The Regular Inverse Galois Problem over Large Fields

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§1 Introduction

There has been recent progress on the Inverse Galois Problem. Most of it consists of results on the absolute Galois group $\mathrm{G}(K(T))$ when K is a field with various good arithmetic properties. This paper is a survey of that recent progress. Our goal is also to try to unify the results and the questions that have arisen in the last few years.

Historically the Inverse Galois Problem (IGP) is: is each finite group the Galois group $\mathrm{G}(E/\mathbb{Q})$ of an extension of $\mathbb{Q}$? The modern approach consists in studying rather the Regular Inverse Galois Problem (RIGP): is each finite group the Galois group $\mathrm{G}(E/\mathbb{Q}(T))$ of *a Galois extension $E/\mathbb{Q}(T)$ with $E/\mathbb{Q}$ a regular extension*? (As usual, we just say in the sequel *regular Galois extension $E/\mathbb{Q}(T)$*). *Regular* means that $E\cap\overline{\mathbb{Q}}=\mathbb{Q}$, or, equivalently, $\mathrm{G}(E/\mathbb{Q}(T))=\mathrm{G}(E\overline{\mathbb{Q}}/\overline{\mathbb{Q}}(T))$. Here are three reasons why considering this problem is more natural:

(1) A positive answer to the RIGP implies a positive answer to the IGP. This classically follows from Hilbert's irreducibility theorem.

(2) Regular Galois extensions $E/\mathbb{Q}(T)$ correspond to Galois covers $f: X\to\mathbb{P}^1$ that are defined over $\mathbb{Q}$ along with their automorphisms. Thus the RIGP essentially consists in studying the action of $\mathrm{G}(\mathbb{Q})$ on covers of the projective line. This fits the general feeling that the action of $\mathrm{G}(\mathbb{Q})$ on geometric objects is expected to reflect much of its structure.

(3) The RIGP can be formulated more generally for an arbitrary field K (possibly of positive characteristic) in place of $\mathbb{Q}$. Given any field K, each group might well be the Galois group of a regular Galois extension of $K(T)$: at least, no counter-example is known. In contrast with the IGP, the RIGP might not depend on the base field K but rest on some universal property of the field $K(T)$ of rational functions.

Because of the regularity condition, solving the RIGP (in an affirmative way) over a given field automatically solves it over each overfield. Therefore it is sufficient to solve the RIGP over each prime field Q. The RIGP has been solved over each algebraically

closed field: the real problem is the descent from $\overline{\mathrm{Q}}$ to Q. Roughly speaking there are then two directions of work. Group theorists fix a group and try to realize it over $\mathrm{Q}(T)$ regularly (*i.e.*, as the Galois group of a regular Galois extension of $\mathrm{Q}(T)$). Thanks to the so-called rigidity criterion and its developments, there have been since the late 70' many results in that direction for simple groups in particular. We refer the reader to the works of Matzat ([Mat], [MatMa]) and his students for that aspect of the question (see also [De1]). On the other hand, arithmetic geometers try to realize regularly *all* finite groups over $K(T)$ with K a given algebraic extension (as small as possible) of Q. There has been more recently some progress in that direction. We will focus on this second aspect of the problem, which was developed in particular by Fried, Harbater and Pop. Of course this distinction is somewhat artificial for in practice there is a strong interaction between the group theory and the arithmetic.

Basically most of the recent progress can be summarized by saying that the RIGP is solved if the base field is "large". By large we actually mean the following precise property, which was introduced by F. Pop: each smooth geometrically irreducible curve defined over K has infinitely many K-rational points provided there is at least one. We will study this property more precisely in §3. Haran and Jarden call it "ample" in [HaJa], which expresses a certain tendency of these fields to develop abundantly through the points of a variety. We note that AMPLE can also be understood as a property of "Automatique Multiplication des Points Lisses Existants". We will use this terminology.

The statement above however does not account for a second series of results of the same spirit. Similarly, the general RIGP does not account for a second series of classical conjectures of the area. Those results and conjectures give, or predict, under certain conditions, the exact structure of some absolute Galois groups. Historically, the starting problem of this second circle is the conjecture of Shafarevich: the absolute Galois group $\mathrm{G}(\mathbb{Q}^{ab})$ is a free profinite group (on countably many generators).

These two circles of the area are closely related. They are both concerned with the structure of absolute Galois groups. The methods use the same kind of arguments, namely some patching and gluing techniques for analytic covers and some specialization arguments for ample fields. Still it was our feeling that the exact connection had never been made completely clear. The goal of this paper is

(1) to state a conjecture that unifies all classical questions. This conjecture is the Main Conjecture. We state it in §1 and show the connection with all other conjectures.

(2) to state a theorem that summarizes most recent results of the area. This theorem is the Main Theorem. It merely asserts that the Main Conjecture is true if the base field K is ample. This contains the above statement that the RIGP is solved if K is ample. The

Main Theorem is precisely stated in §2 where we also show how to deduce most recent results as special cases. A general proof of the Main Theorem was given by F. Pop [Po4]. (3) to explain the main arguments of the proof of Main Theorem. For simplicity, we will restrict to the solution of the RIGP over large fields.

The necessary background on the arithmetic of fields and the theory of covers can be found in the Fried-Jarden book [FrJa] and in the recent books [MatMa] by Matzat-Malle and [Vo] by Völklein.

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§2 Conjectures

2.1 Classical conjectures.

2.1.1 First circle. Recall from the introduction these various conjectures around the Inverse Galois Problem. The Galois group of a Galois extension E/k is denoted by $\mathrm{G}(E/k)$. Given a field K, we denote by K_s (resp. by $\overline{K}$) a separable (resp. algebraic) closure of K and by $\mathrm{G}(K)$ the absolute Galois group $\mathrm{G}(K) = \mathrm{G}(K_s/K)$ of K.

