}$. When we picture the elements of L as points in the affine plane $L_{\mathbf{R}}$, we call them the *integral points* of the plane. When A and B are points of the affine plane $L_{\mathbf{R}}$, we denote by AB the element of the vector space $L_{\mathbf{R}}$ which translates A into B, by $[AB]$ the closed segment in $L_{\mathbf{R}}$ of extremities A, B and by $[AB$ the closed half-line having A as an extremity and directed towards B.

If (u, v) is an ordered basis of $L_{\mathbf{R}}$ and l is a line of $L_{\mathbf{R}}$, its *slope* is the quotient $\beta/\alpha \in \mathbf{R} \cup \{\infty\}$, where $\alpha u + \beta v$ generates l.

Definition 4.1. A (closed convex) triangle ABC in $L_{\mathbf{R}}$ is called **elementary** if its vertices are integral and they are the only intersections of the triangle with the lattice L.

If the triangle ABC is elementary, then each pair of vectors (AB, AC), (BC, BA), (CA, CB) is a basis of the lattice L. Conversely, if one of these pairs is a basis of the lattice, then the triangle is elementary.

We call a line or a half-line in $L_{\mathbf{R}}$ *rational* if it contains at least two integral points. If so, then it contains an infinity of them. If A and B are two integral points, the *integral length* $l_{\mathbf{Z}}[AB]$ of the segment $[AB]$ is the number of subsegments in which it is divided by the integral points it contains. A vector OA of L is called *primitive* if $l_{\mathbf{Z}}[OA] = 1$.

Let σ be a closed strictly convex 2-dimensional cone in the plane $L_{\mathbf{R}}$, that is, the convex "angle" (in the language of plane elementary geometry) delimited by two non-opposing half-lines originating from 0. These half-lines are called the *edges* of σ. The cone σ is called *rational* if its edges are rational. A cone is called *regular* if its edges contain points A, B such that the triangle OAB is elementary. The name is motivated by the fact that the associated toric surface $\mathcal{Z}(L, \sigma)$ is smooth (that is, all its local rings are *regular*) if and only if σ is regular (see Section 6.1).

Let $K(\sigma)$ be the convex hull of the set of lattice points situated inside σ, with the exception of the origin, that is:

$$K(\sigma) := \mathrm{Conv}(\sigma \cap (L - \{0\})).$$

The closed convex set $K(\sigma)$ is unbounded. Denote by $P(\sigma)$ its boundary: it is a connected polygonal line. It has two ends (in the topological sense), each one being asymptotic to (or contained inside) an edge of σ (see Fig. 2). An edge of σ intersects $P(\sigma)$ if and only if it is rational.

Denote by $\mathcal{V}(\sigma)$ the set of vertices of $P(\sigma)$ and by $\mathcal{E}(\sigma)$ the set of its (closed) edges. For example, in Fig. 3 the vertices are the points A_0, A_2, A_5 and the edges are the segments $[A_0A_2]$, $[A_2A_5]$ and two half-lines contained in l_-, l_+, starting from A_0, respectively A_5.

Now *order* arbitrarily the edges of σ. Denote by l_- the first one and by l_+ the second one. This orients the plane $L_{\mathbf{R}}$, by deciding to turn from l_- towards l_+ *inside* σ. If we orient $P(\sigma)$ from the end which is asymptotic to l_- towards the end which is asymptotic to l_+, we get induced orientations of its edges.

Suppose now that the edge l_- of σ is rational. Denote then by $A_- \neq 0$ the integral point of the half-line l_- which lies nearest to 0, and by $V_- \neq A_-$ the vertex of $P(\sigma)$ which lies nearest to A_-. Define in the same way A_+ and V_+ whenever l_+ is rational. Denote by $(A_n)_{n \geq 0}$ the sequence of integral points on $P(\sigma)$, enumerated as they appear when one travels on this polygonal line in the positive direction, starting from $A_0 = A_-$. If l_+ is a rational half-line, then we stop this sequence when we arrive at the point A_+. If l_+ is irrational, then this sequence is

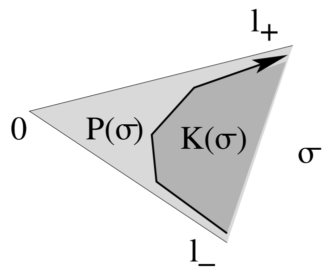

Fig. 2. The polygonal line associated to a convex cone

infinite. Define $r \geq 0$ such that $A_{r+1} = A_+$. So, $r = +\infty$ if and only if l_+ is irrational.

Example 4.2. We consider the lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$ and the cone σ with rational edges, generated by the vectors (1, 0) and (4, 11) (see Fig. 3). The small dots represent integral points in the plane and the bigger ones represent integral points on the polygonal lines $P(\sigma)$. In this example we have $V_+ = V_- = A_2$.

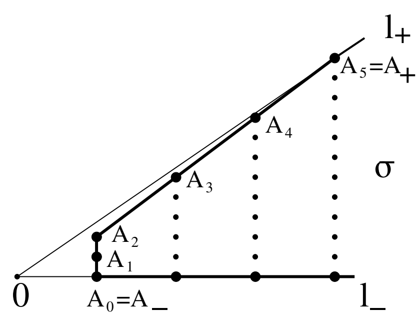

Fig. 3. An illustration of Example 4.2

Each triangle OA_nA_{n+1} is elementary, by the construction of the convex hull $K(\sigma)$, which implies that all the couples (OA_n, OA_{n+1}) are bases of L. This shows that for any $n \in \{1, \ldots, r\}$, one has a relation of the type:

$$OA_{n+1} + OA_{n-1} = \alpha_n \cdot OA_n \tag{10}$$

with $\alpha_n \in \mathbf{Z}$, and the convexity of $K(\sigma)$ shows that:

$$\alpha_n \geq 2, \quad \forall n \in \{1, \dots, r\} \tag{11}$$

Conversely:

Proposition 4.3. *Suppose that $(OA_n)_{n\geq 0}$ is a (finite or infinite) sequence of primitive vectors of L, related by relations of the form (10). Then we have*

$$OA_n = Z^-(\alpha_1, \dots, \alpha_{n-1})OA_1 - Z^-(\alpha_2, \dots, \alpha_{n-1})OA_0, \quad \forall n \geq 1$$

and the slope of the half-line $l_+ = \lim_{n\to\infty}[OA_n)$ in the base $(-OA_0, OA_1)$ is equal to $[\alpha_1, \alpha_2, \dots]^-$.

Proof. Recall that the polynomials Z^- were defined by the recursion formula (1). The first assertion can be easily proved by induction, using the relations (10). The second one is a consequence of formula (2), which shows that the slope of the half-line $[OA_n$ in the base $(-OA_0, OA_1)$ is equal to $[\alpha_1, \dots, \alpha_{n-1}]^-$. Q.E.D.

Proposition 4.4. *Let σ be the closure of the convex hull of the union of the half-lines $([OA_n)_{n\geq 0}$. Then σ is strictly convex and the points $\{A_n\}_{n\geq 1}$ are precisely the integral points on the compact edges of the polygonal line $P(\sigma)$ if and only if the conditions (11) are satisfied and the sequence $(\alpha_n)_{n\geq 1}$ is not infinite and ultimately constant equal to 2.*

Proof. • What remains to be proved about the *necessity* is that if the sequence $(\alpha_n)_{n\geq 1}$ is infinite, then it cannot be ultimately constant equal to 2. If this was the case, by relation (8) we would deduce that $[\alpha_1, \alpha_2, \dots]^-$ is rational, and Proposition 4.3 would imply that l_+ is rational. Then $P(\sigma)$ would contain a finite number of integral points on its compact edges, which would contradict the infinity of the sequence $(\alpha_n)_{n\geq 1}$.

• Let us prove now the *sufficiency.* As $\alpha_n \geq 2$, $\forall n \in \{1, \dots, r\}$, we see that the triangles $(OA_nA_{n+1})_{n\geq 0}$ turn in the same sense. Moreover, Proposition 4.3 shows that σ is a strictly convex cone. The vertices of the polygonal line $P = A_0A_1A_2\dots$ are precisely those points A_n for which $\alpha_n \geq 3$. As all the triangles OA_nA_{n+1} are elementary, we see that the origin O is the only integral point of the connected component of $\sigma - P$ which contains it. Moreover, conditions (11) show that the other component is convex. So, $P \subset P(\sigma)$.

The proposition is proved. Q.E.D.

§5. Geometric comparison of Euclidean and HJ-continued fractions

In Section 5.1 we relate the two preceding interpretations, by describing a duality between two supplementary cones in the plane, an underlying lattice being fixed (see Proposition 5.3). In Section 5.2 we introduce a so-called *zigzag diagram* based on this duality, which makes it very easy to visualize the various relations between continued fractions proved algebraically in Section 2. In Section 5.3 we give a proof of the isomorphism between the supplementary cone (L, σ') and the dual cone $(\check{L}, \check{\sigma})$ of a given cone (L, σ).

5.1. A geometric duality between supplementary cones

Suppose again that σ is *any* strictly convex cone in $L_{\mathbf{R}}$, whose edge l_- is not necessarily rational. Let l'_- be the half-line opposite to l_- and σ' be the closed convex cone bounded by l_+ and l'_-. So, σ and σ' are *supplementary cones*:

Definition 5.1. Two strictly convex cones in a real plane are called **supplementary** if they have a common edge and if their union is a half-plane.

By analogy with what we did in the previous section for σ, orient the polygonal line $P(\sigma')$ from l'_- towards l_+. If l_- is rational, define the point A'_- and the sequence $(A'_n)_{n\geq 0}$, with $A'_0 = A'_-$. They are the analogs for σ' of the points A_- and $(A_n)_{n\geq 0}$ for σ. In particular, $OA_- + OA'_- = 0$.

Example 5.2. Consider the same cone as in Example 4.2. Then the polygonal lines $P(\sigma)$ and $P(\sigma')$ are represented in Fig. 4 using heavy segments.

The basis for our geometric comparison of Euclidean and Hirzebruch-Jung continued fractions is the observation that *the polygonal line $P(\sigma')$ can be constructed in a very simple way once one knows $P(\sigma)$*. Namely, starting from the origin, one draws the half-lines parallel to the oriented edges of $P(\sigma)$. On each half-line, one considers the integer point which is nearest to the origin. Then the polygonal line which joins those points is the union of the compact edges of $P(\sigma')$.

Now we describe this with more precision. If $e \in \mathcal{E}(\sigma)$ is an edge of $P(\sigma)$, denote by $\mathcal{I}(e) \in L$ the integral point such that $O\mathcal{I}(e)$ is a primitive vector of L positively parallel to e (where e is oriented according to the chosen orientation of $P(\sigma)$). Then it is an easy exercise to see that $\mathcal{I}(e) \in \sigma'$ (use the fact that the line containing e intersects l_- and l_+ in

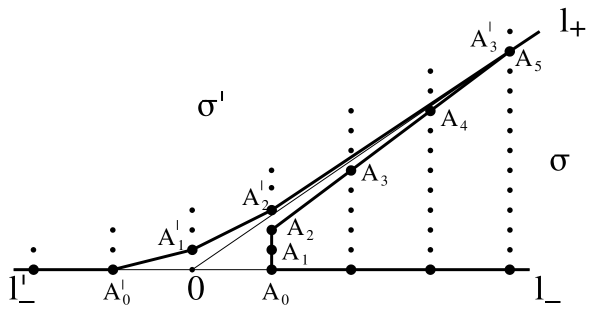

Fig. 4. An illustration of Example 5.2

interior points). We can define a map:

$$\begin{array}{cccc} \mathcal{I}: & \mathcal{E}(\sigma) & \longrightarrow & \sigma' \cap L \\ & e & \longrightarrow & \mathcal{I}(e) \end{array} \tag{12}$$

As the edges of $P(\sigma)$ always turn in the same direction, one sees that the map $\mathcal{I}$ is injective.

Proposition 5.3. *The map $\mathcal{I}$ respects the orientations and the image of $\mathcal{I}$ verifies the double inclusion*

$$\mathcal{V}(\sigma') \subset \mathrm{Im}(\mathcal{I}) \subset P(\sigma') \cap L.$$

The difference $\mathrm{Im}(\mathcal{I}) - \mathcal{V}(\sigma')$ *contains at most the points $\mathcal{I}[A_-V_-]$, $\mathcal{I}[V_+A_+]$. Such a point is a vertex of $P(\sigma')$ if and only if the integral length of the corresponding edge of $P(\sigma)$ is ≥ 2. In particular, one has the equality $\mathcal{V}(\sigma') = \mathrm{Im}(\mathcal{I})$ if and only if $l_{\mathbf{Z}}[A_-V_-] \geq 2$ and $l_{\mathbf{Z}}[V_+A_+] \geq 2$, whenever these segments exist.*

Proof. Denote by $(V_j)_{j\in J}$ the vertices of $P(\sigma)$, enumerated in the positive direction. The indices form a set of consecutive integers, well-defined only up to translations.

For any $j \in J$, denote by V_j^- and V_j^+ respectively the integral points of $P(\sigma)$ which precede and follow V_j. If V_j is an interior point of σ, denote by $W_j \in L$ the point such that $OW_j = OV_j^- + OV_j^+$, and by W_j^- its nearest integral point in the interior of the segment $[OW_j]$ (see Fig. 5).

As $OV_j^- V_j$ and $OV_j V_j^+$ are elementary triangles, it implies that both (OV_j^-, OV_j) and (OV_j, OV_j^+) are bases of L. So, there exists an integer

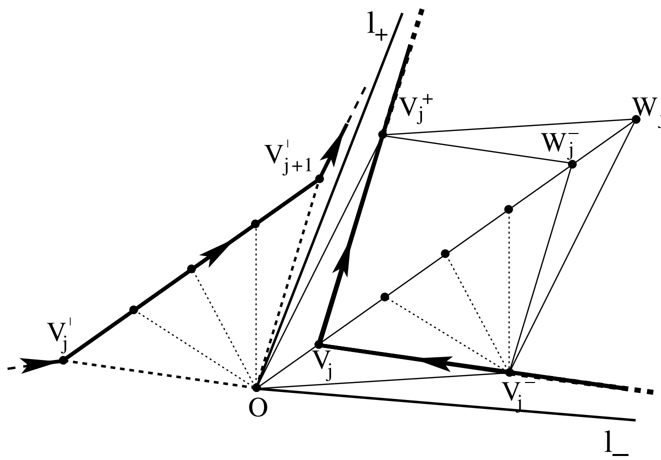

Fig. 5. The first illustration for the proof of Proposition 5.3

n_j such that

$$OV_j^- + OV_j^+ = (n_j+3)OV_j. \tag{13}$$

As V_j is a vertex of $P(\sigma)$, we see that $n_j \geq 0$. We deduce that the points O, V_j, W_j^-, W_j are aligned in this order, that $V_jV_j^- + V_jV_j^+ = V_jW_j^-$ and that $l_{\mathbf{Z}}[V_jW_j^-] = n_j+1$.

Let us join each one of the n_j interior points of $[V_jW_j^-]$ to V_j^-. This gives a decomposition of the triangle $V_j^-V_jW_j^-$ into (n_j+1) triangles. These are necessarily elementary, because the triangle $OV_j^-V_j$ is. Denote

$$V_j' = \mathcal{I}[V_{j-1}V_j] \quad \text{and} \quad V_{j+1}' = \mathcal{I}[V_jV_{j+1}].$$

By the definition of the map $\mathcal{I}$, we see that $OV_j' = V_j^-V_j$ and $OV_{j+1}' = V_jV_j^+ = V_j^-W_j^-$. This implies that *the triangle $OV_j'V_{j+1}'$ is the translated image by the vector V_j^-O of the triangle $V_j^-V_jW_j^-$*. The preceding arguments show that its only integral points are its vertices and n_j other points in the interior of the segment $[V_j'V_{j+1}']$. Indeed:

$$V_j'V_{j+1}' = V_jW_j^- = (n_j+1)OV_j \tag{14}$$

Moreover, the triangle $OV_j'V_{j+1}'$ is included in the cone σ' and the couple of vectors (OV_j', OV_{j+1}') has the same orientation as (l_-', l_+).

This shows that the triangles $(OV_j'V_{j+1}')_{j\in J}$ are pairwise disjoint and that their union does not contain integral points in its interior.

• *If both edges of σ are irrational*, then the closure of the union of the cones $\mathbf{R}_+OV_j' + \mathbf{R}_+OV_{j+1}'$ is the cone σ', as the edges l_- and l_+ are asymptotic to $P(\sigma)$. We deduce from relation (14) that the sequence $(\lambda_j)_{j\in J}$ of slopes of the vectors $(V_j'V_{j+1}')_{j\in J}$, expressed in a base (u_-, u_+) of $L_{\mathbf{R}}$ which verifies $l_\pm = \mathbf{R}_+u_\pm$ is *strictly* increasing, and that $\lim_{j\to-\infty}\lambda_j = 0$, $\lim_{j\to+\infty}\lambda_j = +\infty$. This shows that the closure of the connected component of $\sigma' - \bigcup_{j\in J}[V_j'V_{j+1}']$ which does not contain the origin is *convex*. As a consequence,

$$\bigcup_{j\in J}[V_j'V_{j+1}'] = P(\sigma').$$

Moreover, as $n_j \geq 0$, the strict monotonicity of the sequence $(\lambda_j)_{j\in J}$ implies that the points $(V_j')_{j\in J}$ are precisely the vertices of $P(\sigma')$. The proposition is proved in this case.

• *Suppose now that l_- is rational.* Then choose the index set J such that $V_0 = A_-$ and $V_1 = V_-$. By the construction of the map $\mathcal{I}$, *the triangle $OV_0'V_1'$ is the translated image of $V_0OV_0^+$ by the vector V_0O* (see Fig. 6).

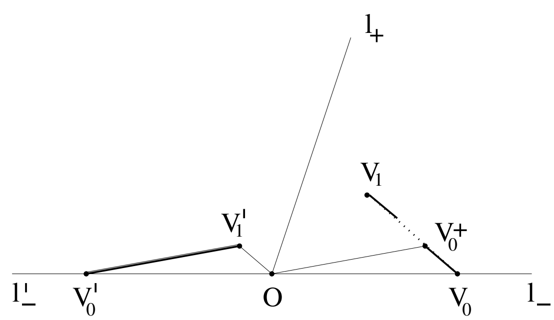

Fig. 6. The second illustration for the proof of Proposition 5.3

In particular, $V_0'V_1' = OV_0^+$. But $V_1'V_2' = (n_1+1)OV_1$ by relation (14), which shows that the vectors $V_0'V_1'$ and $V_1'V_2'$ are proportional if and only if $V_0^+ = V_1$, which is equivalent to $l_{\mathbf{Z}}[A_-V_-] = 1$. Moreover, the property of monotonicity for the slopes of the vectors $(V_j'V_{j+1}')_{j\in J}$ is true as before, if one starts from $j = 0$.

• An analogous reasoning is valid for l_+ *if this edge of σ is rational.* By combining all this, the proposition is proved. Q.E.D.

The previous proposition explains a geometric *duality* between the supplementary cones σ, σ' with respect to the lattice L. We see that, with possible exceptions for the compact edges which intersect the edges of σ and σ', the compact edges of $P(\sigma)$ correspond to the vertices of $P(\sigma')$ interior to σ' and conversely (by permuting the roles of σ and σ'), which is a kind of point-line polarity relation.

The next corollary shows that the involution (9) studied algebraically in Section 2 is closely related to the previous duality.

Corollary 5.4. *Suppose that l_- is rational and that σ is not regular. If (OA'_0, U) is a basis of L with respect to which the slope of l_+ is greater than 1, then $U = OA_1$. If $\lambda > 1$ denotes the slope of the half-line l_+ in the base (OA'_0, OA_1), then $\lambda/(\lambda-1)$ is its slope in the base (OA_0, OA'_1).*

Proof. We leave the first affirmation to the reader (look at Fig. 6).

As the triangles OA_0A_1 and $OA'_0A'_1$ are elementary, we see that (OA_0, OA_1) and (OA'_0, OA'_1) are indeed two bases of the lattice L. Proposition 5.3 shows that $OA'_0 = A_0A_1$, which allows us to relate the two bases:

$$\begin{cases} OA'_0 = -OA_0 \\ OA'_1 = OA_1 - OA_0 \end{cases} \tag{15}$$

Let $v \in L_{\mathbf{R}}$ be a vector which generates the half-line l_+. We want to express it in these two bases. As l_+ lies between the half-lines $[OA'_0$ and $[OA_1$, we see that:

$$v = -qOA_0 + pOA_1, \quad \text{with} \quad p,\, q \in \mathbf{R}^*_+ \tag{16}$$

The equations (15) imply then that:

$$v = -(p-q)OA'_0 + pOA'_1 \tag{17}$$

which shows that $p - q > 0$, as l_+ lies between the half-lines $[OA'_1$ and $[OA_0$. This implies that $\lambda := p/q > 1$. We then deduce the corollary from equation (17). Q.E.D.

The previous corollary shows that the number $\lambda > 1$ can be canonically attached to the pair (L, σ), once a rational edge of σ is chosen as the first edge l_-. This motivates the following definition:

Definition 5.5. Suppose that l_- is rational and that the cone σ is not regular. We say that the pair (L, σ) with the chosen ordering of sides is **of type** $\lambda > 1$ if λ is the slope of the half-line l_+ in the base (OA'_0, OA_1).

