ON THE ANALYTICAL INVARIANCE OF THE SEMIGROUPS OF A QUASI-ORDINARY HYPERSURFACE SINGULARITY

PATRICK POPESCU-PAMPU

Abstract

We associate to any irreducible germ $\mathscr{S}$ of a complex quasi-ordinary hypersurface an analytically invariant semigroup. We deduce a direct proof (without passing through their embedded topological invariance) of the analytical invariance of the normalized characteristic exponents. These exponents generalize the generic Newton-Puiseux exponents of plane curves. Incidentally, we give a toric description of the normalization morphism of the germ $\mathscr{S}$.

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1. Introduction

In this paper we generalize to arbitrary dimension results first obtained in [24] for surfaces (see also [22]).

A classical way to study an irreducible germ C of complex analytical plane curve is to introduce its Newton-Puiseux series in some coordinate system (X, Y), which allows us to define its so-called characteristic (Newton-Puiseux) exponents. If the coordinates are *generic*—which means that the Y-axis and the embedded reduced tangent cone of the curve are transversal—the characteristic exponents do not depend on them and their collection is a complete invariant of the embedded topological type of the curve.

Another way to study the germ C is to associate to it a *semigroup* $\Gamma(C)$. We recall two ways of doing this. Both of them yield isomorphic (abstract) semigroups. (Throughout the paper, by semigroup we mean a finitely generated abelian monoid without torsion.)

(1) Take the values by the canonical valuation of the elements of the local algebra $\mathscr{A}$ of C. In other words, take the orders of vanishing of $\nu^*(h)$ at the base point of $\overline{C}$, where $\nu : \overline{C} \to C$ is the normalization morphism of C and h varies through $\mathscr{A}$. They form a subsemigroup of $(\mathbf{N}, +)$ which obviously depends only on the analytical type of the germ C.

(2) Take the orders of the series $h(\xi)$, where h varies in $\mathbf{C}\{X, Y\}$ and ξ is a Newton-Puiseux series of C in the coordinates (X, Y). They form a subsemigroup of $(\mathbf{Q}_+, +)$ which can be expressed in terms of the characteristic exponents of C.

Seen as an abstract semigroup, $\Gamma(C)$ is also a complete invariant of the embedded topological type of C. For the preceding claims, see [30] and [23]. In [23] we noticed that from the isomorphism of the two semigroups one can deduce the analytical invariance of the (generic) characteristic exponents of C.

Here we extend this idea to a class of higher-dimensional hypersurface germs, the so-called quasi-ordinary ones, for which a generalization of the characteristic exponents can be defined. If $\mathscr{S}$ is an *irreducible* germ of a hypersurface of dimension d, such exponents are associated to any *quasi-ordinary projection*

$$\psi : \mathscr{S} \to \mathbf{C}^d,$$

which is by definition a finite morphism whose discriminant locus is contained in a hypersurface with normal crossings.

Lipman generalized the notion of generic Newton-Puiseux exponents of plane curves by defining the *normalized* characteristic exponents of $\mathscr{S}$ (see Sec. 2). The irreducible quasi-ordinary germ $\mathscr{S}$ of a hypersurface being given, there is always a quasi-ordinary projection that has, moreover, normalized characteristic exponents.

It is a natural question to study the degree of invariance (analytic or topological) of the normalized characteristic exponents. In [9], Gau proves their embedded topological invariance when $\mathscr{S}$ is a germ of surface. Then Gau [10] and Lipman [17] generalize this result to arbitrary dimensions.

In [11], [13], González Pérez generalizes the second of the constructions presented above of the semigroup of a plane curve to the case of an irreducible quasi-ordinary hypersurface germ $\mathscr{S}$. He starts from a fixed quasi-ordinary polynomial f which defines $\mathscr{S}$ (see Def. 2.2). Instead of the order of a series in one variable, he uses the set of vertices of the Newton polyhedron of a series in various variables. He defines the semigroup $\Gamma(f)$ as the set of vertices of the Newton polyhedra of the fractional series $h(\xi)$, where h varies in $\mathbf{C}\{X_1, \dots, X_d\}[Y] - (f)$ and ξ is a fractional series representing a root of f (see Sec. 2). The same semigroup is obtained if one considers only functions h such that $h(\xi)$ has a dominating exponent (see Sec. 8). Using the embedded topological invariance of the normalized characteristic exponents, he shows that, up to isomorphism, this semigroup does not depend on the quasi-ordinary polynomial f. Moreover, it is a complete invariant of the embedded topological type of the germ, and a fortiori it is an *analytical invariant* of the germ.

Here we avoid using Lipman and Gau's results. Instead, we generalize the first definition given above.

As before, the germ $\mathscr{S}$ is supposed to be irreducible. If f is a quasi-ordinary polynomial defining $\mathscr{S}$, we remark that one can take as the root ξ of f the restriction $Y|_{\mathscr{S}}$, as $f(Y)|_{\mathscr{S}} = 0$. With this choice of root, we have the equality

$$\psi^*\big(h(\xi)\big) = h|_{\mathscr{S}}. \tag{1}$$

This remark is the starting point of our method of construction of an intrinsic semigroup using the elements of the algebra $\mathscr{A}$.

In dimension $d \geq 2$, there is no canonical valuation associated to $\mathscr{S}$. We generalize the first definition above by constructing a morphism $\theta : (\overline{\mathscr{R}}, P) \to (\mathscr{S}, 0)$ whose source is a smooth space, and a divisor with normal crossings $\overline{\mathscr{H}}$ on $\overline{\mathscr{R}}$ at P, both of them being determined by the analytical structure of $\mathscr{S}$. Then we restrict to those functions $h \in \mathscr{A}$ such that the components of the divisor $(\theta^*(h))$ either are components of $\overline{\mathscr{H}}$ or do not contain the intersection of its components. We say then that $\theta^*(h)$ *has a dominating exponent (d.e.) with respect to* $\overline{\mathscr{H}}$ *at* P, this exponent being the tuple formed by the orders of vanishing of $\theta^*(h)$ along the components of $\overline{\mathscr{H}}$. The set of these dominating exponents obviously forms a semigroup. We denote it by $\Gamma'_P(\mathscr{S})$. The rank of the associated abelian group is not greater than the number of components of $\overline{\mathscr{H}}$ at P.

To construct the morphism θ, the idea is to use the structure of the couple $(\mathscr{S}, \mathrm{Sing}(\mathscr{S}))$. For $d \geq 2$, one cannot simply take as a morphism θ a normalization $\nu : \overline{\mathscr{S}} \to \mathscr{S}$, as $\overline{\mathscr{S}}$ is not necessarily smooth. So, in order to get a smooth source, we compose ν with a finite morphism $\mu : \mathscr{R} \to \overline{\mathscr{S}}$ (see Sec. 11), which we call an *orbifold map*. We construct the hypersurface with normal crossings by looking at the preimage $(\nu \circ \mu)^{-1}(\mathrm{Sing}(\mathscr{S}))$. This preimage is not necessarily a hypersurface, as $\mathrm{Sing}(\mathscr{S})$ may have components of codimension 2 (see Sec. 7). Therefore we look only at its components of codimension 1 in $\mathscr{S}$. Let s be their number.

If $s = d$, we take $\theta := \nu \circ \mu,\ P := \theta^{-1}(0)$, and $\overline{\mathscr{H}} := \theta^{-1}(\mathrm{Sing}(\mathscr{S}))$. Then $\theta^*(h|_{\mathscr{S}})$ has a d.e. with respect to $\overline{\mathscr{H}}$ at P whenever $h(\xi)$ has a d.e. The relation (1) allows us to construct a morphism $\Phi_P : \Gamma(f) \to \Gamma'_P(\mathscr{S})$. In this case, our main theorem says that Φ_P *realizes an isomorphism of semigroups* (Th. 9.1).

If $s < d$, the situation is more complex. Then (see Sec. 10) we construct a third morphism $\eta : \overline{\mathscr{R}} \to \mathscr{R}$ as a composition of blow-ups of smooth centers, determined canonically by the structure of $\big(\mathscr{R}, (\nu \circ \mu)^{-1}(\mathrm{Sing}(\mathscr{S}))\big)$. We do this third step in order to get more components of codimension 1 of the preimage of $\mathrm{Sing}(\mathscr{S})$ passing through the same point. We arrive at germs $\overline{\mathscr{H}}_P$ having c' components, where c' denotes the *reduced equisingular dimension* of $\mathscr{S}$, defined in Section 12.

In order to get an isomorphism between $\Gamma'_P(\mathscr{S})$ and a semigroup generalizing the second construction above for plane curves, we are obliged to modify González Pérez's definition in order to get, first of all, the equality of the ranks of the associated abelian groups. We define (see Sec. 12) the *reduced semigroup* $\Gamma'(f)$ of f by taking the vertices of the Newton polyhedron of $h(\xi)$, where $h(\xi)$ is now expressed as a fractional series in the first c' variables $X_1, \ldots, X_{c'}$. It is a semigroup of rank c'. Incidentally, we define (see Def. 11.1) a notion of *reduced Newton polyhedron*.

We then define a finite set $\overline{\mathscr{P}} \subset \overline{\mathscr{H}}$ such that for any $P \in \overline{\mathscr{P}}$, the situation is analogous to the one explained before in the case where $s = d$. Namely, the function $\theta^*(h|_{\mathscr{S}})$ has a d.e. with respect to $\overline{\mathscr{H}}$ at P whenever $h(\xi)$ has a d.e. The relation (1) allows us to construct a morphism $\Phi_P : \Gamma'(f) \to \Gamma'_P(\mathscr{S})$ (see Sec. 13).

Our main theorem is the following.

THEOREM 13.2
The morphism Φ_P realizes an isomorphism between the semigroups $\Gamma'(f)$ and $\Gamma'_P(\mathscr{S})$.

The important point for the proof is that *the morphisms ν, μ, η can be constructed in the chosen coordinate system using toric geometry*. In particular, we think our construction of a normalization morphism (see Sec. 5 and Th. 5.1) has independent interest. The needed background of toric geometry is recalled in Section 4.

As $\Gamma'(f)$ depends on f but not on P and $\Gamma'_P(\mathscr{S})$ just the other way round, we see that, up to isomorphism, *the semigroup obtained like this is an analytic invariant of $\mathscr{S}$*. In Section 15, we deduce the following from this fact.

COROLLARY 13.5
The normalized characteristic exponents are analytical invariants of the germ $\mathscr{S}$.

Other proofs of Corollary 13.5 were given in the case of surfaces by Lipman [16], [18] and Luengo Velasco [19], [20]. Their method was to compare the normalized characteristic exponents with the combinatorial type of a canonical resolution of singularities obtained by blowing up smooth centers. Such a canonical resolution by blowing up smooth centers in a way deducible only from the characteristic exponents is presently unavailable in higher dimensions. In the case of surfaces, an embedded resolution of this kind was obtained by Ban and McEwan in [4].

At a first reading, one should understand the proof of Theorem 9.1. The needed material is presented in the sections that precede it. In Sections 10 – 15 we present the modifications needed to prove the general case. We end the paper by a comparison with our initial study [22] in dimension 2.

2. The quasi-ordinary hypersurface germs and their characteristic exponents

In this section we introduce the basic objects of study of this paper.

If P is a point on a complex space $\mathscr{V}$, we denote by $\mathscr{O}_{\mathscr{V},P}$ the local algebra of $\mathscr{V}$ at P. If $\mathscr{H}$ is a hypersurface of an analytical smooth variety $\mathscr{V}$ and P is a point of $\mathscr{H}$, we say that $\mathscr{H}$ has *normal crossings* at P if there are local coordinates for $\mathscr{V}$ at P such that the germ $\mathscr{H}_P$ of $\mathscr{H}$ at P is the union of some hypersurfaces of coordinates. If ϕ is an analytic function defined on a complex space, we denote by $Z(\phi)$ its (reduced) zero-locus.

In the sequel we denote with the same letter a germ and a sufficiently small representative of it. It is deduced from the context if one is dealing with one or the other notion.

If $a \in \mathbf{R}$, we denote by $[a]$ its integral part. If $F \in K[Y]$ is a polynomial with coefficients in a field K, we denote by $d_Y(F)$ its degree and by $R(F)$ the set of its roots in an algebraic closure of the basis field, taken with multiplicities. If E is a finite set, we denote by $|E|$ its cardinal. If $a, b \in \mathbf{Z}$ with $a \leq b$, we denote $[[a, b]] := \{a, a+1, \ldots, b\}$.

Let $d \geq 1$ be an integer. Define the *algebra of fractional series*

$$\widetilde{\mathbf{C}\{X\}} := \lim_{\substack{\rightarrow \\ N \geq 0}} \mathbf{C}\{X_1^{1/N}, \ldots, X_d^{1/N}\},$$

where $X := (X_1, \ldots, X_d)$. If $m = (m_1, \ldots, m_d) \in \mathbf{Q}_+^d$, we denote $X^m := X_1^{m_1} \cdots X_d^{m_d}$. If $\eta \in \widetilde{\mathbf{C}\{X\}}$ can be written $\eta = X^m u(X)$, with $m \in \mathbf{Q}_+^d$ and $u \in \widetilde{\mathbf{C}\{X\}}$, $u(0, \ldots, 0) \neq 0$, we say that η has a *dominating exponent*, this exponent being denoted by $v_X(\eta) := m$.

If $\eta \in \widetilde{\mathbf{C}\{X\}}$, we define its *Newton polyhedron* $\mathscr{N}_X(\eta)$ to be the convex hull in $\mathbf{R}^d$ of the set $\mathrm{Supp}_X(\eta) + \mathbf{R}^d_+$, where $\mathrm{Supp}_X(\eta)$ denotes the support of η written as a series in the variables X. If η has a d.e., then $\mathscr{N}_X(\eta) = \{v_X(\eta)\} + \mathbf{R}^d_+$, which shows that *the Newton polyhedron is a generalization of the dominating exponent*.

Definition 2.1
Let $\mathscr{A}$ be a reduced equidimensional local complex-analytical algebra of dimension d, and let $(\mathscr{S}, 0)$ be a germ of complex space such that $\mathscr{O}_{\mathscr{S},0} \simeq \mathscr{A}$. The algebra $\mathscr{A}$ and the germ $(\mathscr{S}, 0)$ are called *quasi-ordinary* if there exists a finite morphism ψ from $(\mathscr{S}, 0)$ to a smooth space of the same dimension, whose discriminant locus is contained in a hypersurface with normal crossings. Such a morphism ψ is also called *quasi-ordinary*.

All germs of curves are quasi-ordinary with respect to any finite morphism whose target is a smooth curve.

Quasi-ordinary germs appear naturally in the Jung method of resolution of the singularities of a germ by embedded resolution of the discriminant locus of a finite morphism from the germ to a smooth space of the same dimension (see the original paper by Jung [14] and a more recent presentation by Laufer [15] for the case of dimension 2). Zariski [28] gave an alternative method of resolution of singularities of surfaces which needs a study of quasi-ordinary germs. Quasi-ordinary hypersurface germs were first systematically studied by Lipman [18] when $d = 2$ (see also the survey [16]). This study was extended to any $d \geq 2$ in Lipman [17].

In the special case in which $\mathscr{A}$ is of embedding dimension $d + 1$, one can find local coordinates X on the target space of ψ such that the discriminant locus of ψ is contained in $\mathscr{Z}(X_1 \cdots X_d)$, and one can find an element Y in the maximal ideal of $\mathscr{A}$ such that (ψ, Y) embeds $(\mathscr{S}, 0)$ in $\mathbf{C}^d \times \mathbf{C}$. So ψ appears as a map

$$\psi : \mathscr{S} \to \mathbf{C}^d$$

induced by a projection $\mathbf{C}^d \times \mathbf{C} \to \mathbf{C}^d$ and unramified over $(\mathbf{C}^*)^d$. By the Weierstrass preparation theorem, the image of $\mathscr{S}$ by (ψ, Y), identified in the sequel with $\mathscr{S}$, is defined by a unitary polynomial $f \in \mathbf{C}\{X\}[Y]$. The discriminant locus of ψ is defined by the discriminant $\Delta_Y(f)$ of f, which therefore has a d.e.

