On Fields of Moduli of Curves

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Abstract. The field of moduli K of a curve X a priori defined over the separable closure K_s of K need not be a field of definition. This paper shows that the obstruction is essentially the same as the obstruction to K being a field of definition of the cover $X \to X/Aut(X)$. Using previous results of Dèbes-Douai, we then obtain a cohomological measure of the obstruction. This yields concrete criteria for the field of moduli to be a field of definition. An interesting application is the following local-global principle. If a curve X, together with all of its automorphims, is defined over $\mathbb{Q}_p$ for all primes p, then it is defined over $\mathbb{Q}$.

1. Introduction

The field of moduli K of a curve *a priori* defined over the separable closure K_s of K is the smallest field k such that each k-automorphism carries the curve X to an isomorphic copy of itself. The field of moduli need not be a field of definition: this paper is devoted to the obstruction.

If X is of genus 0, then X is isomorphic over K_s to $\mathbb{P}^1$ which is defined over the prime field Q of K. If X is of genus $g = 1$, the field of moduli is $Q(j)$ where j is the modular invariant of X and it is known that for char $\neq 2, 3$, X is isomorphic over K_s to a model defined over $Q(j)$ (see [Si;Ch.III,Prop.1.4]). Thus real problems occur for $g \geq 2$. We will assume throughout that the order of the automorphism group $\mathrm{Aut}(X)$ is relatively prime to the characteristic of K. Our main result (Th.3.1) then essentially asserts that the curve $X/\mathrm{Aut}(X)$ has a K-model B, called the canonical K-model of $X/\mathrm{Aut}(X)$, such that the obstruction to the field of moduli K being a field of definition is the same for the curve X as for the cover $X \to B$.

Using [DeDo1], we then obtain that the obstruction to the field of moduli K of a curve X of genus $g \geq 2$ being a field of definition “lies” in the 2nd cohomological group $H^2(K, Z(\mathrm{Aut}(X)))$ with values in the center of the automorphism group of X and for a certain action of the absolute Galois group $G(K_s/K)$ of K on $Z(\mathrm{Aut}(X))$ (Cor.4.1). The field of moduli K of a curve is then shown to be a field of definition in each of the following situations (Cor.4.3):

(a) the automorphism group $\mathrm{Aut}(X)$ of X has no center and has a complement in the automorphism group of $\mathrm{Aut}(X)$,

(b) the field of moduli K is of cohomological dimension ≤ 1,

(c) the canonical K-model of $X/\mathrm{Aut}(X)$ has K-rational points.

A consequence of (b) is that the field of moduli of a curve defined over $\overline{\mathbb{Q}}$ is the intersection of its fields of definition (Cor.4.4).

A classical result due to Coombes and Harbater [CoHa] asserts that Galois covers of $B = \mathbb{P}^1$ are defined over their field of moduli. This result was generalized in [DeDo1]: the same holds if B is an arbitrary smooth projective curve and the Galois cover $f : X \to B$ is unramified above an affine subset $B^* \subset B$ satisfying the (Seq/Split) condition of [DeDo1], *i.e.*, such that the exact sequence of algebraic fundamental groups

$$1 \to \pi_1(B^* \otimes_K K_s) \to \pi_1(B^*) \to G(K_s/K) \to 1$$

splits. Using this result, we obtain that if the (Seq/Split) condition holds for the K-model B of $X/\mathrm{Aut}(X)$, then the cover $f : X \to X/\mathrm{Aut}(X)$ and so the original curve X are defined over their field of moduli K. The (Seq/Split) condition is automatically satisfied if the curve has K-rational points. As a consequence we obtain that a curve with a marked point is defined over its field of moduli.

The last part of the paper is devoted to a discussion around the (Seq/Split) condition and the Coombes-Harbater theorem. We use examples of Shimura [Sh] and of Couveignes-Granboulan [CouGr] to produce Galois covers $f : X \to B$ for which the conclusion of the Coombes-Harbater theorem does not hold, and so affine curves B^* for which the (Seq/Split) condition is not satisfied.

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2. Field of moduli versus field of definition

Given a field K, we denote by K_s a separable closure of K and by $\overline{K}$ an algebraic closure of K. Given a Galois extension F/K, its Galois group is denoted by $G(F/K)$.

Let F/K be a Galois extension and X be a smooth projective curve *a priori* defined over F. Consider the subgroup $M(X)$ of $G(F/K)$ consisting of all the elements $\tau \in G(F/K)$ such that the curves X and X^τ are isomorphic over F. Then the *field of moduli* of the curve X *relative to the extension* F/K is defined to be the fixed field $F^{M(X)}$ of $M(X)$ in F. The field of moduli relative to the extension K_s/K is called the *absolute field of moduli* (relative to K).