Proposition 4.3 shows that, if (L, σ) is of type $\lambda > 1$, then $\lambda = [\alpha_1, \alpha_2, \dots]^-$, where the sequence $(\alpha_n)_{n\geq 1}$ was defined using relation (10).

Suppose now that *both edges of σ are rational*. Then one can choose $p, q \in \mathbf{N}^*$ with $\gcd(p, q) = 1$ in relation (16), condition which determines them uniquely. So, $\lambda = p/q$. The following proposition describes the type of (L, σ) after changing the ordering of the sides.

Proposition 5.6. *If (L, σ) is of type p/q with respect to the ordering l_-, l_+, then it is of type $p/\overline{q}$ with respect to the ordering l_+, l_-, where $q\overline{q} \equiv 1 (\operatorname{mod} p)$.*

Proof. By relation (16), we have $OA_+ = -qOA_- + pOA_1$. Multiply both sides by $\overline{q}$. By the definition of $\overline{q}$, there exists $k \in \mathbf{N}$ such that $q\overline{q} = 1 + kp$. We deduce that $OA_- = -\overline{q}OA_+ + p(\overline{q}OA_1 - kOA_-)$. So, $(-OA_+, \overline{q}OA_1 - kOA_-)$ is a base of L in which the slope of l_- is $p/\overline{q} > 1$. By the first affirmation of Corollary 5.4, the proposition is proved. Q.E.D.

By combining the previous proposition with Proposition 4.3, we deduce the following classical fact (see [4, section III.5]):

Corollary 5.7. *If $p/q = [\alpha_1, \alpha_2, \dots, \alpha_r]^-$, then $p/\overline{q} = [\alpha_r, \alpha_{r-1}, \dots, \alpha_1]^-$.*

Another immediate consequence of Corollary 5.4 is:

Proposition 5.8. *If (L, σ) is of type p/q with respect to the ordering l_-, l_+, then (L, σ') is of type $p/(p-q)$ with respect to the ordering l'_-, l_+.*

The previous proposition describes the relation between the types of two supplementary cones. In Section 5.2, we describe more precisely the relation between numerical invariants attached to the edges and the vertices of $P(\sigma)$ and $P(\sigma')$.

5.2. A diagram relating Euclidean and HJ-continued fractions

We introduce now a diagram which allows one to "see" the duality between $P(\sigma)$ and $P(\sigma')$, as well as the relations between the various numerical invariants attached to these polygonal lines.

• *Suppose first that both l_- and l_+ are irrational.* Consider two consecutive vertices V_j, V_{j+1} of $P(\sigma)$. Let us *attach the weight $n_j + 3$ to the vertex V_j*, where $n_j \geq 0$ was defined by relation (13). Introduce also the integer $m_{j+1} \geq 0$ such that $l_{\mathbf{Z}}[V_j V_{j+1}] = m_{j+1} + 1$. The relation (14) shows that $l_{\mathbf{Z}}[V'_j V'_{j+1}] = n_j + 1$. By reversing the roles of the polygonal lines $P(\sigma')$ and $P(\sigma)$, we deduce that the weight of the vertex V'_{j+1} of $P(\sigma')$ is $m_{j+1}+3$.

We can visualize the relations between the vertices $V_j, V_{j+1}, V'_j, V'_{j+1}$ as well as the numbers associated to them and to the segments $[V_jV_{j+1}]$, $[V'_jV'_{j+1}]$ by using a diagram, in which the heavy lines represent the polygonal lines $P(\sigma)$, $P(\sigma')$, and each vertex V_j is joined to V'_j and V'_{j+1} (see Fig. 7). In this way, the region contained between the two curves representing $P(\sigma)$ and $P(\sigma')$ is subdivided into triangles. Each edge E of $P(\sigma)$, $P(\sigma')$ is contained in only one of those triangles. Look at its opposite vertex. We say that E is the *opposite edge* of that vertex in the zigzag diagram. We see that *the weight of a vertex is equal to the length of the opposite edge augmented by* 2.

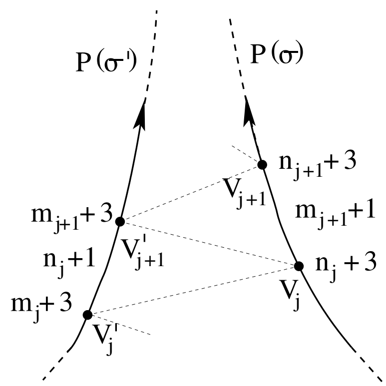

Fig. 7. Local aspect of the zigzag diagram

As an edge and its opposite vertex are dual through the morphism $\mathcal{I}$ (see Proposition 5.3) and its analog $\mathcal{I}'$ attached to the cone σ', the triangles appearing in the zigzag diagram are a convenient graphical representation of the duality explained in Section 5.1.

• *When l_- is rational and l_+ is irrational*, we draw a little differently the diagram (see Fig. 8). The curves representing $P(\sigma)$ and $P(\sigma')$ start from points V_0 and V'_0 of a horizontal line representing the line which contains l_-. We represent the integral point V'_1 differently from the points V'_2, V'_3, …, because it may not be a vertex of $P(\sigma')$, as explained in Proposition 5.3. The length of $[V'_0V'_1]$ is always 1. The relation between the length of an edge and the weight of the opposite vertex is the same as before, with the exception of the triangle $V'_1V_0V_1$, where the weight of V'_1 is equal to $l_{\mathbf{Z}}[V_0V_1]+1$.

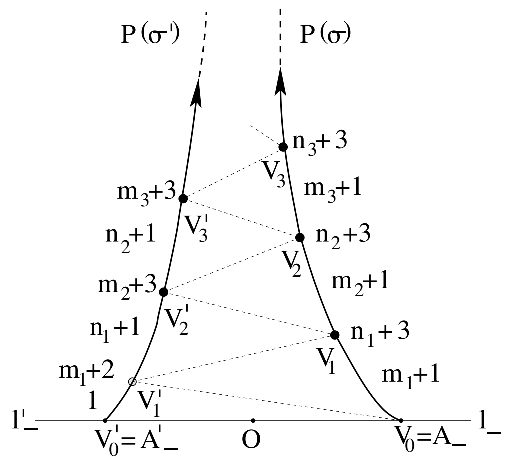

Fig. 8. The zigzag diagram when l_- is rational

• *When both l_- and l_+ are rational and there is at least one vertex on $P(\sigma)$ lying strictly between A_- and A_+ (that is, $s \geq 1$)*, the curves representing $P(\sigma)$ and $P(\sigma')$ start again from a horizontal line, but now they join in a point A_+ (see Fig. 9).

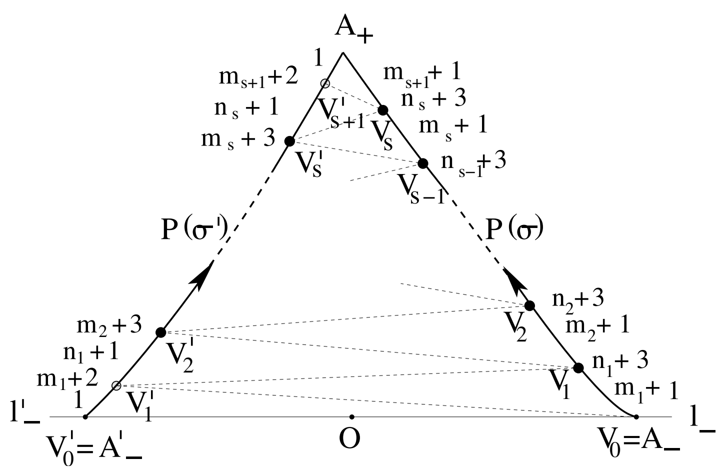

Fig. 9. The zigzag diagram when both l_- and l_+ are rational

• *When both l_- and l_+ are rational and $[A_- A_+]$ is an edge of $P(\sigma)$* (*that is,* $s = 0$), the diagram is represented in Fig. 10.

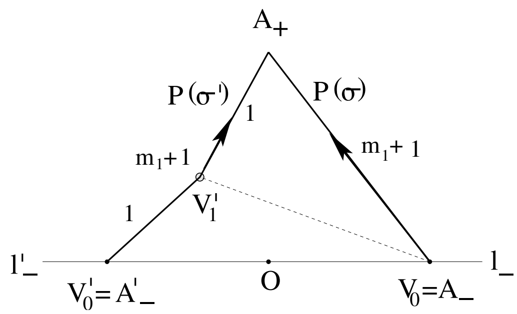

Fig. 10. The zigzag diagram when $P(\sigma)$ has only one compact edge

To summarize, we have the following procedure for constructing and decorating the diagram when l_- is rational:

Procedure. *Suppose that l_- is rational. Then draw a horizontal line with three marked points $V_0' = A_-'$, O, $V_0 = A_-$ in this order, V_0' on the left and V_0 on the right. Starting from V_0' and V_0, draw in the upper half-plane two curves $P(\sigma')$, respectively $P(\sigma)$, concave towards* 0 *and coming closer and closer from one another. If l_+ is rational, join them in a point A_+. Draw a zigzag line starting from V_0 and going alternatively from $P(\sigma)$ to $P(\sigma')$. Denote its successive vertices by V_1', V_1, V_2', ... and stop at the point V_{s+1}'. Decorate the edges $V_0'V_1'$ and $V_{s+1}'A_+$ by* 1*. The other edges and vertices will be decorated using the initial data* (*discussed in the sequel*)*, by respecting the following rule*:

Rule. *The weight of a vertex is equal to the length of the opposite edge augmented by the number of its vertices distinct from the points A_-, A_-', A_+.*

Initial data. *If σ is of type λ, write the HJ-continued fraction expansion of λ in the form*:

(18) $$\lambda = [(2)^{m_1},\, n_1 + 3,\, (2)^{m_2},\, n_2 + 3,\, \ldots]^-$$

Then decorate the edges of $P(\sigma)$ with the numbers $m_1 + 1$, $m_2 + 1$, ... and the vertices with the numbers $n_1 + 3$, $n_2 + 3$, ...

Definition 5.9. We call the previous diagram the **zigzag diagram** associated to the pair (L, σ) and to the chosen ordering of the edges of σ, or to the number $\lambda > 1$, where (L, σ) is of type λ with respect to this ordering. We denote it by $ZZ(\lambda)$.

The zigzag diagrams allow one to visualize the relations between Euclidean and Hirzebruch-Jung continued fractions, proved algebraically in Section 2. Indeed, one can read the HJ-continued fraction expansion of $\lambda > 1$ on the right-hand curved line of $ZZ(\lambda)$. By Corollary 5.4, we can read the HJ-continued fraction expansion of $\lambda/(\lambda - 1)$ on the left-hand curved line $P(\sigma)$ of $ZZ(\lambda)$. So, by looking at Fig. 9, which can be easily constructed from the initial data by respecting the rule, we get:

$$\frac{\lambda}{\lambda - 1} = [m_1 + 2,\, (2)^{n_1},\, m_2 + 3,\, (2)^{n_2},\, m_3 + 3,\, \ldots]^- \tag{19}$$

which gives a geometric proof of Proposition 2.7.

Now, by Klein's geometric interpretation of E-continued fractions (see Section 3), we see that the E-continued fraction expansion of $\lambda/(\lambda - 1)$ can be obtained by writing alternatively the integral lengths of the edges of the polygonal lines $P(\sigma)$ and $P(\sigma') - [V_0'V_1']$ (indeed, $\lambda/(\lambda - 1)$ is the slope of l_+ in the base $(OV_0,\, OV_1')$):

$$\frac{\lambda}{\lambda - 1} = [m_1 + 1,\, n_1 + 1,\, m_2 + 1,\, n_2 + 1,\, m_3 + 1,\, \ldots]^+. \tag{20}$$

This proves geometrically Proposition 2.3.

In order to read the E-continued fraction expansion of λ on the diagram, one has to look at $ZZ(\lambda)$ from left to right instead of from right to left and draw a new zigzag line starting from V_0'. The important point here is that one has to discuss according to the alternative $m_1 = 0$ or $m_1 > 0$. In the first case, the zigzag line joins V_0' to V_1 and V_1 to V_2'. In the second case, it joins V_0' to a new point representing A_1 and A_1 to V_1'. Compare this with Lemma 2.5.

Example 5.10. Take $\lambda = 11/7$. After computing $\lambda = [2, 3, 2, 2]^-$, we can construct the associated zigzag diagram $ZZ(11/7)$. We see that the extreme points V_1', V_2' are vertices of $P(\sigma')$. One can read on it the results of the Examples 2.4, 2.6, 2.9.

If one had starts instead from $\lambda = 11/4 = [3,\, 4]^-$, the corresponding diagram would be $ZZ(11/4)$. In this case the extreme points are not vertices of $P(\sigma')$, because their weights are equal to 2.

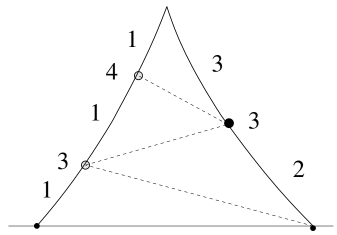

Fig. 11. The first illustration for Example 5.10: $ZZ(11/7)$

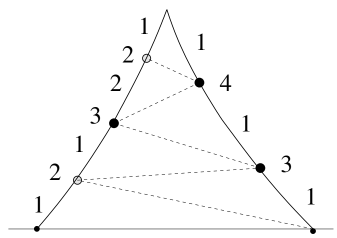

Fig. 12. The second illustration for Example 5.10: $ZZ(11/4)$

5.3. Relation with the dual cone

Denote by $\check{L} := \mathrm{Hom}(L, \mathbf{Z})$ the *dual lattice* of L. Inside the associated vector space $\check{L}_{\mathbf{R}}$ lives the *dual cone* $\check{\sigma}$ of σ, defined by:

$$\check{\sigma} := \{\check{u} \in \check{L}_{\mathbf{R}} \mid \check{u}.u \geq 0, \ \forall u \in \sigma\}.$$

Let ω be the volume form on $L_{\mathbf{R}}$ which verifies $\omega(u_1, u_2) = 1$ for any basis (u_1, u_2) of L defining the *opposite* orientation to (l_-, l_+). It is a symplectic form, that is, a non-degenerate alternating bilinear form on $L_{\mathbf{R}}$. But we prefer to look at it as a morphism (obtained by making interior products with the elements of L):

$$\omega \colon L \longrightarrow \check{L}.$$

Proposition 5.11. *The mapping ω realizes an isomorphism between the pairs (L, σ') and $(\check{L}, \check{\sigma})$.*

Proof. Indeed we have:

$$\omega^{-1}(\check{\sigma}) = \{u \in L \mid \omega(u) \in \check{\sigma}\} = \{u \in L \mid \omega(u, v) \geq 0, \ \forall v \in L\} = \sigma'.$$

While writing the last equality, we used our convention on the orientation of ω. Notice that the dual cone $\check{\sigma}$ can be defined without the help of any orientation, in contrast with the morphism ω. Q.E.D.

The previous proposition shows that the construction of the polygonal line $P(\sigma')$ explained in Proposition 5.3 describes also the polygonal line $P(\check{\sigma})$. This observation is crucial when one wants to use zigzag diagrams for understanding computations with invariants of toric surfaces (see next section).

It also helps to understand geometrically the duality between the convex polygons $K(\sigma)$ and $K(\check{\sigma})$ explained in Gonzalez-Sprinberg [30] and in Oda [60, pages 27-29]. As Dimitrios Dais kindly informed us after seeing a version of this paper on ArXiv, a better algebraic understanding of that duality is explained in Dais, Haus & Henk [14, Section 3]. In particular, modulo Proposition 5.11, the Theorem 3.16 in the previous reference leads easily to an algebraic proof of our Proposition 5.3.

(Added in proof) Emmanuel Giroux has informed us that he had realized the existence of a duality between supplementary cones (see [25, section 1.G]).

§**6. Relations with toric geometry**

First we introduce elementary notions of toric geometry (see Section 6.1). In Section 6.2 we explain how to get combinatorially various invariants of a normal affine toric surface and of the corresponding Hirzebruch-Jung analytic surface singularities. In Section 6.3 we explain how to read the combinatorics of the minimal embedded resolution of a plane monomial curve on an associated zigzag diagram.

The basics about resolutions of surface singularities needed in order to understand this section are recalled in Section 8.1.

6.1. Elementary notions of toric geometry

For details about toric geometry, general references are the books of Oda [60] and Fulton [23], as well as the first survey of it by Kempf, Knudson, Mumford & St. Donat [44].

In the previous section, our fundamental object of study was a pair (L, σ), where L is a lattice of rank 2 and σ is a strictly convex cone in the 2-dimensional vector space $L_{\mathbf{R}}$.

Suppose now that the lattice L has arbitrary finite rank $d \geq 1$ and that σ is a strictly convex *rational* cone in $L_{\mathbf{R}}$. The pair (L, σ) gives rise canonically to an affine algebraic variety:

$$\mathcal{Z}(L, \sigma) := \operatorname{Spec} \mathbf{C}[\check{\sigma} \cap \check{L}].$$

This means that the algebra of regular functions on $\mathcal{Z}(L,\, \sigma)$ is generated by the monomials whose exponents are elements of the semigroup $\check{\sigma} \cap \check{L}$ of integral points in the dual cone of σ. If $v \in \check{\sigma} \cap \check{L}$, we formally write such a monomial as X^v. One can show that the variety $\mathcal{Z}(L,\, \sigma)$ is *normal* (see the definition at the beginning of Section 8.1).

The closed points of $\mathcal{Z}(L,\, \sigma)$ are the morphisms of semigroups $(\check{\sigma} \cap \check{L},\, +) \to (\mathbf{C},\, \cdot\,)$. Among them, those whose image is contained in $\mathbf{C}^*$ form a d-dimensional *algebraic torus* $\mathcal{T}_L = \operatorname{Spec} \mathbf{C}[\check{L}]$, that is, a complex algebraic group isomorphic to $(\mathbf{C}^*)^d$. The elements of L correspond to the *1-parameter subgroups* of $\mathcal{T}_L$, that is, the group morphisms $(\mathbf{C}^*,\, \cdot\,) \to (\mathcal{T}_L,\, \cdot\,)$. The action of $\mathcal{T}_L$ on itself by multiplication extends canonically to an algebraic action on $\mathcal{Z}(L,\, \sigma)$, such that $\mathcal{T}_L$ is the unique open orbit. If $(\overline{L},\, \overline{\sigma})$ is a second pair and $\phi\colon \overline{L} \to L$ is a morphism such that $\phi(\overline{\sigma}) \subset \sigma$, one gets an associated *toric morphism*:

$$\phi_*\colon \mathcal{Z}(\overline{L},\, \overline{\sigma}) \to \mathcal{Z}(L,\, \sigma)$$

It is birational if and only if ϕ realizes an isomorphism between $\overline{L}$ and L. In this case ϕ_* identifies the tori contained inside $\mathcal{Z}(\overline{L},\, \overline{\sigma})$ and $\mathcal{Z}(L,\, \sigma)$.

In general:

Definition 6.1. Given an algebraic torus $\mathcal{T}$, a **toric variety** $\mathcal{Z}$ is an algebraic variety containing $\mathcal{T}$ as a dense Zariski open set and endowed with an action $\mathcal{T} \times \mathcal{Z} \to \mathcal{Z}$ which extends the group multiplication of $\mathcal{T}$.

Oda [60] and Fulton [23] study mainly the *normal* toric varieties. For an introduction to the study of non-necessarily normal toric varieties, one can consult Sturmfels [76] and González Pérez & Teissier [29].

A normal toric variety can be described combinatorially using *fans*, that is finite families of rational strictly convex cones, closed under the operations of taking faces or intersections. If L is a lattice and $\mathcal{F}$ is a fan in $L_{\mathbf{R}}$, we denote by $\mathcal{Z}(L,\, \mathcal{F})$ the associated normal toric variety. It is obtained by glueing the various affine toric varieties $\mathcal{Z}(L,\, \sigma)$ when σ varies among the cones of the fan $\mathcal{F}$. As glueing maps, one uses the toric birational maps $\mathcal{Z}(\overline{L},\, \overline{\sigma}) \to \mathcal{Z}(L,\, \sigma)$ induced by the inclusion morphisms $(L,\, \overline{\sigma}) \to (L,\, \sigma)$, for each pair $\overline{\sigma} \subset \sigma$ of cones of $\mathcal{F}$.

The variety $\mathcal{Z}(L,\, \mathcal{F})$ is smooth if and only if each cone of the fan $\mathcal{F}$ is *regular*, that is, generated by a subset of a basis of the lattice L.

6.2. Toric surfaces

We restrict now to the case of surfaces. Consider a 2-dimensional normal toric surface $\mathcal{Z}(L,\, \sigma)$, where σ is a strictly convex cone with non-empty interior. There is a unique 0-dimensional orbit O, whose maximal ideal is generated by the monomials with exponents in the semigroup $\check{\sigma} \cap \check{L} - O$. The surface is smooth outside O, and O is a smooth point of it if and only if σ is a regular cone. Supposing that σ is not regular, we explain how to describe combinatorially the minimal resolution morphism of $\mathcal{Z}(L, \sigma)$ and the effect of blowing-up the point O. We also give a formula for the embedding dimension of the germ $(\mathcal{Z}(L, \sigma), O)$, which is a so-called *Hirzebruch-Jung singularity.*

With the notations of Section 4, let us subdivide σ by drawing the half-lines starting from O and passing through the points A_k, $\forall k \in \{1, \dots, r\}$. In this way we decompose σ in a finite number of regular subcones. They form *the minimal regular subdivision* of σ, in the sense that any subdivision of σ by regular cones is necessarily a refinement of the preceding one.