Definition 2.2
Let $f \in \mathbf{C}\{X\}[Y]$ be unitary. If $\Delta_Y(f)$ has a d.e., we say that f is *quasi-ordinary*. If $\mathscr{A} \simeq \mathbf{C}\{X\}[Y]/(f)$ with f quasi-ordinary, we say that f is a *qo-defining polynomial* of $\mathscr{S}$ and of the algebra $\mathscr{A}$.

The following theorem (see [1], [17]) generalizes the theorem of Newton and Puiseux for plane curves.

THEOREM 2.3 (Jung and Abhyankar)
If $f \in \mathbf{C}\{X\}[Y]$ *is quasi-ordinary, then the set* $R(f)$ *of roots of* f *embeds canonically in the algebra* $\widetilde{\mathbf{C}\{X\}}$.

In the sequel we consider $R(f)$ as a subset of $\widetilde{\mathbf{C}\{X\}}$.

Suppose now that f is *irreducible*. Then all the differences of roots of f have d.e.'s, which are totally ordered for the componentwise order (see [17], [18]). Denote them by $A_1 < \cdots < A_G,\ A_i = (A_i^1, \ldots, A_i^d),\ \forall i \in \{1, \ldots, G\}$.

Definition 2.4
Let $f \in \mathbf{C}\{X\}[Y]$ be an irreducible quasi-ordinary polynomial. We say that the vectors $A_1, \ldots, A_G \in \mathbf{Q}_+^d$ are the *characteristic exponents* and the monomials $X^{A_1}, \ldots, X^{A_G}$ are the *characteristic monomials* of f.

After possibly permuting the variables $X_1, \ldots, X_d$, we can ensure that

$$(A_1^1, \ldots, A_G^1) \geq_{\text{lex}} \cdots \geq_{\text{lex}} (A_1^d, \ldots, A_G^d). \tag{2}$$

Here $\geq_{\text{lex}}$ denotes the lexicographic ordering. In what follows, we suppose that this condition is always verified.

Definition 2.5
We say that f is a *normalized* qo-defining polynomial of $\mathscr{S}$ if (2) is verified and either $A_1^2 \neq 0$ or $A_1^1 > 1$.

Lipman [18] proved that any irreducible quasi-ordinary germ of a hypersurface has normalized qo-defining polynomials (see also [16], [17], [13]).

Following Lipman [17], we define inductively the abelian groups $M = M_0 := \mathbf{Z}^d,\ M_i := M_{i-1} + \mathbf{Z}A_i,\ \forall i \in \{1, \ldots, G\}$, and the successive indices $N_i := \operatorname{card}(M_i/M_{i-1}),\ \forall i \in \{1, \ldots, G\}$. Following González Pérez [11], [13], we define the vectors $\overline{A}_1, \ldots, \overline{A}_G \in \mathbf{Q}_+^d$ as

$$\overline{A}_1 := A_1, \qquad \overline{A}_i := N_{i-1}\overline{A}_{i-1} + A_i - A_{i-1}, \quad \forall i \in \{2, \ldots, G\}, \qquad \overline{A}_{G+1} := \infty. \tag{3}$$

It can be easily seen that M_G is also generated by M_0 and $\overline{A}_1, \ldots, \overline{A}_G$. Moreover, one has a canonical way of writing the elements of M_G (see [11], [13]).

LEMMA 2.6
Every element of M_G *can be written in a unique way as a sum* $A + i_0\overline{A}_1 + \cdots + i_{G-1}\overline{A}_G$*, where* $A \in M_0$ *and* $0 \leq i_k \leq N_{k+1} - 1,\ \forall k \in \{0, \ldots, G-1\}$.

Proof
As $N_k = \mathrm{card}(M_k/M_{k-1})$, we deduce that $N_k\overline{A}_k \in M_{k-1},\ \forall k \in \{1, \ldots, G\}$. From this we deduce immediately the *existence* of an expression verifying the asked property.

In order to prove the *uniqueness*, remark first that if $i \in \{1, \ldots, G\}$, one has $N_i = \min\{k \in \mathbf{N}^*,\ kA_i \in M_{i-1}\} = \min\{k \in \mathbf{N}^*,\ k\overline{A}_i \in M_{i-1}\}$. Then suppose by contradiction that $\exists(i_0, \ldots, i_{G-1}) \neq (j_0, \ldots, j_{G-1})$ and $A, B \in M$ so that $A+i_0\overline{A}_1+\cdots+i_{G-1}\overline{A}_G = B+j_0\overline{A}_1+\cdots+j_{G-1}\overline{A}_G$. Define $p := \min\{k \in \{0, \ldots, G-1\}, i_l = j_l, \forall l > k\}$. Then $p \geq 0$ and $(i_p - j_p)\overline{A}_{p+1} = (B-A)+\sum_{k=0}^{p-1}(j_k - i_k)\overline{A}_{k+1} \in M_p$. But $0 < |i_p - j_p| \leq N_{p+1} - 1$, which contradicts the previous remark. □

3. The singular locus of the germ

In this section we suppose that $f \in \mathbf{C}\{X\}[Y]$ is an irreducible qo-defining polynomial of $\mathscr{S}$.

The characteristic exponents of f allow us to describe precisely the singular locus $\mathrm{Sing}(\mathscr{S})$ of $\mathscr{S}$. Before stating this description, we introduce some notation taken from [17].

If $I \subset \{1, \ldots, d\}$, let $\mathscr{D}_I$ be the linear subspace of $\mathbf{C}^d$ defined by $\{X_i = 0,\ \forall i \in I\}$. Its codimension is $|I|$. Denote $\mathscr{Z}_I := \mathscr{S} \cap \psi^{-1}(\mathscr{D}_I)$. In [17], Lipman shows that the spaces $\mathscr{Z}_I$ are *irreducible*. For simplicity, we denote $\mathscr{Z}_i := \mathscr{Z}_{\{i\}},\ \forall i \in \{1, \ldots, d\}$.

Definition 3.1
The minimal number $c \in \{1, \ldots, d\}$ with the property that $A_i^k = 0,\ \forall i \in \{1, \ldots, G\},\ \forall k \in \{c+1, \ldots, d\}$, is called the *equisingular dimension* of the quasi-ordinary projection ψ.

Recalling that the characteristic exponents verify condition (2), we see that c represents the number of variables appearing with nonzero exponents among the monomials $X^{A_1}, \ldots, X^{A_G}$. The name, suggested to us by González Pérez, is motivated by the fact that ψ is then an equisingular deformation of a c-dimensional quasi-ordinary germ but not of a smaller-dimensional germ (see Ban [3]).

The following theorem is a reformulation of [17, Th. 7.3].

THEOREM 3.2 (Lipman)
If $\mathscr{S}$ *is a germ defined by an irreducible quasi-ordinary polynomial, then the ir-*

reducible components of $\mathrm{Sing}(\mathscr{S})$ *are of the form* $\mathscr{Z}_I$, *with* $I \subset \{1, \dots, c\}$ *and* $|I| \in \{1, 2\}$. *Moreover,*

(1) *if* $i \in \{1, \dots, c\}$, *then* $\mathscr{Z}_i$ *is a component of* $\mathrm{Sing}(\mathscr{S})$ *if and only if one does not simultaneously have* $A^i_k = 0$, $\forall k \in \{1, \dots, G-1\}$, *and* $A^i_G = 1/N_G$;

(2) *if* $\{i, j\} \subset \{1, \dots, c\}$ *with* $i \neq j$, *then* $\mathscr{Z}_{\{i,j\}}$ *is a component of* $\mathrm{Sing}(\mathscr{S})$ *if and only if neither* $\mathscr{Z}_i$ *nor* $\mathscr{Z}_j$ *are components of* $\mathrm{Sing}(\mathscr{S})$;

(3) *if* $\{i, j\} \subset \{1, \dots, c\}$ *with* $i \neq j$ *and* $\mathscr{Z}_{\{i,j\}}$ *is a component of* $\mathrm{Sing}(\mathscr{S})$, *then the germ of* $\mathscr{S}$ *at any point* P *of* $\mathscr{Z}_{\{i,j\}} - \bigcup_{k \notin \{i,j\}} \mathscr{Z}_k$ *is isomorphic to the subgerm of* $(\mathbf{C}^{d+1}, 0)$ *defined by the equation* $Y^{N_G} = T_1 T_2$.

Let $s \in \{0, \dots, c\}$ be such that $\mathscr{Z}_i \subset \mathrm{Sing}(\mathscr{S})$ for $i \in \{1, \dots, s\}$ and $\mathscr{Z}_i \not\subseteq \mathrm{Sing}(\mathscr{S})$ otherwise. Our ordering convention (2) and the previous theorem imply that there is always such an s, which is equal to the *number of components of* $\mathrm{Sing}(\mathscr{S})$ *having codimension* 1 *in* $\mathscr{S}$. Then $\mathscr{Z}_{\{i,j\}}$ is a component of $\mathrm{Sing}(\mathscr{S})$ if and only if $i \neq j$ and $i, j \in \{s+1, \dots, c\}$. We get

$$\mathrm{Sing}(\mathscr{S}) = \bigcup_{1 \leq i \leq s} \mathscr{Z}_i \cup \bigcup_{\substack{j,l \in [[s+1,c]] \\ j \neq l}} \mathscr{Z}_{\{j,l\}}. \tag{4}$$

Examples (where $G = 1$ and $c = 3$)

(1) If $\mathscr{S} = \{(X, Y),\ Y^N - X_1 X_2 X_3 = 0\}$ with $N > 1$, then $s = 0$ and $\mathrm{Sing}(\mathscr{S}) = \mathscr{Z}_{\{1,2\}} \cup \mathscr{Z}_{\{2,3\}} \cup \mathscr{Z}_{\{3,1\}}$.

(2) If $\mathscr{S} = \{(X, Y),\ Y^N - X_1^{B^1} X_2 X_3 = 0\}$ with $N > 1,\ B^1 > 1$, then $s = 1$ and $\mathrm{Sing}(\mathscr{S}) = \mathscr{Z}_1 \cup \mathscr{Z}_{\{2,3\}}$.

(3) If $\mathscr{S} = \{(X, Y),\ Y^N - X_1^{B^1} X_2^{B^2} X_3 = 0\}$ with $N > 1,\ B^1 > 1,\ B^2 > 1$, then $s = 2$ and $\mathrm{Sing}(\mathscr{S}) = \mathscr{Z}_1 \cup \mathscr{Z}_2$.

(4) If $\mathscr{S} = \{(X, Y),\ Y^N - X_1^{B^1} X_2^{B^2} X_3^{B^3} = 0\}$ with $N > 1,\ B^1 > 1,\ B^2 > 1,\ B^3 > 1,\ \gcd(N, B^1, B^2, B^3) = 1$, then $s = 3$ and $\mathrm{Sing}(\mathscr{S}) = \mathscr{Z}_1 \cup \mathscr{Z}_2 \cup \mathscr{Z}_3$.

4. A reminder of toric geometry

For the constructions of Sections 5, 6, and 10, we need some elementary results of toric geometry. For details, one can consult Fulton's book [8] or Oda's book [21].

A *lattice* $\mathscr{W}$ is a finitely generated free abelian group. An element $v \neq 0$ of $\mathscr{W}$ is called *primitive* if it cannot be written $v = av'$ with $a \in \mathbf{N}^* - \{1\}$, $v' \in \mathscr{W}$. The dual lattice $\mathscr{M}$ of $\mathscr{W}$ is by definition $\mathrm{Hom}(\mathscr{W}, \mathbf{Z})$. If $\phi : \overline{\mathscr{W}} \to \mathscr{W}$ is a morphism of lattices and K denotes a field containing $\mathbf{N}$, denote by $\mathscr{W}_K$ the K-vector space generated by $\mathscr{W}$, and denote by ϕ_K the corresponding morphism of vector spaces.

The elements of $\mathscr{M}$ should be thought about here as exponents of *monomials* and those of $\mathscr{W}$ as *weights* of those monomials.

Let σ be a *strictly convex rational polyhedral (s.c.r.p.) cone* in $\mathscr{W}_{\mathbf{R}}$, and let $\check{\sigma} := \{u \in \mathscr{M}_{\mathbf{R}},\ (u,v) \geq 0,\ \forall v \in \sigma\}$ be its dual cone inside $\mathscr{M}_{\mathbf{R}}$. This dual cone is again s.c.r.p. by Gordan's lemma. The *affine toric variety* $\mathscr{Z}(\mathscr{W},\sigma)$ *of weight lattice* $\mathscr{W}$ *and cone* σ is, by definition,

$$\mathscr{Z}(\mathscr{W},\sigma) := \operatorname{Spec} \mathbf{C}[\check{\sigma} \cap \mathscr{M}],$$

where $\mathbf{C}[\check{\sigma} \cap \mathscr{M}]$ denotes the $\mathbf{C}$-algebra of the additive semigroup $\check{\sigma} \cap \mathscr{M}$. Denote by χ^m the monomial of $\mathbf{C}[\check{\sigma} \cap \mathscr{M}]$ which has the exponent m.

If $v_1, \ldots, v_k \in \mathscr{W}$, we denote by $\mathbf{R}_+(v_1, \ldots, v_k)$ the cone generated by them. A s.c.r.p. cone is called *regular* if it is generated by a subset of a basis of the lattice $\mathscr{W}$. The variety $\mathscr{Z}(\mathscr{W},\sigma)$ is smooth if and only if the cone σ is regular.

If $\phi : \overline{\mathscr{W}} \to \mathscr{W}$ is a morphism and $\overline{\sigma} \subset \overline{\mathscr{W}}_{\mathbf{R}}$, $\sigma \subset \mathscr{W}_{\mathbf{R}}$ are s.c.r.p. cones such that $\phi_{\mathbf{R}}(\overline{\sigma}) \subset \sigma$, then there is a canonical induced *toric morphism* $\phi_* : \mathscr{Z}(\overline{\mathscr{W}},\overline{\sigma}) \to \mathscr{Z}(\mathscr{W},\sigma)$.

If $\phi = \mathrm{id}_{\mathscr{W}}$, then ϕ_* is an embedding. As a particular case of this, by taking $\phi = \mathrm{id}_{\mathscr{W}}$ and $\overline{\sigma} = \{0\}$, we obtain a canonical embedding of the *complex torus* $T_{\mathscr{W}} := \mathscr{Z}(\mathscr{W},\{0\}) \simeq (\mathbf{C}^*)^{\dim \mathscr{W}_{\mathbf{R}}}$ in any affine toric variety $\mathscr{Z}(\mathscr{W},\sigma)$. Moreover, there is a canonical action of the torus $T_{\mathscr{W}}$ on $\mathscr{Z}(\mathscr{W},\sigma)$, such that $\mathscr{Z}(\mathscr{W},\{0\})$ is the only open orbit. With respect to these actions, the preceding morphisms are *equivariant*.

If ϕ is an inclusion of finite index and $\phi_{\mathbf{R}}(\overline{\sigma}) = \sigma$, then ϕ_* is a finite map. More precisely, we have the following proposition (see [21, Cor. 1.16]).

PROPOSITION 4.1
If $\phi : \overline{\mathscr{W}} \to \mathscr{W}$ presents $\overline{\mathscr{W}}$ as a $\mathbf{Z}$-submodule of finite index of $\mathscr{W}$, then $\phi_* : \mathscr{Z}(\overline{\mathscr{W}},\overline{\sigma}) \to \mathscr{Z}(\mathscr{W},\sigma)$ coincides with the projection for the quotient of $\mathscr{Z}(\overline{\mathscr{W}},\overline{\sigma})$ with respect to the natural action of the finite group $\mathscr{W}/\phi(\overline{\mathscr{W}})$.