Proposition 2.1 — *Let K_m the field of moduli of X. Then the subgroup $M(X)$ is a closed subgroup of $G(F/K)$ for the Krull topology. That is,*

$$M(X) = G(F/K_m)$$

The field of moduli of X is contained in each field of definition intermediate between K and F (in particular, it is a finite extension of K). Hence if the field of moduli is a field of definition, it is the smallest field of definition intermediate between K and F. Finally the field of moduli of X relative to the extension F/K_m is K_m.

Proof. The subgroup $G(F/K_m)$ is the Krull topological closure of $M(X)$. The point of Prop.2.1 is the containment $M(X) \supset G(F/K_m)$. Let $\sigma \in G(F/K_m)$. Let F_o/K be a finite Galois sub-extension of F such that X is defined over F_o. The Galois group $G(F/F_o)$ is a normal subgroup of finite index of $G(F/K)$. Thus $\sigma \cdot G(F/F_o)$ is an open neighborhood of σ. Hence we have

$$\sigma \cdot G(F/F_o) \cap M(X) \neq \emptyset$$

Therefore there exists $\widetilde{\sigma} \in M(X)$ such that $\sigma^{-1}\widetilde{\sigma} \in G(F/F_o)$. Since X is defined over F_o and that $\widetilde{\sigma} \in M(X)$, we have $X^\sigma = X^{\widetilde{\sigma}} \simeq X$. Therefore σ is in $M(X)$. The rest of the proof readily follows. □

The final observation of Prop.2.1 — the field of moduli relative to the extension F/K_m equals K_m — generally allows one to reduce to the situation where the base field K is the field of moduli of the given curve X (by extension of scalars from K to K_m).

More generally, the field of moduli can be defined for other structures (*e.g.* covers, abelian varieties): the definitions are the same but the isomorphisms involved should be understood as isomorphisms for the given structure. The structure (or category) should only be assumed to be of *finite type*; by this we mean a structure for which objects and morphisms can be defined over an extension of finite type of the prime field.

We will mainly work with curves on one hand and covers with fixed K-base on the other hand. Unless otherwise specified, curves are smooth and projective. We refer to [DeDo1] for all definitions relative to covers, Galois actions, algebraic fundamental groups, etc.

Let B be a geometrically irreducible curve defined over a subfield K of F. Fix a K-model B_K of B. The action of $G(F/K)$ on covers over F with K-base B_K should be understood as follows. Let $\sigma \in G(F/K)$. *A priori*, σ transforms a cover $f : X \to B$ into the cover $f^\sigma : X^\sigma \to B^\sigma$. Attached to the K-model B_K of B, there is a canonical isomorphism $\chi_\sigma : B^\sigma \to B$. Within the category of covers with fixed K-base B_K, the conjugate cover of $f : X \to B$ by σ should be understood as the cover $\chi_\sigma f^\sigma : X^\sigma \to B$. In particular, the field of moduli (just as the fields of definition) depends on the chosen K-model B_K of B.

Remark 2.2. In this paper, a Galois extension F/K is fixed, objects are *a priori* defined over F and the question of concern is the algebraic descent from F to the field of moduli relative to the extension F/K. Others notions of field of moduli for which the extension F/K is not necessarily algebraic exist in the literature (*e.g.* $F/K = \mathbb{C}/\mathbb{Q}$ in [Sh], [Wo]). These several notions of field of moduli will be considered and unified in a subsequent paper. We will prove that, under suitable assumptions, the main part of the obstruction is algebraic. That is, if the field of moduli is K, then the obstruction to K being a field of definition arises in the algebraic part of the descent, *i.e.*, from $\overline{K}$ to K. An important step consists in proving that the object can be defined over $\overline{K}$. For curves and in the case that F/K is the extension $\mathbb{C}/\mathbb{Q}$, a nice proof of that can be found in [Wo].

3. From curves to covers: the main result

Let F/K be a Galois extension and X be a smooth projective curve over F of genus $g \geq 2$ and with K as field of moduli. The group $\mathrm{Aut}(X)$ of all automorphisms of X defined over F is finite. Assume that the order of this group is relatively prime to the characteristic of K. Consider then the curve $B = X/\mathrm{Aut}(X)$. Basically, our main result shows that the obstruction to the field of moduli K being a field of definition is the same for the curve X as for the cover $X \to B$. The idea of reducing to covers was suggested to us by D. Harbater.

Theorem 3.1 — *Under the assumptions above there exists a model B_K of the curve $B = X/\mathrm{Aut}(X)$ defined over the field of moduli K such that the cover $X \to B$ with K-base B_K is of field of moduli equal to K. Furthermore, a field E such that $K \subset E \subset F$ is a field of definition of the curve X if and only if it is a field of definition of the cover $X \to B$ with K-base B_K.*

Th.3.1 asserts in particular that $B = X/\mathrm{Aut}(X)$ can be defined over K. This will be part of our argument but was known before (*e.g.* [Ba]). Here we show that a certain K-model B_K has some further properties. This K-model B_K, which is precisely defined in the proof of Th.3.1, is intrinsically attached to X. We will call it the *canonical model of* $X/\mathrm{Aut}(X)$ *over the field of moduli of* X (see Remark 3.2).