The family consisting of the 2-dimensional cones in the subdivision, of their edges and of the origin form a fan $\mathcal{F}(\sigma)$. For each such subcone σ' of σ, there is a canonical birational morphism $\mathcal{Z}(L, \sigma') \to \mathcal{Z}(L, \sigma)$, which realizes an isomorphism of the tori. Using these morphisms, one can glue canonically the tori contained in the surfaces $\mathcal{Z}(L, \sigma')$ when σ' varies, and obtain a new toric surface $\mathcal{Z}(L, \mathcal{F}(\sigma))$, endowed with a morphism:

$$\mathcal{Z}(L, \mathcal{F}(\sigma)) \xrightarrow{p_\sigma} \mathcal{Z}(L, \sigma)$$

Proposition 6.2. *The morphism p_σ is the minimal resolution of singularities of the surface $\mathcal{Z}(L, \sigma)$. Moreover, its exceptional locus E_σ is a normal crossings divisor and the dual graph of E_σ is topologically a segment.*

Proof. For details, see [23]. Here we outline only the main steps. The morphism p_σ is proper, birational and realizes an isomorphism over $\mathcal{Z}(L, \sigma) - O$. As $\mathcal{Z}(L, \mathcal{F}(\sigma))$ is smooth, p_σ is a a resolution of singularities of $\mathcal{Z}(L, \sigma)$ (see Definition 8.2). There is a canonical bijection between the irreducible components E_k of the exceptional divisor $E_\sigma = p_\sigma^{-1}(0)$ and the half-lines $[OA_k$, for $k \in \{1, \dots, r\}$. Moreover, E_k is a smooth compact rational curve and

$$E_k^2 = -\alpha_k, \quad \forall k \in \{1, \dots, r\} \tag{21}$$

where the numbers α_k were introduced in relation (10).

Using the inequality (11), we deduce that no component of E_σ is exceptional of the first kind (see the comments which follow Definition 8.2). This implies that p_σ is the minimal resolution of singularities of $\mathcal{Z}(L, \sigma)$. The proposition is proved. Q.E.D.

Notice that relation (21) gives an intersection-theoretical interpretation of the weights attached through relation (10) to the integral points situated on $P(\sigma)$ which are interior to σ.

Conversely (see [4] and [64]):

Proposition 6.3. *Suppose that a smooth surface $\mathcal{R}$ contains a compact normal crossings divisor E whose components are smooth rational curves of self-intersection ≤ -2 and whose dual graph is topologically a segment. Denote by $\alpha_1, \dots, \alpha_r$ the self-intersection numbers read orderly along the segment. Then E can be contracted by a map $p\colon (\mathcal{R}, E) \to (\mathcal{S}, 0)$ to a normal surface $\mathcal{S}$ and the germ $(\mathcal{S}, 0)$ is analytically isomorphic to a germ of the form $(\mathcal{Z}(L, \sigma), O)$, where σ is of type $\lambda := [\alpha_1, \dots, \alpha_r]^-$.*

This motivates:

Definition 6.4. A normal surface singularity $(\mathcal{S}, 0)$ isomorphic to a germ of the form $(\mathcal{Z}(L, \sigma), O)$ is called a **Hirzebruch-Jung singularity**.

Hirzebruch-Jung singularities can also be defined as *cyclic quotient singularities* (see [4] and [64]). They appear naturally in the so-called *Hirzebruch-Jung method* of studying an arbitrary surface singularity. Namely, one projects the given singularity by a finite morphism on a smooth surface, then one makes an embedded resolution of the discriminant curve and takes the pull-back of the initial surface by this morphism. In this case, the normalization of the new surface has only Hirzebruch-Jung singularities (see Laufer [47], Lipman [52], Brieskorn [8] for details and Popescu-Pampu [64] for a generalization to higher dimensions).

The proof of Proposition 6.2 shows that the germs $(\mathcal{Z}(L, \sigma), O)$ and $(\mathcal{Z}(\overline{L}, \overline{\sigma}), O)$ are analytically isomorphic if and only if there exists an isomorphism of the lattices L and $\overline{L}$ sending σ onto $\overline{\sigma}$. The same is true for strictly convex cones in arbitrary dimensions, as proved by González Pérez & Gonzalez-Sprinberg [28]. Previously we had proved this for simplicial cones in [64].

A Hirzebruch-Jung singularity isomorphic to $(\mathcal{Z}(L, \sigma), O)$ is said to be *of type* $\mathcal{A}_{p,q}$, with $1 \leq q < p$ and $\gcd(p, q) = 1$ if (using Definition 5.5) the pair (L, σ) is of type p/q with respect to one of the orderings of the sides of σ. Then, by Proposition 4.3, we have $p/q = [\alpha_1, \dots, \alpha_r]^-$. By Proposition 5.6, one has $\mathcal{A}_{p,\,q} \simeq \mathcal{A}_{p',\,q'}$ if and only if $p = p'$ and $q' \in \{q, \overline{q}\}$, where $q\overline{q} \equiv 1 \pmod{p}$.

The singularities of type $\mathcal{A}_{n+1,\,n}$ are also called *of type* $\mathbf{A}_n$. They are those for which the polygonal line $P(\sigma)$ has only one compact edge, as $(n+1)/n = [(2)^n]^-$ (a case emphasized in Section 5.2), and also the only Hirzebruch-Jung singularities of embedding dimension 3 (more precisely, they can be defined by the equation $z^{n+1} = xy$). Indeed:

Proposition 6.5. *If* $p/q = [\alpha_1, \ldots, \alpha_r]^- = [(2)^{m_1}, n_1 + 3, \ldots, n_s + 3, (2)^{m_{s+1}}]^-$*, then*:

$$\mathrm{embdim}(\mathcal{A}_{p,q}) = 3 + \sum_{i=1}^{r}(\alpha_i - 2) = 3 + s + \sum_{k=1}^{s} n_k.$$

Proof. If S is a generating system of the semigroup $\check{L} \cap \check{\sigma} - O$, then the monomials $(X^v)_{v \in S}$ form a generating system of the Zariski cotangent space $\mathcal{M}/\mathcal{M}^2$ of the germ at the singular point, where $\mathcal{M}$ is the maximal ideal of the local algebra of the singularity $\mathcal{A}_{p,q}$. By taking a minimal generating system, one gets a basis of this cotangent space. But such a minimal generating system is unique, and consists precisely of the integral points of $P(\check{\sigma})$ interior to $\check{\sigma}$. By Propositions 5.11 and 2.7, we see that this number is as given in the Proposition. Q.E.D.

Hirzebruch-Jung singularities are particular cases of *rational singularities*, introduced by M. Artin [2], [3] in the 60's (see also [4]). In [79], Tjurina proved that the blow-up of a rational surface singularity is a normal surface which has again only rational singularities (see also the comments of Lê [50, 4.1]). As any surface can be desingularized by a sequence of blow-ups of its singular points followed by normalizations (Zariski [87], see also Cossart [13] and the references therein), this shows that a rational singularity can be desingularized by a sequence of blow-ups of closed points. In particular this is true for a Hirzebruch-Jung singularity. As the operation of blow-up is analytically invariant, we can describe the blow-up of O in the model surface $\mathcal{Z}(L, \sigma)$. We use notations introduced at the beginning of the proof of Proposition 5.3.

Proposition 6.6. *Suppose that the cone σ is not regular. Subdivide it by drawing the half-lines starting from O and passing through the points $A_1, V_1, V_2, \ldots, V_s, A_r$. Denote by $\mathcal{F}_0(\sigma)$ the fan obtained in this way. Then the natural toric morphism $\mathcal{Z}(L, \mathcal{F}_0(\sigma)) \xrightarrow{p_0} \mathcal{Z}(L, \sigma)$ is the blow-up of O in $\mathcal{Z}(L, \sigma)$.*

Proof. A proof is sketched by Lipman in [52]. Here we give more details.

Let $(\mathcal{S}, 0)$ be any germ of normal surface. Consider its minimal resolution $p_{\min}\colon (\mathcal{R}_{\min}, E_{\min}) \to (\mathcal{S}, 0)$ and its exceptional divisor $E_{\min} = \sum_{k=1}^{r} E_k$. The divisors $Z \in \sum_{k=1}^{r} \mathbf{Z} E_k$ which satisfy $Z \cdot E_k \leq 0,\ \forall k \in$

$\{1, \ldots, r\}$ form an additive semigroup with a unique minimal element Z_{top}, called the *fundamental cycle* of the singularity. It verifies

$$Z_{\mathrm{top}} \geq \sum_{k=1}^{r} E_k \tag{22}$$

for the componentwise order on the set of cycles with integral coefficients. *In the case of a rational singularity*, Tjurina [79] showed that the divisors E_k which appear in the blow-up of 0 on $\mathcal{S}$ can be characterized using the fundamental cycle: they are precisely those for which $Z_{\mathrm{top}} \cdot E_k < 0$.

In our case, where $(\mathcal{S}, 0) = (\mathcal{Z}(L, \sigma), O)$, Proposition 6.2 shows that $p_{\min} = p_\sigma$. Using the relations (21) and (22), we see that $Z_{\mathrm{top}} = \sum_{k=1}^{r} E_k$. Again using relation (21), we get:

$$Z_{\mathrm{top}} \cdot E_k < 0 \Longleftrightarrow \text{ either } k \in \{1, r\} \text{ or } \alpha_k \geq 3.$$

This shows that the components of E_σ which appear when one blows-up the origin, are precisely those which correspond to the half-lines $[OA_1$, $[OV_1$, $[OV_2, \ldots, [OV_s$, $[OA_r$. But the surface obtained by blowing-up the origin is again normal, by Tjurina's theorem, which shows that it coincides with $\mathcal{Z}(L, \mathcal{F}_0(\sigma))$. Q.E.D.

One sees that after the first blow-up, the new surface has only singularities of type $\mathbf{A}_n$, where n varies in a finite set of positive numbers. The singular points are contained in the set of 0-dimensional orbits of the toric surface $\mathcal{Z}(L, \mathcal{F}_0(\sigma))$, which in turn correspond bijectively to the 2-dimensional cones of the fan $\mathcal{F}_0(\sigma)$. The germs of the surface at those points are Hirzebruch-Jung singularities of types $\mathbf{A}_{n_0}, \ldots, \mathbf{A}_{n_s}$, where $n_0 = l_{\mathbf{Z}}[A_1V_1]$, $n_1 = l_{\mathbf{Z}}[V_1V_2], \ldots, n_s = l_{\mathbf{Z}}[V_sA_r]$.

We have spoken until now of algebraic aspects of Hirzebruch-Jung singularities. We discuss their topology in Section 8.3.

6.3. Monomial plane curves

Suppose that $(\mathcal{S}, 0)$ is a germ of *smooth* surface and that $(\mathcal{C}, 0) \subset (\mathcal{S}, 0)$ is a germ of reduced curve. A proper birational morphism $p\colon \mathcal{R} \to \mathcal{S}$ is called an *embedded resolution* of the germ $(\mathcal{C}, 0)$ if $\mathcal{R}$ is smooth, p is an isomorphism above $\mathcal{S} - 0$ and the *total transform* $p^{-1}(\mathcal{C})$ of $\mathcal{C}$ is a divisor with normal crossings on $\mathcal{R}$ in a neighborhood of the *exceptional divisor* $E := p^{-1}(0)$. The closure in $\mathcal{R}$ of the difference $p^{-1}(\mathcal{C}) - p^{-1}(0)$ is called the *strict transform* of $\mathcal{C}$ by the morphism p.

It is known since the XIX-th century that any germ of plane curve can be resolved in an embedded way by a sequence of blow-ups of points (see Enriques & Chisini [19], Laufer [47], Brieskorn & Knörrer [9]). The combinatorics of the exceptional divisor of the resolution can be determined starting from the Newton-Puiseux exponents of the irreducible components of the curve and from their intersection numbers using E-continued fraction expansions. We explain here how to read the sequence of self-intersection numbers of the components of the exceptional divisor of the minimal embedded resolution of a monomial plane curve by using a zigzag diagram, instead of just doing blindly computations with continued fractions.

If $p, q \in \mathbf{N}^*$, $1 \leq q < p$ and $\gcd(p, q) = 1$, consider the plane curve $C_{p/q}$ defined by the equation:

$$x^p - y^q = 0 \tag{23}$$

It can be parametrized by:

$$\begin{cases} x = t^q \\ y = t^p \end{cases} \tag{24}$$

As p and q are relatively prime, one sees that (24) describes the normalization morphism for $C_{p/q}$ (see its definition at the beginning of Section 8.1). As t^p and t^q are monomials, one says that $C_{p/q}$ is a *monomial curve*. There is a natural generalization to higher dimensions (see Teissier [77]).

If one identifies the plane $\mathbf{C}^2$ of coordinates (x, y) with the toric surface $\mathcal{Z}(L_0, \sigma_0)$, where $L_0 = \mathbf{Z}^2$ and σ_0 is the first quadrant, then it is easy to see (look at equation (24)) that $C_{p/q}$ is the closure in $\mathbf{C}^2$ of the image of the 1-parameter subgroup of the complex torus $\mathcal{T}_{L_0} = (\mathbf{C}^*)^2$ corresponding to the point (q, p).

Consider again the notations introduced before Lemma 3.1. Let $l_- := [O(1, 0)$ and $l_+ := [O(q, p)$ be the edges of the cone $\sigma_x(p/q)$. We leave to the reader the proof of the following lemma, which is very similar to the proof of Lemma 3.1. Recall that the *type* of a cone was introduced in Definition 5.5.

Lemma 6.7. *With respect to the chosen ordering of its edges, the cone $\sigma_x(p/q)$ is of type $p/(p-q)$. Moreover, with the notations of* Section 5*, $A_1 = (1, 1)$, $A'_1 = (0, 1)$ and $A_+ = (q, p)$.*

Even if the proof is very easy, it is important to be conscious of this result, as it allows to apply the study done in Section 5 to our context.

Given the pair (p, q), we want to describe the process of embedded resolution of the curve $C_{p/q}$ by blow-ups, as well as the final exceptional divisor, the self-intersections of its components and their orders of appearance during the process.

Lemma 6.8. *The blow-up* $\pi_0 \colon \mathcal{R}_0 \to \mathbf{C}^2$ *of* 0 *in* $\mathbf{C}^2$ *is a toric morphism corresponding to the subdivision of* σ_0 *obtained by joining* O *to* $A_1 = (1, 1)$. *The strict transform of* $C_{p/q}$ *passes through the* 0-*dimensional orbit of* $\mathcal{R}_0$ *associated to the cone* $\mathbf{R}_+ OA_1 + \mathbf{R}_+ OA'_1$.

Proof. With the notations of Section 3, we consider the fan $\mathcal{F}_0$ subdividing σ_0 which consists of the cones $\sigma_x(1)$, $\sigma_y(1)$, their edges and the origin. Let $\pi_{\mathcal{F}_0} \colon \mathcal{Z}(L, \mathcal{F}_0) \to \mathcal{Z}(L, \sigma_0)$ be the associated toric morphism. It is obtained by gluing the maps $\pi_x \colon \mathcal{Z}(L, \sigma_x(1)) \to \mathcal{Z}(L, \sigma_0)$ and $\pi_y \colon \mathcal{Z}(L, \sigma_y(1)) \to \mathcal{Z}(L, \sigma_0)$ over $(\mathbf{C}^*)^2$. With respect to the coordinates given by the monomials associated to the primitive vectors of L situated on the edges of the cones σ_0, $\sigma_x(1)$, $\sigma_y(1)$, the maps π_x and π_y are respectively described by:

$$\begin{cases} x = x_1 y_1 \\ y = y_1 \end{cases} \quad \text{and} \quad \begin{cases} x = x_2 \\ y = x_2 y_2 \end{cases}$$

One recognizes the blow-up of 0 in $\mathbf{C}^2$. Now, in order to compute the strict transform of $C_{p/q}$, one has to make the previous changes of variables in equation (19). The lemma follows immediately. Q.E.D.

Starting from Lemma 6.7 and using the previous lemma as an induction step, we get:

Proposition 6.9. *The following procedure constructs the dual graph of the total transform of* $C_{p/q}$ *by the minimal embedded resolution morphism, starting from the zigzag diagram* $ZZ(p/(p-q))$:

- *On each edge of integral length* $l \geq 1$, *add* $(l-1)$ *vertices of weight* 2. *Then erase the weights of the edges* (*that is, their integral length*).
- *Attach the weight* 1 *to the vertex* A_+. *Then change the signs of all the weights of the vertices.*
- *Label the vertices by the symbols* $E_1, E_2, E_3, \ldots$ *starting from* A_1 *on* $P(\sigma)$ *till arriving at* V_1, *continuing from the first vertex which follows* V'_1 *on* $P(\sigma')$ *till arriving at* V'_2, *coming then back to* $P(\sigma)$ *at the first vertex which follows* V_1 *and so on, till labelling the vertex* A_+.
- *Erase the horizontal line, the zigzag line and the curved segment between* V'_0 *and the first vertex which follows* V'_1.
- *Add an arrow to the vertex* A_+ *and keep only the weights of the vertices and their labels* E_n.

The arrowhead vertex represents the strict transform of the curve $C_{p/q}$ *and the indices of the components* E_i *correspond to the orders of appearance during the process of blow-ups.*

It is essential to remark that in the previous construction one starts from $ZZ(p/(p-q))$ and not from $ZZ(p/q)$ (look again at Lemma 6.7).

Example 6.10. Consider the curve $x^{11} - y^4 = 0$. Then $\lambda = 11/(11-4) = 11/7$. Its zigzag diagram $ZZ(11/7)$ was constructed in Example 5.10. So, the dual graph of the total transform of $C_{11/4}$ by the minimal embedded resolution morphism has 6 vertices, of easy computable weights (see Fig. 13).

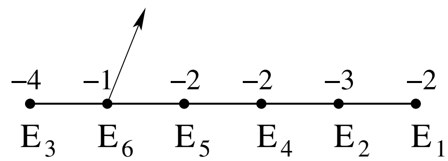

Fig. 13. The dual graph of the total transform of $C_{11,\,4}$

Proposition 6.9 endows us with an easy way of remembering the following classical description of the minimal embedded resolution of a monomial plane curve (see Jurkiewicz [43], who attributes it to Hirzebruch; Spivakovsky [74] extends it to the case of monomial-type valuations on function-fields of surfaces):

Proposition 6.11. *If* $p/q = [m_1+1, n_1+1, m_2+1, \ldots, n_s+1, m_{s+1}+1]^+$, *then the dual graph of the total transform of the monomial curve* $C_{p/q}$ *is the one which appears in* Fig. 14.

Proof. Combine formulae (20) and (18) with Fig. 9 and Proposition 6.9. Q.E.D.

In Fig. 14 we have indicated only the orders of appearance of the components of the exceptional divisor corresponding to the extremities of the graph. We leave as an exercise for the reader to complete the diagram with the sequence $(E_k)_{k\geq 1}$.

Notice that in the E-continued fraction expansion of p/q used in the previous proposition, there is the possibility that $m_{s+1} = 0$. In this case, the canonical expansion is obtained using relation (7). But in order to express in a unified form the result of the application of the algorithm, it was important for us to use an expansion of $\frac{p}{q}$ with an *odd* number of partial quotients (which is always possible, precisely according to formula (7)).

One can use the combinatorics of the embedded resolution of monomial plane curves as building blocks for the description of the combinatorics of the resolution of any germ of plane curve. A detailed description of the passage between *the Eggers tree*, which encodes the Newton-Puiseux exponents of the components of the curve, and the dual graph

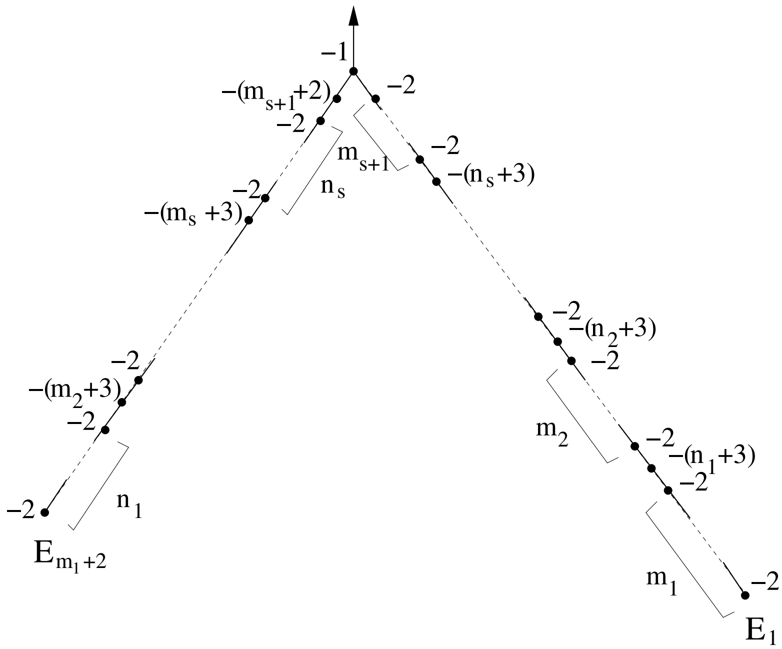

Fig. 14. The dual graph of the total transform of $C_{p/q}$

of the total transform of the curve by its embedded resolution morphism can be found in García Barroso [24] (see also Brieskorn & Knörrer [9, section 8.4] and Wall [85]). A topological interpretation of the trees appearing in these two encodings was given in Popescu-Pampu [62, chapter 4].