The natural action alluded to comes from the following action of $\mathscr{W}$ on the monomials of $\mathbf{C}[\check{\overline{\sigma}}]$:

$$v \cdot \chi^{\overline{u}} = e^{2i\pi(\overline{u},v)}\chi^{\overline{u}}. \tag{5}$$

Let us express the morphism ϕ_* using coordinates in the case in which $\dim \mathscr{W}_{\mathbf{R}} = \dim \overline{\mathscr{W}}_{\mathbf{R}} = d$ and $\sigma, \overline{\sigma}$ are regular cones of the maximal dimension. Let $v_1, \ldots, v_d \in \mathscr{W}$ and $\overline{v}_1, \ldots, \overline{v}_d \in \overline{\mathscr{W}}$ be the unique primitive elements situated on the edges of σ, respectively, $\overline{\sigma}$. Write $\phi(\overline{v}_j) := \sum_{i=1}^{d} \alpha_i^j v_i,\ \forall\ j \in \{1, \ldots, d\}$. The hypothesis $\phi(\overline{\sigma}) \subset \sigma$ implies that $\alpha_i^j \in \mathbf{N},\ \forall (i,j) \in \{1, \ldots, d\}^2$. Let $u_1, \ldots, u_d \in \mathscr{M}$ and $\overline{u}_1, \ldots, \overline{u}_d \in \overline{\mathscr{M}}$ be the dual bases of $v_1, \ldots, v_d$, respectively, $\overline{v}_1, \ldots, \overline{v}_d$. The adjoint morphism $\check{\phi} : \mathscr{M} \to \overline{\mathscr{M}}$ then verifies $\check{\phi}(u_i) = \sum_{j=1}^{d} \alpha_i^j \overline{u}_j,\ \forall i \in \{1, \ldots, d\}$. The monomials $U_i := \chi^{u_i},\ 1 \leq i \leq d$, and $\overline{U}_i := \chi^{\overline{u}_i},\ 1 \leq i \leq d$, are free generators of the group algebras $\mathbf{C}[\check{\sigma} \cap \mathscr{M}]$, respectively, $\mathbf{C}[\check{\overline{\sigma}} \cap \mathscr{M}]$. Then we have the following.

LEMMA 4.2
The morphism $\phi^* : \mathbf{C}[\check{\sigma} \cap \mathscr{M}] \to \mathbf{C}[\check{\overline{\sigma}} \cap \mathscr{M}]$ *can be expressed as*

$$\begin{cases} U_1 = \overline{U}_1^{\alpha_1^1} \cdots \overline{U}_d^{\alpha_1^d}, \\ \quad \vdots \\ U_d = \overline{U}_1^{\alpha_d^1} \cdots \overline{U}_d^{\alpha_d^d}. \end{cases}$$

This shows that, with respect to the coordinates $U, \overline{U}$ of the two algebras, the morphism ϕ_* is *monomial*. Let us look at its effect on the Newton polyhedron of a fractional series in the coordinates U. If $\eta \in \widetilde{\mathbf{C}\{U\}}$, $\eta = \sum_{m \in \mathrm{Supp}_U(\eta)} c_m U^m$, where $\mathrm{Supp}_U(\eta) \subset M_{\mathbf{Q}}$, one has

$$\phi^*(\eta) = \sum_{m \in \mathrm{Supp}_U(\eta)} c_m \overline{U}^{\check{\phi}(m)}.$$

So $\mathrm{Supp}_{\overline{U}}(\phi^*(\eta)) \subset \check{\phi}(\mathrm{Supp}_U(\eta))$. If $m, m' \in M$ and $m' \in \{m\} + \check{\sigma}$, then from the condition $\phi^*(\check{\sigma}) \subset \check{\overline{\sigma}}$ we immediately get $\check{\phi}(m') \in \{\check{\phi}(m)\} + \check{\overline{\sigma}}$. We deduce the following.

LEMMA 4.3
The vertices of $\mathscr{N}_{\overline{U}}(\phi^*(\eta))$ *are images by* $\check{\phi}$ *of vertices of* $\mathscr{N}_U(\eta)$.

Define a *fan* Σ in $\mathscr{W}_{\mathbf{R}}$ to be a finite collection of s.c.r.p. cones such that, for any $\sigma \in \Sigma$, all the faces of σ are also in Σ, and the intersection of any two elements of Σ is also in Σ. For example, to any s.c.r.p. cone σ is associated canonically a fan, the set of the faces of σ. The *support* $|\Sigma|$ of the fan Σ is by definition the union of the cones composing it.

Using the affine toric morphisms defined before, the affine toric varieties $\mathscr{Z}(\mathscr{W}, \sigma)$ for $\sigma \in \Sigma$ can be equivariantly glued in a new variety $\mathscr{Z}(\mathscr{W}, \Sigma)$ called the *toric variety of weight lattice* $\mathscr{W}$ *and fan* Σ. It is always normal. It is smooth if and only if the fan Σ is *regular*, that is, if and only if its constituting cones are all regular.

The *orbits* of the action of $T_{\mathscr{W}}$ on $\mathscr{Z}(\mathscr{W}, \Sigma)$ are in one-to-one correspondence with the cones of Σ. Denote by O_σ the orbit corresponding to $\sigma \in \Sigma$ and by V_σ its closure. One has the equality $\dim V_\sigma + \dim \sigma = \dim \mathscr{W}_{\mathbf{R}}$.

The notion of toric morphism can be extended to this more general setting, starting from a morphism $\phi : \overline{\mathscr{W}} \to \mathscr{W}$ verifying $\forall\, \overline{\sigma} \in \overline{\Sigma}$, $\exists \sigma \in \Sigma$ such that $\phi_{\mathbf{R}}(\overline{\sigma}) \subset \sigma$. Then one obtains an associated (equivariant) toric morphism $\phi_* : \mathscr{Z}(\overline{\mathscr{W}}, \overline{\Sigma}) \to \mathscr{Z}(\mathscr{W}, \Sigma)$. A particular case of this construction is obtained if one considers a *subdivision* of the fan Σ, that is, a second fan $\overline{\Sigma}$ in $\mathscr{W}_{\mathbf{R}}$ such that $|\overline{\Sigma}| = |\Sigma|$ and $\forall\, \overline{\sigma} \in \overline{\Sigma}$, $\exists \sigma \in \Sigma$ with $\overline{\sigma} \subset \sigma$. The associated toric morphism is then proper and birational.

Let us specialize this even more. Consider a regular fan $\Sigma \subset \mathscr{W}_{\mathbf{R}}$, and let σ_0 be one of its cones of dimension $e \geq 2$. Let $V_1, \dots, V_e$ be the primitive elements of $\mathscr{W}$ situated on the edges of σ_0, and let $V_0 := V_1 + \cdots + V_e$. Denote $\sigma_0^j := \mathbf{R}_+(V_0, \dots, V_{j-1}, V_{j+1}, \dots, V_e)$, $\forall j \in \{1, \dots, e\}$. Each $\sigma \in \Sigma$ with $\sigma_0 \subset \sigma$ can be written uniquely as $\sigma = \sigma_0 + \tau$ with $\tau \in \Sigma$ and $\sigma_0 \cap \tau = \{0\}$. Then denote $\sigma^j := \sigma_0^j + \tau$, $\forall j \in \{1, \dots, e\}$.

Definition 4.4
The *star subdivision* $\Sigma^*(\sigma_0)$ of Σ with respect to σ_0 is the fan $(\Sigma - \{\sigma \in \Sigma,\ \sigma_0 \subset \sigma\}) \cup \{\text{the faces of } \sigma^j,\ \sigma \in \Sigma, \sigma_0 \subset \sigma,\ 1 \leq j \leq e\}$. In the particular case in which Σ is the fan composed of the faces of σ_0, we write σ_0^* instead of $\Sigma^*(\sigma_0)$.

The importance of this particular subdivision comes from the following proposition (see [21]), which shows that the associated toric morphism is intrinsic from an analytical viewpoint.

PROPOSITION 4.5
The equivariant morphism obtained by passing from Σ to the star subdivision $\Sigma^*(\sigma_0)$ is isomorphic to the blow-up of $\mathscr{Z}(\mathscr{W}, \Sigma)$ along V_{σ_0}.

The previous result is used in Section 10.

5. A toric normalization of the germ

We suppose that $\mathscr{S}$ is an irreducible quasi-ordinary germ (not necessarily of a hypersurface).

In this section we show how one can construct, using toric geometry, a canonical normalization morphism of the germ $(\mathscr{S}, 0)$ once the map $\psi : \mathscr{S} \to \mathbf{C}^d$ unramified over $(\mathbf{C}^*)^d$ is fixed (Th. 5.1). These results were first obtained in [24].

We denote $W = \mathbf{Z}^d$. Let σ_0 be the cone generated by the canonical basis of $\mathbf{Z}^d$. It is a regular cone. We identify the space $\mathbf{C}^d$ of coordinates X with the toric variety $\mathscr{Z}(W, \sigma_0)$. Denote by $u_1, \dots, u_d$ the primitive elements of $M = \mathrm{Hom}(W, \mathbf{Z})$ situated on the edges of $\check{\sigma}_0$, such that $X_i = \chi^{u_i}$, $\forall i \in \{1, \dots, d\}$.

Coming back to the quasi-ordinary morphism ψ, denote $\mathscr{D} := \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_d$ and $\mathscr{Z} := \mathscr{Z}_1 \cup \cdots \cup \mathscr{Z}_d$ (see the beginning of Sec. 3 for the notation). Then the map $\mathscr{S} - \mathscr{Z} \xrightarrow{\psi} \mathbf{C}^d - \mathscr{D}$ is an unramified covering. As the fundamental group $\pi_1(\mathbf{C}^d - \mathscr{D})$ is abelian, this covering is Galoisian, its Galois group being $W/W(\psi)$. Here $W(\psi)$

denotes the following subsemigroup of W:

$$W(\psi) := \psi_* \pi_1(\mathscr{S} - \mathscr{Z}) \hookrightarrow \pi_1(\mathbf{C}^d - \mathscr{D}) = W.$$

Apart from $\mathbf{C}^d = \mathscr{Z}(W, \sigma_0)$, consider also the affine d-dimensional toric variety $\mathscr{Z}(W(\psi), \sigma_0)$ and the canonical finite toric morphism

$$\gamma_{W:W(\psi)} : \mathscr{Z}\big(W(\psi), \sigma_0\big) \to \mathscr{Z}(W, \sigma_0),$$

induced by the inclusion $W(\psi) \subset W$. In both cases we denote by 0 the point that is the unique closed orbit of the corresponding toric variety.

By Proposition 4.1, $\mathscr{Z}(W, \sigma_0) \simeq \mathscr{Z}(W(\psi), \sigma_0)/(W/W(\psi))$. The restriction

$$\gamma_{W:W(\psi)} : \mathscr{Z}\big(W(\psi), \sigma_0\big) - \gamma^{-1}_{W:W(\psi)}(\mathscr{D}) \longrightarrow \mathscr{Z}(W, \sigma_0) - \mathscr{D}$$

is an unramified covering with automorphism group precisely $W/W(\psi)$, which shows that one can complete a commutative diagram

$$\begin{array}{ccc} \mathscr{Z}\big(W(\psi), \sigma_0\big) - \gamma^{-1}_{W:W(\psi)}(\mathscr{D}) & \xrightarrow{\ \nu\ } & \mathscr{S} - \mathscr{D} \\ & {\scriptstyle \gamma_{W:W(\psi)}} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & \mathbf{C}^d & \end{array}$$

The morphism ν is analytic on $\big(\mathscr{Z}(W(\psi), \sigma_0), 0\big) - \gamma^{-1}_{W:W(\psi)}(\mathscr{D})$ and bounded in the neighborhood of $\gamma^{-1}_{W:W(\psi)}(\mathscr{D})$. As the variety $\big(\mathscr{Z}(W(\psi), \sigma_0), 0\big)$ is normal, by the Riemann extension theorem the morphism ν is, in fact, everywhere analytic, which shows that it is a *normalization* of $(\mathscr{S}, 0)$. We get the following.

THEOREM 5.1

For any morphism $\psi : \mathscr{S} \to \mathbf{C}^d$, unramified over $(\mathbf{C}^*)^d$ and such that $\mathscr{S}$ is irreducible, one has the following commutative diagram, in which ν is a normalization morphism:

$$\begin{array}{ccc} \big(\mathscr{Z}(W(\psi), \sigma_0), 0\big) & \xrightarrow{\ \nu\ } & (\mathscr{S}, 0) \\ & {\scriptstyle \gamma_{W:W(\psi)}} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & (\mathbf{C}^d, 0) & \end{array}$$

In the special case in which $\mathscr{S}$ is a hypersurface germ, we can express the lattice $W(\psi)$ using the characteristic exponents of ψ. In order to do this, following [11], [13], let us introduce the dual lattices W_k of the lattices M_k defined in Section 2. One has the inclusions $M = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_G$, $W = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_G$.

Let us also introduce the sequence of field extensions (see [17], [13]) $\mathrm{Frac}(\mathbf{C}\{X\}) = L = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_G$, where $L_k := L(X^{A_1}, \ldots, X^{A_k})$, $\forall k \in \{0, \ldots, G\}$. One has the following lemma, proved in [13] (see also [24]).

LEMMA 5.2 (Lipman)

(1) *For every* $i, j \in \{0, \ldots, G\}$, $i < j$, *the field extension* $L_j : L_i$ *is Galoisian and* $\mathrm{Gal}(L_j : L_i) \simeq W_j / W_i$.

(2) *If* $\xi \in R(f)$, *then* $\mathrm{Frac}(\mathscr{A}) = L(\xi) = L_G$.

(3) *If* $N = d_Y(f)$, *then* $N = N_1 \cdots N_G$.

The action of the group $W/W(\psi) = \mathrm{Gal}(L(\xi) : L)$ on the field $L(\xi)$ can be canonically lifted as an action of the group W by multiplication with roots of unity on the monomials of $L(\xi)$ (cf. relation (5)),

$$v \cdot X^u := e^{2i\pi \langle v, u\rangle} X^u, \quad \forall v \in W.$$

Here u varies through the set of exponents of the monomials in $L(\xi)$, that is, through the lattice M_G. So $W(\psi) = \{v \in W,\ \langle v, u\rangle \in \mathbf{Z}, \forall u \in M_G\} = W \cap \mathrm{Hom}(M_G, \mathbf{Z}) = W_G$. We get the following.

PROPOSITION 5.3

Let $f \in \mathbf{C}\{X\}[Y]$ *be an irreducible quasi-ordinary polynomial, and let* ψ *be the associated quasi-ordinary projection. Then* $W(\psi) = W_G$.

Using this identification, Theorem 5.1 becomes the following.

COROLLARY 5.4 (González Pérez)

If f *is an irreducible quasi-ordinary polynomial defining the germ* $\mathscr{S}$, *then one has the following commutative diagram, in which* ν *is a normalization morphism:*

$$\begin{array}{ccc} \left(\mathscr{Z}(W_G, \sigma_0), 0\right) & \xrightarrow{\ \nu\ } & (\mathscr{S}, 0) \\ & {\scriptstyle \gamma_{W:W_G}} \searrow \qquad \swarrow {\scriptstyle \psi} & \\ & (\mathbf{C}^d, 0) & \end{array}$$

This theorem was first proved algebraically by González Pérez in [13] without passing through Proposition 5.3 (see also [12]). It inspired our Theorem 5.1.

Remark. In the case of surfaces, the normalization of an irreducible quasi-ordinary germ has a Hirzebruch-Jung singularity (see Barth, Peters, and Van de Ven [5]). In

[25] we classify analytically the higher-dimensional normal quasi-ordinary germs, which we also call Hirzebruch-Jung singularities. Important tools are Theorem 5.1 and the orbifold map constructed in the next section.

6. The canonical orbifold map

We suppose that $\mathscr{S}$ is an arbitrary irreducible quasi-ordinary germ and that $\psi : \mathscr{S} \to \mathbf{C}^d$ is unramified over $(\mathbf{C}^*)^d$.

In this section we show that there is a canonical finite morphism μ whose target is the normalization $\overline{\mathscr{S}}$ of $\mathscr{S}$, its source being a smooth germ $\mathscr{R}$. It is a particular case of the *orbifold maps* defined by Deligne and Mostow in [7], and also of a construction described by Prill in [26] using Grauert-Remmert's existence theorem (see Bell and Narasimhan [6] for a presentation of this last theorem).