Conjecture ($\mathbf{RIGP}/_{K(T)}$) — *For every field K and for every finite group G, there exists a regular Galois extension $E_G/K(T)$ such that $\mathrm{G}(E_G/K(T)) = G$.*

Conjecture ($\mathbf{IGP}/_{K\ hilb.}$) — *For every hilbertian field K and for each finite group G, there exists a Galois extension E_G/K such that $\mathrm{G}(E_G/K) = G$. Or, equivalently, each finite group G is a quotient of the absolute Galois group $\mathrm{G}(K)$ of K.*

Conjecture (IGP) — *For each finite group G, there exists a Galois extension $E_G/\mathbb{Q}$ such that $\mathrm{G}(E_G/\mathbb{Q}) = G$.*

We have: $$\mathbf{RIGP}/_{K(T)} \Longrightarrow \mathbf{IGP}/_{K\ hilb.} \Longrightarrow \mathbf{IGP}$$

Proof. Indeed, it follows from the hilbertian property that, if $E_T/K(T)$ is a regular Galois extension of Galois group G, then there exists some specialization $t \in K$ of T such that the specialized extension E_t/K is a Galois extension of Galois group G. Whence $\mathbf{RIGP}/_{K(T)} \Rightarrow \mathbf{IGP}/_{K\ hilb.}$. From Hilbert's irreducibility theorem, $\mathbb{Q}$ is a hilbertian field. So $\mathbf{IGP}/_{K\ hilb.} \Rightarrow \mathbf{IGP}$. □

2.1.2 Second circle. Conjectures of this second circle are conjectures about embedding problems. Recall that an *embedding problem* for a group Γ is a diagram of group homomorphisms

$$\begin{array}{ccccccccc} & & & & & & \Gamma & & \\ & & & & & & \downarrow f & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\alpha} & H & \longrightarrow & 1 \end{array}$$

where the horizontal sequence is exact and the map $f : \Gamma \to H$ is surjective. A *proper solution* is a surjective group homomorphism $g : \Gamma \to G$ such that $\alpha g = f$. Without the condition "g surjective", such a map g is said to be a *weak solution*. The embedding problem is said to be *finite* if G is finite. It is said to be *split* if $\alpha : G \twoheadrightarrow H$ has a group-theoretic section.

A profinite group Γ is said to be *projective* if each finite embedding problem for Γ is weakly solvable. Free profinite groups are some examples of projective groups. Finally recall that, given a weakly solvable embedding problem for a group Γ, there exists a standard procedure that generally allows to reduce to a split situation. More precisely, this procedure (e.g. [Po4;§1 B) 2)]) uses a weak solution to construct a *split* embedding problem for Γ such that existence of a proper solution for it implies existence of a proper solution for the original one. We will use this reduction in several occasions. For simplicity, we call it the *weak→split* reduction.

Conjecture (Fried-Völklein [FrVo2]) — *Let K be a hilbertian countable field such that* $\mathrm{G}(K)$ *is projective (e.g. cohdim$(K) \leq 1$), then* $\mathrm{G}(K)$ *is pro-free.*

Conjecture (Shafarevich) — $\mathrm{G}(\mathbb{Q}^{ab})$ *is pro-free.*

These conjectures are more or less classical. We will denote them respectively by **FrVo** and **SHA**. The latter is actually a special case of the former. Indeed, from a result of Kuyk, $\mathbb{Q}^{ab}$ is hilbertian, see [FrJa,Th.15.6]; and $\mathbb{Q}^{ab}$ is of cohomological dimension ≤ 1: this follows from the fact that any division algebra over every number field has a cyclotomic splitting field [CaFr]. In turn, **FrVo** is a consequence of the two following equivalent conjectures about embedding problems.

Conjecture (Split $\mathbf{EP}/_{K(T)}$) — *Let K be an arbitrary field. Then each split embedding problem for $\mathrm{G}(K(T))$ has a proper solution.*

Conjecture (Split $\mathbf{EP}/_{K\ hilb.}$) — *Let K be a hilbertian field. Then each split embedding problem for $\mathrm{G}(K)$ has a proper solution.*

We call these conjectures the *Split Embedding Problem conjecture* over $K(T)$ (resp. over hilbertian fields). We have:

$$\textbf{Split EP}/_{K(T)} \Longleftrightarrow \textbf{Split EP}/_{K\ hilb.} \Longrightarrow \textbf{FrVo} \Longrightarrow \textbf{SHA}$$

Proof of Split $\mathbf{EP}/_{K(T)}$ $\Leftrightarrow$ Split $\mathbf{EP}/_{K\ hilb.}$: ($\Leftarrow$) follows from the fact that for every field K, the field $K(T)$ is hilbertian [FrJa;Th.12.10].

($\Rightarrow$): Given a split embedding problem for $\mathrm{G}(K)$ with K hilbertian, consider the embedding problem for $\mathrm{G}(K(T))$ obtained by composition with the map $\mathrm{G}(K(T)) \to \mathrm{G}(K)$. The **Split $\mathbf{EP}/_{K(T)}$** conjecture provides a proper solution. Then, as in the proof of $\mathbf{RIGP}/_{K(T)} \Rightarrow \mathbf{IGP}/_{K\ hilb.}$ above, use the hilbertian property to specialize this proper solution to a proper solution of the original embedding problem for $\mathrm{G}(K)$. □

Proof of Split $\mathbf{EP}/_{K\ hilb.}$ $\Rightarrow$ FrVo. We will show that any given embedding problem for $\mathrm{G}(K)$ has a proper solution. Conclusion "$\mathrm{G}(K)$ pro-free" will follow then from Iwasawa's theorem recalled below. Since $\mathrm{G}(K)$ is assumed to be projective, the given embedding problem has a weak solution. From the weak→split reduction, one may assume that the embedding problem is split. Therefore, from the **Split $\mathbf{EP}/_{K\ hilb.}$** conjecture, this split embedding problem has a proper solution. □

Theorem 2.1 (Iwasawa [Iw],[FrJa;Cor.24.2]) — *Let K be a countable field, $\mathrm{G}(K)$ is pro-free if and only if each finite embedding problem for $\mathrm{G}(K)$ has a proper solution.*