In higher dimensions, González Pérez [27] used toric geometry in order to describe embedded resolutions of quasi-ordinary hypersurface singularities. Again, the building blocks are monomial varieties. A prototype for his study is the method of resolution of an irreducible germ of plane curve by only one toric morphism, developed by Goldin & Teissier [26].

In the classical treatise of Enriques & Chisini [19], resolutions of curves by blow-ups of points are not studied using combinatorics of divisors, but instead using the *infinitely near points* through which the strict transforms of the curve pass during the process of blowing ups. Those combinatorics were also encoded in a diagram, called nowadays *Enriques diagram* (see Casas-Alvero [10]). Enriques diagrams are very easily constructed using the knowledge of the orders of appearance of the divisors during the process of blowing ups. For this reason, zigzag diagrams combined with Proposition 6.9 give an easy way to draw them for a monomial plane curve. We leave the details to the interested reader. Then one uses this again as building blocks for the analysis of general plane curve singularities (see [10]).

§7. Graph structures and plumbing structures on 3-manifolds

This section contains preparatory material for the topological study of the 3-manifolds appearing as abstract boundaries of normal surface singularities, done in sections 8 and 9.

We recall general facts about Seifert, graph and plumbing structures on 3-manifolds, as well as about JSJ theory. We also define particular classes of plumbing structures on thick tori and solid tori, starting from naturally arising pairs (L, σ), where L is a 2-dimensional lattice and σ is a rational strictly convex cone in $L_{\mathbf{R}}$. Namely, given a pair of essential curves on the boundary of a thick torus M, their classes generate two lines in the lattice $L := H_1(M, \mathbf{Z})$. A choice of orientations of these lines distinguishes one of the four cones in which the lines divide the plane...

7.1. Generalities on manifolds and their splittings

We denote by $\mathbf{I}$ the interval $[0, 1]$, by $\mathbf{D}$ the closed disc of dimension 2 and by $\mathbf{S}^n$ the sphere of dimension n. An *annulus* is a surface diffeomorphic to $I \times \mathbf{S}^1$.

A simple closed curve on a 2-dimensional torus is called *essential* if it is non-contractible. It is classical that an oriented essential curve on a torus T is determined up to isotopy by its image in $H_1(T, \mathbf{Z})$ (see [21, Section 2.3]). Moreover, the vectors of $H_1(T, \mathbf{Z})$ which are homology classes of essential curves are precisely the primitive ones.

We say that a manifold is *closed* if it is compact and without boundary. If M is a manifold with boundary, we denote by $\overset{\circ}{M}$ its interior and by ∂M its boundary. If moreover M is oriented, we orient ∂M in such a way that at a point of ∂M, an outward pointing tangent vector to M, followed by a basis of the tangent space to ∂M, gives a basis of the tangent space to M (this is the convention which makes Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$ true). We say then that ∂M is oriented *compatibly with* M.

If M is an oriented manifold, we denote by $-M$ the same manifold with reversed orientation. If M is a closed oriented surface, then $-M$ is orientation-preserving diffeomorphic to M. This fact is no longer true in dimension 3, that is why it is important to describe carefully the choice of orientation. In this sense, see Theorem 8.11, as well as Propositions 9.3 and 9.6.

We denote by $\mathrm{Diff}(M)$ the group of self-diffeomorphisms of M, by $\mathrm{Diff}^\circ(M)$ the subgroup of self-diffeomorphisms which are isotopic to the identity and by $\mathrm{Diff}^+(M)$ the subgroup of diffeomorphisms which preserve the orientation of M (when M is orientable).

Definition 7.1. Let M be a 3-manifold with boundary. We say that M is a **thick torus** if it is diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{I}$. We say that M is a **solid torus** if it is diffeomorphic to $\mathbf{D} \times \mathbf{S}^1$. We say that M is a **thick Klein bottle** if it is diffeomorphic to a unit tangent circle bundle to the Möbius band.

In the definition of a thick Klein bottle M we use an arbitrary riemannian metric on a Möbius band. The manifold obtained like this is independent of the choices up to diffeomorphism. Moreover, it is orientable, because any tangent bundle is orientable and the manifold we define appears as the boundary of a unit tangent disc bundle. The preimage of a central circle of the Möbius band by the fibration map is a Klein bottle, and the manifold M appears then as a tubular neighborhood of it, which explains the name. For details, see [82, Section 3] and [21, Section 10.11].

On the boundary of a solid torus M there exists an essential curve which is contractible in M. Such a curve, which is unique up to isotopy (see [21]), is called a *meridian* of M. A 3-manifold M is called *irreducible* if any embedded sphere bounds a ball. A surface embedded in M is called *incompressible* if its π_1 injects in $\pi_1(M)$. Two tori embedded in M are called *parallel* if they are disjoint and they cobound a thick torus embedded in M. The manifold M is called *atoroidal* if any embedded incompressible torus is parallel to a component of ∂M.

Definition 7.2. Let M be an orientable manifold and S be an orientable closed (not necessarily connected) hypersurface of M. A manifold with boundary M_S endowed with a map $M_S \xrightarrow{r_{M,S}} M$ is called **a splitting of M along S** if:

- $r_{M,S}$ is a local embedding;
- $\partial M_S = (r_{M,S})^{-1}(S)$ and the restriction $r_{M,S}|_{\partial M_S}$ is a trivial double covering of S;
- the restriction $(r_{M,S})|_{\overset{\circ}{M}_S} : \overset{\circ}{M}_S \longrightarrow M - S$ is a diffeomorphism.

If this is the case, the map $r_{M,S}$ is called the **reconstruction map** associated to the splitting. We say that S **splits** M **into** M_S and that the connected components of M_S are the **pieces** of the splitting. If N

is a piece of M_S and $P \subset M$ is a set, we say that P **contains** N if $r_{M,S}(N) \subset P$.

It can be shown easily that splittings of M along S exist and are unique up to unique isomorphism. The idea is very intuitive, one simply thinks at M being split open along each connected component of S. A way to realize this is to take the complement of an open tubular neighborhood of S in M and to deform the inclusion mapping in an arbitrarily small neighborhood of the boundary in order to push it towards S (see Waldhausen [83] and Jaco [39]).

If $\phi \in \mathrm{Diff}^+(M)$, one can also canonically split ϕ and get a diffeomorphism ϕ_S of manifolds with boundary (we leave the axiomatic definition of ϕ_S to the reader):

$$\phi_S \colon M_S \longrightarrow M_{\phi(S)}$$

Among closed 3-manifolds, two particular classes will be especially important for us, the lens spaces and the torus fibrations. The reason why we treat them simultaneously will appear clearly in Section 8.3.

Definition 7.3. Let M be an orientable 3-manifold. We say that M is a **lens space** if it contains an embedded torus T such that M_T is the disjoint union of two solid tori whose meridians have non-isotopic images on T. We say that M is a **torus fibration** if it contains an embedded torus T such that M_T is a thick torus.

Lens spaces can also be defined as quotients of $\mathbf{S}^3$ by linear free cyclic actions or - and this explains the name - as manifolds obtained by gluing in a special way the faces of a lens-shaped polyhedron (see [71] or [21, Section 4.3]). We impose the condition on the meridians in order to avoid the manifold $\mathbf{S}^1 \times \mathbf{S}^2$, which can also be split into two solid tori, but whose universal cover is not the 3-dimensional sphere, a difference which makes it to be excluded from the set of lens spaces by most authors. There exists a classical encoding of oriented lens spaces by positive integers. We recall it at the end of Section 9.1 (see Proposition 9.4).

If M is a torus fibration and $T \subset M$ splits it into a thick torus, then a trivial foliation of M_T by tori parallel to the boundary components is projected by $r_{M,T}$ onto a foliation by pairwise parallel tori. The space of leaves is topologically a circle and the projection $\pi \colon M \to \mathbf{S}^1$ is a locally trivial fibre bundle whose fibres are tori, which explains the name.

Definition 7.4. Let $\pi \colon M \to \mathbf{S}^1$ be a locally trivial fibre bundle whose fibres are tori. Fix a fibre of π (for example the initial torus T) and also an orientation of the base space $\mathbf{S}^1$. The **algebraic monodromy**

operator m is by definition the first return map of the natural parallel transport on the first homology fibration over $\mathbf{S}^1$, when one travels in the positive direction.

The map m is a well-defined linear automorphism $m \in SL(H_1(T, \mathbf{Z}))$, once an orientation of $\mathbf{S}^1$ was chosen. Its conjugacy class in $SL(2, \mathbf{Z})$ is independent of the choice of the fibre. If one changes the orientation of $\mathbf{S}^1$, then m is replaced by m^{-1}. This shows that the trace of m is independent of the choice of T and of the orientation of $\mathbf{S}^1$. Remark that no choice of orientation of M is needed in order to define it.

For more information about torus fibrations, see Neumann [57] and Hatcher [33]. We come back to them in Section 9.2, with special emphasis on subtleties related to their orientations.

7.2. Seifert structures

Seifert manifolds are special 3-manifolds whose study can be reduced in some way to the study of lower-dimensional spaces.

Definition 7.5. A **Seifert structure** on a 3-manifold M is a foliation by circles such that any leaf has a compact orientable saturated neighborhood. A leaf with trivial holonomy is called a **regular fibre**. A leaf which is not regular is called an **exceptional fibre**. The space of leaves is called the **base** of the Seifert structure. We say that a Seifert structure is **orientable** if there is a continuous orientation of all the leaves of the foliation. If such an orientation is fixed, one says that the Seifert structure is **oriented**. If there exists a Seifert structure on M, we say that M is a **Seifert manifold**.

The condition on the leaves to have compact saturated neighborhoods is superfluous if the ambient manifold M is compact, it is enough then to ask that any leaf be orientation-preserving, as was shown by Epstein [20]. This is no longer true on non-compact manifolds, as was shown by Vogt [81].

The initial definition of Seifert [70] was slightly different:

a) He did not speak of “foliation”, but of “fibration”.

b) He gave models for the possible neighborhoods of the leaves.

In what concerns point a), Seifert’s definition is one of the historical sources of the concept of fibration and fibre bundle. For him a fibration is a decomposition of a manifold into “fibres”; only in a second phase can one try to construct the associated “orbit space”, or the “base” with our vocabulary. This shows that his definition is closer to the present notion of foliation; in fact his “fibration” is a foliation, but this can be seen only by using the required condition on model neighborhoods. We prefer to speak about “Seifert structure” and not “Seifert fibration” precisely because what is important to us is to see the structure as living *inside* the manifold, which makes possible to speak about isotopies. For details about the historical development of different notions of fibrations, see Zisman [88].

In what concerns point b), the possible orientable saturated neighborhoods of foliations by circles coincide up to a leaf-preserving diffeomorphism with Seifert's model neighborhoods. If one drops the orientability condition, appears a new model which was not considered by Seifert, but which is very useful in the classification of *non-orientable* 3-manifolds (see Scott [69], Bonahon [6]). Some general references about Seifert manifolds are Orlik [61], Neumann & Raymond [58] (where the base was defined as an orbifold), Scott [69], Fomenko & Matveev [21] and Bonahon [6].

In the sequel, we are interested in Seifert structures only *up to isotopy.*

Definition 7.6. Two Seifert structures $\mathcal{F}_1$ and $\mathcal{F}_2$ on M are called **isotopic** if there exists $\phi \in \mathrm{Diff}^{\circ}(M)$ such that $\phi(\mathcal{F}_1) = \mathcal{F}_2$.

The following proposition is proved in Jaco [39] and Fomenko & Matveev [21].

Proposition 7.7. *The only orientable compact connected 3-manifolds with non-empty boundary which admit more than one Seifert structure up to isotopy are the thick torus, the solid torus and the thick Klein bottle.*

a) If M is a thick torus*, any essential curve on one of its boundary components is the fibre of a Seifert structure on* M*, unique up to isotopy, and devoid of exceptional fibres. Moreover,* M *appears like this as the total space of a trivial circle bundle over an annulus.*

b) If M is a solid torus *and* γ *is a meridian of it, an essential curve* c *on its boundary is a fibre of a Seifert structure on* M *if and only if their homological intersection number* $[c] \cdot [\gamma]$ (*once they are arbitrarily oriented*) *is non-zero. In this case, the associated structure is unique up to isotopy and has at most one exceptional fibre. All fibres are regular if and only if* $[c] \cdot [\gamma] = \pm 1$*. In this last case,* M *appears as the total space of a trivial circle bundle over a disc.*

c) If M is a thick Klein bottle*, it admits up to isotopy two Seifert structures. One of them is devoid of exceptional fibres and its space of orbits is a Möbius band. The other one has two exceptional fibres with holonomy of order* 2 *and its space of orbits is topologically a disc.*

The closed orientable 3-manifolds which admit more than one Seifert structure up to isotopy are also classified (see Bonahon [6] and the references therein). In this paper we need only the following less general result, which can be deduced by combining [6] with [57] (see Definition 8.1):

Proposition 7.8. *The only* 3*-manifolds which are diffeomorphic to abstract boundaries of normal surface singularities and which admit non-isotopic Seifert structures are the lens spaces.*

7.3. Graph structures and JSJ decomposition theory

If one glues various Seifert manifolds along components of their boundaries, one obtains so-called *graph-manifolds*:

Definition 7.9. A **graph structure** on a 3-manifold M is a pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T}$ is an embedded surface in M whose connected components are tori and where $\mathcal{F}$ is a Seifert structure on $M_{\mathcal{T}}$ (see Definition 7.2). We say that a graph structure is **orientable** if $\mathcal{F}$ is an orientable Seifert structure on $M_{\mathcal{T}}$. If there exists a graph structure on M, we say that M is a **graph manifold**.

Notice that no particular graph structure is specified when one speaks about a graph manifold. One only supposes that there exists one. In the sequel we are interested in graph structures on a given manifold only *up to isotopy*:

Definition 7.10. Two graph structures $(\mathcal{T}_1, \mathcal{F}_1)$, $(\mathcal{T}_2, \mathcal{F}_2)$ on M are called **isotopic** if there exists $\phi \in \mathrm{Diff}^{\circ}(M)$ such that $\phi(\mathcal{T}_1) = \mathcal{T}_2$ and $\phi_{\mathcal{T}_1}(\mathcal{F}_1)$ is isotopic to $\mathcal{F}_2$.

Graph manifolds were introduced by Waldhausen [82], generalizing von Randow's *tree manifolds* (see their definition in the next paragraph) studied in [65]. Following Mumford [55] who proved Poincaré conjecture for the abstract boundaries of normal surface singularities (see Definition 8.1), von Randow proved it for tree manifolds; his proof contained a gap which was later filled by Scharf [68].

Waldhausen's definition was different from Definition 7.9. On one side he did not allow exceptional fibres in the Seifert structure on $M_{\mathcal{T}}$ and on another side he did not fix (up to isotopy) a precise fibration by circles, but only supposed that such a fibration existed. He represented a graph structure by a finite graph with decorated vertices and edges (corresponding respectively to the pieces of $M_{\mathcal{T}}$ and to the components of $\mathcal{T}$), which explains the name. Tree manifolds are then the graph manifolds which admit a graph structure $(\mathcal{T}, \mathcal{F})$ such that the corresponding graph is a tree and the base of the Seifert structure on $\mathcal{F}$ has genus 0. With our definition, graph structures can also be encoded by graphs. One has only to add more decorations to the vertices, in order to keep in memory the exceptional fibres of the corresponding Seifert fibred pieces.

With his definition, Waldhausen solved the homeomorphism problem for graph-manifolds, by giving normal forms for the graph structures on a given manifold and by showing that with exceptions in a finite explicit list, any irreducible graph-manifold has a graph-structure in normal form which is unique *up to isotopy*.

Later, Jaco & Shalen [40] and Johannson [41] showed that there remains no exception in the classification up to isotopy if one modifies the notion of graph-structure by allowing exceptional fibres, that is, when one works with Definition 7.9. More generally, they proved:

Theorem 7.11. *Let M be a compact, connected, orientable and irreducible* 3*-manifold* (*with possible non-empty boundary*). *Then M contains an embedded surface $\mathcal{T}$ whose connected components are incompressible tori and such that any piece of $M_{\mathcal{T}}$ is either a Seifert manifold or is atoroidal. Moreover, if $\mathcal{T}$ is minimal for the inclusion among surfaces with this property, then it is well-defined up to isotopy.*

We say that a minimal family $\mathcal{T}$ as in the previous theorem is a *JSJ family of tori* in M.

A variant of the previous theorem considers also embedded annuli. These various theorems of canonical decomposition are called nowadays *Jaco-Shalen-Johannson* (*JSJ*) *decomposition theory*, and were the starting point of Thurston's geometrization program, as well as of the theory of JSJ decompositions for groups. For details about JSJ decompositions, in addition to the previously quoted books one can consult Jaco [39], Neumann & Swarup [59], Hatcher [33] and Bonahon [6]. In [62] and [63], we showed that also knot theory inside an irreducible 3-manifold reflects the ambient JSJ decomposition.

We define now a notion of *minimality* for graph structures on a given manifold.

Definition 7.12. Suppose that $(\mathcal{T}, \mathcal{F})$ is a graph structure on M. We say that it is **minimal** if the following conditions are verified:

- No piece of $M_{\mathcal{T}}$ is a thick torus or a solid torus.
- One cannot find a Seifert structure $\mathcal{F}'$ on $M_{\mathcal{T}}$ such that the images of its leaves by the reconstruction mapping $r_{M,\mathcal{T}}$ coincide on a component of $\mathcal{T}$.

As a corollary of Theorem 7.11, if $(\mathcal{T}, \mathcal{F})$ is a minimal graph structure on M, then $\mathcal{T}$ is the minimal JSJ system of tori in M. But one can prove more:

Theorem 7.13. *Each closed orientable irreducible graph manifold which is not a torus fibration with* $|\operatorname{tr} m| \geq 3$ *admits a minimal graph structure. Moreover, the family* $\mathcal{T}$ *of tori associated to a minimal graph structure coincides with the JSJ family of tori. In particular, it is unique up to an isotopy.*

Suppose that $(\mathcal{T}, \mathcal{F})$ is a given graph structure without thick tori and solid tori among its pieces. In view of Proposition 7.7, its only pieces which can have non-isotopic Seifert structures are the thick Klein bottles. This shows that, in order to check whether $(\mathcal{T}, \mathcal{F})$ is minimal or not, one has only to consider the possible choices of Seifert structures on them up to isotopy (that is 2^n possibilities, where n is the number of such pieces).

Suppose that M is a graph manifold which is neither a torus fibration with $|\operatorname{tr} m| \geq 3$, nor a Seifert manifold which admits non-isotopic Seifert structures. Then, if $\mathcal{T}$ is a family of tori associated to a minimal graph structure, there is a unique Seifert structure on $M_{\mathcal{T}}$ up to isotopy, such that each piece which is a thick Klein bottle has an orientable base.

Definition 7.14. Suppose that M is an orientable graph manifold which is neither a torus fibration with $|\operatorname{tr} m| \geq 3$ nor a Seifert manifold which admits non-isotopic Seifert structures. We say that a minimal graph structure is the **canonical graph structure** on M if each piece which is a thick Klein bottle has an orientable base.

7.4. Plumbing structures

Plumbing structures are special types of graph structures:

Definition 7.15. A **plumbing structure** on a 3-manifold M is a graph structure without exceptional fibres $(\mathcal{T}, \mathcal{F})$ on M, such that for any component T of $\mathcal{T}$, the homological intersection number on T of two fibres of $\mathcal{F}$ coming from opposite sides is equal to ± 1.

Plumbing structures are the ancestors of graph structures. They were introduced by Mumford [55] in the study of singularities of complex analytic surfaces (see Hirzebruch [36], Hirzebruch, Neumann & Koh [38], as well as our explanations in Section 8.2). In fact Mumford does not speak about "plumbing structure". Instead, he describes a way to construct the abstract boundary of a normal surface singularity (see Definition 8.1) by gluing total spaces of circle-bundles over real surfaces using "plumbing fixtures".

Later on, "plumbing" was more used as a verb than as a noun. That is, one concentrated more on the operations needed to construct a new object from elementary pieces, than on the structure obtained on the manifold resulting from the construction. The fact that we are interested precisely in this structure up to isotopy and not on the graph which encodes it, is a difference with Neumann [57] for example.

In [57], Neumann describes an algorithm for deciding if two manifolds obtained by plumbing are diffeomorphic. He uses as an essential ingredient Waldhausen's classification theorem of graph manifolds (according to the definition which does not allow exceptional fibres, see the comments made in Section 7.3). In fact, by using the uniqueness up to isotopy of the JSJ-tori, we can deduce the uniqueness up to isotopy for special plumbing structures on singularity boundaries. This is the subject of Section 9.

Even if Definition 7.15 seems to suggest the opposite, the class of graph manifolds is the same as the class of manifolds which admit a plumbing structure. A way to see this is to use the construction of plumbing structures on thick tori and solid tori described in Section 7.5. For a detailed comparison of graph structures and plumbing structures, as well as for a study of the elementary operations on them, one can consult Popescu-Pampu [62, Chapter 4].

7.5. Hirzebruch-Jung plumbing structures on thick tori and solid tori

In this section we define special classes of plumbing structures on thick tori and solid tori, which will be used in Section 9. The starting point is in both cases a pair (L, σ) of a 2-dimensional lattice and a rational strictly convex cone $\sigma \subset L_{\mathbf{R}}$, naturally attached to essential curves on the boundary of the 3-manifold.