We denote again by 0 the base point $\nu^{-1}(0)$ of $\overline{\mathscr{S}}$. Consider the extrinsic isomorphism $\overline{\mathscr{S}} \simeq \mathscr{Z}(W(\psi), \sigma_0)$ given by Theorem 5.1. Let $\tilde{W}$ be the sublattice of $W(\psi)$ generated by the primitive elements of $W(\psi)$ situated on the edges of σ_0. Then σ_0 is regular with respect to $\tilde{W}$, and so $\mathscr{Z}(\tilde{W}, \sigma_0)$ is smooth. Consider the toric map

$$\mu : \mathscr{Z}(\tilde{W}, \sigma_0) \to \mathscr{Z}\big(W(\psi), \sigma_0\big)$$

induced by the finite index inclusion $\tilde{W} \subset W(\psi)$. Denote by $\tilde{u}_1, \ldots, \tilde{u}_d$ the primitive elements of $\tilde{M} = \mathrm{Hom}(\tilde{W}, \mathbf{Z})$ situated on the edges of $\check{\sigma}_0$, such that the image of u_i by the canonical finite index inclusion $M \subset \tilde{M}$ is proportional to $\tilde{u}_i$ in $M_{\mathbf{R}}$. Denote $\tilde{U}_i := \chi^{\tilde{u}_i}$, $\forall i \in \{1, \ldots, d\}$. Then the composition $\nu \circ \mu : \mathscr{Z}(\tilde{W}, \sigma_0) \to \mathscr{Z}(W, \sigma_0)$ is given in the coordinates $(\tilde{U}_i)_{1 \leq i \leq d}$, $(X_i)_{1 \leq i \leq d}$ by equations of the form $X_i = \tilde{U}_i^{m_i}$, $m_i \in \mathbf{N}^*$, $\forall i \in \{1, \ldots, d\}$. Denote also by $\tilde{v}_1, \ldots, \tilde{v}_d$ the dual basis of $\tilde{u}_1, \ldots, \tilde{u}_d$.

By Proposition 4.1, the morphism μ is the quotient map of $\mathscr{Z}(\tilde{W}, \sigma_0)$ by the natural action of the finite group $W(\psi)/\tilde{W}$. It can be easily seen that $W(\psi)/\tilde{W}$ does not contain complex reflections. This shows that the locus F_μ of the fixed points of the elements of $W(\psi)/\tilde{W}$ distinct from the identity has codimension at least 2 in $\mathscr{Z}(\tilde{W}, \sigma_0)$. Moreover, $\mu^{-1}\big(\mathrm{Sing}(\mathscr{Z}(W(\psi), \sigma_0))\big) \subset F_\mu$. As $\mathscr{Z}(\tilde{W}, \sigma_0)$ is smooth, the complement $\mathscr{Z}(\tilde{W}, \sigma_0) - \mu^{-1}\big(\mathrm{Sing}(\mathscr{Z}(W(\psi), \sigma_0))\big)$ is simply connected, and so the restriction of μ over the smooth part of $\mathscr{Z}(W(\psi), \sigma_0)$ is the universal covering map. This shows its uniqueness by the same arguments as in [26] or [7]. More precisely, we have the following result, which is a particular case of [7, Prop. 14.3], proved there algebraically.

LEMMA 6.1

For $i = 1, 2$, let $\mu_i : (\mathscr{R}_i, P_i) \to (\overline{\mathscr{S}}, 0)$ be two finite maps unramified in codimension 1, with smooth sources $\mathscr{R}_i$. Then there exists an isomorphism $(\mathscr{R}_1, P_1) \to$

$(\mathscr{R}_2, P_2)$ *making the following diagram commutative:*

$$\begin{array}{ccc} (\mathscr{R}_1, P_1) & \xrightarrow{\hspace{3em}} & (\mathscr{R}_2, P_2) \\ & {\scriptstyle \mu_1}\searrow \quad \swarrow{\scriptstyle \mu_2} & \\ & (\overline{\mathscr{S}}, 0) & \end{array}$$

We denote $\mathscr{R} := \mathscr{Z}(\tilde{W}, \sigma_0)$. The preceding lemma shows that the morphism $\mu : \mathscr{R} \to \overline{\mathscr{S}}$ is independent of the particular isomorphism $\overline{\mathscr{S}} \simeq \mathscr{Z}(W(\psi), \sigma_0)$ under consideration.

Definition 6.2
We call the canonical map μ, described before, the *orbifold map* associated to $\overline{\mathscr{S}}$.

The vocabulary is motivated by the fact that μ is locally the quotient map of the action of a finite group on $\mathscr{R}$.

Denote $P_0 := (\nu \circ \mu)^{-1}(0)$. So we have constructed a canonical finite map of germs of analytical spaces $\nu \circ \mu : (\mathscr{R}, P_0) \to (\mathscr{S}, 0)$.

7. Expansions according to semiroots

From now on, $\mathscr{S}$ is supposed to be defined by an irreducible quasi-ordinary polynomial $f \in \mathbf{C}\{X\}[Y]$.

Semiroots were introduced for arbitrary irreducible quasi-ordinary polynomials by González Pérez in [13] by analogy with the known constructions for curves. For their use in the study of plane curves, see [23]. Here we need them as an essential tool in the proofs of Theorem 9.1 and its generalization Theorem 13.2.

Definition 7.1
Let us fix $\xi \in R(f)$. Take any $k \in \{0, \ldots, G\}$. A unitary polynomial $f_k \in \mathbf{C}\{X\}[Y]$ is called a *k-semiroot* of f if f_k is of degree $N_1 \cdots N_k$, and f_k has a d.e. with $v_X(f_k(\xi)) = \overline{A}_{k+1}$. A $(G+1)$-tuple $(f_0, \ldots, f_G)$ such that, $\forall k \in \{0, \ldots, G\}$, f_k is a k-semiroot of f is called a *complete system of semiroots* for f.

These objects are independent of the choice of ξ.

Let $(f_0, \ldots, f_G)$ be a complete system of semiroots for f (which always exists: e.g., the minimal polynomials of suitable truncations of ξ or the characteristic approximate roots of f; see [13]). Immediately generalizing Abhyankar [2] (see also [23]), we have the following.

LEMMA 7.2
Any $h \in \mathbf{C}\{X\}[Y]$ *can be uniquely written as a finite sum* $h = \sum c_{i_0 \cdots i_G} f_0^{i_0} \cdots f_G^{i_G}$ *with* $c_{i_0 \cdots i_G} \in \mathbf{C}\{X\}$, *where the* $(G+1)$*-tuples* $(i_0, \ldots, i_G) \in \mathbf{N}^{G+1}$ *verify* $0 \leq i_k \leq N_{k+1} - 1$, $\forall k \in \{0, \ldots, G-1\}$, *and* $i_G \leq [d_Y(h)/N]$.

Proof
Make the Euclidean division of h by f_G and of the successive quotients by f_G, until one obtains a quotient of degree less than $d_Y(f_G)$. Then one gets the f_G*-adic expansion* of h by f_G, which has the form $h = \sum c_{i_G} f_G^{i_G}$, where $i_G \leq [d(h)/d(f_G)]$. Then iterate this, making at each step the f_{k-1}-adic expansions of the coefficients $c_{i_k \cdots i_G}$.

The unicity comes from the remark that the Y-degrees of the terms $c_{i_0 \cdots i_G} f_0^{i_0} \cdots f_G^{i_G}$ are pairwise distinct. To see this, suppose by contradiction that $\exists (i_0, \ldots, i_G) \neq (j_0, \ldots, j_G)$ and $d_Y(c_{i_0 \ldots i_G} f_0^{i_0} \cdots f_G^{i_G}) = d_Y(c_{j_0 \ldots j_G} f_0^{j_0} \cdots f_G^{j_G})$. Then $\exists p \in \{0, \ldots, G\}$ such that $i_k = j_k, \forall k > p$, and $i_p \neq j_p$. Suppose, for example, that $i_p > j_p$. Then

$$(i_p - j_p) N_1 \cdots N_p = \sum_{k=0}^{p-1} (j_k - i_k) N_1 \cdots N_k$$
$$\leq \sum_{k=0}^{p-1} (N_{k+1} - 1) N_1 \cdots N_k = N_1 \cdots N_p - 1,$$

and so $i_p - j_p < 1$, which is a contradiction. □

Definition 7.3
The preceding equality is called the $(f_0, \ldots, f_G)$*-adic expansion of* h. The finite set $\{(i_0, \ldots, i_G),\ c_{i_0 \cdots i_G} \neq 0\}$ is called the $(f_0, \ldots, f_G)$*-adic support of* h, denoted $\mathrm{Supp}_{(f_0, \ldots, f_G)}(h)$.

The following lemma, which generalizes the properties of Abhyankar's expansions in terms of semiroots in the plane branch case (see [2] and [23]), is a simple consequence of Lemma 2.6.

LEMMA 7.4
If $h = \sum c_{i_0 \cdots i_G} f_0^{i_0} \cdots f_G^{i_G}$ *is the* $(f_0, \ldots, f_G)$*-adic expansion of* $h \in \mathbf{C}\{X\}[Y]$, *then for every* $\xi \in R(f)$, *the sets of vertices of the Newton polyhedra* $\mathcal{N}_X\big(c_{i_0 \cdots i_G}(f_0(\xi))^{i_0} \cdots (f_G(\xi))^{i_G}\big)$ *are pairwise disjoint when* $(i_0, \ldots, i_G)$ *varies through the* $(f_0, \ldots, f_G)$*-adic support of* h.

In the sequel, the previous lemma is important, combined with the following one.

LEMMA 7.5
If $h_1, \ldots, h_p \in \widetilde{\mathbf{C}\{X\}}$ *and the sets of vertices of the Newton polyhedra* $\mathcal{N}_X(h_1), \ldots, \mathcal{N}_X(h_p)$ *are pairwise disjoint, then* $\mathcal{N}_X(h_1+\cdots+h_p)$ *is the convex hull of the union* $\mathcal{N}_X(h_1) \cup \cdots \cup \mathcal{N}_X(h_p)$*. In particular, each vertex of* $\mathcal{N}_X(h_1+\cdots+h_p)$ *is a vertex of one of the polyhedra* $\mathcal{N}_X(h_1), \ldots, \mathcal{N}_X(h_p)$*.*

Proof
Denote by $\mathrm{Conv}(E)$ the convex hull of a set E. If $h := h_1 + \cdots + h_p$, one always has the inclusion $\mathcal{N}_X(h) \subset \mathrm{Conv}(\mathcal{N}_X(h_1) \cup \cdots \cup \mathcal{N}_X(h_p))$, as each monomial of h is a monomial of one of the h_i.

Conversely, each vertex of $\mathrm{Conv}(\mathcal{N}_X(h_1) \cup \cdots \cup \mathcal{N}_X(h_p))$ is a vertex of one of the $\mathcal{N}_X(h_i)$. The hypothesis of the lemma implies that it is necessarily also a vertex of $\mathcal{N}_X(h)$, which proves the converse inclusion. □

8. Various definitions of semigroups

A difficulty for extending the second definition presented in the introduction of the semigroup of a plane irreducible curve is that in dimension greater than 1, a fractional series may not have a dominating exponent.

In [11], [13], González Pérez considers a more general notion by using instead of the dominating exponents the Newton polyhedra $\mathcal{N}_X(h(\xi))$ for varying $h \in \mathbf{C}\{X\}[Y] - (f)$ and the following set of their vertices:

$$\Gamma_{\mathcal{N}}(f) := \left\{A \in \mathbf{Q}_+^d,\ A \text{ is a vertex of } \mathcal{N}_X(h(\xi)),\ h \in \mathbf{C}\{X\}[Y] - (f)\right\}.$$

Here ξ denotes again an arbitrary root of f.

One sees immediately the independence of the set $\Gamma_{\mathcal{N}}(f)$ from the choice of ξ. In [11], [13], González Pérez proves the following.

PROPOSITION 8.1
One has the following equality of sets:

$$\Gamma_{\mathcal{N}}(f) = \mathbf{N}^d + \mathbf{N}\overline{A}_1 + \cdots + \mathbf{N}\overline{A}_G.$$

Proof
This is an immediate consequence of Lemmas 7.4 and 7.5. □

So the set $\Gamma_{\mathcal{N}}(f)$ has the structure of a semigroup for the addition, a fact that was not a priori clear from the definition.

Now we modify the previous definition by considering only those functions that have a dominating exponent, as we find it convenient for stating Theorem 13.2. We introduce the following subset of $\mathbf{Q}_+^d$:

$$\Gamma_{\mathscr{D}}(f) := \big\{ v_X(h(\xi)),\ h \in \mathbf{C}\{X\}[Y] - (f),\ h(\xi) \text{ has a d.e.} \big\}.$$

This time it is clear that it is a semigroup. In fact, one obtains the same semigroup as before, again an immediate consequence of Lemmas 7.4 and 7.5.

PROPOSITION 8.2
One has the following equality of semigroups:

$$\Gamma_{\mathscr{D}}(f) = \mathbf{N}^d + \mathbf{N}\overline{A}_1 + \cdots + \mathbf{N}\overline{A}_G.$$

This motivates the following definition.

Definition 8.3
The semigroup $\Gamma_{\mathscr{N}}(f) = \Gamma_{\mathscr{D}}(f)$ is called the *semigroup of $\mathscr{A}$ with respect to f*, denoted $\Gamma(f)$.

Let us introduce now some notions needed to give in Sections 9 and 13 another definition of a semigroup associated to f, generalizing the first one of the introduction.

Let $\mathscr{V}$ be a complex analytical smooth variety of dimension n, and let $\mathscr{H}$ be a hypersurface of $\mathscr{V}$. Let $P \in \mathscr{H}$ be a point such that the germ of $\mathscr{H}$ at P has normal crossings. Let $(\mathscr{H}_1, P), \dots, (\mathscr{H}_r, P)$ be the irreducible components of $(\mathscr{H}, P)$, where $1 \leq r \leq n$. From now on, their ordering is supposed to be fixed.

Definition 8.4
The germ of r-codimensional submanifold $C(\mathscr{H}, P) := (\mathscr{H}_1, P) \cap \cdots \cap (\mathscr{H}_r, P)$ of $\mathscr{V}$ is called the *center* of $\mathscr{H}$ at P.

Let $(x_1, \dots, x_n)$ be local coordinates of $\mathscr{V}$ at P, such that $\mathscr{H}_i = \mathscr{Z}(x_i)$, $\forall i \in \{1, \dots, r\}$. We say in this case that they are *adapted* to $\mathscr{H}$ at P. Take $h \in \mathscr{O}_{\mathscr{V},P}$. Write $h = x_1^{m_1} \cdots x_r^{m_r} u$ with $u \in \mathscr{O}_{\mathscr{V},P}$ not divisible by any of $x_1, \dots, x_r$. As m_i is the multiplicity of $\mathscr{H}_i$ as an irreducible component of the principal divisor (h), we have the following.

LEMMA 8.5
The r-tuple $(m_1, \dots, m_r)$ depends only on the ordering of the components of the germ $(\mathscr{H}, P)$ and not on the choice of the adapted coordinates $x_1, \dots, x_n$.

Let us define a subclass of elements of $\mathscr{O}_{\mathscr{V},P}$, distinguished by their relation with $\mathscr{H}$.

Definition 8.6
We say that $h \in \mathscr{O}_{\mathscr{V},P}$ *has a dominating exponent* with respect to the germ $(\mathscr{H}, P)$ if $C(\mathscr{H}, P)$ is not included in the closure of $\mathscr{Z}(h) - \mathscr{H}$. The r-tuple $(m_1, \dots, m_r) \in \mathbf{N}^r$, written $\upsilon_{\mathscr{H},P}(h)$, is called the *dominating exponent* of h with respect to $(\mathscr{H}, P)$.