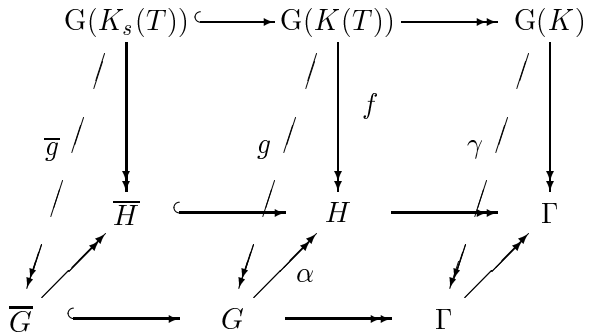
2.1.3 Conclusion. Finally we have **Split $\mathbf{EP}/_{K\ hilb.}$ $\Rightarrow$ $\mathbf{IGP}/_{K\ hilb.}$**. Indeed realizing a group G over K amounts to solving the split embedding problem for $\mathrm{G}(K)$ in which the exact sequence is $1 \to G \to G \to 1 \to 1$. The following diagram summarizes §2.1.

$$\begin{array}{ccccccc} \textbf{RIGP}/_{K(T)} & \Longrightarrow & \textbf{IGP}/_{K\ hilb.} & \Longrightarrow & \textbf{IGP} \\ & & \Uparrow & & \\ \textbf{Split EP}/_{K(T)} & \Longleftrightarrow & \textbf{Split EP}/_{K\ hilb.} & \Longrightarrow & \textbf{FrVo} & \Longrightarrow & \textbf{SHA} \end{array}$$

Remark 2.2. Conjectures of the second circle seem to be stronger since they give results on the structure of absolute Galois groups whereas those of the first circle only deal with the finite quotients of absolute Galois groups. But there is in fact no other obvious implication than **Split EP**$/_{K(T)} \Rightarrow$ **IGP**$/_{K(T)}$ between both these sets of conjectures. For example there is no obvious implication between **FrVo** and **IGP** [1], nor there is between **Split EP**$/_{K(T)}$ and **RIGP**$/_{K(T)}$. The motivation of the following section is to state a more general conjecture that unifies both these circles of the area. The difference between the two conjectures **Split EP**$/_{K(T)}$ and **RIGP**$/_{K(T)}$ will then become clear (see Remark 2.4).

2.2 Main Conjecture

2.2.1 Statement of the Main Conjecture. Suppose given a commutative diagram of group homomorphisms



- where the sequences in which the arrows are lined up are exact,

[1] The main point is that $\mathrm{G}(\mathbb{Q})$ is not projective (since $\mathrm{G}(\mathbb{Q})$ has elements of order 2) and so **FrVo** cannot be applied directly to $\mathrm{G}(\mathbb{Q})$.

- where $\hookrightarrow$ means that the homomorphism in question is injective,
- where $\twoheadrightarrow$ means that the homomorphism in question is surjective.

Such a diagram is called an *embedding problem of exact sequences over* $K(T)$. A *proper solution* is a triple $(\bar{g}, g, \gamma)$ of surjective maps as in the diagram above, such that the enlarged diagram commutes. If the condition "surjective" is removed, the triple $(\bar{g}, g, \gamma)$ is said to be a *weak solution*.

Main Conjecture — *Let K be an arbitrary field. Then for each embedding problem EP of exact sequences over $K(T)$, if EP has a weak solution, then EP has a proper solution.*

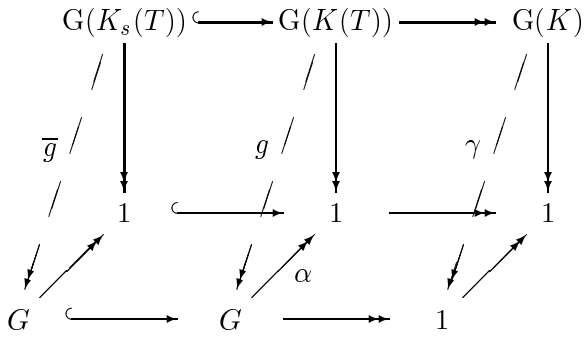
Remark 2.3. (a) Weak solutions $(\bar{g}, g, \gamma)$ actually correspond in a one-one way with the maps $g : \mathrm{G}(K(T)) \to G$ such that $\alpha g = f$. Indeed, given g, take for $\bar{g}$ the restriction of g to $\mathrm{G}(K_s(T))$. The containment $g(\mathrm{G}(K_s(T)) \subset \overline{H}$ holds because the two down-right groups in the diagram are equal (to Γ). There exists then a unique map γ that makes the diagram commute. The triple $(\bar{g}, g, \gamma)$ is a weak solution.

(b) We will say that the embedding problem of exact sequences over $K(T)$ is split if the map α splits. In that case, the embedding problem has a weak solution. Conversely, it is an exercise to show if an embedding problem of exact sequences has a weak solution, then there exists a *split* embedding problem of exact sequences such that existence of a proper solution for it implies existence of a proper solution for the original one: the weak→split reduction mentioned above for classical embedding problems generalizes with no difficulties. Therefore, in the Main Conjecture, the condition that the embedding problem has a weak solution can be replaced by the condition that it is split.