• *Suppose first that M is an* oriented *thick torus.*

On each component of its boundary, we consider an essential curve. Denote by γ, δ these curves. We suppose that their homology classes (once they are arbitrarily oriented) in $H_1(M, \mathbf{R}) \simeq \mathbf{R}^2$ are non-proportional. So, we are in presence of a 2-dimensional lattice $L = H_1(M, \mathbf{Z})$ and of two distinct rational lines in it, generated by the homology classes $[\gamma]$, $[\delta]$.

Orient ∂M compatibly with M. Then *order* in an arbitrary way the components of ∂M: call the first one T_- and the second one T_+. Denote by γ_- the simple closed curve drawn on T_- and by γ_+ the one drawn on T_+. Then *orient* γ_- and γ_+. By hypothesis, their homology classes $[\gamma_-]$, $[\gamma_+]$ are non-proportional primitive vectors in the 2-dimensional lattice $L = H_1(M, \mathbf{Z})$. This shows that $([\gamma_-], [\gamma_+])$ is a basis of $L_{\mathbf{R}} =$

$H_1(M, \mathbf{R})$ which induces an orientation of this vector space. As T_+ is a deformation retract of M, one has canonically $H_1(T_+, \mathbf{Z}) = L$, and so the ordered pair (γ_-, γ_+) induces an orientation of T_+.

Definition 7.16. We say that γ_- and γ_+ are oriented **compatibly with the orientation of** M if, when taken in the order (γ_-, γ_+), they induce on T_+ an orientation which coincides with its orientation as a component of ∂M.

Of course, a priori there is no reason for choosing this notion of compatibility rather than the opposite one. Our choice was done in order to get a more pleasant formulation for Lemma 8.5.

Let σ be the cone generated by $[\gamma_-]$ and $[\gamma_+]$ in $L_{\mathbf{R}}$. As these homology classes were supposed non-proportional, the cone σ is strictly convex and has non-empty interior. Denote by $l_\pm$ the edge of σ which contains the integral point $[\gamma_\pm]$. Then, with the notations of Section 4, $A_\pm = [\gamma_\pm]$. Indeed, as $\gamma_\pm$ is an essential curve of $T_\pm$, its homology class is a primitive vector of L.

Let $(A_n)_{0 \leq n \leq r+1}$ be the integral points on the compact edges of $P(\sigma)$, defined in Section 4. So, $OA_0 = [\gamma_-]$ and $OA_{r+1} = [\gamma_+]$. Let $(T_n)_{0 \leq n \leq r+2}$ be a sequence of pairwise parallel tori in M, such that $T_0 = T_-$ and $T_{r+2} = T_+$. Moreover, we number them in the order in which they appear between T_- and T_+. Denote $\mathcal{T} := \bigsqcup_{n=1}^{r+1} T_n$. If M_n denotes the piece of $M_{\mathcal{T}}$ whose boundary components are T_n and T_{n+1}, where $n \in \{0, \ldots, r+1\}$, we consider on it a Seifert structure such that the homology class of its fibres in L is OA_n.

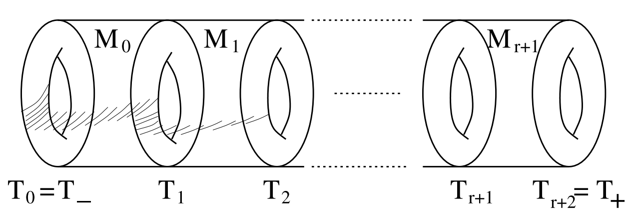

Fig. 15. Hirzebruch-Jung plumbing structures on thick tori

We get like this a plumbing structure on M, well-defined up to isotopy, and depending only on the triple (M, γ_-, γ_+). We see that the simultaneous change of the orientations of γ_- and γ_+ or the change of their ordering (in order to respect the compatibility condition of Definition 7.16) leads to the same (unoriented) plumbing structure.

Definition 7.17. We say that the previous unoriented plumbing structure on the oriented thick torus M is **the Hirzebruch-Jung plumbing structure** associated to (γ, δ) and we denote it by $\mathcal{P}(M, \gamma, \delta)$.

• *Suppose now that* M *is an* oriented *solid torus.*

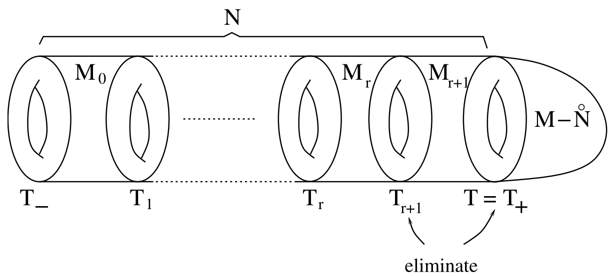

Fig. 16. Hirzebruch-Jung plumbing structures on solid tori

We consider an essential curve γ on ∂M *which is not a meridian.* Take a torus T embedded in $\overset{\circ}{M}$ and parallel to ∂M. Denote by N the thick torus contained between ∂M and T. Put $T_- = \partial M$, $T_+ = T$, $\gamma_- = \gamma$ and let γ_+ be an essential curve on T_+ which is a meridian of the solid torus $M- \overset{\circ}{N}$ (see Fig. 16). Consider the Hirzebruch-Jung plumbing structure $\mathcal{P}(N, \gamma_-, \gamma_+)$. With the notations of the construction done for thick tori, denote $\mathcal{T}(M, \gamma) := \bigsqcup_{n=1}^{r} T_n$. Then the pieces of $M_{\mathcal{T}(M, \gamma)}$ are the thick tori $M_0, M_1, \ldots, M_{r-1}$ and a solid torus which is the "union" of M_r, M_{r+1} and $M- \overset{\circ}{N}$. On $M_0, \ldots, M_{r-1}$ we keep the Seifert structure of $\mathcal{P}(N, \gamma_-, \gamma_+)$. On the solid torus we extend the Seifert structure of M_r. By Proposition 7.7 b), we see that this Seifert structure has no exceptional fibres. This shows that we have constructed a plumbing structure on M. It is obviously well-defined up to isotopy, once the isotopy class of γ is fixed.

Definition 7.18. We say that the previous unoriented plumbing structure on the oriented solid torus M is **the Hirzebruch-Jung plumbing structure** associated to γ and we denote it by $\mathcal{P}(M, \gamma)$.

§8. Generalities on the topology of surface singularities

In this section we look at the boundaries $M(\mathcal{S})$ of normal surface singularities $(\mathcal{S}, 0)$. We explain how to associate to any normal crossings resolution p of $(\mathcal{S}, 0)$ a plumbing structure on $M(\mathcal{S})$. Then we explain how to pass from the plumbing structure associated to the minimal normal crossings resolution of $(\mathcal{S}, 0)$ to the canonical graph structure on $M(\mathcal{S})$ (see Definition 7.14).

We recommend the survey articles of Némethi [56] and Wall [84] for an introduction to the classification of normal surface singularities.

8.1. Resolutions of normal surface singularities and their dual graphs

First we recall basic facts about normal analytic spaces. Let $\mathcal{V}$ be a reduced analytic space. It is called *normal* if for any point $P \in \mathcal{V}$, the germ $(\mathcal{V}, P)$ is irreducible and its local algebra is integrally closed in its field of fractions. If $\mathcal{V}$ is not normal, then there exists a finite map $\nu\colon \tilde{\mathcal{V}} \to \mathcal{V}$ which is an isomorphism over a dense open set of $\mathcal{V}$ and such that $\tilde{\mathcal{V}}$ is normal. Such a map, which is unique up to unique isomorphism, is called a *normalization* map of $\mathcal{V}$.

A reduced analytic curve is normal if and only if it is smooth. If a germ $(\mathcal{S}, 0)$ of reduced surface is normal, then there exists a representative of it, which we keep calling $\mathcal{S}$, such that $\mathcal{S} - 0$ is smooth. The converse is not true.

Let $(\mathcal{S}, 0)$ be a germ of normal complex analytic surface. We say also that $(\mathcal{S}, 0)$ is a *normal surface singularity* (even if the point 0 is regular on $\mathcal{S}$). In the sequel, we use the same notation $(\mathcal{S}, 0)$ for the germ and for a sufficiently small representative of it. If $e\colon (\mathcal{S}, 0) \to (\mathbf{C}^N, 0)$ is any local embedding, denote by $\mathcal{S}_{e,r}$ the intersection of $\mathcal{S}$ with a euclidean ball of $\mathbf{C}^N$ of radius $r \ll 1$ and by $M_{e,r}(\mathcal{S})$ the boundary of $\mathcal{S}_{e,r}$.

By general transversality theorems due to Whitney, when $r > 0$ is small enough, $M_{e,r}(\mathcal{S})$ is a smooth manifold, *naturally oriented* as the boundary of the complex manifold $\mathcal{S}_{e,r}$. It does not depend on the choices of embedding e and radius $r \ll 1$ made to define it (see Durfee [16]).

Definition 8.1. An oriented 3-manifold $M(\mathcal{S})$ orientation-preserving diffeomorphic with the manifolds $M_{e,r}(\mathcal{S})$, where $r > 0$ is small enough, is called the (**abstract**) **boundary** or the **link** of the singularity $(\mathcal{S}, 0)$.

It is important to keep in mind that *in the sequel* $M(\mathcal{S})$ *is supposed naturally oriented as explained before*. In order to understand better this remark, look at Theorem 8.11.

The easiest way to describe the topological type of the manifold $M(\mathcal{S})$ is (as first done by Mumford [55]) by retracting it to the exceptional divisor of a resolution of $(\mathcal{S}, 0)$. Let us first define this last notion.

Definition 8.2. An analytic map $p\colon (\mathcal{R}, E) \to (\mathcal{S}, 0)$ is called **a resolution** of the singularity $(\mathcal{S}, 0)$ with **exceptional divisor** $E = p^{-1}(0)$ if the following conditions are simultaneously satisfied:

- $\mathcal{R}$ is a smooth surface;
- p is a proper morphism;
- the restriction of p to $\mathcal{R} - E = \mathcal{R} - f^{-1}(0)$ is an isomorphism onto $\mathcal{S} - 0$.

We say that $p\colon (\mathcal{R}, E) \to (\mathcal{S}, 0)$ is **a normal crossings resolution** if one has moreover:

- E is a divisor with normal crossings.

Recall that, by definition, a divisor on a smooth complex surface has *normal crossings* if in the neighborhood of any of its points, its support is either smooth, or the union of transverse smooth curves.

Normal crossings resolutions always exist (see Laufer [47] and Lipman [52] for a careful presentation of the Hirzebruch-Jung method of resolution, as well as Cossart [13] for Zariski's method of resolution by normalized blow-ups).

There is a unique *minimal* resolution, which we denote $p_{\min}\colon (\mathcal{R}_{\min}, E_{\min}) \to (\mathcal{S}, 0)$. The minimality property means that any other resolution $p\colon (\mathcal{R}, E) \to (\mathcal{S}, 0)$ can be factorized as $p = p_{\min} \circ q$, where $q\colon \mathcal{R} \to \mathcal{R}_{\min}$ is a proper bimeromorphic map. The minimal resolution $p_{\min}$ is characterized by the fact that $E_{\min}$ contains no component E_i which is smooth, rational and of self-intersection -1 (classically called an *exceptional curve of the first kind*).

Analogously, there is a unique resolution which is minimal among normal crossings ones. We denote it:

$$p_{\mathrm{mnc}}\colon (\mathcal{R}_{\mathrm{mnc}}, E_{\mathrm{mnc}}) \to (\mathcal{S}, 0)$$

It is characterized by the fact that E_{mnc} has normal crossings and each component E_i of E_{mnc} which is an exceptional curve of the first kind contains at least 3 points which are singular on E_{mnc}.

If a normal crossings resolution has moreover only smooth components, one says usually that the resolution is *good*; there exists also a unique minimal good resolution, but in this paper we don't consider it.

The following criterion allows one to recognize the divisors which are exceptional with respect to some resolution of a normal surface singularity.

Theorem 8.3. *Let E be a reduced compact connected divisor in a smooth surface $\mathcal{R}$. Denote by $(E_i)_{1\leq i\leq n}$ its components. Then E is the exceptional divisor of a resolution of a normal surface singularity if and only if the intersection matrix $(E_i \cdot E_j)_{i,j}$ is negative definite.*

The necessity is classical (see [38, Section 9], where is presented Mumford's proof of [55] and where the oldest reference is to Du Val [80]). The sufficiency was proved by Grauert [31] (see also Laufer [47]). If E verifies the conditions which are stated to be equivalent in the theorem, one also says that E *can be contracted* on $\mathcal{R}$.

From now on we suppose that $p\colon (\mathcal{R}, E) \to (\mathcal{S}, 0)$ *is a normal crossings resolution of* $(\mathcal{S}, 0)$.

Denote by $\Gamma(p)$ its *weighted dual graph.* Its set of vertices $\mathcal{V}(p)$ is in bijection with the irreducible components of E. Depending on the context, we think about E_i as a curve on $\mathcal{R}$ or a vertex of $\Gamma(p)$. The vertices which represent the components E_i and E_j are joined by as many edges as E_i and E_j have intersection points on $\mathcal{R}$. In particular, there are as many *loops* based at the vertex E_i as singular points (that is, self-intersections) on the curve E_i (see Fig. 17). Each vertex E_i is decorated by two weights, the *geometric genus* g_i of the curve E_i (that is, the genus of its normalization) and its self-intersection number $e_i \leq -1$ in $\mathcal{R}$. Denote also by δ_i the *valency* of the vertex E_i, that is, the number of edges starting from it (where each loop counts for 2). For example, in Fig. 17 one has $\delta_1 = 9$, $\delta_2 = 5$, etc.

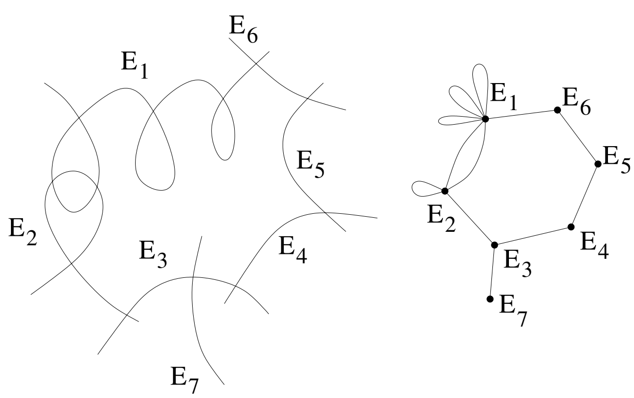

Fig. 17. A normal crossings divisor and its dual graph

8.2. The plumbing structure associated to a normal crossings resolution

By Definition 8.1, $M(\mathcal{S})$ is diffeomorphic to $M_{e,\,r}(\mathcal{S})$, where $e\colon (\mathcal{S}, 0) \to (\mathbf{C}^N, 0)$ is an embedding and $r \ll 1$. But $M_{e,\,r}(\mathcal{S})$ is the level-set at level r of the function $\rho_e\colon (\mathcal{S}, 0) \to (\mathbf{R}_+, 0)$, the restriction to $e(\mathcal{S})$ of the distance-function to the origin in $\mathbf{C}^N$.

As the resolution p realizes by definition an isomorphism between $\mathcal{R} - E$ and $\mathcal{S} - 0$, it means that $M_{e,r}(\mathcal{S}) = \rho_e^{-1}(r)$ is diffeomorphic to $\psi_e^{-1}(r)$, where $\psi_e := \rho_e \circ p$. The advantage of this changed viewpoint on $M(\mathcal{S})$ is that it appears now orientation-preserving diffeomorphic to the boundary of a "tubular neighborhood" of the curve E in the smooth manifold $\mathcal{R}$. As in general E has singularities, one has to discuss the precise meaning of the notion of tubular neighborhood. We quote Mumford [55, pages 230–231]:

> Now the general problem, given a complex $K \subset E^n$, Euclidean n-space, to define a tubular neighborhood, has been attacked by topologists in several ways although it does not appear to have been treated definitively as yet. J.H.C. Whitehead [86], when K is a subcomplex in a triangulation of E^n, has defined it as the boundary of the star of K in the second barycentric subdivision of the given triangulation. I am informed that Thom [78] has considered it more from our point of view: for a suitably restricted class of positive C^∞ fcns. f such that $f(P) = 0$ if and only if $P \in K$, define the tubular neighborhood of K to be the level manifolds $f = \epsilon$, small ϵ. The catch is how to suitably restrict f; here the archtype for f^{-1} may be thought of as the potential distribution due to a uniform charge on K.

In [34] M. W. Hirsch has constructed a theory of tubular neighborhoods well-adapted to complexes K appearing in analytic singularity theory.

Let us come back to the normal crossings divisor E in the smooth surface $\mathcal{R}$.

If E is smooth, then one can construct a diffeomorphism between a tubular neighborhood $U(E)$ of E in $\mathcal{R}$ and of E in the total space $N_{\mathcal{R}}E$ of its normal bundle in $\mathcal{R}$. As $N_{\mathcal{R}}E$ is naturally fibred by discs, this is also true for $U(E)$. The fibration of $U(E)$ can be chosen in such a way that the levels $\psi_e^{-1}(r)$ are transversal to the fibres for $r \ll 1$. In this way one gets a Seifert structure without singular fibres on $\psi_e^{-1}(r) \simeq M(\mathcal{S})$.

Suppose now that *E is not smooth, but that its irreducible components are so.* One can also define in this situation a notion of tubular neighborhood $U(E)$ of E in $\mathcal{R}$. One way to do it is to take the union of conveniently chosen tubular neighborhoods $U(E_i)$ of E's components E_i. Abstractly, one has to glue the 4-manifolds with boundary $U(E_i)$ by identifying well-chosen neighborhoods of the points which get identified on E. This procedure is what is called the "plumbing" of disc-bundles over surfaces (see Hirzebruch [36], Hirzebruch & Neumann & Koh [38], Brieskorn [8]). Its effect on the boundaries $\partial U(E_i)$ is to take out saturated filled tori and to identify their boundaries, by a diffeomorphism which permutes fibres and meridians in an orientation-preserving way. This is the 3-dimensional "plumbing" operation introduced by Mumford [55], alluded to in Section 7.4.

In order to understand what happens near a singular point of E, it is convenient to choose local coordinates (x, y) on E in the neighborhood of the singular point, such that E is defined by the equation $xy = 0$. So, $y = 0$ defines locally an irreducible component E_i of E and similarly $x = 0$ defines E_j. It is possible that $E_i = E_j$, a situation excluded in the previous paragraph for pedagogical reasons. If this equality is true, then the same plumbing procedure can be applied, this time by identifying well-chosen neighborhoods of points of the same 4-manifold with boundary $U(E_i)$.

At this point appears a subtlety: the 4-manifold $U(E_i)$ to be considered is no longer a tubular neighborhood of E_i in $\mathcal{R}$, but instead of the normalization $\tilde{E}_i$ of E_i inside the modified normal bundle $\nu_i^* T\mathcal{R}/T\tilde{E}_i$.

Here $\nu_i \colon \tilde{E}_i \to \mathcal{R}$ denotes the normalization map of E_i and $T\mathcal{R}$, respectively $T\tilde{E}_i$ denote the holomorphic tangent bundles to the smooth complex manifolds $\mathcal{R}$ and $\tilde{E}_i$. As a real differentiable bundle of rank 2, this vector bundle over $\tilde{E}_i$ is characterized by its Euler number $\tilde{e}_i$, which is equal to the self-intersection number of $\tilde{E}_i$ inside the total space of the bundle. This number is related to the self-intersection of E_i inside $\mathcal{R}$ in the following way (see Neumann [57, page 333]):

Lemma 8.4. *If $\tilde{e}_i$ is the Euler number of the real bundle $\nu_i^* T\mathcal{R}/T\tilde{E}_i$ over $\tilde{E}_i$, where $\nu_i \colon \tilde{E}_i \to \mathcal{R}$ is the normalization map of E_i, then $\tilde{e}_i = e_i - \delta_i$.*

Proof. In order to understand this formula, just think at the effect of a small isotopy of E_i inside $\mathcal{R}$. Near each self-crossing point of E_i, the intersection point of one branch of E_i with the image of the other branch after the isotopy is not counted when one computes $\tilde{e}_i$. Q.E.D.

Notice that Theorem 8.3 is *true* if one takes as diagonal entries of the matrix the numbers $e_i = E_i^2$, but is *false* if one takes instead the numbers $\tilde{e}_i$. The easiest example is given by an irreducible divisor $E = E_1$, with $e_1 = 1 > 0$ and $\delta_1 = 2$ which, by Lemma 8.4 implies that $\tilde{e}_1 = -1 < 0$.

In Fig. 18 we represent in two ways the local situation near the chosen singular point of E. On the left we simply draw the union of the two neighborhoods $U(E_i)$ and $U(E_j)$. On the right, "the corners are smoothed". This is precisely what happens when we look at the levels of the function ψ_e. Moreover, we represent by interrupted lines the real analytic set defined by the equation $|x| = |y|$. Its intersection with $\psi_e^{-1}(r) \simeq \partial U(E) \simeq M(\mathcal{S})$ is a two-dimensional torus T. *This is the way in which such tori appear naturally* as structural elements of the 3-manifolds $M(\mathcal{S})$. One also sees how the complement of T in $\partial U(E)$ is fibred by boundaries of discs transversal to E_i or E_j.