Proof. Let $\sigma \in G(F/K)$. It is readily checked that the group

$$\mathrm{Aut}(X)^\sigma = \{\varphi^\sigma | \varphi \in \mathrm{Aut}(X)\}$$

is the automorphism group of X^σ and that $X^\sigma/Aut(X^\sigma)$ is canonically isomorphic to $(X/Aut(X))^\sigma$: merely map each element $Aut(X^\sigma) \cdot x^\sigma \in X^\sigma/Aut(X^\sigma)$ to the element $Aut(X)^\sigma \cdot x^\sigma \in (X/Aut(X))^\sigma$. Here we use the notation $\mathrm{Aut}(X) \cdot x$ to denote the class of $x \in X$ modulo the action of $\mathrm{Aut}(X)$.

Since K is the field of moduli of X, there exists an isomorphism $f_\sigma : X \to X^\sigma$ defined over F. This isomorphism induces a map $\widetilde{f_\sigma} : X/\mathrm{Aut}(X) \to X^\sigma/\mathrm{Aut}(X^\sigma)$ that makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{f_\sigma} & X^\sigma \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_\sigma} \\ X/\mathrm{Aut}(X) & \xrightarrow{\widetilde{f_\sigma}} & X^\sigma/\mathrm{Aut}(X^\sigma) \end{array}$$

Namely, what should only be noticed is that, if $y \in \mathrm{Aut}(X)\cdot x$, *i.e.*, if $y = \varphi(x)$ for some $\varphi \in \mathrm{Aut}(X)$, then $f_\sigma(y) \in \mathrm{Aut}(X^\sigma)\cdot f_\sigma(x)$: we have indeed $f_\sigma(y) = f_\sigma(\varphi(x)) = \left(f_\sigma\varphi f_\sigma^{-1}\right)(f_\sigma(x))$ with $f_\sigma\varphi f_\sigma^{-1} \in \mathrm{Aut}(X^\sigma)$.

The map $\widetilde{f_\sigma}$ is an isomorphism defined over F. Indeed, with obvious notation, the inverse of $\widetilde{f_\sigma}$ is $\widetilde{f_{\sigma^{-1}}}$, and, $\widetilde{f_\sigma}$ is by construction defined over F (since f_σ is).

Compose $\widetilde{f_\sigma}$ with the canonical isomorphism

$$i_\sigma : X^\sigma/\mathrm{Aut}(X^\sigma) \to (X/\mathrm{Aut}(X))^\sigma$$

to obtain an isomorphism

$$\overline{f_\sigma} : X/\mathrm{Aut}(X) \to (X/\mathrm{Aut}(X))^\sigma$$

such that $\overline{f_\sigma}p = p^\sigma f_\sigma$ (use the identity $i_\sigma p_\sigma = p^\sigma$ which follows straighforwardly from the definitions). The map $\overline{f_\sigma}$ is in addition uniquely determined by this relation.

We next check that the family $(\overline{f_\tau})$ $(\tau \in G(F/K))$ satisfies the Weil's cocycle condition $\overline{f_\tau}^\sigma\overline{f_\sigma} = \overline{f_{\sigma\tau}}$. In view of the uniqueness property of the f_σs, this amounts to checking that, for all $\sigma, \tau \in G(F/K)$, we have

$$\overline{f_\tau}^\sigma\overline{f_\sigma}p = p^{\sigma\tau}f_{\sigma\tau}$$

This straightforwardly reduces to the condition

$$p^{\sigma\tau}f_\tau^\sigma f_\sigma = p^{\sigma\tau}f_{\sigma\tau}$$

which holds true, since $f_{\sigma\tau}^{-1}f_\tau^\sigma f_\sigma \in \mathrm{Aut}(X)$.

Conclude from Weil's descent criterion [We] that there exists a model B_K of $B = X/\mathrm{Aut}(X)$ over K and an isomorphism $\theta : B \to B_K \otimes_K F$ such that

$\theta^{-1}\theta^{\tau} = \overline{f_\tau}^{-1}$ for each $\tau \in G(F/K)$. Denote by $\psi : X \to B$ the cover with K-base B_K obtained by composing the original cover $p : X \to X/\mathrm{Aut}(X)$ with $\theta : B \to B_K \otimes_K F$. Then it is readily checked that $\psi^\tau = \psi f_\tau^{-1}$, for each $\tau \in G(F/K)$. That is, K is the field of moduli of the cover $\psi : X \to B_K \otimes_K F$ relative to the extension F/K.