2.2.2 Relation with other conjectures. The Main Conjecture contains all conjectures of §2.1. In summary we have

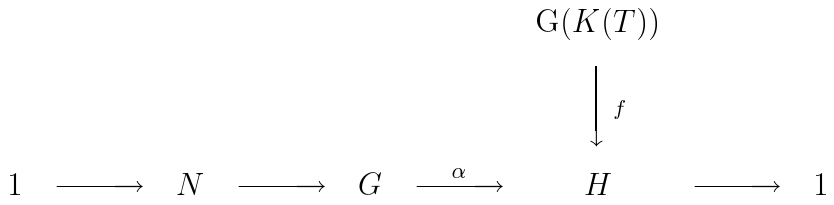
$$\begin{array}{lccccccc} & \Longrightarrow & \mathbf{RIGP}/_{K(T)} & \Longrightarrow & \mathbf{IGP}/_{K\ hilb.} & \Longrightarrow & \mathbf{IGP} & \\ \mathbf{Main\ Conj.} & & & & \Uparrow & & & \\ & \Longrightarrow & \mathbf{Split\ EP}/_{K(T)} & \Longleftrightarrow & \mathbf{Split\ EP}/_{K\ hilb.} & \Longrightarrow & \mathbf{FrVo} & \Longrightarrow\ \mathbf{SHA} \end{array}$$

Proof of Main Conj. $\Rightarrow \mathbf{RIGP}/_{K(T)}$: From the Main Conjecture, the following embedding problem of exact sequences over $K(T)$,

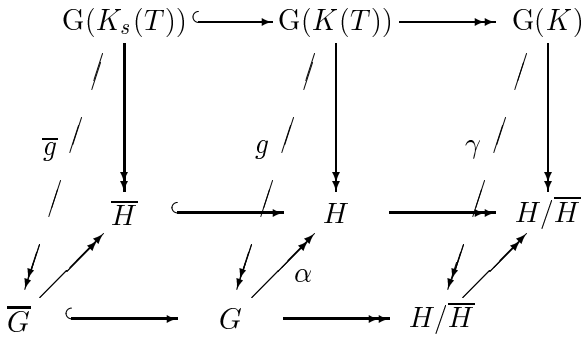


which is obviously split, has a proper solution. This exactly means that there exists an extension $E_G/K(T)$ such that $\mathrm{G}(E_G/K(T)) = \mathrm{G}(E_G K_s/K_s(T)) = G$.

Proof of Main Conj. $\Rightarrow$ Split EP$/_{K(T)}$: Consider a split embedding problem



Denote the group $f(\mathrm{G}(K_s(T)))$ by $\overline{H}$, the natural map $H \to H/\overline{H}$ by p and the kernel of the surjective map $p\alpha$ by $\overline{G}$. It is readily checked that α maps surjectively $\overline{G}$ onto $\overline{H}$. Thus we can form the following embedding problem of exact sequences over $K(T)$



This embedding problem is split. From the Main Conjecture there exists a proper solution $(\bar{g}, g, \gamma)$. In particular the map g is a proper solution of the original embedding problem for $\mathrm{G}(K(T))$. $\square$

Remark 2.4. Roughly speaking the Main Conjecture asserts that each split embedding problem for $\mathrm{G}(K(T))$ given with a certain constraint over K_s has a proper solution. Conjectures **Split EP**$/_{K(T)}$ and **RIGP**$/_{K(T)}$ correspond to the following special cases. The **Split EP**$/_{K(T)}$ conjecture is obtained by leaving out the constraint over K_s. While the **RIGP**$/_{K(T)}$ has the constraint over K_s but is concerned only with the split embedding problem for which the quotient is $H = 1$.

2.2.3 Further comments. (a) The Main Conjecture can be generalized in two directions. First the field $K(T)$ can be replaced by the function field $K(C)$ of any smooth projective K-curve. That is, the upper exact sequence of the diagram in §2.2.1 is replaced by the exact sequence

$$1 \to \mathrm{G}(K_s(C)) \to \mathrm{G}(K(C) \to \mathrm{G}(K) \to 1$$

Harbater [Har4] and Pop [Po2] work in this more general context. An extra difficulty is that this exact sequence need not be split in general, in contrast with the special case $C = \mathbb{P}^1$.

A second generalization is to replace the absolute Galois group $\mathrm{G}(K(C))$ by the algebraic fundamental group $\pi_1(C-D)$ where D is a reduced $\mathrm{G}(K)$-invariant divisor of C. That is, the upper exact sequence of the diagram in §2.2.1 is replaced by the exact sequence

$$1 \to \pi_1(\overline{C}-\overline{D}) \to \pi_1(C-D) \to \mathrm{G}(K) \to 1$$

where $\overline{C}-\overline{D} = (C-D) \otimes_K K_s$. However, with that change, the conjecture is false. Take $K = \mathbb{C}$, $C = \mathbb{P}^1$ and $|D| > 0$. Since $\pi_1(\mathbb{P}^1-D)$ is pro-free of rank $|D| - 1$, an embedding problem

$$\begin{array}{ccccccccc} & & & & & & \pi_1(\mathbb{P}^1 - D) & & \\ & & & & & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\alpha} & H & \longrightarrow & 1 \end{array}$$

with G of rank $\geq |D|$ cannot have any proper solution. But the Main Conjecture predicts that the embedding problem has a proper solution if some extra branch points are allowed. More precisely, the Main Conjecture can be rephrased as follows:

(*) given a split embedding problem for the exact sequence above of fundamental groups, there exists a finite set $D' \subset C$ such that the embedding problem for the exact sequence

$$1 \to \pi_1(\overline{C} - (\overline{D} \cup \overline{D'})) \to \pi_1(C - (D \cup D')) \to \mathrm{G}(K) \to 1$$

obtained by composing with the map $\pi_1(C - (D \cup D') \to \pi_1(C - D)$ has a proper solution.

The equivalence "Statement (*) $\Leftrightarrow$ **Main Conj.**" follows from:

$$\mathrm{G}(K(C)) = \varprojlim_{D} \pi_1(C - D)$$

Further questions of interest then arise: how should the extra banch points be selected? how many of them are needed?, etc.

(b) Conjecturally, if K is an arbitrary field, both the Split Embedding Problem conjecture and the Inverse Galois Problem hold over $K(T)$. This is not true anymore if $K(T)$ is replaced by K. Specifically, there exist fields K such that each finite group is a Galois group over K but for which there exist some split embedding problems for $\mathrm{G}(K)$ with no proper solution. An example was given by Fried and Völklein [FrVo2;§2 Example] (see also [Ja1;Ex.3.5]).