Lemma 8.5 shows that the dominating exponent is independent of the chosen adapted coordinates once we have fixed an ordering of the components of $\mathscr{H}$ at P. In coordinates $(x_1, \dots, x_n)$ adapted to $\mathscr{H}$ at P, we see that h has a d.e. with respect to $(\mathscr{H}, P)$ if and only if the series $h \in \mathbf{C}\{x_1, \dots, x_n\}$ has a d.e. with respect to $(x_1, \dots, x_r)$, in the sense of Section 11, that is, if and only if $u(x_1, \dots, x_r, 0, \dots, 0) \neq 0$.

If $\mathscr{B}$ is a subalgebra of $\mathscr{O}_{\mathscr{V},P}$, let us introduce the following subsemigroup of the multiplicative semigroup $(\mathscr{B}, \cdot)$:

$$\mathscr{E}_{\mathscr{H},P}(\mathscr{B}) := \left\{h \in \mathscr{B} - \{0\}, h \text{ has a d.e. with respect to } (\mathscr{H}, P)\right\}.$$

This allows us to define the following.

Definition 8.7
The *semigroup of $\mathscr{B}$ with respect to* $(\mathscr{H}, P)$ is the following subsemigroup of $(\mathbf{N}^r, +)$:

$$\Gamma_{\mathscr{H},P}(\mathscr{B}) := \left\{\upsilon_{\mathscr{H},P}(h),\ h \in \mathscr{E}_{\mathscr{H},P}(\mathscr{B}) - \{0\}\right\}.$$

This semigroup consists simply of the dominating exponents with respect to $(\mathscr{H}, P)$ of the elements of $\mathscr{B}$ which have d.e's.

9. The simplest case of the main theorem

We suppose in this section that $s = d$. Then, by relation (4), $\mathrm{Sing}(\mathscr{S}) = \bigcup_{1 \leq i \leq d} \mathscr{Z}_i$.

Let $\nu : \overline{\mathscr{S}} \to \mathscr{S}$ be the normalization morphism of $\mathscr{S}$, and let $\mu : \mathscr{R} \to \overline{\mathscr{S}}$ be the orbifold map of $\overline{\mathscr{S}}$. Denote $\theta := \nu \circ \mu : \mathscr{R} \to \mathscr{S}$, and let θ^* be the corresponding morphism of sheaves of local algebras. Define $\overline{\mathscr{H}} := \theta^{-1}(\mathrm{Sing}(\mathscr{S}))$. Then, using the toric constructions of the morphisms ν and μ given in Sections 5 and 6, we see that $\overline{\mathscr{H}}$ is a divisor with normal crossings whose center at $P := \theta^{-1}(0)$ is zero-dimensional and reduced to P itself. Denote

$$\Gamma'_P(\mathscr{S}) := \Gamma_{\overline{\mathscr{H}},P}\big(\theta^*(\mathscr{A})_P\big).$$

This subsemigroup of $(\mathbf{N}^d, +)$ is obviously an *analytical invariant* of $\mathscr{S}$. The following theorem, which is the main one of this paper specialized to the case where $s = d$, shows that this semigroup is isomorphic to $\Gamma(f)$. The general case is stated in Theorem 13.2.

THEOREM 9.1
Let f be a quasi-ordinary defining polynomial of $\mathscr{S}$. Suppose that $s = d$. If h varies through $\mathbf{C}\{X\}[Y]$ such that $h(\xi)$ has a d.e., then $\theta^*(h|_{\mathscr{S}})$ has a d.e. with respect to $\overline{\mathscr{H}}$ at $P := \theta^{-1}(0)$ and one obtains a well-defined mapping

$$\begin{aligned} \Phi_P : \Gamma(f) &\longrightarrow \Gamma'_P(\mathscr{S}), \\ \upsilon_X\big(h(\xi)\big) &\longrightarrow \upsilon_{\overline{\mathscr{H}},P}\big(\theta^*(h|_{\mathscr{S}})_P\big) \end{aligned}$$

which realizes an isomorphism of semigroups.

Proof
Denote $\overline{\psi} := \psi \circ \theta : \mathscr{R} \to \mathbf{C}^d$.

As the image of $Y \in \mathbf{C}\{X\}[Y]$ in $\mathscr{A}$ verifies the equation $f(X, Y) = 0$, one sees that $Y|_{\mathscr{S}}$ can be thought of as an element of $R(f)$. Denoting it by ξ, one has the equality (1): $\overline{\psi}^*(h(\xi)) = \theta^*(h|_{\mathscr{S}})$, $\forall h \in \mathbf{C}\{X\}[Y]$.

Taking the representatives of the morphisms μ and ν, constructed before using toric geometry, the point P is the zero-dimensional orbit of $\mathscr{R} \simeq \mathscr{Z}(\tilde{W}, \sigma_0)$, and one can choose canonical toric coordinates adapted to $\overline{\mathscr{H}}$ at P. With such coordinates, the morphism $\overline{\psi}^*$ is monomial, and using formula (1), we see that Φ_P *is injective*.

In order to prove the surjectivity of Φ_P, we must show that if $h \in \mathbf{C}\{X\}[Y]$ is such that $\theta^*(h|_{\mathscr{S}}) \in \mathscr{E}_{\overline{\mathscr{H}},P}(\theta^*(\mathscr{A}))$, then one can find another element $h' \in \mathbf{C}\{X\}[Y]$ such that $h(\xi)$ has a d.e. and $\upsilon_{\overline{\mathscr{H}},P}(\theta^*(h'|_{\mathscr{S}})_P) = \upsilon_{\overline{\mathscr{H}},P}(\theta^*(h|_{\mathscr{S}})_P)$. As $f|_{\mathscr{S}} = 0$, we can suppose that $\deg(h) < \deg(f)$, after possibly making the Euclidean division of h by f. Then we consider a complete system $(f_0, \dots, f_G)$ of semiroots of f and the $(f_0, \dots, f_G)$-adic expansion of h, which by our hypothesis is of the form $h = \sum c_{i_0 \cdots i_{G-1}} f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}$. Using Lemma 7.4, we see that there exists a tuple $(i_0, \dots, i_{G-1}, 0) \in \operatorname{Supp}_{(f_0,\dots,f_G)}(h)$ such that

$$\begin{aligned} \upsilon_{\overline{\mathscr{H}},P}\big(\theta^*(h|_{\mathscr{S}})_P\big) &= \upsilon_{\overline{\mathscr{H}},P}\big(\theta^*(c_{i_0 \cdots i_{G-1}} f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}|_{\mathscr{S}})_P\big) \\ &= \upsilon_{\overline{\mathscr{H}},P}\big(\theta^*(X^m f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}|_{\mathscr{S}})_P\big), \end{aligned}$$

where m is one of the vertices of the Newton polyhedron $\mathscr{N}_X(c_{i_0 \cdots i_{G-1}})$. But the term $X^m f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}$ has a d.e., which proves that Φ_P *is surjective*. □

10. A canonical sequence of blow-ups

We no longer suppose that $s = d$. In Sections 10 – 12 we generalize some of the ingredients of Theorem 9.1. We use those generalizations in order to state Theorem 13.2.

In this section we consider $(\nu \circ \mu)^{-1}(\operatorname{Sing}(\mathscr{S}))$ as a subspace of $\mathscr{R}$, and we construct from it a canonical map $\eta : \overline{\mathscr{R}} \to \mathscr{R}$ obtained as a composition of blow-ups of smooth centers. This construction is used in order to replace the morphism θ of Theorem 9.1 by $\nu \circ \mu \circ \eta$ (see Sec. 13).

Define

$$c' := \begin{cases} c-2 & \text{if } s = c-2, \\ c & \text{if } s \neq c-2, \end{cases}$$

$$\mathscr{D}' := \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_{c'},$$

$$\mathscr{Z}' := \begin{cases} \bigcup_{1 \leq i \leq s} \mathscr{Z}_i & \text{if } s \in \{c-2, c-1\}, \\ \bigcup_{1 \leq i \leq s} \mathscr{Z}_i \cup \mathscr{Z}_{[[s+1,c]]} & \text{if } s \notin \{c-2, c-1\}. \end{cases}$$

We say that c' is the *reduced equisingular dimension* of ψ (see also Def. 3.1). When $s \in \{c-2, c-1, c\}$, the space $\mathscr{Z}'$ is precisely the union of the components of $\mathrm{Sing}(\mathscr{S})$ which have codimension 1 in $\mathscr{S}$ (see formula (4)). As $\mathscr{Z}_{[[s+1,c]]} = \bigcap_{\substack{j,l \in [[s+1,c]] \\ j \neq l}} \mathscr{Z}_{\{j,l\}}$, using formula (4) we see that $\mathscr{Z}' \hookrightarrow \mathrm{Sing}(\mathscr{S})$ is determined by the analytical structure of $\mathscr{S}$.

If $I \subset \{1, \dots, d\}$, denote $\mathscr{U}_I := (\nu \circ \mu)^{-1}(\mathscr{Z}_I)$. Using the toric construction of the composition $\nu \circ \mu$ presented in Sections 5 and 6, one sees that in the canonical toric coordinates $\tilde{U}_1, \dots, \tilde{U}_d$ of $\mathscr{R}$, one has $\mathscr{U}_I = \bigcap_{i \in I} \mathscr{Z}(\tilde{U}_i)$. Formula (4) shows that

$$(\nu \circ \mu)^{-1}\big(\mathrm{Sing}(\mathscr{S})\big) = \bigcup_{1 \leq i \leq s} \mathscr{U}_i \cup \bigcup_{\substack{j,l \in [[s+1,c]] \\ j \neq l}} \mathscr{U}_{\{j,l\}}. \tag{6}$$

In order to construct η, we consider several cases according to the values of s, the number of components of $\mathrm{Sing}(\mathscr{S})$ of codimension 1 in $\mathscr{S}$, and of the equisingular dimension c.

(1) *Suppose that* $s \leq c-3$.

The indexing of the objects we introduce from now on respects the following rule: lower indices refer to steps of blowing up, and upper ones refer to strict transforms of $\mathscr{U}_1, \dots, \mathscr{U}_d$. The only exception to the rule is constituted by $\mathscr{U}_1, \dots, \mathscr{U}_d$ themselves.

Consider first the case in which $c = d$ *and* $s = 0$. Then $\mathrm{Sing}(\mathscr{S})$ has only components of codimension 2 in $\mathscr{S}$: $\mathrm{Sing}(\mathscr{S}) = \bigcap_{\substack{I \subset [[1,d]] \\ |I|=2}} \mathscr{Z}_I$.

Then the axis of the coordinates $\tilde{U}_1, \dots, \tilde{U}_d$ can be obtained analytically from $(\nu \circ \mu)^{-1}(\mathrm{Sing}(\mathscr{S}))$. Indeed, if $\mathscr{L}^i$ is the axis of the coordinate $\tilde{U}_i$, one has $\mathscr{L}^i = \bigcap_{\substack{|I|=2 \\ i \notin I}} \mathscr{U}_I$, and by formula (6), one has $(\nu \circ \mu)^{-1}(\mathrm{Sing}(\mathscr{S})) = \bigcup_{\substack{I \subset [[1,d]] \\ |I|=2}} \mathscr{U}_I$.

Let $\pi_1 : \mathscr{R}_1 \to \mathscr{R}$ be the blow-up of $\mathscr{R}$ at $P_0 = (\nu \circ \mu)^{-1}(0)$. Denote by $\mathscr{H}_1 := \pi_1^{-1}(P_0)$ the exceptional divisor of π_1, and denote by P_1^i the point where the strict transform of $\mathscr{L}^i$ meets the exceptional divisor $\mathscr{H}_1$ for all $i \in \{1, \dots, d\}$.

Let $\pi_2 : \mathscr{R}_2 \to \mathscr{R}_1$ be the blow-up of $\mathscr{R}_1$ at all the points P_1^i for $i \in \{1, \dots, d\}$. Denote by $\mathscr{H}_2^i := \pi_2^{-1}(P_1^i)$ the components of the exceptional divisor of π_2, and denote by $\mathscr{H}_{1,2}$ the strict transform of $\mathscr{H}_1$ by π_2. Denote by $\mathscr{L}^{\{i,j\}}$ the line of the $(c-1)$-dimensional projective space $\mathscr{H}_1$ which joins P_1^i and P_1^j. Define $P_2^{i,j}$ to be the point where the strict transform of $\mathscr{L}^{\{i,j\}}$ meets the divisor $\mathscr{H}_2^i$.

More generally, suppose that $\mathscr{R}_{k-1}$ is already constructed with $d \geq k \geq 2$. Let $\pi_k : \mathscr{R}_k \to \mathscr{R}_{k-1}$ be the blow-up of $\mathscr{R}_{k-1}$ at all the points $P_{k-1}^{i_1,\dots,i_{k-1}}$ for $i_1, \dots, i_{k-1} \in [[1, d]]$ pairwise distinct. The components of the exceptional divisor of π_k are $\mathscr{H}_k^{i_1,\dots,i_{k-1}} := \pi_k^{-1}(P_{k-1}^{i_1,\dots,i_{k-1}})$. For $l \in \{1, \dots, k-1\}$ and $j_1, \dots, j_{l-1} \in \{1, \dots, d\}$ pairwise distinct, the strict transform by π_k of $\mathscr{H}_{l,k-1}^{j_1,\dots,j_{l-1}}$ is denoted $\mathscr{H}_{l,k}^{j_1,\dots,j_{l-1}}$. Denote by $\mathscr{L}^{i_1,\dots,i_{k-2},\{i_{k-1},i_k\}}$ the line joining the points $P_{k-1}^{i_1,\dots,i_{k-2},i_{k-1}}$ and $P_{k-1}^{i_1,\dots,i_{k-2},i_k}$ of the $(d-1)$-dimensional projective space $\mathscr{H}_{k-1}^{i_1,\dots,i_{k-2}}$. Define $P_k^{i_1,\dots,i_k}$ to be the point where the strict transform of the line $\mathscr{L}^{i_1,\dots,i_{k-2},\{i_{k-1},i_k\}}$ meets the exceptional divisor $\mathscr{H}_k^{i_1,\dots,i_{k-1}}$.

Denote $\overline{\mathscr{R}} := \mathscr{R}_d$ and $\eta := \pi_1 \circ \pi_2 \circ \cdots \circ \pi_d : \mathscr{R}_d \to \mathscr{R}$. Let

$$\overline{\mathscr{H}} := \eta^{-1}(P_0) = \bigcup_{\substack{k,i_1,\dots,i_{k-1} \in [[1,d]] \\ |\{i_1,\dots,i_{k-1}\}| = k-1}} \mathscr{H}_{k,d}^{i_1,\dots,i_{k-1}}$$

be the exceptional divisor of η. By construction, it is a divisor with normal crossings. The minimal dimension of the centers (see Def. 8.4) of $\overline{\mathscr{H}}$ is zero and is attained precisely at the points $P_d^{i_1,\dots,i_d}$, where $(i_1, \dots, i_d)$ varies among the $d!$ permutations of $(1, \dots, d)$. Denote this set by $\overline{\mathscr{P}}$.

The morphism η is isomorphic with a toric morphism obtained from $\mathscr{Z}(\tilde{W}, \sigma_0)$ by a sequence of star subdivisions of σ_0 with respect to cones of dimension d. One begins by taking the star subdivision σ_0^* of σ_0 with respect to itself. This gives the morphism π_1. Then one star subdivides each cone σ_0^i, $1 \leq i \leq d$ (see the notation preceding Def. 4.4), and one gets π_2. At each one of the next steps, one subdivides only part of the cones of the maximal dimension. The point is that at each step it is possible to number canonically the edges of each cone from 1 to d. (The numbering depends on the cone looked upon.) Then one subdivides only cones of the form $\left(\cdots((\sigma_0^{i_1})^{i_2})\cdots\right)^{i_k}$ with $i_1, \dots, i_k$ pairwise distinct.

If $\tilde{\Sigma}$ denotes the fan obtained by composing all these subdivisions, the morphism η is isomorphic with the toric morphism $\mathscr{Z}(\tilde{W}, \tilde{\Sigma}) \to \mathscr{Z}(\tilde{W}, \sigma_0)$ induced by the identity and $\overline{\mathscr{P}}$ is the union of the orbits of dimension zero of the action of $T_{\tilde{W}}$ on $\mathscr{Z}(\tilde{W}, \tilde{\Sigma})$.