It remains to prove that the curve X and the cover $X \to B$ with K-base B_K have the same fields of definition (between K and F). We will use the following observation. For each $\sigma \in G(F/K)$, the isomorphism $\overline{f}_\sigma$ does not depend on the particular isomorphism f_σ between X and X^σ.

A field of definition of the cover $f : X \to B$ is automatically a field of definition of X. Conversely let E be a field of definition of the curve X such that $K \subset E \subset F$. Thus there exists a E-model X_E of X and an isomorphism $\chi : X \to X_E \otimes_E F$ defined over F. For each $\sigma \in G(F/E)$, denote the map $(\chi^\sigma)^{-1}\chi : X \to X^\sigma$ by g_σ. Because of the observation just above, we have $\overline{g_\sigma} = \overline{f_\sigma}$ for all $\sigma \in G(F/E)$. It can be now easily checked that the cover $\theta p \chi^{-1} : X_E \otimes_E F \to B_K \otimes_K F$ satisfies $\left(\theta p \chi^{-1}\right)^\sigma = \theta p \chi^{-1}$ for all $\sigma \in G(F/E)$. That is, E is a field of definition of the cover $X \to B$ with K-base B_K. □

Remark 3.2. (a) The K-curve B_K is *the* K-model of $X/\mathrm{Aut}(X)$ determined by the descent data $(\overline{f_\tau})$ $(\tau \in G(F/K))$. As noted in the proof, these descent data come from the field of moduli condition and are intrinsically attached to the original curve X. The K-curve B_K is called the *canonical model of* $X/\mathrm{Aut}(X)$ *over the field of moduli of* X.

(b) Th.3.1 can be generalized to other types of objects in place of curves (*e.g.* marked curves, covers, etc.). More precisely, the result extends to categories of finite type with the following property: given an object X defined over a field F, the quotient $X/\mathrm{Aut}(X)$ is defined and can be endowed with a structure which makes it an object of the category defined over F. We will elaborate on this in a subsequent paper. As an illustration we state the result for the category of covers with fixed base. We will use it in §5. Some version of it is already used in [Cou]. We leave the reader adjust the proof of Th.3.1 to the situation of covers.

Let F/K be a Galois extension. Fix a smooth projective K-curve B_K and set $B = B_K \otimes_K F$. Let $f : X \to B$ be a cover over F with K-base B_K. The group $\mathrm{Aut}(f)$ of all automorphisms of the cover defined over F is finite.

The quotient cover $X/\mathrm{Aut}(f) \to B$ is still a cover over F with K-base B_K.

THEOREM 3.3 — *Assume in addition that K is the field of moduli of the cover $f : X \to B$ with K-base B_K. Then there exists a K-model $Y_K \to B_K$ of the cover $X/\mathrm{Aut}(f) \to B$ with K-base B_K with the following properties. The field of moduli of the cover $X \to X/\mathrm{Aut}(f)$ with K-base Y_K is equal to K. Furthermore, a field E such that $K \subset E \subset F$ is a field of definition of the cover $X \to X/\mathrm{Aut}(f)$ with K-base Y_K if and only if it is a field of definition of the cover $X \to B$ with K-base B_K.*

As for curves, the K-model $Y_K \to B_K$ can be shown to be *the* K-model of $X/\mathrm{Aut}(f) \to B$ determined by the descent data provided by the field of moduli condition and is intrinsically attached to the original cover $f : X \to B$. The K-cover $Y_K \to B_K$ is called the *canonical model of the quotient cover $X/\mathrm{Aut}(f) \to B$ over the field of moduli of the cover $f : X \to B$.*

4. Cohomological nature of the obstruction

Keep the notation and hypotheses of §3. We now conjoin Th.3.1 with the Main Theorem of [DeDo1], which gives a cohomological measure of the obstruction to the field of moduli of a cover being a field of definition.

Consider the canonical model B_K of $X/\mathrm{Aut}(X)$ over the field of moduli K of X and the cover $X \to X/\mathrm{Aut}(X)$ with K-base B_K constructed in the previous section. Set $\Gamma = G(F/K)$ and $G = \mathrm{Aut}(X)$. From the Main Theorem of [DeDo1], to the cover $X \to X/G$ is explicitly attached an action L of Γ on the center $Z(G)$ of the group G and a family $(\Omega_\delta)_{\delta\in\Delta}$ of elements $\Omega_\delta \in H^2(\Gamma, Z(G), L)$ indexed by a certain set Δ with the following property:

(*) *The field of moduli K is a field of definition of the cover $X \to X/G$ with K-base B_K if and only if at least one out of the Ω_δs with $\delta \in \Delta$ is trivial in $H^2(\Gamma, Z(G), L)$.*

Thus we obtain

COROLLARY 4.1 — *Under the assumptions above, the field of moduli K is a field of definition of the curve X if and only if at least one out of the Ω_δs is trivial in $H^2(\Gamma, Z(G), L)$.*

REMARK 4.2. The set Δ may be empty, in which case of course the field of moduli is not a field of definition. Condition "$\Delta \neq \emptyset$" is equivalent to condition (λ/Lift) of [DeDo1], which requires that a certain embedding problem for $\Gamma = G(F/K)$ has a weak solution. In particular, from Prop.3.1 of [DeDo1], this condition automatically holds here in each of the following situations:

• The Galois group $G(F/K)$ is a projective profinite group (*e.g.* $F = K_s$ and K is of cohomological dimension ≤ 1),

• The inner automorphism group $\mathrm{Inn}(G)$ of G has a complement in the automorphism group $\mathrm{Aut}(G)$ of G.