Fried and Völklein say a field K is RG-hilbertian if the Hilbert specialization property is true for all polynomials $P(T, Y) \in K(T)[Y]$ such that the associated extension of $K(T)$ is Galois and regular. They showed that there exists a countable field K of characteristic 0 that is RG-hilbertian but not hilbertian and that is PAC (Cf. §3.1 Ex.1). It follows from Cor.3.5 below and the RG-hilbertianity that each finite group is a Galois group over

K. (For PAC fields of characteristic 0, the RG-hilbertianity and the property that each finite group is a Galois group over K are actually equivalent [FrVo2;Th.B]).

Assume now that each split embedding problem for $\mathrm{G}(K)$ has a proper solution. Then from the weak→split reduction, each weakly solvable embedding problem for $\mathrm{G}(K)$ has a proper solution. But from a result of Ax, since K is PAC, $\mathrm{G}(K)$ is projective [FrJa,Th.10.17] and each embedding problem for $\mathrm{G}(K)$ is weakly solvable. Finally, this proves that any embedding problem for $\mathrm{G}(K)$ has a proper solution. From Iwasawa's theorem, $\mathrm{G}(K)$ is pro-free. But Roquette showed that a PAC field such that $\mathrm{G}(K)$ is pro-free is necessarily hilbertian [FrJa;Cor.24.38]. A contradiction. Conclude that there exists some split embedding problem for $\mathrm{G}(K)$ with no proper solution.

§3 Results

3.1 Ample fields

Definition 3.1 — *A field K is said to be ample if for each geometrically irreducible smooth curve C defined over K we have*

$$C(K) \neq \emptyset \Rightarrow C(K) \text{ infinite}$$

This definition is due to F. Pop who calls it *large*. We prefer to follow Jarden who calls it *ample* and to keep the word *large* for informal contexts. Before giving some examples of ample fields, we state an nice observation of F. Pop, which is used in the proof of the Main Theorem (§4): ample fields are *existentially closed* in the field $K((x))$ of formal Laurent series over K.

Definition 3.2 — *Let Ω/K be a regular extension. The field K is said to be existentially closed in Ω if the following equivalent properties hold:*

(1) For each smooth geometrically irreducible K-variety V,

$$V(\Omega) \neq \emptyset \Rightarrow V(K) \neq \emptyset$$

(2) For each smooth geometrically irreducible K-variety V,

$$K(V) \subset \Omega \Rightarrow V(K) \neq \emptyset$$

(3) For each smooth geometrically irreducible K-variety V,

$$V(\Omega) \text{ Zariski-dense} \Rightarrow V(K) \text{ Zariski-dense}$$

[(3) $\Rightarrow$ (1): Pick $\xi \in V(\Omega)$. Consider the Zariski closure of ξ in the K-variety V: it is an irreducible closed subscheme W of V. It follows from $K(W) = K(\xi) \subset \Omega$ and the regularity of the extension Ω/K that the extension $K(W)/K$ is regular. Hence W is a geometrically irreducible sub-variety of V. It follows from (3) that $W(K) \neq \emptyset$ and so $V(K) \neq \emptyset$.
(1) $\Rightarrow$ (2): If $K(V) \subset \Omega$, the generic point ξ of V is in $V(\Omega)$ (since $K(\xi) = K(V)$). It follows from (1) that $V(K) \neq \emptyset$.
(2) $\Rightarrow$ (3): Pick $\xi \in V(\Omega)$. Consider the Zariski closure of ξ in the K-variety V. As explained above, W is a geometrically irreducible sub-variety of V. It follows from (2) that $W(K) \neq \emptyset$ and so $V(K) \neq \emptyset$. The same argument holds with V replaced by an open subset of V. Conclude that $V(K)$ is Zariski-dense.]

Theorem 3.3 (Pop, [Po4;Prop.1.1]) — *A field K is ample if and only if it is existentially closed in its formal Laurent series field $K((x))$*

Examples: (1) PAC fields are ample. This follows from the definition of PAC: a field K is P(seudo) A(lgebraically) C(losed) if $V(K) \neq \emptyset$ for each geometrically irreducible variety defined over K) [FrJa;Ch.10]. Clearly, algebraically closed fields are PAC. So are separably closed fields. There are many other examples of PAC fields, even inside $\overline{\mathbb{Q}}$. For example, from Pop's theorem stated below (Th.3.4), the field $\mathbb{Q}^{tr}(\sqrt{-1})$ is PAC, where $\mathbb{Q}^{tr}$ is the field of totally real algebraic numbers.

(2) Complete valued fields (*e.g.* $K = \mathbb{Q}_p$, $\mathbb{R}$, $k((x))$, etc.) are ample. This follows immediately from the Implicit Function Theorem. The following argument shows directly (*i.e.*, without Th.3.3) that a complete valued field (K, v) is existentially closed in $K((x))$. Let V be a variety defined over K. Assume that condition (2) of the definition holds, *i.e.*, that $K(V) \subset K((x))$. Denote the Henselian closure of $K(x)$ in $K((x))$ by $\widetilde{K(x)}$. The field $\widetilde{K(x)}$ is known to be existentially closed in $K((x))$ [Ja1; Lemmas 2.2 and 2.3]. Therefore $V(\widetilde{K(x)}) \neq \emptyset$. Now $\widetilde{K(x)}$ is the algebraic closure of $K(x)$ in $K((x))$. Consequently elements of $\widetilde{K(x)}$ are formal power series with a positive radius of convergence (*e.g.* [De3;Prop.p.387]). Pick a point $M_X \in V(\widetilde{K(x)})$. Specializing x to some element $\xi \neq 0$ in the disk of convergence of the series involved in the coordinates of M_x yields a point $M_\xi \in V(K)$.

(3) For each prime number p, denote by $\mathbb{Q}^{tp}$ the field of totally p-adic algebraic numbers (*i.e.*, such that all conjugates over $\mathbb{Q}$ lie in a given copy of $\mathbb{Q}_p$). We use the notation $\mathbb{Q}^{tr}$ for the field of totally real algebraic numbers, which corresponds to the case $p = \infty$. Then for each prime number p (including $p = \infty$), the field $\mathbb{Q}^{tp}$ is ample. This immediately follows

from the preceding example and the following result of Pop [Po5] (see also [GrPoRo], [Po4;App.I], [Ja3]).