By considering model neighborhoods of the singular points of E structured as in the right-hand side of Fig. 18 and conveniently extending them to a tubular neighborhood of all of E, one gets a retraction

$$\Phi \colon U(E) \to E$$

which restricts to a locally trivial disc-fibration over the smooth locus of E and whose fibre over each singular point of E is a cone over a 2-dimensional torus. By considering the restriction $\Phi|_{\partial U(E)}$, we see that the fibres over the singular points of E are embedded tori, and that their complement gets fibred by circles.

As $\partial U(E)$ is orientation-preserving diffeomorphic to $M(\mathcal{S})$, we see that $M(\mathcal{S})$ gets endowed with a graph structure $(\mathcal{T}(p), \mathcal{F}(p))$ well-defined up to isotopy. It is a good test of the understanding of the complexifications of Fig. 18 to show that $(\mathcal{T}(p), \mathcal{F}(p))$ *is in fact a plumbing structure* (see Definition 7.15).

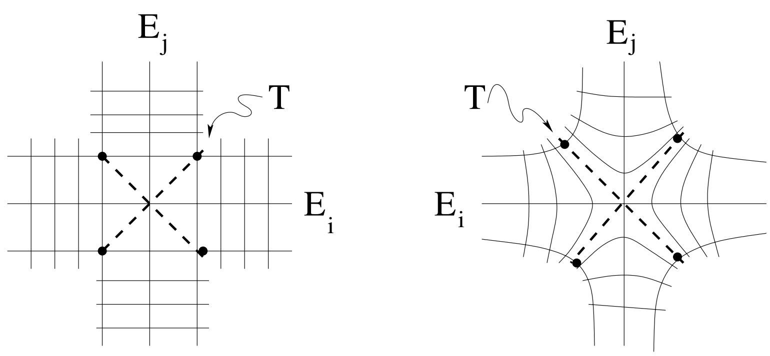

Fig. 18. The local configuration which leads to plumbing

The pieces of $M(\mathcal{S})_{\mathcal{T}(p)}$ correspond to the irreducible components of E, that is to the vertices of $\Gamma(p)$. Denote by $M(E_i)$ the piece which corresponds to E_i. The fibres of $M(E_i)$ are obtained up to isotopy by cutting the boundary of the chosen sufficiently small tubular neighborhood of E with smooth holomorphic curves transversal to E at smooth points of E_i. So, the plumbing structure $(\mathcal{T}(p), \mathcal{F}(p))$ is naturally oriented.

Lemma 8.5. *With their natural orientations, the fibres on both sides of any component of $\mathcal{T}(p)$ are oriented compatibly with the orientation of $M(\mathcal{S})$.*

Proof. The notion of compatibility we speak about is the one of Definition 7.16. We mean that, if we take an arbitrary component T of $\mathcal{T}(p)$, and a tubular neighborhood $N(T)$ such that its preimage in $M(\mathcal{S})_{\mathcal{T}(p)}$ is saturated by the leaves of the foliation $\mathcal{F}(p)$, then two fibres, one in each boundary component of $N(T)$, are oriented compatibly with the orientation of $N(T)$. Now, this is an instructive exercise on the geometrical understanding of the relations between the orientations of various objects in the neighborhood of a normal crossing on a smooth surface. Just think of the complexification of Fig. 18. Q.E.D.

Corollary 8.6. *The orientation of the fibres of $(\mathcal{T}(p), \mathcal{F}(p))$ is determined by the associated unoriented plumbing structure up to a simultaneous change of orientation of all the fibres.*

Proof. Consider the unoriented plumbing structure. Start from an arbitrary piece $M(E_i)$, and choose one of the two continuous orientations of its fibres. Then propagate this orientations farther and farther through the components of $\mathcal{T}(p)$, by respecting the compatibility condition on the neighboring orientations. As $M(\mathcal{S})$ is connected, we know that after a finite number of steps one has oriented the fibres of all the pieces. As one orientation exists which is compatible in the neighborhood of all the tori, we see that our process cannot arrive at a contradiction (that is, a non-trivial monodromy around a loop of $\Gamma(p)$ in the choice of orientations). Q.E.D.

The following lemma is a particular case of the study done in Mumford [55, page 11] and Hirzebruch [36, page 250-03].

Lemma 8.7. *Suppose that E_i is a component of E which is smooth, rational and whose valency in the graph $\Gamma(p)$ is 2. In the thick torus $M(E_i)$ which corresponds to it in the plumbing structure $(\mathcal{T}(p), \mathcal{F}(p))$, consider an oriented fibre f of $M(E_i)$, as well as oriented fibres f', f'' of the two (possibly coinciding) adjacent pieces. Then one has the following relation in the homology group $H_1(M(E_i), \mathbf{Z})$:*

$$[f'] + [f''] = |e_i| \cdot [f].$$

8.3. The topological characterization of HJ and cusp singularities

We want now to understand how to pass from the plumbing structure $(\mathcal{T}(p), \mathcal{F}(p))$ on $M(\mathcal{S})$ to the canonical graph structure on it (see Definition 7.14). We see that the pieces of $M(\mathcal{S})_{\mathcal{T}(p)}$ which are thick tori correspond to components E_i which are smooth and rational with $\delta_i = 2$, and those which are solid tori correspond to components E_i which are smooth and rational with $\delta_i = 1$. It is then natural to introduce the following:

Definition 8.8. We say that a vertex E_i of $\Gamma(p)$ is a **chain vertex** if E_i is smooth, $g_i = 0$ and $\delta_i \leq 2$. If moreover $\delta_i = 2$, we call it an **interior chain vertex**, otherwise we call it a **terminal chain vertex**. We say that a vertex of $\Gamma(p)$ is a **node** if it is not a chain vertex.

In [51], Lê, Michel & Weber used the name “rupture vertex” for a node in the dual graph associated to the minimal embedded resolution of a plane curve singularity. In their situation, where all the vertices represent smooth rational curves, nodes are simply those of valency ≥ 3. In our case this is no longer true, as one can have also vertices of valency ≤ 2, if they correspond to curves E_i which are either not smooth or of genus $g_i \geq 1$.

Denote by $\mathcal{N}(p)$ the set of nodes of $\Gamma(p)$. It is an empty set if and only if $\Gamma(p)$ is topologically a segment or a circle and all the components E_i are smooth rational curves. The first situation occurs precisely for the Hirzebruch-Jung singularities, defined in Section 6.2 (see Proposition 6.2), and the second one for cusp singularities, introduced by Hirzebruch [37] in the number-theoretical context of the study of Hilbert modular surfaces.

Definition 8.9. A germ $(\mathcal{S}, 0)$ of normal surface singularity is called a **cusp singularity** if it has a resolution p such that $\Gamma(p)$ is topologically a circle and $\mathcal{N}(p) = \emptyset$.

For other definitions and details about them, see Hirzebruch [37], Laufer [49] (where they appear as special cases of *minimally elliptic singularities*), Ebeling & Wall [17] (where they appear as special cases of *Kodaira singularities*), Oda [60], Wall [84] and Némethi [56]. They were generalized to higher dimensions by Tsuchihashi (see Oda [60, Chapter 4]).

In the previous definition it is not possible to replace the resolution p by the minimal normal crossings one. Indeed:

Lemma 8.10. *If* $(\mathcal{S}, 0)$ *is a cusp singularity, then* $\Gamma(p_{\mathrm{mnc}})$ *is topologically a circle and either* $\mathcal{N}(p_{\mathrm{mnc}}) = \emptyset$*, or* E_{mnc} *is irreducible, rational, with one singular point where it has normal crossings.*

Proof. One passes from p to p_{mnc} by successively contracting components F which are smooth, rational and verify $F^2 = -1$ (that is, exceptional curves of the first kind, by a remark which follows Definition 8.2). The new exceptional divisor verifies the same hypothesis as the one of p, except when one passes from a divisor with 2 components to a divisor with one component. In this last situation, this second irreducible divisor is rational, as its strict transform F is so. Moreover, it has one singular point with normal crossing branches passing through it, as by hypothesis F cuts transversely the other component of the first divisor in exactly two points. Q.E.D.

We would like to emphasize the following theorem due to Neumann [57, Theorem 3], which characterizes Hirzebruch-Jung and cusp singularities among normal surface singularities.

Theorem 8.11. *Let* $(\mathcal{S}, 0)$ *be a normal surface singularity. The manifold* $-M(\mathcal{S})$ *is orientation-preserving diffeomorphic to the abstract boundary of a normal surface singularity if and only if* $(\mathcal{S}, 0)$ *is either a Hirzebruch-Jung singularity or a cusp-singularity.*

Recall that $-M(\mathcal{S})$ denotes the manifold $M(\mathcal{S})$ with reversed orientation.

We will bring more light on this theorem with Propositions 9.3 and 9.6, which show that for both Hirzebruch-Jung and cusp singularities, the involutions $M(\mathcal{S}) \rightsquigarrow -M(\mathcal{S})$ are manifestations of the duality described in Section 5.

As Hirzebruch-Jung singularities, cusp singularities can also be defined using toric geometry (see Oda [60, Chapter 4]). In the same spirit, as a particular case of Laufer's [48] classification of *taut* singularities, we have:

Theorem 8.12. *Hirzebruch-Jung and cusp singularities are taut, that is, their analytical type is determined by their topological type.*

For this reason, it is natural to ask which 3-manifolds are obtained as abstract boundaries of Hirzebruch-Jung singularities and cusp singularities. This question is answered by:

Proposition 8.13. 1) $(\mathcal{S}, 0)$ *is a Hirzebruch-Jung singularity if and only if* $M(\mathcal{S})$ *is a lens space. Moreover, each oriented lens space appears like this.*

2) $(\mathcal{S}, 0)$ *is a cusp singularity if and only if* $M(\mathcal{S})$ *is a torus fibration with algebraic monodromy of trace* ≥ 3*. Moreover, each oriented torus fibration of this type appears like this.*

Proof. This proposition is a particular case of Neumann [57, Corollary 8.3]. Here we sketch the proofs of the necessities, in order to develop tools for sections 9.1 and 9.2.

Let $p\colon (\mathcal{R}, E) \to (\mathcal{S}, 0)$ be the minimal normal crossings resolution of $(\mathcal{S}, 0)$ (for notational convenience, we drop the index "mnc"). Denote by $U(E)$ a (closed) tubular neighborhood of E in $\mathcal{R}$ and by $\Phi\colon U(E) \to E$ a preferred retraction, as defined in Section 8.2. Denote also by

$$\Psi\colon \partial U(E) \to E$$

the restriction of Φ to $\partial U(E) \simeq M(\mathcal{S})$.

1) *Suppose that* $(\mathcal{S}, 0)$ *is a Hirzebruch-Jung singularity.*

Orient the segment $\Gamma(p)$. Denote then by $E_1, \ldots, E_r$ the components of E in the order in which they appear along $\Gamma(p)$ in the positive direction. For each $i \in \{1, \ldots, r-1\}$, denote by $A_{i,\,i+1}$ the intersection point of E_i and E_{i+1}. Consider also two other points $A_{0,\,1} \in E_1$, $A_{r,\,r+1} \in E_r$ which are smooth points of E. Then consider on each component E_i a Morse function

$$\Pi_i\colon E_i \to \left[\frac{i-1}{r}, \frac{i}{r}\right]$$

having as its only critical points $A_{i-1,i}$ (where Π_i attains its minimum) and $A_{i,\,i+1}$ (where Π_i attains its maximum). As $\Pi_i(A_{i,\,i+1}) = \Pi_{i+1}(A_{i,\,i+1})$ for all $i \in \{1, \ldots, r-1\}$, we see that the maps Π_i can be glued together in a continuous map

$$\Pi\colon E \to [0, 1].$$

Consider the composed continuous map $\Pi \circ \Psi\colon M(\mathcal{S}) \to [0, 1]$ (see Fig. 19).

Our construction shows that its fibres over 0 and 1 are circles and that those over interior points of $[0, 1]$ are tori. Moreover, each such torus splits M into two solid tori. By Definition 7.3, we see that M *is a lens space.*

It remains now to prove that each oriented lens space appears like this.

Denote $L := H_1(M(\mathcal{S}) - (\Pi \circ \Psi)^{-1}\{0, 1\}, \mathbf{Z})$. As $M(\mathcal{S}) - (\Pi \circ \Psi)^{-1}\{0, 1\}$ is the interior of a thick torus foliated by the tori $(\Pi\circ\Psi)^{-1}(c)$, where $c \in (0, 1)$, we see that L is a 2-dimensional lattice. With the

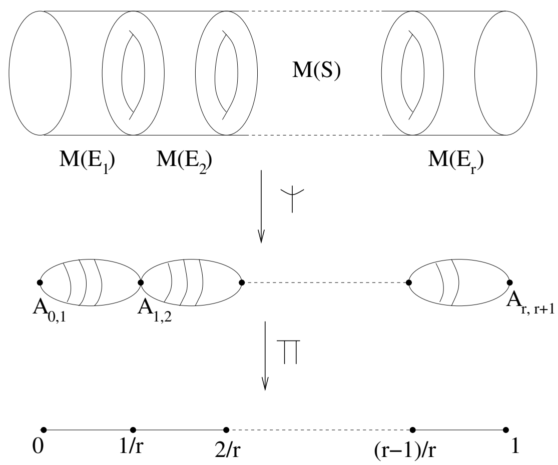

Fig. 19. The maps Π and Ψ for a Hirzebruch-Jung singularity

notations of Section 8.2, let f_i be an oriented fibre in the piece $M(E_i)$ of the plumbing structure $(\mathcal{T}(p), \mathcal{F}(p))$ which corresponds to E_i. Consider also f_0 and f_{r+1}, canonically oriented meridians on the boundaries of tubular neighborhoods of $(\Pi \circ \Psi)^{-1}(0)$, respectively $(\Pi \circ \Psi)^{-1}(1)$.

For each $i \in \{0, \dots, r+1\}$, denote by $v_i := [f_i] \in L$ the homology class of f_i. Recall that $e_i := E_i^2$. By Lemma 8.7, we see that

$$v_{i+1} = |e_i| \cdot v_i - v_{i-1}, \quad \forall i \in \{0, \dots, r\}. \tag{25}$$

By Proposition 6.2, p is also the minimal resolution of $(\mathcal{S}, 0)$, which shows that $|e_i| \geq 2$, $\forall i \in \{1, \dots, r\}$. Now apply Proposition 4.4. We deduce that the numbers e_i are determined by the oriented topological type of the lens space $M(\mathcal{S})$, once the isotopy class of the tori $(\Pi \circ \Psi)^{-1}(c)$ is fixed.

This shows that, starting from any oriented lens space M and torus $T \subset M$ which splits M into two solid tori, one can construct a Hirzebruch-Jung singularity $(\mathcal{S}, 0)$ such that $M(\mathcal{S}) \simeq M$ only by looking at the classes of the meridians of the two solid tori in the lattice $L = H_1(T, \mathbf{Z})$. One has only to be careful to orient them compatibly with the orientation of M (as explained at the beginning of the proof of Lemma 8.5).

2) *Suppose that* $(\mathcal{S}, 0)$ *is a cusp singularity.*

• *Consider first the case where* $r \geq 2$. Orient the circle $\Gamma(p)$ and choose one of its vertices. Denote then by $E_1, \dots, E_r$ the components of E in the order in which they appear along $\Gamma(p)$ in the positive direction, starting from E_1. For each $i \in \{1, \ldots, r\}$, denote by $A_{i,\,i+1}$ the intersection point of E_i and E_{i+1}, where $E_{r+1} = E_1$. Consider then functions $\Pi_i \colon E_i \to [(i-1)/r,\, i/r]$ with the same properties as in the case of Hirzebruch-Jung singularities. By passing to the quotient $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$, we can glue the previous maps into a continuous map:

$$\Pi \colon E \to \mathbf{R}/\mathbf{Z}.$$

Consider then the map $\Pi \circ \Psi \colon M(\mathcal{S}) \to \mathbf{R}/\mathbf{Z}$ (see Fig. 20).

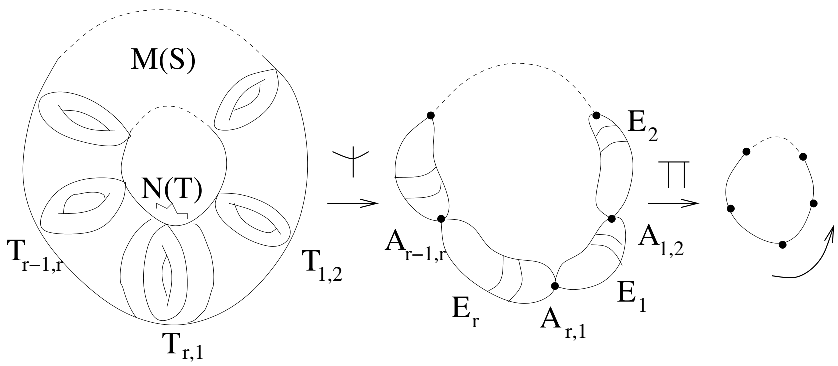

Fig. 20. The maps Π and Ψ for a cusp singularity

Our construction shows that Π realizes $M(\mathcal{S})$ as the total space of a torus fibration over $\mathbf{R}/\mathbf{Z}$.

Denote by $T_{i,\,i+1} := \Psi^{-1}(A_{i,\,i+1})$ the torus of $\mathcal{T}(p)$ which corresponds to the intersection point of E_i and E_{i+1}. Denote $T := T_{r,\,1}$ and let $N(T)$ be a (closed) tubular neighborhood of T, which does not intersect any other torus $T_{i,\,i+1}$, for $i \in \{1, \ldots, r-1\}$ (see Fig. 20).

Denote $L := H_1(M(\mathcal{S}) - N(T),\, \mathbf{Z})$. As $M(\mathcal{S}) - N(T)$ is the interior of a thick torus, we see that L is a 2-dimensional lattice. With the notations of Section 8.2, let f_i be an oriented fibre in the piece $M(E_i)$. We suppose moreover that f_1 and f_r are situated on the boundary of $N(T)$. Consider two other circles f_0 and f_{r+1} on $\partial N(T)$, such that f_0, f_r are isotopic *inside* $N(T)$ and situated on distinct boundary components and such that the same is true for the pair f_1, f_{r+1}.

For each $i \in \{0, \ldots, r+1\}$, denote by $v_i := [f_i] \in L$ the homology class of f_i. By Lemma 8.7, we see that:

$$v_{i+1} = |e_i| \cdot v_i - v_{i-1} = -e_i \cdot v_i - v_{i-1}, \quad \forall i \in \{0, \ldots, r\}, \tag{26}$$

where $E_0 := E_r$.

Denote by $n \in GL(L)$ the automorphism which sends the basis (v_0, v_1) of L into the basis (v_r, v_{r+1}). The relations (26) show that its matrix in the basis (v_0, v_1) is:

$$\begin{pmatrix} 0 & -1 \\ 1 & e_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & e_r \end{pmatrix}$$

A little thinking shows that n is the inverse of the algebraic monodromy $m \in GL(L)$ in the positive direction along $\mathbf{R}/\mathbf{Z}$. So, the matrix of m in the basis (v_0, v_1) is:

$$\begin{pmatrix} e_r & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e_{r-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} e_1 & 1 \\ -1 & 0 \end{pmatrix}$$

We have reproved like this Theorem 6.1 IV in Neumann [57]. We deduce by induction the following expression for its trace, where the polynomials Z^- were defined by formula (1):

$$\text{(27)} \qquad \operatorname{tr} m = Z^-(|e_1|, \ldots, |e_r|) - Z^-(|e_2|, \ldots, |e_{r-1}|).$$

The negative definiteness of the intersection matrix of E (see Theorem 8.3) shows that there exists $i \in \{1, \ldots, r\}$ such that $|e_i| \geq 3$. As p is supposed to be the minimal resolution of $(\mathcal{S}, 0)$, we have also $e_j \geq 2$, $\forall j \in \{1, \ldots, r\}$. Using equation (27), we deduce then easily by induction on r that $\operatorname{tr} m \geq 3$.

• *Consider now the case* $r = 1$. Then, by Lemma 8.10, E is a rational curve with one singular point P, where E has normal crossings. Let $p' \colon (\mathcal{R}', E') \to (\mathcal{S}, 0)$ be the resolution of $(\mathcal{S}, 0)$ obtained by blowing up $P \in \mathcal{R}$. Then E' is a normal crossings resolution with smooth components E_1, E_2, where $E_1^2 = -1$ and E_2 is the strict transform of E. As $(p')^*E = 2E_1 + E_2$ and $((p')^*E)^2 = E^2$, we deduce that $E_2^2 = E^2 - 4 \leq -5$. Now we apply the same argument as in the case $r \geq 2$, but for the resolution p'.

An alternative proof could use Lemma 8.4.

The fact that each oriented torus fibration with $\operatorname{tr} m \geq 3$ appears like this is a consequence of the study done in Section 9.2. Indeed, there we show how to extract the numbers $(e_1, \ldots, e_r)$ from the oriented topological type of $M(\mathcal{S})$. Q.E.D.

By Neumann [57], there exist also abstract boundaries $M(\mathcal{S})$ which are torus fibrations with algebraic monodromy of trace 2. But in that case the exceptional divisor of the minimal resolution is an elliptic curve (then, following Saito [67], one speaks about *simple elliptic singularities*, which are other particular cases of minimally elliptic ones).

8.4. Construction of the canonical graph structure

Consider again an arbitrary normal surface singularity $(\mathcal{S}, 0)$ and a normal crossings resolution p of it.