We obtain the following practical criteria for the field of moduli of a curve to be a field of definition.

COROLLARY 4.3 — *Let F/K be a Galois extension and X be a smooth projective curve over F of genus $g \geq 2$ and with automorphism group $\mathrm{Aut}(X)$ of order relatively prime to p. Assume that K is the field of moduli of X. Then K is a field of definition in each of the following situations:*
(a) $F = K_s$ and K is of cohomological dimension ≤ 1,
(b) The group $G = \mathrm{Aut}(X)$ has no center and has a complement in $\mathrm{Aut}(G)$,
(c) $F = K_s$ and the canonical K-model of $X/\mathrm{Aut}(X)$ has at least one K-rational point.

In both situations (a) and (b), the set Δ is non-empty and the group $H^2(\Gamma, Z(\mathrm{Aut}(X)), L)$ of Cor.4.1 is trivial. Situation (c) is considered more generally below (see Cor.5.3).

COROLLARY 4.4 — *Let X be a smooth projective curve over $\overline{\mathbb{Q}}$. Then the field of moduli of X is the intersection of its fields of definition. In particular, if there is a minimal field of definition, then it necessarily is the field of moduli.*

Cor.4.4 follows from Cor.4.3 (a) and Artin-Schreier's theorem, which allows to write every number field as the intersection of fields of cohomological dimension ≤ 1. This argument is due to Coombes and Harbater who proved the analog of Cor.4.4 for G-covers in place of curves [CoHa]. The same result but for mere covers was proved in [DeDo1].

5. The Coombes-Harbater theorem and the (Seq/Split) condition

The following result is proved in [CoHa].

THEOREM 5.1 (Coombes-Harbater) — *Assume* $\mathrm{char}(K) = 0$. *Then Galois covers* $f : X \to \mathbb{P}^1$ *over* $\overline{K}$ *with* K*-base* $\mathbb{P}^1$ *are defined over their field of moduli (relative to the extension* $\overline{K}/K$*).*

This result was generalized in [DeDo1] in the following way. Let F/K be a Galois extension. Suppose given a smooth geometrically irreducible projective K-curve B_K. Set $B = B_K \otimes_K F$. Fix a divisor D of $B \otimes_K K_s$ defined over K with only simple components. Denote then by B_K^* the affine K-curve B_K with the support of D removed and set $B^* = B_K^* \otimes_K F$. The condition that the exact sequence of algebraic fundamental groups

$$1 \to \pi_1(B^*) \to \pi_1(B_K^*) \to G(F/K) \to 1$$

splits, is denoted by (Seq/Split) in [DeDo1]. Here we will rather use the "prime to p version" (where $p = \mathrm{char}(K)$), denoted by (Seq/Split)', for which fundamental groups π_1 are replaced by their prime to p part π_1'. Clearly we have (Seq/Split)⇒(Seq/Split)'. The (Seq/Split)' condition holds in particular in each of the following situations:

- $F = K_s$ and $B_K(K) \neq \emptyset$: classically, a section can be obtained from any K-rational base point (possibly a tangential base point), or, in other words, from any embedding of function fields in some field of power series (possibly of Puiseux series).

- $G(F/K)$ is a projective profinite group (*e.g.* $F = K_s$ and K is of cohomological dimension ≤ 1).

THEOREM 5.2 [DeDo1;Cor.3.4] — *Let* $f : X \to B$ *be a Galois cover over* F *with* K*-base* B_K *and of degree relatively prime to the characteristic of* K*. Assume that condition (Seq/Split)' holds with* K *taken to be the field of moduli of the cover* f *and* D *taken to be the reduced ramification divisor* D *of* f *(which is automatically defined over the field of moduli of* f*). Then the cover* $f : X \to B$ *is defined over its field of moduli.*

Th.5.2 was originally stated with the (Seq/Split) condition (instead of the (Seq/Split)' condition) but with no assumption on the degree of the cover. The proof of the variant given here is completely similar to the original one.

Cor.5.3 below is a straightforward consequence of Th.3.1 and Th.5.2. Conclusion (c) of Cor.4.3 is a special case of Cor.5.3.