Theorem 3.4 (Pop) — *Let V be a smooth geometrically irreducible variety defined over $\mathbb{Q}^{tp}$, then $V(\mathbb{Q}^{tp}) \neq \emptyset$ provided that $V(\mathbb{Q}_p^{\sigma}) \neq \emptyset$ for each $\sigma \in \mathrm{G}(\mathbb{Q}_p)$. (This last condition can be replaced by $V(\mathbb{Q}_p) \neq \emptyset$ if V is defined over $\mathbb{Q}$).*

This example can be generalized to consider the field K^S of totally S-adic elements of a global field K. Here S is a finite set of places of K and K^S is the subfield of K_s of all elements such that, for all $v \in S$, all conjugates over K lie in a given copy of the completion K_v. Pop's result is shown to hold in this generality. The field K^S is ample.

(4) The field $\mathbb{Q}$ is not ample. More generally, from Faltings's theorem, no number field is ample. Jarden mentioned to us that, using Faltings's theorem and the theorem of Manin and Grauert, one could show that no field which is finitely generated over its prime field is ample.

3.2 Main results

Main Theorem — *The Main Conjecture is true if the base field K is ample.*

The Main Theorem is built from a series of results. The main contributions are due to Fried, Harbater and Pop. The idea of working with families of covers goes back to Fried [Fr]. Harbater [Har2] introduced some very efficient patching and gluing techniques for covers, especially over complete and algebraically closed fields. Pop developed these gluing techniques and the arithmetic aspect of the method. In particular he introduced and studied the notion of ample fields. An equivalent form of the Main Theorem along with a proof can be found in Pop's papers ([Po2],[Po4]). Others took an active part to the Main Theorem, in particular, Dèbes, Jarden, Haran, Liu, Völklein. The Main Theorem has two main consequences (Cor.3.5 and Cor.3.6). We show below that they contain most recent results on this topic.

Corollary 3.5 — *The* $\mathbf{RIGP}_{/K(T)}$ *conjecture holds if K ample. That is, if K is ample, each finite group is the Galois group of a regular Galois extension of $K(T)$.*

This result contains the following special cases:

- $K = \mathbb{C}$ (Riemann),

- $K = \mathbb{R}$ (Hurwitz (1890)),
- K algebraically closed (Harbater (1984) [Har1]),
- $K = \mathbb{Q}_p$ (Harbater (1985) [Har2]),
- K PAC (Fried-Völklein for *char* $K = 0$ (1991) [FrVo1], Pop in general (1993) [Po4]),
- $K = \mathbb{Q}^{tr}$ (Dèbes-Fried (1991) [DeFr]),
- $K = \mathbb{Q}^{tp}$ (Dèbes (1993) [De2]; Pop (1993) [Po4]),
- K ample (Pop (1995) [Po4]).

Corollary 3.6 — *The* **Split EP**$/_{K(T)}$ *conjecture holds if the base field* K *is ample. In particular, the Fried-Völklein conjecture* **FrVo** *is true if* K *is ample.*

Corollary 3.7 — *For every countable field* K, $\mathrm{G}(\overline{K}(T))$ *is pro-free.*

Proof: From Iwasawa's theorem (Th.2.1), we need to show that each embedding problem for $\mathrm{G}(\overline{K}(T))$ has a proper solution. From Tsen's theorem, $\mathrm{G}(\overline{K}(T))$ is projective. Therefore a given embedding problem for $\mathrm{G}(\overline{K}(T))$ has a weak solution. From the weak→split reduction, one may then assume that the embedding problem is split. But since the algebraically closed field $\overline{K}$ is ample, the Split Embedding conjecture holds over $\overline{K}(T)$. Hence, the given embedding problem has indeed a proper solution. □

Cor.3.7 is due to:

- Douady in *char* $K = 0$ [Do] (1964),
- Harbater [Har4] and Pop [Po2] in general (1993).

Corollary 3.8 — *For every countable Hilbertian and PAC field* K, $\mathrm{G}(K)$ *is pro-free.*

Proof: From a result of Ax [FrJa;Th.10.17], if K is PAC then $\mathrm{G}(K)$ is projective. Furthermore PAC fields are ample. The Fried-Völklein conjecture, which is true for ample fields, gives the conclusion. □

Cor.3.8 is due to:

- Fried-Völklein [FrVo2] in *char* $K = 0$ (1992),
- Pop in general [Po4] (1993).

Remark 3.9. (a) Conjecture **(RIGP$_{K(T)}$)** predicts in particular that, given a finite group G, then, for every finite field F, G is the Galois group of a regular Galois extension $E/F(T)$. Cor.3.5 allows to show that at least this is known for all but finitely many finite fields F. The proof uses the following model-theoretical argument. Suppose on the contrary that, for some finite group G, the set $\mathfrak{L}$ of all finite fields F for which G is not the Galois group of a regular Galois extension $E/F(T)$ is infinite. Consider a non-principal ultraproduct K of the set of finite fields in $\mathfrak{L}$ [FrJa;Ch.6]. Then [FrJa, Cor.10.6] says that K is a PAC field (essentially this rests on the Lang-Weil estimate for the number of rational points on a curve over a finite field [FrJa;Th.4.9]). So, from Cor.3.5, if G is a finite group, then G can be realized regularly as a Galois group over $K(T)$. But every first order statement true about K is true about all but finitely many F in $\mathfrak{L}$ [FrJa;Cor.6.12]. A contradiction.

This consequence of Cor.3.5 first appeared in the 1991 paper of Fried-Völklein [FrVo1; Cor.2], in a slightly weaker form where only finite prime fields are considered. In that context, the ultraproduct K above is a PAC field of characteristic 0 and the argument works with the special case of Cor.3.5 proved by Fried and Völklein. In their paper they also give an alternate more geometrical exposition of the argument above. The general case of the above argument requires the version of Cor.3.5 for PAC fields of arbitrary characteristic proved by Pop [Po4]. This result of Pop already appeared in some form in a letter of Roquette of 1991.