Consider then the case in which $c = d$ and $0 < s \leq d - 3$.

The codimension 1 part of $(\nu \circ \mu)^{-1}(\mathrm{Sing}(\mathscr{S}))$ is, by formula (6), the space $\bigcup_{1 \leq i \leq s} \mathscr{U}_i$. Its center at the point P_0 is the space $\bigcap_{1 \leq i \leq s} \mathscr{U}_i = \mathscr{U}_{[[1,s]]}$, which in the toric setting explained in Section 4 is the subspace of the coordinates $\tilde{U}_{s+1}, \dots, \tilde{U}_d$. The intersections $\mathscr{U}_{\{j,l\}} \cap \mathscr{U}_{[[1,s]]} = \mathscr{U}_{[[1,s]] \cup \{j,l\}}$ are analytically determined for any $j \neq l,\ j, l \in [[s+1, d]]$, and they allow us to repeat the construction done in the previous case, looking inside $\mathscr{U}_{[[1,s]]}$. For example, the axis $\mathscr{L}^j$ of the coordinate $\tilde{U}_j$ for $s+1 \leq j \leq d$ can be obtained as $\mathscr{L}^j = \bigcap_{\substack{I \subset [[s+1,d]] \\ |I|=2,\ j \notin I}} \mathscr{U}_{[[1,s]] \cup I}$. Then, instead of blowing up points, one blows up analytically defined smooth varieties of dimension s, starting with $\mathscr{U}_{[[s+1,d]]} = \bigcap_{\substack{j,l \in [[s+1,d]] \\ j \neq l}} \mathscr{U}_{\{j,l\}}$.

After $(d-s)$ blow-ups, one gets a morphism $\eta := \pi_1 \circ \cdots \circ \pi_{d-s} : \mathscr{R}_{d-s} \to \mathscr{R}$. Denote $\overline{\mathscr{R}} := \mathscr{R}_{d-s}$. Then $\eta^{-1}(\mathscr{U}_{[[s+1,d]]})$ is the exceptional divisor of η. Again, by construction, it is a divisor with normal crossings. The minimal dimension of its centers is s. Among the points at which the dimension of the center is s, a discrete set $\overline{\mathscr{P}}$ of $(d-s)!$ elements is analytically distinguished, as formed by the points that are, moreover, situated on the fiber $\eta^{-1}(P_0)$. Denote

$$\overline{\mathscr{H}} := \eta^{-1}\Big(\bigcup_{1 \leq i \leq s} \mathscr{U}_i \cup \mathscr{U}_{[[s+1,d]]} \Big).$$

Then $\overline{\mathscr{H}}$ is also a divisor with normal crossings, and at the points of $\overline{\mathscr{P}}$, the dimension of its center is zero.

In this case, *the morphism η is isomorphic to a toric morphism $\mathscr{Z}(\tilde{W}, \tilde{\Sigma}) \to \mathscr{Z}(\tilde{W}, \sigma_0)$ obtained by a sequence of star subdivisions with respect to cones of dimension $(d-s)$.*

Consider finally the case in which $c < d$.

The discussion of this case is comparable to the one done in the previous case. Indeed, one has simply to add $(d-c)$ to the dimensions of all the varieties that are blown up. Using the fibers of the normalization morphism $\nu : \overline{\mathscr{S}} \to \mathscr{S}$, one has at one's disposal, at each step of blowing up, canonical families of projective spaces in which to join points. This gives the canonical smooth varieties whose strict transforms meet the exceptional divisors at the new centers of blowing up.

At the end, one obtains again a map $\eta : \overline{\mathscr{R}} \to \mathscr{R}$ having as exceptional divisor $\eta^{-1}(\mathscr{U}_{[[s+1,c]]}) \hookrightarrow \overline{\mathscr{R}}$. The minimal dimension of its centers is now $(s+d-c)$. Among the points at which the dimension of the center is $(s+d-c)$, a discrete set $\overline{\mathscr{P}}$ of $(c-s)!$ elements is analytically determined, as formed by the points that are, moreover, situated on the fiber $\eta^{-1}(P_0)$. We define

$$\overline{\mathscr{H}} := \eta^{-1}\Big(\bigcup_{1 \leq i \leq s} \mathscr{U}_i \cup \mathscr{U}_{[[s+1,c]]} \Big).$$

Then $\overline{\mathscr{H}}$ is again a divisor with normal crossings, and at the points of $\overline{\mathscr{P}}$, the dimension of its center is $d-c$. Now η is isomorphic to a toric morphism obtained by a sequence of star subdivisions with respect to cones of dimension $(c-s)$.

(2) *Suppose that* $s = c - 1$. Then we define $\eta : \overline{\mathscr{R}} \to \mathscr{R}$ to be the blow-up of $\mathscr{U}_{[[1,c]]}$ and $\overline{\mathscr{H}} := \eta^{-1}\big(\bigcup_{1\leq i\leq s} \mathscr{U}_i\big)$. The strict transform of $\mathscr{U}_{[[1,s]]} = \bigcap_{1\leq i\leq s} \mathscr{U}_i$ cuts the fiber $\eta^{-1}(P_0)$ in a unique point P_1. We define $\overline{\mathscr{P}} := \{P_1\}$.

(3) *Suppose that* $s \in \{c, c-2\}$. Then we do not modify $\mathscr{R}$. So $\eta = \mathrm{id}_{\mathscr{R}}$ and $\overline{\mathscr{R}} = \mathscr{R}$. We then define $\overline{\mathscr{H}} := \bigcup_{1\leq i\leq s} \mathscr{U}_i$ and $\overline{\mathscr{P}} := \{P_0\}$.

All the previous constructions are analytically canonical presentations of toric constructions, as we have explained it case by case. So we have obtained the following proposition.

PROPOSITION 10.1
If $\overline{\mathscr{R}}, \eta, \overline{\mathscr{H}}, \overline{\mathscr{P}}$ *are constructed as before, there is a fan* $\tilde{\Sigma}$ *obtained from* σ_0 *by a sequence of star subdivisions with respect to cones of dimension* $d - c' + s$*, such that* η *is isomorphic to the restriction of the toric map* $\eta_T : \mathscr{Z}(\tilde{\mathscr{W}}, \tilde{\Sigma}) \to \mathscr{Z}(\tilde{\mathscr{W}}, \sigma_0)$ *induced by the identity to a neighborhood of* $\eta_T^{-1}(0)$. *Moreover, the components of* $\overline{\mathscr{H}}$ *correspond by this isomorphism to orbit closures of codimension* 1, *the points of* $\overline{\mathscr{P}}$ *to orbits of dimension zero, and at each point of* $\overline{\mathscr{P}}$*, the hypersurface* $\overline{\mathscr{H}}$ *has exactly* c' *components.*

Analyzing the different cases of our construction, we see that the following description of $\overline{\mathscr{H}}$ is always true:

$$\overline{\mathscr{H}} = (\nu \circ \mu \circ \eta)^{-1}(\mathscr{Z}'). \tag{7}$$

This is the reason why we defined $\mathscr{Z}'$ in the way presented at the beginning of the section.

11. Reduced Newton polyhedra

The notions developed in this section are needed in the next one in order to define the reduced semigroup of an irreducible quasi-ordinary polynomial.

Let $\hat{X}$ denote a subset with $\hat{d} \geq 1$ elements of the set of variables $X = \{X_1, \ldots, X_d\}$. By $X - \hat{X}$ we denote the set of the remaining variables.

If $\eta \in \widetilde{\mathbf{C}\{X\}}$ can be written $\eta = (\hat{X})^{\hat{m}} u(X)$ with $\hat{m} \in \mathbf{Q}_+^{\hat{d}}$, and $u \in \widetilde{\mathbf{C}\{X\}}$ verifying $u(\hat{X}, 0) \neq 0$, we say that η *has a dominating exponent with respect to* $\hat{X}$, an exponent denoted by $v_{\hat{X}}(\eta) := \hat{m}$. Here $u(\hat{X}, 0)$ denotes the series obtained by annulating all the variables $X_i \in X - \hat{X}$.

In fact, one can also extend to this setting the notion of a Newton polyhedron.

Definition 11.1
If $\eta \in \widetilde{\mathbf{C}\{X\}}$, we define the *reduced Newton polyhedron* $\mathscr{N}_{\hat{X}}(\eta)$ of η with respect to $\hat{X}$ to be the convex hull in $\mathbf{R}^{\hat{d}}$ of the set $\mathrm{Supp}_{\hat{X}}(\eta) + \mathbf{R}_+^{\hat{d}}$, where $\mathrm{Supp}_{\hat{X}}(\eta)$ denotes

the support of η written as a series in the variables $\hat{X}$, with coefficients in the algebra $\mathbf{C}\{X - \hat{X}\}$.

One can obtain the reduced Newton polyhedron $\mathcal{N}_{\hat{X}}(\eta)$ from the knowledge of the usual Newton polyhedron $\mathcal{N}_X(\eta)$. In order to see it, let $M = \mathbf{Z}^d$ be the lattice of exponents of the monomials in $\mathbf{C}\{X\}$, and let $\hat{M} \simeq \mathbf{Z}^{\hat{d}}$ be the sublattice of exponents of the monomials in $\mathbf{C}\{\hat{X}\}$. Denote by $\hat{p} : M \to \hat{M}$ the canonical projection of M on $\hat{M}$.

LEMMA 11.2
One has the following equality: $\mathcal{N}_{\hat{X}}(\eta) = \hat{p}_{\mathbf{R}}(\mathcal{N}_X(\eta))$.

Proof
We can suppose, without loss of generality, that $\hat{X}$ is composed of the first $\hat{d}$ variables $X_1, \ldots, X_{\hat{d}}$.

Let $\hat{A} \in \hat{M}_{\mathbf{R}}$ be a vertex of $\mathcal{N}_{\hat{X}}(\eta)$. Then there is a monomial of η of the form $\hat{X}^{\hat{A}} X_{\hat{d}+1}^{j_1} \cdots X_d^{j_{d-\hat{d}}}$, and so $(\hat{A}, j_1, \ldots, j_{d-\hat{d}}) \in \mathcal{N}_X(\eta) \Rightarrow \hat{A} \in \hat{p}_{\mathbf{R}}(\mathcal{N}_X(\eta))$. This implies that $\mathcal{N}_{\hat{X}}(\eta) \subset \hat{p}_{\mathbf{R}}(\mathcal{N}_X(\eta))$.

Take now a vertex $\hat{A}$ of $\hat{p}_{\mathbf{R}}(\mathcal{N}_X(\eta))$. It is immediate to see that $\hat{A}$ is the projection by $\hat{p}$ of a vertex of $\mathcal{N}_X(\eta)$. Choose one such vertex $(\hat{A}, j_1, \ldots, j_{d-\hat{d}}) \in M_{\mathbf{Q}}$. Then $(\hat{A}, j_1, \ldots, j_{d-\hat{d}}) \in \mathrm{Supp}_X(\eta) \Rightarrow \hat{A} \in \mathrm{Supp}_{\hat{X}}(\eta) \Rightarrow \hat{A} \in \mathcal{N}_{\hat{X}}(\eta)$. This implies the reverse inclusion $\hat{p}_{\mathbf{R}}(\mathcal{N}_X(\eta)) \subset \mathcal{N}_{\hat{X}}(\eta)$. □

Even if we do not need it later in this work, we indicate here a generalization of the dominating exponent, as we think that it has independent interest. This generalization gives an *intrinsic meaning* to the notion of reduced Newton polyhedron.

Consider a germ $(\mathcal{H}, P)$ of a hypersurface with normal crossings on a smooth variety $\mathcal{V}$. We suppose that the r components $\mathcal{H}_1, \ldots, \mathcal{H}_r$ of $\mathcal{H}$ at P are taken in a fixed order. We recall that the notion of local coordinates adapted to $\mathcal{H}$ at P was defined before Lemma 8.5.

PROPOSITION 11.3
Let $x = (x_1, \ldots, x_n)$ *be a system of local coordinates of* $\mathcal{V}$ *adapted to* $\mathcal{H}$ *at* P. *Denote* $\hat{x} := (x_1, \ldots, x_r)$. *If* $h \in \mathcal{O}_{\mathcal{V},P}$, *then its reduced Newton polyhedron* $\mathcal{N}_{\hat{x}}(h)$ *depends only on the pair* $(h, \mathcal{H})$ *and on the ordering of the components of* $\mathcal{H}$ *at* P *but not on the choice of the coordinates* x.

Proof
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two coordinate systems of $\mathcal{V}$,

adapted to $\mathscr{H}$ at P. Then, $\forall k \in \{1, \ldots, r\}$, $x_k = y_k u_k$ with $u_k \in \mathscr{O}^*_{\mathscr{V},P}$. If $h \in \mathscr{O}_{\mathscr{V},P}$ and $h = \sum_{\mathrm{Supp}_{\hat{x}}(h)} c_{i_1 \cdots i_r} x_1^{i_1} \cdots x_r^{i_r}$ is the expression of h as a series in the variables $x_1, \ldots, x_r$ with coefficients in $\mathbf{C}\{x_{r+1}, \ldots, x_n\}$, one deduces that $h = \sum_{\mathrm{Supp}_{\hat{x}}(h)} (c_{i_1 \cdots i_r} u_1^{i_1} \cdots u_r^{i_r}) y_1^{i_1} \cdots y_r^{i_r}$.

Notice that $c_{i_1 \cdots i_r} u_1^{i_1} \cdots u_r^{i_r}$, seen as an element of $\mathbf{C}\{y_1, \ldots, y_n\}$, is not necessarily an element of $\mathbf{C}\{y_{r+1}, \ldots, y_n\}$.

Let $m \in \mathbf{N}^r$ be a vertex of $\mathscr{N}_{\hat{y}}(h)$ (where $\hat{y} := (y_1, \ldots, y_r)$). It is the exponent of a $(y_1, \ldots, y_r)$-monomial appearing in the series expansion of one of the terms $(c_{i_1 \cdots i_r} u_1^{i_1} \cdots u_r^{i_r}) y_1^{i_1} \cdots y_r^{i_r}$ introduced before. So, for this exponent $(i_1, \ldots, i_r)$, we have $m \in \{(i_1, \ldots, i_r)\} + \mathbf{R}^r_+$. But $(i_1, \ldots, i_r) \in \mathrm{Supp}_{\hat{x}}(h) \Rightarrow m \in \mathscr{N}_{\hat{x}}(h)$. As this is true for any vertex m of $\mathscr{N}_{\hat{y}}(h)$, it implies that $\mathscr{N}_{\hat{y}}(h) \subset \mathscr{N}_{\hat{x}}(h)$. Permuting now the roles of x and y, we get the desired equality $\mathscr{N}_{\hat{x}}(h) = \mathscr{N}_{\hat{y}}(h)$. □

This proposition motivates the following.

Definition 11.4
We call the invariant Newton polyhedron of the previous proposition the *Newton polyhedron of h with respect to* $(\mathscr{H}, P)$, and we denote it by $\mathscr{N}_{(\mathscr{H},P)}(h)$.

The function h has a d.e. with respect to $(\mathscr{H}, P)$ (see Def. 8.6) if and only if this polyhedron has only one vertex, which is then equal to $v_{(\mathscr{H},P)}(h)$. So this notion generalizes that of dominating exponent.

12. The reduced semigroup

Recall that the reduced equisingular dimension c' of $\mathscr{S}$ was defined in Section 10. If E is a set and $V \in E^d$, we denote by V' the c'-tuple of the first c'-coordinates of V. Now we particularize the constructions of Section 11 to the case where $\hat{X} = X'$.

The following lemma is an immediate consequence of Theorem 3.2. We use the notation $(a)^j := (a, \ldots, a) \in \mathbf{R}^j$.