Corollary 5.3 — *Let X be a smooth projective curve of genus $g \geq 2$ defined over F and of field of moduli K. Assume that the order of the group $G = \mathrm{Aut}(X)$ is relatively prime to the characteristic of K. Assume in addition that condition (Seq/Split)' holds with the field K and with D taken to be the reduced ramification divisor of the cover $f : X \rightarrow X/\mathrm{Aut}(X)$ (with K-base the canonical K-model B_K of $X/\mathrm{Aut}(X)$). Then K is a field of definition of X.*

Corollary 5.4 — *Let (X, a) be a smooth projective curve of genus ≥ 2 with a marked point defined over F. Assume that the subgroup $\mathrm{Aut}_a(X)$ of $\mathrm{Aut}(X)$ of all automorphisms of X fixing a is of order relatively prime to the characteristic of K. Then (X, a) is defined over its field of moduli (as marked curve).*

Proof. Denote the field of moduli of (X, a) by K_m. We apply Th.3.1 in its version for marked curves (Cf. Remark 3.2 (b)). Denote by (B_{K_m}, b) the canonical K-model of of $(X, a)/\mathrm{Aut}_a(X)$. The point b is K_m-rational on B_{K_m}. Thus condition (Seq/Split)' holds with K taken to be K_m and D the reduced ramification divisor of $X \rightarrow X/\mathrm{Aut}_a(X)$. From Th.5.2, the cover $X \rightarrow X/\mathrm{Aut}_a(X)$ with K_m-base B_{K_m} is defined over K_m. It follows that (X, a) is defined over K_m. □

Remark 5.5. For curves of genus 1, the result classically holds if $F = K_s$ and $\mathrm{char}(K) \neq 2, 3$ (without any assumption on the order of $\mathrm{Aut}_a(X)$) [Si;Ch.III;Prop.1.4].

The second part of this section has the two goals (a) and (b) below. We will use an example of Shimura and an example of Granboulan-Couveignes.

(a) We will show that it is false that Galois covers are defined over their field of moduli in general. That is, condition (Seq/Split)' cannot be removed in Th.5.2.

(b) As a consequence, we will obtain some examples of affine curves B_K^* for which the (Seq/Split) condition does not hold.

Using Shimura's example. Shimura [Sh] gives very explicit examples of curves that are not defined over their field of moduli. Specifically he considers an hyperelliptic curve X of genus $g = m - 1$ with m odd:

$$y^2 = a_o x^m + \sum_{r=1}^{m} (a_r x^{m+r} + (-1)^r a_r^c x^{m-r})$$

with $a_m = 1$ and $a_o \in \mathbb{R}$ and c is the complex conjugation. He observes that the map μ defined by

$$\mu(x, y) = (-x^{-1}, ix^{-m} y)$$

is an isomorphism between the curve X and the complex conjugate curve X^c. Therefore the complex conjugation c fixes the field of moduli of X. On the other hand, he shows that if the curve has no other automorphisms than the two obvious ones Id and i (where $i(x, y) = (x, -y)$), than the curve cannot have a $\mathbb{R}$-model.

Namely, if $\mathrm{Aut}(X) = \{Id, i\}$, there are only two isomorphisms χ_c between X and X^c, namely μ and μi. It is readily checked that both of them satisfy $\chi_c^c \chi_c = i \neq 1$ (note that $\mu^c \mu = i$). Therefore the Weil's cocycle condition does not hold.

Taking the coefficients a_is such that $a_o, a_1, \ldots, a_{m-1}, a_1^c, \ldots, a_{m-1}^c$ are algebraically independent over $\mathbb{Q}$ insures that there are no non trivial automorphisms. But the argument below shows that the same can be achieved for "most" choices of the coefficients a_is in $\overline{\mathbb{Q}}$ thus providing examples of hyperelliptic curves defined over $\overline{\mathbb{Q}}$ which are not defined over their field of moduli.

Namely, the subset of hyperelliptic curves with trivial automorphism group $\{Id, i\}$ is a dense open subset U defined over $\overline{\mathbb{Q}}$ of the moduli space H_g of hyperelliptic curves of genus g. Furthermore, the morphism $\Lambda : \mathbb{A}^{2m+1} \to H_g$ associated with Shimura's equation is defined over $\overline{\mathbb{Q}}$. Thus the preimage $\Lambda^{-1}(U)$ of U is defined over $\overline{\mathbb{Q}}$ and is non-empty as Λ is surjective (essentially this follows from the fact that every hyperelliptic curve of genus g has an equation $y^2 = P(x)$ with $\deg(P) \leq 2g + 2$). Therefore $\Lambda^{-1}(U)$ contains $\overline{\mathbb{Q}}$-rational points.