(b) For $K = \mathbb{Q}^{ab}$, conclusion "G(K) pro-free" of Cor.3.8 would yield Shafarevich's conjecture. But Frey noted that $\mathbb{Q}^{ab}$ is not PAC [FrJa;Cor.10.15]. So Cor.3.8 cannot be applied to $K = \mathbb{Q}^{ab}$. On the other hand, it follows from Cor.3.7 that G($\overline{\mathbb{F}_p}(T)$) is pro-free. This can be regarded as the functional analog of Shafarevich's conjecture: $\overline{\mathbb{F}_p}(T))$ is indeed the maximal cyclotomic extension $\mathbb{F}_p(T)^{cycl}$ of $\mathbb{F}_p(T)$ (just as $\mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$). Similarly the absolute Galois group of the maximal cyclotomic extension of $K = \mathbb{Q}^{tr}$ is pro-free. Indeed $(\mathbb{Q}^{tr})^{cycl} = \mathbb{Q}^{tr}(\sqrt{-1})$; from Cor.3.8, G($\mathbb{Q}^{tr}(\sqrt{-1})$) is pro-free (since $\mathbb{Q}^{tr}(\sqrt{-1})$ is PAC (by Th.3.4) and is hilbertian (by Weissauer's theorem [FrJa;Prop.12.14])).

(c) Cor.3.7 also holds in the uncountable case: for any field K, the group G($\overline{K}(T)$) is pro-free (of rank $card(K)$). This can be proved by the same methods but some adjustments are needed. Namely, one should use the following generalization of Iwasawa's theorem. If m is an infinite cardinal, then an iff condition for a profinite group F to be pro-free of rank m is that each finite embedding problem for F with a non-trivial kernel has exactly m proper solutions (see [Ja2;Lemma 2.1] where this generalization is credited to Z. Chatzidakis). So, for the general case of Cor.3.7, we have to prove that each embedding problem for G($\overline{K}(T)$) with a non-trivial kernel has $card(K)$ proper solutions, which

requires more precise statements than the mere existence result of a proper solution given by the Main Theorem (see [Har4] and [Po2]).

(d) A field K is said to be ω-free if every finite embedding problem for $\mathrm{G}(K)$ has a proper solution. From Iwasawa's theorem, "ω-free" is equivalent to "pro-free" if K is countable. M. Jarden [Ja3;Remark 11.3] observed that if the conclusion "$\mathrm{G}(K)$ pro-free" is replaced by "$\mathrm{G}(K)$ ω-free" then Cor.3.8 also holds for uncountable fields. That is, we have "PAC + hilbertian $\Rightarrow$ ω-free" in general. On the other hand, the implication "PAC + hilbertian $\Rightarrow$ pro-free" is false in general. There are fields which are PAC and Hilbertian but with a non pro-free absolute Galois group [Ja2;Example 3.2].

§4 Main arguments

In this section we sketch the proof of Cor.3.5. That is, we show that, if K is an ample field, then each finite group G is the Galois group of a regular Galois extension of $K(T)$. The proof of this special case of the Main Theorem contains the main arguments of the method. There are two stages. The first one consists in solving the problem over the field $K((x))$ of formal power series with coefficients in K. The second one uses a specialization argument to descend from $K((x))$ to K.

4.1 The Regular Inverse Galois Problem over $K((x))(T)$. The main tool is this patching and gluing result.

Theorem 4.1 — *Let k be a local field. Let G be a finite group generated by two subgroups G_1 and G_2. Assume that for $i = 1, 2$, there exists a regular Galois extension $F_i/k(T)$ of Galois group G_i and with an unramified prime of degree 1. Then there exists a regular Galois extension $F/k(T)$ of Galois group G and with an unramified prime of degree 1.*

This result reduces the problem to the realization of cyclic groups (regularly over $k(T)$ and with an unramified prime of degree 1). In [Har2] Harbater uses some results of Saltman to handle this case (see also [Li] and [Vo;Ch.11]). In characteristic 0, Deschamps recently gave a simpler proof, which, in addition, provides some precise information on the ramification of the constructed extension [Des2;Lemme 2.1.2].

Th.4.1 is due to Harbater [Har2]. The main tools of his proof are a patching and gluing result for formal analytic spaces along with a formal GAGA theorem. Harbater's result was revisited from the point of view of rigid geometry by Liu ([Li], see also [Des1]), following a suggestion of Serre [Se;p.93]. Pop and Harbater developed then independently

the patching and gluing procedure: in particular they showed it could be used not only to realize groups but also to solve embedding problems. Pop also managed to keep some arithmetic control on the procedure. That led him to his so-called "1/2-Riemann's existence theorem" [Po1]. The patching and gluing procedure is also an essential ingredient of the proof of Abhyankar's conjecture on Galois groups over curves: the case of the affine line was first proved by Raynaud [Ra]; using formal patching, Harbater could then prove the general case [Har3]; an alternate proof of the general case from Raynaud's result, using rigid patching, was given a little later by Pop [Po3]. There is now an elementary exposition of the proof of Th.4.1 which does not need any geometric background by Haran and Völklein ([HaVo], [Vo;Ch.11]).

4.2 Specialization argument (see also [Ja4;Prop.2.2]). Let Q be a prime field. From the first stage, we know that, for each finite group G, there exists a regular Galois extension $E_X/\mathrm{Q}((x))(T)$ of Galois group G.

Let F/Q be an extension of finite type with $F \subset \mathrm{Q}((x))$ such that the irreducible polynomial of y over $\mathrm{Q}((x))(T)$ lies in $F(T)[Y]$ and such that all conjugates of y over $\mathrm{Q}((x))(T)$ lie in $F(T,y)$. Set $E = F(T,y)$. The extension $E/F(T)$ is a regular Galois extension of $F(T)$ such that $\mathrm{G}(E/F(T)) = G$.