LEMMA 12.1
For $i \in \{1, \ldots, G-1\}$, *one has* $A_i = A'_i \oplus (0)^{d-c'}$ *and*

$$A_G = \begin{cases} A'_G \oplus (0)^{d-c'} & \text{if } s \neq c-2, \\ A'_G \oplus \left(\frac{1}{N_G}\right)^2 \oplus (0)^{d-c} & \text{if } s = c-2. \end{cases}$$

The relations (3) remain true if one considers the vectors A'_i *and* $\overline{A}'_i$ *instead of the vectors* A_i *and* $\overline{A}_i$.

We mimic now the construction of the semigroups $\Gamma_{\mathscr{N}}(f)$ and $\Gamma_{\mathscr{D}}(f)$ (see Sec. 8) by introducing the following subsets of $\mathbf{Q}_+^{c'}$:

$$\Gamma'_{\mathscr{D}}(f) := \left\{v_{X'}(h(\xi)), h \in \mathbf{C}\{X\}[Y] - (f),\ h(\xi) \text{ has a d.e. with respect to } X'\right\},$$
$$\Gamma'_{\mathscr{N}}(f) := \left\{A' \in \mathbf{Q}_+^{c'},\ A' \text{ is a vertex of } \mathscr{N}_{X'}(h(\xi)),\ h \in \mathbf{C}\{X\}[Y] - (f)\right\}.$$

For the same reasons as in Section 8, they are additive semigroups.

We have the following proposition, to be compared with Propositions 8.1 and 8.2.

PROPOSITION 12.2
One has the following equality of semigroups:

$$\Gamma'_{\mathscr{N}}(f) = \Gamma'_{\mathscr{D}}(f) = \mathbf{N}^{c'} + \mathbf{N}\overline{A}'_1 + \cdots + \mathbf{N}\overline{A}'_G.$$

Proof
Take any $A \in \mathbf{N}^{c'}$ and $(i_0, \ldots, i_{G-1}) \in \mathbf{N}^G$. As

$$v_{X'}\big((X')^A(f_0(\xi))^{i_0} \cdots (f_{G-1}(\xi))^{i_{G-1}}\big) = A + i_0\overline{A}'_1 + \cdots + i_{G-1}\overline{A}'_G,$$

one gets the inclusions $\mathbf{N}^{c'} + \mathbf{N}\overline{A}'_1 + \cdots + \mathbf{N}\overline{A}'_G \subset \Gamma'_{\mathscr{D}}(f) \subset \Gamma'_{\mathscr{N}}(f)$.

Suppose now that $A' \in \Gamma'_{\mathscr{N}}(f)$. Then there is an element $h \in \mathbf{C}\{X\}[Y] - (f)$ such that A' is a vertex of $\mathscr{N}_{X'}(h(\xi))$. By Lemma 11.2, there is a vertex of $\mathscr{N}_X(h(\xi))$ of the form $(A', j_1, \ldots, j_{d-c'})$. But, by Proposition 8.1, this vertex is an element of $\mathbf{N}^d + \mathbf{N}\overline{A}_1 + \cdots + \mathbf{N}\overline{A}_G$, which implies that A' is an element of $\mathbf{N}^{c'} + \mathbf{N}\overline{A}'_1 + \cdots + \mathbf{N}\overline{A}'_G$. □

This motivates the following definition.

Definition 12.3
The semigroup $\Gamma'_{\mathscr{N}}(f) = \Gamma'_{\mathscr{D}}(f)$ is called the *reduced semigroup of $\mathscr{A}$ with respect to f*, denoted $\Gamma'(f)$.

If $(\Gamma, +)$ is a finitely generated abelian semigroup without torsion, let $\hat{\Gamma}$ be the lattice generated by Γ. Denote by $\sigma(\Gamma)$ the convex cone generated by Γ in $\hat{\Gamma}_{\mathbf{R}}$.

Particularize this to the reduced semigroup of f. As $\Gamma'(f) \subset \mathbf{Q}_+^{c'}$, the cone $\sigma(\Gamma'(f))$ is strictly convex. Let $u^1, \ldots, u^{c'}$ be the smallest nonzero elements of $\Gamma'(f)$ situated on the edges of $\sigma(\Gamma'(f))$. The following lemma shows that almost all the vectors $\overline{A}'_1, \ldots, \overline{A}'_G$ of $\mathbf{Q}_+^{c'}$ are determined by the isomorphism type of $\Gamma'(f)$.

LEMMA 12.4
For any $j \geq 1$, if $\alpha_1, \ldots, \alpha_{j-1}$ are already defined and verify $\Gamma'(f) \neq \mathbf{N}u^1 + \cdots + \mathbf{N}u^{c'} + \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_{j-1}$, there exists a unique smallest element α_j of $\Gamma'(f)$ not

contained in the semigroup $\mathbf{N}u^1 + \cdots + \mathbf{N}u^{c'} + \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_{j-1}$. *Define* $g \in \mathbf{N}$ *by* $\Gamma'(f) = \mathbf{N}u^1 + \cdots + \mathbf{N}u^{c'} + \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_g$ *and* $\epsilon \in \{0, 1\}$ *to be either* 0 *if* $s \neq c-2$ *or* 1 *if* $s = c-2$. *Then* $g \in \{G - \epsilon, G\}$ *and, after possibly permuting* $u^1, \ldots, u^{c'}$, *the components of* $\alpha_1, \ldots, \alpha_{G-\epsilon}$ *written in the basis* $u^1, \ldots, u^{c'}$ *coincide with the vectors* $\overline{A}'_1, \ldots, \overline{A}'_{G-\epsilon}$ *of* $\mathbf{Q}_+^{c'}$ *associated to any normalized qo-defining polynomial of* $\mathscr{S}$.

Proof

By Lemma 12.1, $\forall k \in \{1, \ldots, G - \epsilon\}$, one has $N_k = \min\{j \in \mathbf{N}^*,\ j\overline{A}'_k \in \mathbf{Z}^{c'} + \mathbf{Z}\overline{A}'_1 + \cdots + \mathbf{Z}\overline{A}'_{k-1}\}$. As $N_k > 1$, we get $\overline{A}'_k \notin \mathbf{N}^{c'} + \mathbf{N}\overline{A}'_1 + \cdots + \mathbf{N}\overline{A}'_{k-1}$. But if $\overline{A}' \in \Gamma'(f)$ and $\overline{A}' \notin \mathbf{N}^{c'} + \mathbf{N}\overline{A}'_1 + \cdots + \mathbf{N}\overline{A}'_{k-1}$, then $\overline{A}' = A'_0 + \sum_{j=1}^{G} i_j \overline{A}'_j$ with $A'_0 \in \mathbf{N}^{c'}$ and at least one of $i_k, \ldots, i_G$ is nonzero. This implies that $\overline{A}' \geq \overline{A}'_k$, and so $\overline{A}'_k$ is the unique smallest element of $\Gamma'(f)$ not contained in the semigroup $\mathbf{N}^{c'} + \mathbf{N}\overline{A}'_1 + \cdots + \mathbf{N}\overline{A}'_{k-1}$. As $(u^1, \ldots, u^{c'})$ obviously form a permutation of the canonical generators of the semigroup $(\mathbf{N}^{c'}, +)$, this proves the lemma. □

This lemma is used in the passage from the analytical invariance of the reduced semigroup to the analytical invariance of the normalized characteristic exponents (see Cor. 13.5).

13. The main results

In the sequel we suppose that the germ $(\mathscr{S}, 0)$ is irreducible, quasi-ordinary of dimension $d \geq 2$ and of embedding dimension $d + 1$. Moreover, we suppose that $\mathscr{S}$ is not smooth at 0. Let f be a qo-defining polynomial of $\mathscr{S}$. With our hypothesis, f is irreducible.

Define $\nu : \overline{\mathscr{S}} \longrightarrow \mathscr{S}$ to be the normalization morphism of $\mathscr{S}$, studied in Section 5. Let $\mu : \mathscr{R} \longrightarrow \overline{\mathscr{S}}$ be the orbifold map of $\overline{\mathscr{S}}$, introduced in Section 6, and let $\eta : \overline{\mathscr{R}} \longrightarrow \mathscr{R}$ be the canonical modification of $\mathscr{R}$, defined in Section 10. We also use the hypersurface $\overline{\mathscr{H}}$ and the finite set $\overline{\mathscr{P}}$, defined in the same section.

Denote $\theta := \nu \circ \mu \circ \eta : \overline{\mathscr{R}} \to \mathscr{S}$. By construction, the morphism θ depends only on the analytical type of $\mathscr{S}$ and not on any particular qo-defining polynomial. Let θ^* be the corresponding morphism of sheaves of local algebras. By formula (7), one has $\overline{\mathscr{H}} = \theta^{-1}(\mathscr{L}')$. Let P be a point of $\overline{\mathscr{P}}$. By Proposition 10.1, the hypersurface $\overline{\mathscr{H}}$ has c' components at P. The following definition generalizes the first one presented in the introduction for plane curves.

Definition 13.1

If $P \in \overline{\mathscr{P}}$, the *reduced semigroup of* $\mathscr{S}$ *with respect to* P, denoted $\Gamma'_P(\mathscr{S})$, is the following subsemigroup of $(\mathbf{N}^{c'}, +)$: $\Gamma'_P(\mathscr{S}) := \Gamma_{\overline{\mathscr{H}}, P}(\theta^*(\mathscr{A})_P)$.

The semigroup $\Gamma'_P(\mathscr{S})$ obviously depends only on the choice of the point P. Theorem 13.2 shows that, in fact, up to isomorphism, this semigroup is independent of $P \in \overline{\mathscr{P}}$, and so it is an *analytical invariant* of $\mathscr{S}$. In order to state it, we introduce a notation for the set of polynomials used to define the semigroup $\Gamma'_{\mathscr{D}}(f)$:

$$\mathscr{E}'(f) := \big\{h \in \mathbf{C}\{X\}[Y],\ h \text{ has a d.e. with respect to } X'\big\}.$$

Recall that the semigroup of an algebra with respect to a germ of a hypersurface with normal crossings was defined in Section 8.

THEOREM 13.2
Let f be a quasi-ordinary defining polynomial of $\mathscr{S}$. For every point $P \in \overline{\mathscr{P}}$, the image of the composition $\mathscr{E}'(f) \to \mathscr{A} \to \mathscr{O}_{\overline{\mathscr{R}},P}$ of the restriction mapping and of θ^* is contained in the set $\mathscr{E}_{\overline{\mathscr{H}},P}(\theta^*(\mathscr{A})_P)$. It induces a well-defined mapping Φ_P which realizes an isomorphism of semigroups:

$$\begin{aligned} \Phi_P : \Gamma'(f) &\longrightarrow \Gamma'_P(\mathscr{S}), \\ v_{X'}\big(h(\xi)\big) &\longrightarrow v_{\overline{\mathscr{H}},P}\big(\theta^*(h|_{\mathscr{S}})_P\big). \end{aligned}$$

Here h varies through $\mathscr{E}'(f)$.

We prove this theorem in the next section. It obviously generalizes Theorem 9.1.

As the left-hand semigroup does not depend on the choice of the point $P \in \overline{\mathscr{P}}$, and the right-hand one does not depend on the choice of qo-defining polynomial f, we get the following.

COROLLARY 13.3
As an abstract semigroup, $\Gamma'(f)$ does not depend on the chosen defining polynomial f of $\mathscr{S}$. We call it the reduced semigroup of $\mathscr{S}$, *denoted* $\Gamma'(\mathscr{S})$.

As a by-product of the proof of Theorem 13.2, we get a way to associate to some elements of $\mathscr{A}$ a value in the semigroup $\Gamma'(\mathscr{S})$.

COROLLARY 13.4
Let f be a qo-defining polynomial of the germ $\mathscr{S}$, and let $\xi \in R(f)$. If $h \in \mathscr{E}'(f)$, then the dominating exponent $v_{X'}(h(\xi))$, seen as an element of the abstract semigroup $\Gamma'(\mathscr{S})$, depends only on the image $h|_{\mathscr{S}} \in \mathscr{A}$ and not on the choice of f.

Our main application of Theorem 13.2 is the following one.

COROLLARY 13.5 (Lipman, Gau)
The characteristic exponents of a normalized qo-defining polynomial f of the germ $\mathscr{S}$ are analytical invariants of $\mathscr{S}$. (We recall that we suppose that condition (2) is verified.)

We postpone the proof to Section 15.

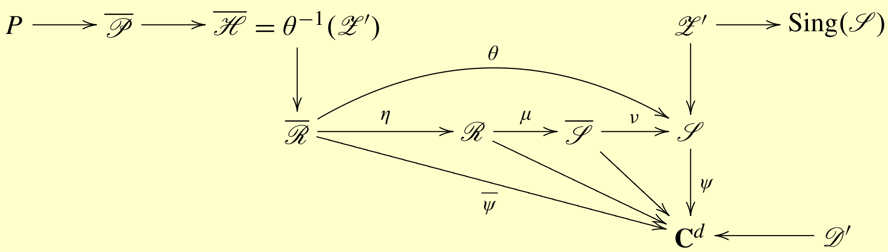
Before, the only available proof in arbitrary dimension was through the embedded topological invariance of the characteristic exponents, obtained by Lipman [17] and Gau [10]. We gave more details on this in the introduction.

Remark. From Corollary 13.5 one can also deduce the analytical invariance of the semigroup $\Gamma(f)$, as was done in [13] using the inversion formulae, expressing arbitrary characteristic exponents in terms of normalized ones.

14. Proof of Theorem 13.2

Recall from Section 3 that s denotes the number of components of $\mathrm{Sing}(\mathscr{S})$ which have codimension 1 in $\mathscr{S}$.

Denote $\overline{\psi} := \psi \circ \theta : \overline{\mathscr{R}} \to \mathbf{C}^d$. We have the following commutative diagram:



As in Section 9, take $\xi := Y|_{\mathscr{S}}$. One uses again formula (1).

One can choose as representative of $\overline{\psi}$ a localization to an open set of a toric morphism. Indeed, following Sections 5, 6, and 10, one can realize $\overline{\mathscr{S}}$ as a germ of toric variety $\overline{\mathscr{S}} \simeq \mathscr{Z}(W_G, \sigma_0)$ and representatives of μ, η as toric morphisms $\mu : \mathscr{Z}(\tilde{W}, \sigma_0) \to \mathscr{Z}(W_G, \sigma_0)$, $\eta : \mathscr{Z}(\tilde{W}, \tilde{\sigma}) \to \mathscr{Z}(\tilde{W}, \sigma_0)$. With such representatives of the morphisms, the point P is an orbit of dimension zero and $\overline{\mathscr{H}}$ is a union of closures of orbits of codimension 1. Let $T := (T_1, \dots, T_d)$ be the toric coordinates of $\overline{\mathscr{R}}$ centered at P. They are adapted to $\overline{\mathscr{H}}$ at P. So $\overline{\mathscr{H}}_P = Z(T_1) \cup \cdots \cup Z(T_{c'})$. With respect to such coordinates, the morphism $\overline{\psi}^*$ is monomial.

Let $\tau_1, \dots, \tau_d \in \tilde{M}$ be such that $T_i = \chi^{\tau_i}$, $\forall i \in \{1, \dots, d\}$. Denote by T' the set of coordinates $T_1, \dots, T_{c'}$, and denote by p' the projection of $\tilde{M}$ on the sublattice $\tilde{M}'$ generated by $\tau_1, \dots, \tau_{c'}$. If $\phi : \tilde{W} \to W$ is the morphism obtained by composing the changes of lattices of Sections 5 and 6, we see that $p' \circ \check{\phi} : M \to \tilde{M}'$ depends only on the restriction of $p' \circ \check{\phi}$ to M' and that the restriction $p' \circ \check{\phi} : M' \to \tilde{M}'$ is an isomorphism. Using formula (1), we see that Φ_P *is injective*.