As in Shimura's example, consider an hyperelliptic curve X and a field K (*e.g.* $K = \mathbb{R}$ or $K = \mathbb{Q}$) such that X is defined over $\overline{K}$ but is not defined over its field of moduli K_m (relative to the extension $\overline{K}/K$). Denote the canonical K_m-model of the curve $X/\mathrm{Aut}(X)$ by B_{K_m}. From Th.3.1, the cover $X \to X/\mathrm{Aut}(X)$ with K_m-base B_{K_m} is of field of moduli equal to K_m; on the other hand, this Galois cover is not defined over K_m (for otherwise X would be). It follows from Th.5.2 that the (Seq/Split) condition does not hold with K taken to be K_m, $F = \overline{K}$ and $B^*_{K_m}$ taken to be the curve B_{K_m} with the ramification divisor of the cover $X \to X/\mathrm{Aut}(X)$ removed.

Remark 5.6. Z. Wojtkowiak suggested to us that some examples for which condition (Seq/Split) does not hold could also be found by using an (unpublished) result of Sullivan [Su]. Namely, this result is that, for an affine curve X defined over $\mathbb{R}$, the (Seq/Split) condition for $K = \mathbb{R}$ and $F = \mathbb{C}$ is actually equivalent to $X(\mathbb{R}) \neq \emptyset$. It follows that if $X(\mathbb{R}) \neq \emptyset$, the (Seq/Split) condition does not hold for X over any field of definition K of X contained in $\mathbb{R}$ (and $F = \overline{K}$). Our approach however proves more generally that the conclusion of the Coombes-Harbater theorem does not hold unconditionally.

Using the Couveignes-Granboulan example. Couveignes and Granboulan [CouGr] give an example of a *dessin d'enfant* called *double rabbit* with the following properties. The corresponding cover $X \to \mathbb{P}^1$ is of field of moduli $K \subset \mathbb{R}$ (relative to extension $\overline{\mathbb{Q}}/\mathbb{Q}$), but is not defined over $\mathbb{R}$.

Denote the automorphism group of the double rabbit viewed as a cover by G. It is a group of order 2. Consider the canonical K-model $Y_K \to \mathbb{P}^1$ of the quotient cover $X/G \to \mathbb{P}^1$ (Cf. Remark 3.2 (b)): from Th.3.3, K is still the field of moduli of the cover $X \to X/G$ with K-base Y_K and the Galois cover $X \to X/G$ (with K-base Y_K) is not defined over its field of moduli K (otherwise the initial cover $X \to \mathbb{P}^1$ would be).

Denote the reduced ramification locus of this cover by D and the affine curve Y_K with D removed by Y^*_K. The Galois cover $X \to X/G$ (with K-base Y_K) is not defined over its field of moduli K. Consequently the (Seq/Split) condition does not hold with $F = \overline{K}$ and with B^*_K taken to be the affine curve Y^*_K.

The cover $X \to X/G$ is of degree 2 and so can be equivalently viewed as a G-cover. It is not defined over its field of moduli. Thus this example also shows that in general G-covers with an abelian Galois group are not defined over their field of moduli. It is known they are if in addition condition (Seq/Split) holds [DeDo1;Cor.3.4].

6. Local-global results

In this section, we combine techniques of this paper and previous local-global results of [DeDo1] and [DeDo2] to prove a local-to-global principle and a global-to-local principle.

6.1. The local-to-global principle.

THEOREM 6.1 — *Let X be a smooth projective curve defined over $\overline{\mathbb{Q}}$. Assume that the curve X and all of its automorphisms are defined over $\mathbb{Q}_p$ for all primes p (including the prime at ∞). Then the curve X and its automorphisms are defined over $\mathbb{Q}$. More generally, the same conclusion holds with $\mathbb{Q}$ replaced by any number field K such that the following special case does not hold.*

SPECIAL CASE. The special case comes from the special case of Grunwald's theorem [ArTa]. For each integer $r > 0$, ζ_r is a primitive 2^rth root of 1 and $\eta_r = \zeta_r + \zeta_r^{-1}$. Then denote by s the smallest integer such that $\eta_s \in K$ and $\eta_{s+1} \notin K$. The special case is defined by these three simultaneous conditions:

1. -1, $2 + \eta_s$, $-(2 + \eta_s)$ are non-squares in K.
2. For each finite place v of K above the prime 2, at least one out of the elements -1, $2 + \eta_s$, $-(2 + \eta_s)$ is a square in the completion K_v.
3. The group $Z(G)$ contains an element of order a multiple of 2^t with $t > s$.

If $K = \mathbb{Q}$, then $s = 2$ and $\eta_s = 0$. Since -1, 2 and -2 are non-squares in $\mathbb{Q}_2$, condition 2. cannot be satisfied. Therefore the special case does not occur if $K = \mathbb{Q}$. Similarly the special case does not occur if K contains $\sqrt{-1}$ or if K contains $\sqrt{-2}$ [DeDo1].