The containment $F \subset \mathrm{Q}((x))$ implies in particular that $F \cap \overline{\mathrm{Q}} = \mathrm{Q}$. Thus F is the function field of a geometrically irreducible algebraic variety V defined over Q. The equality $\mathrm{G}(E/F(T)) = G$ rewrites $\mathrm{G}(E/\mathrm{Q}(V)(T)) = G$.

The extension $E/\mathrm{Q}(V)(T)$ is a regular extension. So the Bertini-Noether theorem (*e.g.* [FrJa;Prop.8.8]) applies. There exists a Zariski closed subset Z of V such that, for each $v \in V(\overline{\mathrm{Q}}) - Z$, the extension $E/\mathrm{Q}(V)(T)$ specializes to a regular Galois extension $E_v/\mathrm{Q}(v)(T)$ of degree $[E_v : \mathrm{Q}(v)(T)] = [E : \mathrm{Q}(V)(T)]$. Up to enlarging the Zariski closed subset Z, we may:

- conclude that the specialized extension $E_v/\mathrm{Q}(v)(T)$ is also Galois. Then we have $\mathrm{G}(E_v/\mathrm{Q}(v)(T)) = G$ (for $v \in V(\overline{\mathrm{Q}}) - Z$),

- assume that the variety V is smooth.

Finally since $F = \mathrm{Q}(V) \subset \mathrm{Q}((x))$, the set $V(\mathrm{Q}((x)))$ is Zariski-dense. We have proved the following.

Theorem 4.2 — *Let* Q *be a prime field. Then to each finite group* G *can be attached a smooth irreducible variety* V *defined over* Q *such that*

(1) For each field K *containing* Q*, if* $V(K)$ *is Zariski-dense then the group* G *is the Galois group of a regular extension of* $K(T)$*.*

(2) $V(\mathrm{Q}((x)))$ is Zariski-dense.

Assume now that K is an ample field. From Th.3.3, K is existentially closed in $K((x))$. Therefore, from (2) above, $V(K)$ is Zariski-dense. Conclude from (1) that G is the Galois group of a regular extension of $K(T)$, thus completing the proof of Cor.3.5.

Remark 4.3. From Th.4.1, the extension $E_X/Q((x))(T)$ in the argument above can be taken to have an unramified prime $\mathbf{p}$ of degree 1. The field F can next be enlarged to contain the field of definition $Q(\mathbf{p}) \subset Q((x)$ of $\mathbf{p}$ (viewed as a point). The specialization argument then goes through to show this more precise form of Cor.3.5. If K is an ample field, then each finite group is the Galois group of a regular extension of $K(T)$ with an unramified prime of degree 1.

From §3.1, complete valued fields are ample. Thus Th.4.2 has this other consequence.

Corollary 4.4 — *To each finite group G can be attached a smooth irreducible variety V defined over $\mathbb{Q}$ such that*

(1) If $V(\mathbb{Q}) \neq \emptyset$ then G is the Galois group of a regular extension of $\mathbb{Q}(T)$.

(2) $V(\mathbb{Q}_p) \neq \emptyset$ for each prime p (including the prime $p = \infty$).

In condition (2) the field $\mathbb{Q}_p$ can actually be replaced by any complete valued field of characteristic 0. Condition (2) however is not sufficient in general to conclude that $V(\mathbb{Q}) \neq \emptyset$ (see [De2;Ex.4.2]).

Cor.4.4 is due to B. Deschamps [Des2]. His proof uses a different approach based on the Hurwitz space theory of M. Fried [Fr]. This theory had been developed by Fried and Völklein to reduce the Regular Inverse Galois Problem to finding rational points over irreducible varieties ([FrVo1], [Em], [Vo;Ch.10]). More specifically, to a given finite group G and an integer r larger than the rank of G, can be attached a specific smooth algebraic variety $\mathfrak{H} = \mathfrak{H}_{G,r}$ — called Hurwitz space — defined over $\mathbb{Q}$ such that, for every field K of characteristic 0, points in $\mathfrak{H}(K)$ correspond to regular Galois extensions $E/K(T)$ of group G with at most r branch points. Under certain conditions, one has a good control on the field of definition of the irreducible components of the algebraic variety $\mathfrak{H}_{G,r}$. That indeed reduces the problem to finding such components defined over $\mathbb{Q}$ with $\mathbb{Q}$-rational points.

A major application was the original proof of Fried and Völklein of the Regular Inverse Galois Problem over PAC fields [FrVo1] and of the implication "PAC + hilbertian $\Rightarrow$ ω-free" [FrVo2] in characteristic 0. The algebraic variety V constructed in §4 is replaced in the Fried-Völklein method by some irreducible component of some Hurwitz space $\mathfrak{H}_{G,r}$. It

is worth noting that the variety V constructed in §4 really appears as a subvariety of some Hurwitz space. So while the techniques are somewhat different, there is some relationship between the two methods.

Deschamps also used the Hurwitz space approach. He managed to show that for some suitably large integer r, there was some irreducible component of $\mathfrak{H}_{G,r}$ defined over $\mathbb{Q}$ and with $\mathbb{Q}_p$-points for all primes p. Furthermore he obtains this extra conclusion in Cor.4.4:

(3) For each prime p, there exists some point in $V(\mathbb{Q}_p)$ *such that the corresponding regular extension* $E_p/\mathbb{Q}_p(T)$ *of group G has the property that its branch points lie in* $\mathbb{P}^1(\mathbb{Q}^{ab})$ *and are* $\mathrm{G}(\mathbb{Q})$*-invariant as a set.*

The Hurwitz space method is more intricate but has the advantage of being much more explicit. The variety V and the Zariski closed subset Z involved in Th.4.2 are precisely described: V is a specific component of a certain Hurwitz space $\mathfrak{H}_{G,r}$ and Z is the branch locus of a certain cover $\mathfrak{H}_{G,r} \to \mathbb{P}_r$. Some further investigation on these spaces might be the key to a general solution of the Regular Inverse Galois Problem.

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