In order to prove the surjectivity of Φ_P, we must show that if $h \in \mathbf{C}\{X\}[Y]$ verifies $\theta^*(h|_{\mathscr{S}})_P \in \mathscr{E}_{\overline{\mathscr{H}},P}(\theta^*(\mathscr{A})_P)$, then one can find another element $\tilde{h} \in \mathscr{E}'(f)$ such that $v_{\overline{\mathscr{H}},P}(\theta^*(\tilde{h}|_{\mathscr{S}})_P) = v_{\overline{\mathscr{H}},P}(\theta^*(h|_{\mathscr{S}})_P)$. As $f|_{\mathscr{S}} = 0$, we can suppose that $\deg(h) < \deg(f)$, after possibly making the Euclidean division of h by f. We consider then a complete system $(f_0, \dots, f_G)$ of semiroots of f and the $(f_0, \dots, f_G)$-adic expansion of h, which by our hypothesis is of the form $h = \sum c_{i_0 \cdots i_{G-1}} f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}$.

Formula (1) implies $v_{\overline{\mathscr{H}},P}(\theta^*(\tilde{h}|_{\mathscr{S}})_P) = v_{T'}\big(\overline{\psi}^*(h(\xi))\big)$. By Lemma 7.5, this vector is the image by p' of a vertex μ of $\mathscr{N}_T\big(\overline{\psi}^*(h(\xi))\big)$. By Lemma 4.3, $\mu = \check{\phi}(m)$, where m is a vertex of $\mathscr{N}_X(h(\xi))$. Now Lemma 7.4 shows that m is a vertex of one of the polyhedra $\mathscr{N}_X\big(c_{i_0 \cdots i_{G-1}}(f_0(\xi))^{i_0} \cdots (f_{G-1}(\xi))^{i_{G-1}}\big)$. In particular, it is of the form $v_X\big(X^A(f_0(\xi))^{i_0} \cdots (f_{G-1}(\xi))^{i_{G-1}}\big)$, where A is a vertex of $\mathscr{N}_X(c_{i_0 \cdots i_{G-1}})$. So $\mu = p'\big(v_{\overline{\mathscr{H}},P}(\theta^*(X^A f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}|_{\mathscr{S}})_P)\big)$. But $X^A f_0^{i_0} \cdots f_{G-1}^{i_{G-1}} \in \mathscr{E}'(f)$, showing that we can take $\tilde{h} := X^A f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}$. This proves that Φ_P *is surjective*. □

15. Proof of Corollary 13.5

We need first some classical results about germs of plane curves and about the equisingularity of plane sections of a germ of a hypersurface.

One has the following classical theorem, attributed sometimes to M. Noether (see [23, Prop. 6.5]).

PROPOSITION 15.1
Let x, y be two indeterminates. Let $f, g \in \mathbf{C}[[x]][y]$ be irreducible unitary polynomials. Denote by (f, g) their intersection number, and denote by

$$K(f,g) := \max\big\{v_x(\xi - \eta),\ \xi \in R(f),\ \eta \in R(g)\big\}$$

the exponent of coincidence *of f and g, where their roots are seen as Newton-Puiseux series. If $A_1, \dots, A_G$ are the characteristic exponents of f, if $A_{G+1} := +\infty$, and if $k \in \{0, \dots, G\}$ is the smallest integer so that $K(f,g) < A_{k+1}$, then*

$$\frac{(f,g)}{d_Y(f) d_Y(g)} = \frac{\overline{A}_k}{N_1 \cdots N_{k-1}} + \frac{K(f,g) - A_k}{N_1 \cdots N_k}.$$

The following proposition is an easy consequence of the also classical *inversion formulas* for plane germs (see [23, Prop. 4.3]).

PROPOSITION 15.2
Let C be a nonregular germ of irreducible plane curve, and let $a_1 > 1$ be its

first characteristic exponent in generic coordinates. Then the possible values of the first characteristic exponent, when the coordinate system varies, span the finite set $\{a_1, 1/a_1\} \cup \{1/m,\ m \leq [a_1]\}$. *Moreover, if the first characteristic exponents of* C *in two coordinate systems coincide, then the whole sequence of such exponents coincide.*

The following proposition is also classical (see [29], [30]).

PROPOSITION 15.3

Let C *and* C' *be two germs of reduced plane curves. The following conditions are equivalent.*

(1) *There is a one-to-one correspondence* $C_i \leftrightarrow C'_i$ *between the components of* C *and* C' *such that* $\forall i$, C_i *and* C'_i *have the same sequence of characteristic Newton-Puiseux exponents and such that* $\forall i \neq j$, *one has the equality* $(C_i, C_j) = (C'_i, C'_j)$ *of intersection numbers.*

(2) *The germs* C *and* C' *have isomorphic processes of embedded resolution by blow-ups.*

(3) *The dual graphs of their total transforms by the minimal embedded resolution morphisms are isomorphic.*

This motivates the following definition.

Definition 15.4

The germs C and C' are said to be *equisingular* if the equivalent conditions of Proposition 15.3 are verified.

Let $(\mathscr{V}, 0)$ be a germ of irreducible complex analytical space of dimension d and of embedding dimension $d+1$. Let $\mathscr{H}$ be an irreducible germ of a hypersurface on $\mathscr{V}$. Consider an embedding $E : (\mathscr{V}, 0) \to (\mathbf{C}^{d+1}, 0)$, and denote by ψ_E the projection on the first d coordinates. Suppose that ψ_E is finite. Let P be a smooth point of $\psi_E(\mathscr{H})$, and let L be a smooth germ of curve in $\mathbf{C}^d \times \{0\}$, transversal to $\psi_E(\mathscr{H})$ at P. Let Q be any point of $\psi_E^{-1}(P) \cap \mathscr{V}$. Zariski [29] proves the following.

THEOREM 15.5

The equisingularity type of the germ of a plane curve $(\psi_E^{-1}(L) \cap \mathscr{V}, Q)$ *depends only on the analytical type of* $(\mathscr{V}, \mathscr{H}, 0)$ *and not on the choices of* E, P, Q, L.

For more information about the notion of equisingularity, one can consult Teissier [27].

Let us pass now to the proof of the corollary.

Suppose that f is a *normalized* qo-defining polynomial of $\mathscr{S}$. Lemma 12.1 shows that the characteristic exponents $A_1, \dots, A_G$ of f are known once $A'_1, \dots, A'_G$ and N_G are known. The same lemma also shows that the knowledge of these last quantities is equivalent to the one of $\overline{A}'_1, \dots, \overline{A}'_{G-1}$, A'_G and N_G. By Lemma 12.4, the abstract semigroup $\Gamma'(\mathscr{S})$ allows us to determine the vectors $\overline{A}'_1, \dots, \overline{A}'_{G-1}$ of $\mathbf{Q}_+^{c'}$. The same lemma also shows that $\forall\, i \in \{1, \dots, G-1\}$, the number N_i is the index of $\mathbf{Z}^{c'} + \mathbf{Z}\overline{A}'_1 + \cdots + \mathbf{Z}\overline{A}'_{i-1}$ in $\mathbf{Z}^{c'} + \mathbf{Z}\overline{A}'_1 + \cdots + \mathbf{Z}\overline{A}'_i$, which proves its analytical invariance. Moreover, the same is true for $\overline{A}'_G$ and N_G, except when $s = c-2$. So the corollary is proved for $s \neq c-2$.

Suppose now that $s = c-2$. We show that, in addition to $\overline{A}'_1, \dots, \overline{A}'_{G-1}$, one can also determine A'_G and N_G from the analytical type of $(\mathscr{S}, 0)$. In order to do it, we no longer use the analytical invariance of the semigroup $\Gamma'(f)$. Instead, we use the results presented at the beginning of this section.

In what follows, if P is a point of $\mathbf{C}^d$ and $I \subset [[1, d]]$, we denote by $\mathscr{D}_I^P$ the $|I|$-codimensional affine subspace of $\mathbf{C}^d$ passing through P and parallel to $\mathscr{D}_I$.

By Theorem 3.2, the variety $\mathscr{Z}_{\{c-1,c\}}$ is analytically distinguished as the only component of $\mathrm{Sing}(\mathscr{S})$ of codimension 2 in $\mathscr{S}$. We can obtain N_G from the analytical structure of $\mathscr{S}$ as *the least number of blow-ups of the strict transforms of* $\mathscr{Z}_{\{c-1,c\}}$ *needed to desingularize* $\mathscr{S}$ *at their generic points*. This is an immediate consequence of Theorem 3.2(3).

It remains to show that A'_G is analytically determined by $\mathscr{S}$. We do it by showing that $A^i_G \in \mathbf{Q}_+$ is analytically determined for any $i \in \{1, \dots, c-2\}$.

(1) *Consider the case where* $c = 3$ *and* $G = 1$. Then $A_1 = (A^1_1, 1/N_1, 1/N_1) \oplus (0)^{d-3}$ with $B^1_1 := N_1 A^1_1 \in \mathbf{N}^* - \{1\}$ (see Th. 3.2). Take $P \in \mathscr{D}_1 - (\mathscr{D}_2 \cup \mathscr{D}_3)$. Denote $C^P := \psi^{-1}(\mathscr{D}^P_{[[2,d]]}) \cap \mathscr{S}$. The intersection $\psi^{-1}(P) \cap \mathscr{S}$ is reduced to one point Q. By Theorem 15.5, the equisingularity type of (C^P, Q) is an analytical invariant of $(\mathscr{S}, 0)$. The germ (C^P, Q) has $\gcd(N_1, B^1_1)$ irreducible components, and so this number is also analytically determined by $(\mathscr{S}, 0)$.

If $\gcd(N_1, B^1_1) = 1$, then B^1_1/N_1 is the unique Newton-Puiseux characteristic exponent of $C^P \hookrightarrow \psi^{-1}(\mathscr{D}^P_{[[2,d]]})$ with respect to the coordinates (X_1, Y) of the plane $\psi^{-1}(\mathscr{D}^P_{[[2,d]]})$. As $A^1_1 \neq 1/N_1$, neither A^1_1 nor $1/A^1_1$ is an integer. By Proposition 15.2, we deduce that A^1_1 is determined as the unique first characteristic exponent of (C^P, Q), for varying coordinate systems, having N_1 as the denominator of its irreducible form.

If $\gcd(N_1, B^1_1) > 1$, then take two irreducible components of C^P at Q. By Proposition 15.1, their intersection number at Q is $(N_1/\gcd(N_1, B^1_1)) \cdot (B^1_1/\gcd(N_1, B^1_1))$. As N_1 and $\gcd(N_1, B^1_1)$ are already determined, this determines B^1_1 and, consequently, $A^1 = B^1_1/N_1$.

(2) *Consider the case where* $c = 3$ *and* $G > 1$. Then the characteristic mono-

mials of f are $X_1^{A_1^1}, \ldots, X_1^{A_{G-1}^1}, X_1^{A_G^1} X_2^{1/N_G} X_3^{1/N_G}$. We take the same notation as in the previous case. If A_G^1 is a characteristic exponent of the components of the germ (C^P, Q) in the coordinates (X_1, Y), then Proposition 15.2 shows that it is determined by the equisingularity type of (C^P, Q), as $A_1^1, \ldots, A_{G-1}^1$ are already known. If A_G^1 is not a characteristic exponent of the components of (C^P, Q), then there are at least two such components. We look at the intersection number I of any two of them. By Proposition 15.1, we have $I/(N_1 \cdots N_{G-1}) = N_{G-1}\overline{A}_{G-1}^1 - A_{G-1}^1 + A_G^1$, which determines A_G^1 from the knowledge of $A_1^1, \ldots, A_{G-1}^1, I$. Indeed, Lemma 12.1 shows that $N_1, \ldots, N_{G-1}$ can be deduced from $A_1^1, \ldots, A_{G-1}^1$.

(3) *Consider the case where $c \geq 4$ and $G \geq 1$.* Then the characteristic monomials of f are $X_1^{A_1^1} \cdots X_{c-2}^{A_1^{c-2}}, \ldots, X_1^{A_{G-1}^1} \cdots X_{c-2}^{A_{G-1}^{c-2}}$ and $X_1^{A_G^1} \cdots X_{c-2}^{A_G^{c-2}} X_{c-1}^{1/N_G} X_c^{1/N_G}$. We want to show that for all $i \in \{1, \ldots, c-2\}$, $A_G^i \in \mathbf{Q}_+$ is determined by the analytical structure of $(\mathscr{S}, 0)$.

Let $P(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_d)$ be a point of $\mathscr{D}_i - \bigcup_{j \neq i} \mathscr{D}_j$. Define $C_i^P := \psi^{-1}(\mathscr{D}^P_{[[1,d]]-\{i\}}) \cap \mathscr{S}$. It is a curve embedded in the plane $\psi^{-1}(\mathscr{D}^P_{[[1,d]]-\{i\}})$. Localize it at any point $Q \in \psi^{-1}(P) \cap \mathscr{S}$, and look at the equisingularity type of the germ (C^P, Q). In particular, look at the characteristic exponents of the irreducible components and at the intersection numbers of the pairs of components, as in the previous two cases. Each component has the same characteristic exponents as the curve with Newton-Puiseux series $X_i^{A_1^i} + \cdots + X_i^{A_G^i}$. These characteristic exponents form a (possibly strict) subset of $\{A_1^i, \ldots, A_G^i\}$. We subdivide this case into two subcases, analogous, respectively, to case (1) and case (2), treated before.

Suppose that $A_1^i, \ldots, A_{G-1}^i \in \mathbf{N}$. Then $B_G^1 := N_G A_G^1 \in \mathbf{N}$ and we determine A_G^1 as in case (1), using the fact that $s = c-2$ implies $B_G^1 > 1$.

Suppose that at least one of $A_1^i, \ldots, A_{G-1}^i$ *is in* $\mathbf{Q} - \mathbf{N}$. Then, as $A_1^i, \ldots, A_{G-1}^i$ are known, we can determine the characteristic exponents among them. Then we determine A_G^1 as in case (2). □

16. Comparison with the 2-dimensional case

Let us now compare this work with the paper [22], in which we had obtained the analytical invariance of the semigroup and of the normalized characteristic exponents in the case of quasi-ordinary surfaces.

In [22] the strategy of proof was the same. The morphism $\theta : \overline{\mathscr{R}} \to \mathscr{S}$ was obtained also as a composition $\theta = \nu \circ \mu \circ \eta$ with $\eta : \overline{\mathscr{S}} \to \mathscr{S}$ being the normalization morphism of $\mathscr{S}$. The proof of the isomorphism of the semigroups was basically the same as here.

The main difference is that we defined μ to be *the minimal resolution* of $\overline{\mathscr{S}}$. The last morphism η was either the identity or the blow-up of a point. We took as hypersurface $\overline{\mathscr{H}}$ the full preimage $\theta^{-1}(\mathrm{Sing}(\mathscr{S}))$ of the singular locus of $\mathscr{S}$ and as set $\overline{\mathscr{P}}$ the union of its singular points. With the exception of the case when $\mathscr{S}$ was analytically isomorphic with the germ of a quadratic cone at its vertex, we showed that $\overline{\mathscr{P}} \neq \emptyset$. At the points of $\overline{\mathscr{P}}$, the curve $\overline{\mathscr{H}}$ had two local components, and so *it was not needed to consider reduced semigroups*, as was done in the present paper.

The obstruction to extending that method to higher dimensions was that, in general, *one no longer has unicity of the minimal resolution of* $\overline{\mathscr{S}}$. Even the existence of some canonical nonminimal resolution would have been enough if it could have been obtained by a toric morphism once $\overline{\mathscr{S}}$ was presented as a germ of toric variety in the way explained in Section 5.

The solution of our problem of extension to higher dimensions came once we looked for a morphism μ *without asking it to be birational*. Indeed, the attributes of μ which were important for us were that its source was smooth and that it could be presented as a toric morphism once $\overline{\mathscr{S}}$ was presented as a germ of toric variety in the way explained in Section 5. We could then realize this by considering the orbifold map introduced in Section 6.

At this stage, after composing ν and μ, we saw that we did not have at our disposal canonical hypersurfaces with normal crossings having enough components. In order to get more components, we introduced the construction of Section 10.

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Institut de Mathématiques, Université Paris 7 Denis Diderot, UMR CNRS 7586, Equipe "Géométrie et dynamique," Case 7012, 2 place Jussieu, 75251 Paris CEDEX 05, France;
ppopescu@math.jussieu.fr