Definition 8.14. Suppose that the set of nodes $\mathcal{N}(p)$ is non-empty. Conceive the graph $\Gamma(p)$ as a 1-dimensional CW-complex and take the complement $\Gamma(p)-\mathcal{N}(p)$. This complement is the disjoint union of segments, which we call **chains**. If a chain is open at both extremities we call it an **interior chain**. If it is half-open we call it a **terminal chain**.

In Fig. 20 we represent the chains of Fig. 17, with the hypothesis that $E_4, E_5, E_7 \notin \mathcal{N}(p)$ and $E_6 \in \mathcal{N}(p)$. That is, we suppose that E_4, E_5, E_6, E_7 are smooth and that $g(E_4) = g(E_5) = g(E_7) = 0$, $g(E_6) \geq 1$. There is only one terminal chain, which contains the terminal chain vertex E_7.

Denote by $\mathcal{C}(p)$ the set of chains. This set can be written as a disjoint union

$$\mathcal{C}(p) = \mathcal{C}_i(p) \sqcup \mathcal{C}_t(p)$$

where $\mathcal{C}_i(p)$ denotes the set of interior chains and $\mathcal{C}_t(p)$ the set of terminal chains. The edges of $\Gamma(p)$ contained in a chain $C \in \mathcal{C}(p)$ correspond to a set of parallel tori in $M(\mathcal{S})$. Choose one torus T_C among them and define:

$$\mathcal{T}'(p) := \bigsqcup_{C \in \mathcal{C}_i(p)} T_C.$$

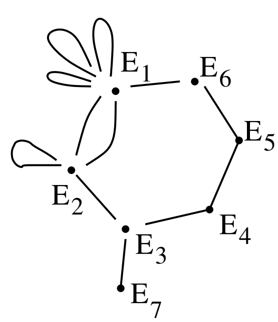

Fig. 21. The chains of Fig. 17 when E_6 is a node

By construction, each piece of $M(\mathcal{S})_{\mathcal{T}'(p)}$ contains a unique piece $M(E_i)$ of $M(\mathcal{S})_{\mathcal{T}(p)}$ such that E_i is a node of $\Gamma(p)$. If $E_i \in \mathcal{N}(p)$, denote by $M'(E_i)$ the piece of $M(\mathcal{S})_{\mathcal{T}'(p)}$ which contains $M(E_i)$. One can extend in a unique way up to isotopy the natural Seifert structure without exceptional fibres on $M(E_i)$ to a Seifert structure on $M'(E_i)$. One obtains like this a graph structure $(\mathcal{T}'(p), \mathcal{F}'(p))$ on $M(\mathcal{S})$.

Till now we have worked with any normal crossings resolution p. We consider now a special one, the minimal normal crossings resolution p_{mnc}.

Proposition 8.15. *Suppose that $(\mathcal{S}, 0)$ is neither a Hirzebruch-Jung singularity, nor a cusp singularity. Then the graph structure $(\mathcal{T}'(p_{\mathrm{mnc}}), \mathcal{F}'(p_{\mathrm{mnc}}))$ is the canonical graph structure on $M(\mathcal{S})$.*

Proof. If $\mathcal{T}'(p_{\mathrm{mnc}})$ *is empty*, as $(\mathcal{S}, 0)$ is not a cusp singularity we deduce that $(\mathcal{T}'(p_{\mathrm{mnc}}), \mathcal{F}'(p_{\mathrm{mnc}}))$ is a Seifert structure. By Proposition 7.8, we see that it is the canonical graph structure on $M(\mathcal{S})$.

Suppose now that $\mathcal{T}'(p_{\mathrm{mnc}})$ *is non-empty.* One has to verify two facts (see Definition 7.14):

• first, that all the fibrations induced by $\mathcal{F}'(p_{\mathrm{mnc}})$ on the pieces which are thick Klein bottles have orientable basis;

• second, that by taking the various choices of Seifert structures on the pieces of $M(\mathcal{S})_{\mathcal{T}'(p_{\mathrm{mnc}})}$, one does not obtain isotopic fibres coming from different sides on one of the tori of $\mathcal{T}'(p_{\mathrm{mnc}})$.

The first fact is immediate, as one starts from Seifert structures with orientable basis on the pieces of $M(\mathcal{S})_{\mathcal{T}(p_{\mathrm{mnc}})}$ before eliminating tori of $\mathcal{T}(p_{\mathrm{mnc}})$ in order to remain with $\mathcal{T}'(p_{\mathrm{mnc}})$.

In what concerns the second fact, the idea is to look at the fibres corresponding to the chain vertices of any interior chain C. The union of the pieces of $M(\mathcal{S})_{\mathcal{T}(p_{mnc})}$ which are associated to those vertices is a thick torus N_R. Take a fibre in each piece (remember that they are naturally oriented as boundaries of holomorphic discs) and look at their images in $L = H_1(N_R, \mathbf{Z})$. One gets like this a sequence of vectors $v_1, \ldots, v_s \in L$. Consider also the images v_0 and v_{s+1} of the fibres coming from the nodes of $\Gamma(p_{\mathrm{mnc}})$ to which C is adjacent, the order of the indices respecting the order of the vertices along the chain.

By Lemma 8.7, $v_{k+1} = \alpha_k v_k - v_{k-1}$ for any $k \in \{1, \ldots, s\}$, where α_k is the absolute value of the self-intersection of the component E_i of $\Gamma(p_{\mathrm{mnc}})$ which gave rise to the vector v_k. Here plays the hypothesis that p_{mnc} is *minimal*: this implies that $\alpha_k \geq 2$. Then one can conclude by using Proposition 7.7.

The analysis of thick Klein bottles is similar. It is based on the fact that a thick Klein bottle can appear only from a portion of the graph $\Gamma(p)$ as in Fig. 21, where E_1, E_2, E_3 are smooth rational curves of self-intersections -2, -2, respectively $-n$ (see Neumann [57, pages 305, 334]). The important point is that $n \geq 2$. Otherwise the complete sub-graph of $\Gamma(p)$ with vertices E_1, E_2, E_3 would have a non-definite intersection matrix, which contradicts Theorem 8.3. Q.E.D.

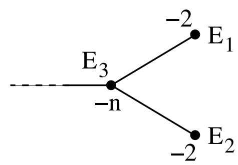

Fig. 22. The appearance of a thick Klein bottle

The plumbing structure $(\mathcal{T}(p_{\text{mnc}}), \mathcal{F}(p_{\text{mnc}}))$ on $N(\mathcal{S})$ is associated to the resolution p_{mnc} of $(\mathcal{S}, 0)$. One can wonder if the canonical graph structure $(\mathcal{T}'(p_{\text{mnc}}), \mathcal{F}'(p_{\text{mnc}}))$ is also associated to some analytic morphism with target $(\mathcal{S}, 0)$.

This is indeed the case. In order to see it, start from p_{mnc} and its exceptional divisor E. Then contract all the components of E which correspond to chain vertices. One gets like this a normal surface with only Hirzebruch-Jung singularities. The image of E on it is a divisor F with again only normal crossings when seen as an abstract curve. Take then as a representative of $M(\mathcal{S})$ the boundary of a tubular neighborhood of F in the new surface and split it into pieces which project into the various components of F. The splitting is done using tori which are associated bijectively to the singular points of F. Namely, in a system of (toric) local coordinates (x, y) such that F is defined by $xy = 0$, one proceeds as for the definition of the plumbing structure associated to a normal crossings resolution (see Section 8.2). Then this system of tori is isotopic to $\mathcal{T}'(p_{\text{mnc}})$.

§9. Invariance of the canonical plumbing structure on the boundary of a normal surface singularity

In this section we describe how to reconstruct the plumbing structure $(\mathcal{T}(p_{\text{mnc}}), \mathcal{F}(p_{\text{mnc}}))$ on $M(\mathcal{S})$ associated to the minimal normal crossings resolution of $(\mathcal{S}, 0)$, only from the abstract oriented manifold $M(\mathcal{S})$. Namely, using the classes of plumbing structures on thick tori defined in Section 7.5, we define a plumbing structure $\mathcal{P}(M(\mathcal{S}))$ on $M(\mathcal{S})$ and we prove:

Theorem 9.1. 1) *When considered as an unoriented structure, the plumbing structure $\mathcal{P}(M(\mathcal{S}))$ depends up to isotopy only on the natural orientation of $M(\mathcal{S})$. We call it the* canonical plumbing structure *on $M(\mathcal{S})$.*

2) *The plumbing structure $(\mathcal{T}(p_{\mathrm{mnc}}), \mathcal{F}(p_{\mathrm{mnc}}))$ associated to the minimal normal crossings resolution of $(\mathcal{S}, 0)$ is isotopic to the canonical plumbing structure $\mathcal{P}(M(\mathcal{S}))$.*

As a corollary we get the theorem of invariance of the plumbing structure $(\mathcal{T}(p_{\mathrm{mnc}}), \mathcal{F}(p_{\mathrm{mnc}}))$ announced in the introduction (see Theorem 9.7). We also explain how the orientation reversal on the boundary of a Hirzebruch-Jung or cusp singularity reflects the duality between supplementary cones explained in Section 5.1 (see Propositions 9.4 and 9.6).

In order to prove Theorem 9.1, we consider three cases, according to the nature of $M(\mathcal{S})$. In the first one it is supposed to be a lens space, in the second one a torus fibration with algebraic monodromy of trace ≥ 3 and in the last one none of the two (so, by Proposition 8.13, this corresponds to the trichotomy: $(\mathcal{S}, 0)$ is a Hirzebruch-Jung singularity/ a cusp singularity/ none of the two).

The idea is to start from some structure on $M(\mathcal{S})$ which is well-defined up to isotopy, and to enrich it by canonical constructions of Hirzebruch-Jung plumbing structures (defined in Section 7.5). When $M(\mathcal{S})$ is neither a lens space nor a torus fibration with algebraic monodromy of trace ≥ 3, this starting structure will be the canonical graph structure (see Definition 7.14). Otherwise we need some special theorems of structure (Theorems 9.2 and 9.5).

9.1. The case of lens spaces

Notice that by Proposition 8.13 1), $M(\mathcal{S})$ is a lens space if and only if $(\mathcal{S}, 0)$ is a Hirzebruch-Jung singularity.

The following theorem was proved by Bonahon [5]:

Theorem 9.2. *Up to isotopy, a lens space contains a unique torus which splits it into two solid tori.*

We say that a torus embedded in a lens space and splitting it into two solid tori is a *central torus.* By the previous theorem, a central torus is well-defined up to isotopy.

Let M be an *oriented* lens space and T a central torus in M. Consider a tubular neighborhood $N(T)$ of T in M, whose boundary components we denote by T_- and T_+, ordered in an arbitrary way. Then $M_{T_- \sqcup T_+}$ has three pieces, one being sent diffeomorphically by the reconstruction map $r_{M, T_- \sqcup T_+}$ on $N(T)$ - by a slight abuse of notations, we keep calling it $N(T)$ - and the others, M_- and M_+, having boundaries sent by $r_{M,\,T_- \sqcup T_+}$ on T_-, respectively T_+ (see Fig. 33). The manifolds M_- and M_+ are solid tori, as T was supposed to be a central torus. Let γ_- and γ_+ be meridians of M_-, respectively M_+, oriented compatibly with the orientation of $N(T)$ (see Definition 7.16). Consider the Hirzebruch-Jung plumbing structure $\mathcal{P}(N(T),\, \gamma_-,\, \gamma_+)$ on $N(T)$, whose tori are denoted by $T_0 = T_-,\, T_1,\, \dots,\, T_{r+2} = T_+$, as explained in Section 7.5.

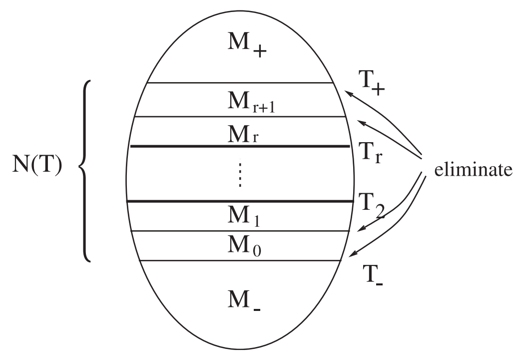

Fig. 23. Construction of the canonical plumbing structure on a lens space

Denote $\mathcal{T}_M := T_2 \sqcup \cdots \sqcup T_r$. Then $M_{\mathcal{T}_M}$ contains four pieces less than the manifold $M_{T_- \sqcup \mathcal{T}_M \sqcup T_+}$. Denote by M'_- and M'_+ the piece which "contains" M_-, respectively M_+. On M'_- we consider the Seifert structure which extends the Seifert structure of M_1 and on M'_+ the one which extends the Seifert structure of M_r. By applying the intersection theoretical criterion of Proposition 7.7 b), we see that those Seifert structures have no exceptional fibres (we used a similar argument to construct in Section 7.5 the Hirzebruch-Jung plumbing structure on solid tori). On the other pieces of $M_{\mathcal{T}_M}$ we consider the Seifert structure coming from the plumbing structure $\mathcal{P}(M,\, \gamma_-,\, \gamma_+)$. Denote by $\mathcal{P}(M)$ the plumbing structure constructed like this on the oriented manifold M.

Proof of Theorem 9.1.

1) This is obvious by construction (we use Theorem 9.2).

2) In the construction of $\mathcal{P}(M(\mathcal{S}))$, one can take as central torus T any torus $(\Pi \circ \Psi)^{-1}(c)$, with $c \in (0,\, 1)$, in the notations of the proof of Proposition 8.13, 1). Then one sees that $[\gamma_-] = [f_0]$ and $[\gamma_+] = [f_{r+1}]$

in the lattice $L = H_1(M(\mathcal{S}) - (\Pi \circ \Psi)^{-1}\{0, 1\}, \mathbf{Z}) = H_1(T, \mathbf{Z})$. Using the relations (25) and the definition of a Hirzebruch-Jung plumbing structure on a thick torus (see Section 7.5), we deduce that the images of the fibres f_i in L are equal to the images of the fibres of $\mathcal{P}(M(\mathcal{S}))$ (see also Proposition 4.4). The proposition follows by the fact that on a 2-torus, any oriented essential curve is well-defined up to isotopy by its homology class. Q.E.D.

Let σ be the strictly convex cone of $L_{\mathbf{R}}$ whose edges are generated by $[\gamma_-]$ and $[\gamma_+]$. If one changes the ordering of the components of $\partial N(T)$, then one gets the same cone σ, and if one changes simultaneously the orientations of γ_- and γ_+, then one gets the opposite cone. But if one *changes the orientation of* M, then the cone σ is replaced by a supplementary cone. So, in view of Section 5.3, the two cones are in duality. In this sense, the canonical plumbing structure $\mathcal{P}(-M(\mathcal{S}))$ is dual to $\mathcal{P}(M(\mathcal{S}))$. We get:

Proposition 9.3. *Let $(\mathcal{S}, 0)$ be a Hirzebruch-Jung singularity. Then the canonical plumbing structures with respect to the two possible orientations of $M(\mathcal{S})$ are dual to each other. More precisely, if $(\mathcal{S}, 0) \simeq (\mathcal{Z}(L, \sigma), 0) \simeq \mathcal{A}_{p,q}$, then $-M(\mathcal{S})$ is orientation preserving diffeomorphic to $M(\check{\mathcal{S}})$, where, with the notations of Section 4, $(\check{\mathcal{S}}, 0) \simeq (\mathcal{Z}(\check{L}, \check{\sigma}), 0) \simeq \mathcal{A}_{p,p-q}$.*

Let $\lambda := p/q$ be the type of the cone (L, σ) in the sense of Definition 5.5, where $0 < q < p$ and $\gcd(p, q) = 1$. The oriented lens space $M(\mathcal{S})$, where $(\mathcal{S}, 0) \simeq (\mathcal{Z}(L, \sigma), 0) \simeq \mathcal{A}_{p,q}$, is said classically to be *of type* $L(p, q)$. By Propositions 5.6 and 5.8, combined with Theorem 9.2, we get the following classical fact:

Proposition 9.4. 1) *The lens spaces $L(p, q)$ and $L(p, q')$ are orientation-preserving diffeomorphic if and only if $p = p'$ and $q' \in \{q, \overline{q}\}$, where $0 < \overline{q} < p$, $q\overline{q} \equiv 1 (\mathrm{mod}\, p)$.*

2) *The lens spaces $L(p, q)$ and $L(p, q')$ are orientation-reversing diffeomorphic if and only if $p = p'$ and $q' \in \{p - q, p - \overline{q}\}$.*

9.2. The case of torus fibrations with $\mathrm{tr}\, m \geq 3$

Notice that by Proposition 8.13 2), $M(\mathcal{S})$ is a torus fibration whose algebraic monodromy verifies $\mathrm{tr}\, m \geq 3$ if and only if $(\mathcal{S}, 0)$ is a cusp singularity. First we study with a little more detail torus fibrations.

Let M be an orientable torus fibration. Take a fibre torus T. Then consider the lattice $L = H_1(T, \mathbf{Z})$ and the algebraic monodromy operator $m \in SL(L)$ (see Definition 7.4) associated with one of the two possible orientations of the base.

The following theorem is a consequence of Waldhausen [83, section 3] (see also Hatcher [33, section 5]):

Theorem 9.5. *Up to isotopy, an orientable torus fibration M such that* $\operatorname{tr} m \geq 3$ *contains a unique torus which splits it into a thick torus* (*see* Definition 7.2).

We say that a torus embedded in an orientable torus fibration whose algebraic monodromy m verifies $\operatorname{tr} m \geq 3$ and which splits it into a thick torus is a *fibre torus*. By the previous theorem, a fibre torus is well-defined up to isotopy.

From now on, we suppose that indeed $\operatorname{tr} m \geq 3$ (see Proposition 8.13, 2)). As M is orientable, m preserves the orientation of L, which shows that $\det m = 1$. This implies that the characteristic polynomial of m is $X^2 - (\operatorname{tr} m)X + 1$. We deduce that m has two strictly positive eigenvalues with product 1, and so the eigenspaces are two distinct real lines in $L_{\mathbf{R}}$.

But the most important point is that these lines are *irrational*. Indeed, the eigenvalues are $\nu_{\pm} := (1/2)(\operatorname{tr} m \pm \sqrt{(\operatorname{tr} m)^2 - 4})$ and $(\operatorname{tr} m)^2 - 4$ is never a square if $\operatorname{tr} m \geq 3$.

Denote by d_- and d_+ the eigenspaces corresponding to ν_-, respectively ν_+. Then m is strictly contracting when restricted to d_- and strictly expanding when restricted to d_+. Choose arbitrarily one of the two half-lines in which 0 divides the line d_-, and call it l_-.

At this point we have not used any orientation of M. *Suppose now that M is oriented.* Then the chosen orientation on the basis of the torus fibration induces an orientation of the fibre torus T, by deciding that this orientation, followed by the transversal orientation which projects on the orientation of the base induces the ambient orientation on M.

Denote by l_+ the half-line bounded by 0 on d_+ into which l_- arrives first when turned in the *negative* direction. Let σ be the strictly convex cone bounded by these two half-lines (see Fig. 24).

We arrive like this at a pair (L, σ) where both edges of σ are irrational. As m preserves L and σ, *it preserves also the polygonal line* $P(\sigma)$.

Let P_1 be an arbitrary integral point of $P(\sigma)$. Consider the sequence $(P_n)_{n\geq 1}$ of integral points of $P(\sigma)$ read in the positive direction along $P(\sigma)$, starting from P_1. There exists an index $t \geq 1$ such that $P_{t+1} = m(P_1)$. It is the *period* of the action of m on the linearly ordered set of integral points of $P(\sigma)$.

Consider t parallel tori $T_1, \ldots, T_t$ inside M, where $T_1 = T$ and the indices form an increasing function of the orders of appearance of the tori when one turns in the positive direction. Denote $\mathcal{T} := \bigsqcup_{1\leq k\leq t} T_k$

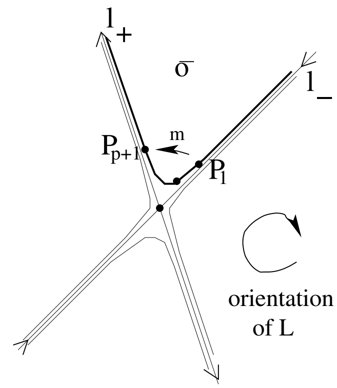

Fig. 24. The case of torus bundles with $\mathrm{tr}\, m \geq 3$

and $T_{t+1} := T_1$. For each $k \in \{1, \ldots, t\}$, denote by M_k the piece of $M_{\mathcal{T}}$ whose boundary components project by $r_{M,\mathcal{T}}$ on T_k and T_{k+1} (see Fig. 25). Then look at the thick torus M_T. Let T_- be its boundary component through which one "enters inside" M_T when one turns in the positive direction, and T_+ be the one by which one "leaves" M_T. Identify then $H_1(M_T, \mathbf{Z})$ with $H_1(T_-, \mathbf{Z})$ through the inclusion $T_- \subset M_{T_1}$ and $H_1(T_-, \mathbf{Z})$ with $H_1(T, \mathbf{Z}) = L$ through the reconstruction mapping $r_{M,T}|_{T_-} : T_- \to T$.