Proof. Suppose that the curve X and all its automorphisms are defined over each completion K_v of a number field K. Then the field of moduli of X relative to the extension $\overline{\mathbb{Q}}/K$ is necessarily equal to K. With no loss, one may assume that the genus of X is ≥ 2. Let B_K be the canonical K-model of the cover $X \to X/\mathrm{Aut}(X)$. Regard it as a G-cover: that is, consider the automorphisms of the cover as part of the data (see *e.g.* [DeDo1] for a formal definition). This G-cover is *a priori* defined over $\overline{\mathbb{Q}}$. Furthermore, from the assumptions, this G-cover is defined over all completions K_v of K. It follows then from Th.3.8 of [DeDo1] that it is necessarily defined over K, except possibly in the special case of Grunwald's theorem. The same conclusion holds *a fortiori* for the curve X. □

6.2. The global-to-local principle.

Theorem 6.2 — *Let X be a smooth projective curve defined over $\overline{\mathbb{Q}}$. Let K be the field of moduli of X. Then for all but finitely many places v of K, the completion K_v of K is a field of definition of X.*

Proof. With no loss, one may assume that the genus of X is ≥ 2. Let B_K be the canonical K-model of the cover $X \to X/\mathrm{Aut}(X)$. From Th.3.1, the cover $X \to X/\mathrm{Aut}(X)$ with K-base B_K is of field of moduli equal to K. It follows then from Th.5.1 of [DeDo2] that the cover $X \to X/\mathrm{Aut}(X)$ with K-base B_K can be defined over all but finitely many completions K_v of K. The same conclusion holds *a fortiori* for the curve X. □

7. Final note

The "field of moduli vs field of definition" question addressed in this paper is classically related to the question of existence of a representing family above a moduli space. Specifically, suppose $\mathcal{H}$ is a moduli space for equivalence classes $(X_h)_{h\in\mathcal{H}}$ of curves (possibly with some extra structure); for example, $\mathcal{H}$ is a modular curve, a Hurwitz space, etc. The question alluded to above is whether there exists a family $\mathcal{F}$ parametrized by $\mathcal{H}$ such that, for each $h \in \mathcal{H}$, the fiber $\mathcal{F}_h$ is a model of the object X_h. The family is required to be defined over a given field of definition K of $\mathcal{H}$; the case K is algebraically closed is referred to as the geometric part of the question and the other more refined case as the arithmetic part.

The special situation for which $\mathcal{H}$ consists of a single point exactly corresponds to the problem considered in this paper. More generally, for each $h \in \mathcal{H}$, the field of definition $K(h)$ of h on the moduli space $\mathcal{H}$ is the field of moduli of the object X_h corresponding to h (relative to the extension

$\overline{K(h)}/K(h)$); this is actually the origin of the phrase “field of moduli”. Consequently, in the case there is a representing family $\mathcal{F}$ above $\mathcal{H}$ defined over K, the object X_h has a model defined over its field of moduli, namely $\mathcal{F}_h$.

The situation where the objects are curves X given with a dominant map $X \to B$, *i.e.*, the situation of covers, is significant; for example this paper shows how other various situations can be reduced to it. Classically there exist coarse moduli spaces for covers of $\mathbb{P}^1$ (with fixed monodromy group and with some additional constraints on the ramification): these are the so-called Hurwitz spaces constructed by M. Fried. In his original paper [Fr], he considers the above question and shows there exists indeed a representing family when the covers parametrized by $\mathcal{H}$ have no non-trivial automorphisms; $\mathcal{H}$ is then a *fine* moduli space.

The question is subtler when the objects do have non-trivial automorphisms. There are some first results in [CoHa]. In his paper [Fr;p.58], Fried also gives some hints about the geometric part of the question in this case. He suggests that the obstruction to existence of a representing family involves the 2-cohomology of $\mathcal{H}$ with values in the center sheaf of automorphism groups of the covers and that the theory of gerbes introduced by Grothendieck and Giraud is an appropriate tool to tackle that non-abelian cohomological question. In a subsequent paper of ours [DeDoEm] we develop these ideas to treat the general question, *i.e.* both geometric and arithmetic parts. We show that in general there is indeed an obstruction to existence of a representing family above a Hurwitz space $\mathcal{H}$ and construct a gerbe that represents this obstruction. Using some reduction techniques from [DeDo1], the obstruction can then be shown to lie in the abelian cohomological group $H^2(\pi_1(\mathcal{H}), Z(G))$ where $Z(G)$ denotes the center of the monodromy group of the covers in question. This study of the obstruction provides new results about existence of Hurwitz families and also leads to a concrete formulation of the connection between the two problems discussed here. Namely, the obstruction to the field of moduli of a cover being a field of definition is shown to be a specialization of the obstruction (viewed as a gerbe or a 2-cocycle) to existence of a representing family above the associated Hurwitz space.

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