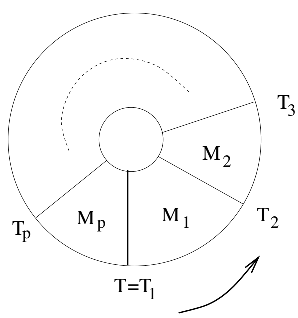

Fig. 25. Construction of the canonical plumbing structure on a torus fibration with $\mathrm{tr}\, m \geq 3$

Consider now on each piece M_k an oriented Seifert fibration $\mathcal{F}_k$ such that the class of a fibre in L (after projection in M_T and identification of $H_1(M_T, \mathbf{Z})$ with L, as explained before) is equal to OP_k. Denote by $\mathcal{F}$ the Seifert structure on $M_{\mathcal{T}}$ obtained by taking the union of the structures $\mathcal{F}_k$. We get like this a plumbing structure on M. Denote it by $\mathcal{P}(M)$.

This plumbing structure does not depend, up to isotopy, on the choice of the initial integral point on $P(\sigma)$. Indeed, by lifting to M a vector field of the form $\partial/\partial\theta$ on the base of the torus fibration and by considering its flow, one sees that one gets isotopic torus fibrations by starting from any integral point of $P(\sigma)$.

Notice that it does neither depend on the choice of the half-line l_-. An opposite choice would lead to the choice of an opposite cone, that is to the same unoriented plumbing structure.

Proof of Theorem 9.1.

1) This is obvious by construction (we use Theorem 9.5).

2) In the construction of $\mathcal{P}(M(\mathcal{S}))$, one can take as fibre torus T the torus $T_{r,\,1}$, with the notations of the proof of Proposition 8.13, 2). Using the relations (26) and Proposition 4.4, we get the Proposition. Q.E.D.

By Theorem 8.12, cusp singularities are determined up to analytic isomorphism by the topological type of the oriented manifold $M(\mathcal{S})$. By Theorem 9.5, this manifold can be encoded by a pair $(T,\, \mu)$, where T is an oriented fibre and μ is a geometric monodromy diffeomorphism of T obtained by turning in the positive direction determined by the chosen orientation of T (recall that this is precisely the point were we use the given orientation of $M(\mathcal{S})$). But it is known that μ can be reconstructed up to isotopy by its action on $L = H_1(T,\, \mathbf{Z})$, that is, by the algebraic monodromy operator $m \in SL(L)$. Moreover, to fix an orientation of T is the same as to fix an orientation of L. As explained in Section 5.3, such an orientation can be encoded in a symplectic isomorphism $\omega\colon L \to \check{L}$.

Denote by $\mathcal{C}(L,\, \omega,\, m)$ the cusp singularity associated to an oriented lattice $(L,\, \omega)$ and an algebraic monodromy operator $m \in SL(L)$ with $\operatorname{tr} m \geq 3$. If one changes the orientation of the base of the torus fibration, one gets the triple $(L,\, -\omega,\, m^{-1})$. This shows that:

$$\mathcal{C}(L,\, \omega,\, m) \simeq \mathcal{C}(L,\, -\omega,\, m^{-1}).$$

When one changes the orientation of $M(\mathcal{S})$, we see that the cone $(L,\, \sigma)$ is replaced by a supplemetary one. In view of Section 5.3, we deduce that the two cones are dual to each other. In this sense, we get the following analog of Proposition 9.3:

Proposition 9.6. *Let $(\mathcal{S},\, 0)$ be a cusp singularity. Then the canonical plumbing structures with respect to the two possible orientations of $M(\mathcal{S})$ are dual to each other. More precisely, if $(\mathcal{S},\, 0) \simeq$*

$(\mathcal{C}(L, \omega, m), 0)$, then $-M(\mathcal{S})$ is orientation preserving diffeomorphic to $M(\check{\mathcal{S}})$, where $(\check{\mathcal{S}}, 0) \simeq (\mathcal{C}(L, -\omega, m), 0)$.

9.3. The other singularity boundaries

As in the two previous cases, we first define the plumbing structure $\mathcal{P}(M(\mathcal{S}))$.

Consider the canonical graph structure $(\mathcal{T}_{\text{can}}, \mathcal{F}_{\text{can}})$ on $M(\mathcal{S})$. We do our construction starting from the neighborhoods of the JSJ tori (the elements of $\mathcal{T}_{\text{can}}$) and the exceptional fibres in $\mathcal{F}_{\text{can}}$.

• For each component T of $\mathcal{T}_{\text{can}}$, consider a saturated tubular neighborhood $N(T)$. We choose them pairwise disjoint. So, each manifold $N(T)$ is a thick torus. We consider on each one of its boundary components a fibre of $\mathcal{F}_{\text{can}}$. Denote these fibres by $\gamma(T)$, $\delta(T)$. We consider on $N(T)$ the restriction of the orientation of $M(\mathcal{S})$. Consider the associated Hirzebruch-Jung plumbing structure $\mathcal{P}(N(T), \gamma(T), \delta(T))$ (see Definition 7.17). Replace the Seifert structure on $N(T)$ induced from $\mathcal{F}_{\text{can}}$ with this plumbing structure. Then eliminate the boundary components of $N(T)$ from the tori present in $M(\mathcal{S})$ (by construction, the fibrations coming from both sides agree on them up to isotopy).

• For each exceptional fibre F, consider a solid torus $N(F)$, which is a saturated tubular neighborhood of F. Choose those neighborhoods pairwise disjoint. On the boundary of $N(F)$, take a fiber $\gamma(F)$ of $\mathcal{F}_{\text{can}}$. Consider the associated Hirzebruch-Jung plumbing structure $\mathcal{P}(N(F), \gamma(F))$ (see Definition 7.18). Replace the Seifert structure on $N(F)$ induced from $\mathcal{F}_{\text{can}}$ with this plumbing structure. Then eliminate the boundary component of $N(F)$ from the tori present inside $M(\mathcal{S})$ (by construction, the fibrations coming from both sides agree on it up to isotopy). Denote by $\mathcal{P}(M(\mathcal{S}))$ the plumbing structure constructed like this on $M(\mathcal{S})$.

Proof of Theorem 9.1. The proof is very similar to the ones explained in the two previous cases, but starting this time from the canonical graph structure on $M(\mathcal{S})$. The main point is Proposition 8.15. We leave the details to the reader. Q.E.D.

9.4. The invariance theorem

Let $(\mathcal{S}, 0)$ be a normal surface singularity. In [57], Neumann proved that the weighted dual graph $\Gamma(p_{\text{mnc}})$ of the exceptional divisor of its minimal normal crossings resolution p_{mnc} is determined by the oriented manifold $M(\mathcal{S})$. But he says nothing about the action of the group $\text{Diff}^+(M(\mathcal{S}))$ on $(\mathcal{T}(p_{\text{mnc}}), \mathcal{F}(p_{\text{mnc}}))$. As a corollary of Theorem 9.1 we get:

Theorem 9.7. *The plumbing structure* $(\mathcal{T}(p_{\mathrm{mnc}}), \mathcal{F}(p_{\mathrm{mnc}}))$ *is invariant up to isotopy by the group* $\mathrm{Diff}^+(M(\mathcal{S}))$.

Proof. Suppose first that $M(\mathcal{S})$ is not a lens space or a torus fibration. As the canonical graph structure on it is invariant by the group $\mathrm{Diff}^+(M(\mathcal{S}))$ up to isotopy, we deduce that the canonical plumbing structure is also invariant up to isotopy by this group. This conclusion is also true when $M(\mathcal{S})$ is a lens space or a torus fibration, as one starts in the construction of $\mathcal{P}(M(\mathcal{S}))$ from tori which are invariant up to isotopy. Then we apply Theorem 9.1. Q.E.D.

An easy study of the fibres of $\mathcal{F}(p_{\mathrm{mnc}})$ in the neighborhoods of the tori of $\mathcal{T}(p_{\mathrm{mnc}})$ which correspond to self-intersection points of components of E_{mnc} show that the analogous statement about the minimal *good* normal crossings resolution of $\mathcal{S}$ is also true.

We arrived at the conclusion that the affirmation of Theorem 9.7 was true while we were thinking about the natural contact structure on $M(\mathcal{S})$ (see Caubel, Némethi & Popescu-Pampu [11]). Indeed, in that paper we prove that for normal surface singularities, the natural contact structure depends only on the topology of $M(\mathcal{S})$ *up to contactomorphisms*. It was then natural to look at the subgroup of $\mathrm{Diff}^+(M(\mathcal{S}))$ which leaves it invariant up to isotopy. Presently, we do not know how to characterize it. But we realized that the homotopy type of the underlying unoriented plane field was invariant by the full group $\mathrm{Diff}^+(M(\mathcal{S}))$, provided that Theorem 9.7 was true (see [11, section 5]).

References

[1] V. I. Arnold, Higher dimensional continued fractions, Regular and chaotic dynamics, **3** (1998), 10–17.

[2] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, Am. J. of Maths., **84** (1962), 485–496.

[3] M. Artin, On isolated rational singularities of surfaces, Am. J. of Maths., **88** (1966), 129–136.

[4] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, Compact complex surfaces, Second enlarged edition, Springer, 2004.

[5] F. Bonahon, Difféotopies des espaces lenticulaires, Topology, **22** (1983), 305–314.

[6] F. Bonahon, Geometric structures on 3-manifolds, In: Handbook of geometric topology, (eds. R. J. Daverman and R. B. Sher), Elsevier, 2002, pp. 93–164.

[7] C. Brezinski, History of Continued Fractions and Padé Approximants, Vol. **12** of Springer Series in Computational Mathematics, Springer, 1991.

[8] E. Brieskorn, Singularities in the work of Friedrich Hirzebruch, Surveys in Differential Geometry, **VII** (2000), 17–60.

[9] E. Brieskorn and H. Knörrer, Plane algebraic curves, Birkhäuser, 1986.

[10] E. Casas-Alvero, Singularities of plane curves, London Mathematical Society Lecture Note Series, **276**, Cambridge U. P., 2000.

[11] C. Caubel, A. Némethi and P. Popescu-Pampu, Milnor open books and Milnor fillable contact 3-manifolds, Topology, **45** (2006), 673–689.

[12] H. Cohn, Support polygons and the resolution of modular functional singularities, Acta Arithmetica, **XXIV** (1973), 261–278.

[13] V. Cossart, Uniformisation et désingularisation des surfaces d'après Zariski, Progress in Math., **181** (2000), 239–258.

[14] D. I. Dais, U. U. Haus and M. Henk, On crepant resolutions of 2-parameter series of Gorenstein cyclic quotient singularities, Results Math., **33** (1998), 208–265.

[15] H. Davenport, The Higher Arithmetic, Seventh edition, Cambridge Univ. Press, 1999.

[16] A. H. Durfee, Neighborhoods of algebraic sets, Trans. Amer. Math. Soc., **276** (1983), 517–530.

[17] W. Ebeling and C. T. C. Wall, Kodaira singularities and an extension of Arnold's strange duality, Compositio Mathematica, **56** (1985), 3–77.

[18] D. Eisenbud and W. Neumann, Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Annals of Math. Studies, **110**, Princeton Univ.Press, 1985.

[19] F. Enriques and O. Chisini, Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, **II**, N. Zanichelli, Bologna, 1918.

[20] D. P. A. Epstein, Periodic flows on 3-manifolds, Ann. of Math., **95** (1972), 66–82.

[21] A. T. Fomenko and S. V. Matveev, Algorithmic and Computer Methods for Three-Manifolds, Mathematics and its Applications, **425**, Kluwer Acad. Publishers, 1997; translated from the russian edition, Moscow Univ. Press, 1991.

[22] D. H. Fowler, The mathematics of Plato's Academy. A new reconstruction, Clarendon Press, 1987.

[23] W. Fulton, Introduction to Toric Varieties, Princeton Univ. Press, 1993.

[24] E. R. García Barroso, Invariants des singularités de courbes planes et courbure des fibres de Milnor, tesis doctoral, Univ. de La Laguna, Tenerife (Spain), 1996.

[25] E. Giroux, Structures de contact en dimension trois et bifurcations des feuilletages de surfaces, Inv. Math., **141** (2000), 615–689.

[26] R. Goldin and B. Teissier, Resolving plane branch singularities with one toric morphism, In: Resolution of singularities, a research textbook in

tribute to Oscar Zariski, Birkhäuser, Progress in Math., **181** (2000), 315–340.

[27] P. D. González Pérez, Toric embedded resolutions of quasi-ordinary hypersurface singularities, Ann. Inst. Fourier, Grenoble, **53** (2003), 1819–1881.

[28] P. D. González Pérez and G. Gonzalez-Sprinberg, Analytical invariants of quasi-ordinary hypersurface singularities associated to divisorial valuations, Kodai Math. J., **27** (2004), 164–173.

[29] P. D. González Pérez and B. Teissier, Toric embedded resolution of non-necessarily normal toric varieties, C. R. Acad. Sci. Paris, **334** (2002), 379–382.

[30] G. Gonzalez-Sprinberg, Éventails en dimension 2 et transformée de Nash, preprint, ENS Paris, 1977.

[31] H. Grauert, Über Modifikationen und exzeptionnelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[32] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, V-th edition, Clarendon Press, Oxford, 1988.

[33] A. Hatcher, Notes on Basic 3-manifold Topology, available on the webpage `http://math.cornell.edu/~hatcher`.

[34] M. W. Hirsch, Smooth regular neighborhoods, Ann. of Math. (2), **76** (1962), 524–530.

[35] F. Hirzebruch, Über vierdimensionale Riemannsche Flächen Mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen, Math. Ann., **126** (1953), 1–22.

[36] F. Hirzebruch, The Topology of Normal Singularities of an Algebraic Surface (d'après un article de D.Mumford), Sem. Bourbaki 1962/63, Exp. 250; Gesammelte Abhandlungen, Band II, Springer-Verlag, 1987, pp. 1–7.

[37] F. Hirzebruch, Hilbert modular surfaces, Enseign. Math., **19** (1973), 183–281.

[38] F. Hirzebruch, W. Neumann and S. S. Koh, Differentiable Manifolds and Quadratic Forms, Marcel Dekker, Inc., 1971.

[39] W. H. Jaco, Lectures on Three-Manifold Topology, Regional Conference Series in Mathematics, AMS, **43** (1980).

[40] W. H. Jaco and P. B. Shalen, Seifert Fibered Spaces in Three-manifolds, Memoirs of the AMS, **21** (1979), 220.

[41] K. Johannson, Homotopy Equivalences of 3-Manifolds with Boundaries, LNM, **761**, Springer-Verlag, 1979.

[42] H. W. E. Jung, Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen x, y in der Umgebung einer Stelle $x = a$, $y = b$, J. Reine Angew. Math., **133** (1908), 289–314.

[43] J. Jurkiewicz, Torus embeddings, polyhedra, k^*-actions and homology, Dissertationes Mathematicae, Rozprawy Matematyczne, **236** (1985).

[44] G. Kempf, F. Knudsen, D. Mumford and B. St. Donat, Toroidal embeddings, Springer LNM, **339** (1973).

[45] F. Klein, Über eine geometrische Auffassung der gewöhlischen Kettenbuchentwicklung, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl., **3** (1895), 357–359; French translation, Sur une représentation géométrique du développement en fraction continue ordinaire, Nouvelles Annales de Mathématiques, **15** (1896), 327–331.

[46] F. Klein, Elementary mathematics from an advanced standpoint. Arithmetic, algebra, analysis, Dover, 2004; translation of the third german edition of 1925.

[47] H. B. Laufer, Normal two-dimensional singularities, Princeton Univ. Press, 1971.

[48] H. B. Laufer, Taut two-dimensional singularities, Math. Ann., **205** (1973), 131–164.

[49] H. B. Laufer, On minimally elliptic singularities, Am. J. of Math., **99** (1977), 1257–1295.

[50] D. T. Lê, Geometry of complex surface singularities, Singularities-Sapporo 1998, Advanced Studies in Pure Maths., **29** (2000), 163–180.

[51] D. T. Lê, F. Michel and C. Weber, Sur le comportement des polaires associées aux germes de courbes planes, Compositio Mathematica, **72** (1989), 87–113.

[52] J. Lipman, Introduction to resolution of singularities, Proceedings of Symposia in Pure Mathematics, **29** (1975), 187–230.

[53] F. Michel and C. Weber, Topologie des germes de courbes planes à plusieurs branches, preprint Univ. Genève, 1985.

[54] J.-O. Moussafir, Voiles et polyèdres de Klein. Géométrie, algorithmes et statistiques, thèse, Univ. Paris IX-Dauphine, 2000; available on the page `http://www.ceremade.dauphine.fr/~msfr/`.

[55] D. Mumford, The Topology of Normal Singularities of an Algebraic Surface and a Criterion for Simplicity, Publ. Math. IHES, **9** (1961), 229–246.

[56] A. Némethi, Invariants of normal surface singularities, Contemporary Mathematics, AMS, **354** (2004), 161–208.

[57] W. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, transactions of the AMS, **268** (1981), 299–344.

[58] W. D. Neumann and F. Raymond, Seifert Manifolds, Plumbing, μ-Invariant and Orientation Reversing Maps, Algebraic and Geometric Topology Proceedings, Santa Barbara, 1977, LNM, **664**, Springer-Verlag, 1978, 162–195.

[59] W. D. Neumann and G. A. Swarup, Canonical Decompositions of 3-Manifolds, Geometry and Topology, **1** (1997), 21–40.

[60] T. Oda, Convex bodies and algebraic geometry, an introduction to the theory of toric varieties, Ergebnisse der Math. **15**, Springer-Verlag, 1988.

[61] P. Orlik, Seifert manifolds, Lecture Notes in Mathematics, **291**, Springer-Verlag, Berlin-New York, 1972.

[62] P. Popescu-Pampu, Arbres de contact des singularités quasi-ordinaires et graphes d'adjacence pour les 3-variétés réelles, thèse, Univ. Paris, **7**, 2001, available on the page `http://tel.ccsd.cnrs.fr/documents/. archives0/00/00/28/00/index_fr.html`

[63] P. Popescu-Pampu, On a canonical placement of knots in irreducible 3-manifolds, C. R. Acad. Sci. Paris, Série I, **334** (2002), 677–682.

[64] P. Popescu-Pampu, On higher dimensional Hirzebruch-Jung singularities, Rev. Mat. Complut., **18** (2005), 209–232.

[65] R. von Randow, Zur Topologie von dreidimensionalen Baummannigfaltigkeiten, Bonner Mathematischer Schriften, **14** (1962).

[66] O. Riemenschneider, Deformationen von Quotientensingularitäten (nach Zyklischen Gruppen), Math. Ann., **209** (1974), 211–248.

[67] K. Saito, Einfach-elliptische Singularitäten, Invent. Math., **23** (1974), 289–325.

[68] A. Scharf, Zur Faserung von Graphenmannigfaltigkeiten, Math. Ann., **215** (1975), 35–45

[69] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc., **15** (1983), 401–487.

[70] H. Seifert, Topologie dreidimensionaler gefaserter Räume, Acta Math., **60** (1933), 147–238; English translation in [71].

[71] H. Seifert and W. Threlfall, A Textbook of Topology, Academic Press, 1980; English translation of Lehrbuch der Topologie, Teubner, 1934.

[72] J. Shallit, Real numbers with bounded partial quotients, L'Enseignement Mathématique, **38** (1992), 151–187.

[73] H. J. S. Smith, Note on continued fractions, Messenger of Mathematics, Ser. II, vol. vi (1876), 1–14; reedited in Collected Papers II, 135–147.

[74] M. Spivakovsky, Valuations in function fields of surfaces, Am. J. of Maths., **112** (1990), 107–156.

[75] H. M. Stark, An introduction to number theory, The MIT Press, 1978.

[76] B. Sturmfels, Gröbner bases and convex polytopes, Univ. Lecture Series, AMS, **8** (1996).

[77] B. Teissier, Monomial ideals, binomial ideals, polynomial ideals, In: Trends in commutative algebra, (eds. L. L. Avramov, M. Green, C. Huneke, K. E. Smith and B. Sturmfels), Cambridge U. P., 2004, pp. 211–246.

[78] R. Thom, Les structures différentiables des boules et des sphères, Centre Belge Rech. Math., Colloque Géom. Differ. Globale, Bruxelles, du 19 au 22 Déc. 1958, 27–35 (1959).

[79] G. N. Tjurina, Absolute isolatedness of rational singularities and triple rational points, Functional Analysis and its Applications, **2** (1968), 324–332.

[80] P. Du Val, On absolute and non-absolute singularities of algebraic surfaces. Revue de la Faculté des Sciences de l'Univ. d'Istanbul, **91** (1944), 159–215.

[81] E. Vogt, A Foliation of $\mathbf{R}^3$ and other Punctured 3-Manifolds by Circles, Publ. Math. IHES, **69** (1989), 215–232.

[82] F. Waldhausen, Eine Klasse von 3-dimensionalen Mannigfaltigkeiten, Inv. Math., **3** (1967), 308–333; Inv. Math., **4** (1967), 87–117.

[83] F. Waldhausen, On Irreducible 3-Manifolds which are Sufficiently Large, Annals of Maths., **87** (1968), 56–88.

[84] C. T. C. Wall, Quadratic forms and normal surface singularities, In: Quadratic forms and their applications, Dublin, 1999, Contemp. Math., AMS, **272** (2000), pp. 293–311.

[85] C. T. C. Wall, Singular points of plane curves, Cambridge UP, 2004.

[86] J. H. C. Whitehead, Simplicial spaces, nuclei, and m-groups, Proc. London Math. Soc., **45** (1939), 243–327.

[87] O. Zariski, The reduction of the singularities of an algebraic surface, Ann. of Math., **40** (1939), 639–689.

[88] M. Zisman, Fibre bundles, fibre maps, In: History of Topology, (ed. I. M. James), Elsevier, 1999, pp. 605–629.

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