# GALOIS COVERS AND THE HILBERT-GRUNWALD PROPERTY

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Abstract. Our main result combines three topics: it contains a Grunwald-Wang type conclusion, a version of Hilbert's irreducibility theorem and a $p$-adic form à la Harbater, but with good reduction, of the Regular Inverse Galois Problem. As a consequence we obtain a statement that questions the RIGP over $\mathbb{Q}$. The general strategy is to study and exploit the good reduction of certain twisted models of the covers and of the associated moduli spaces.

## 1. Introduction

1.1. **The Grunwald problem.** Given a field $K$, a finite set $S$ of independent non-trivial discrete valuations of $K$ and a finite group $G$, a natural question, which we call the *Grunwald problem*, is whether there exist Galois extensions $E/K$ with group $G$ which have prescribed $v$-completions $E_v/K_v$ ($v \in S$). More precisely, given some homomorphisms[1] $\varphi_v : \mathrm{G}_{K_v} \to G$ ($v \in S$), with $\mathrm{G}_{K_v}$ the absolute Galois group of $K_v$, does there exist an epimorphism $\varphi : \mathrm{G}_K \to G$ which, composed with the restriction maps $\mathrm{G}_{K_v} \to \mathrm{G}_K$, yields the local maps $\varphi_v$?

When $K$ is a number field, the answer is known to be positive in the following cases: when $G$ is cyclic of odd order (Grunwald with a correction of Wang, see [NSW08, (9.2.8)]), and when $G$ is solvable of order prime to the number of roots of 1 in $K$ (Neukirch [Neu79], [NSW08, (9.5.5)]). More specifically, in these cases, the following *Grunwald map*

$$\mathrm{Gr}_{K,S} : \mathrm{Epi}(\mathrm{G}_K, G) \to \prod_{v\in S} \mathrm{Hom}(\mathrm{G}_{K_v}, G)^{\equiv}$$

is surjective for every finite set $S$ of finite places; here $\mathrm{Epi}(\mathrm{G}_K, G)$ denotes the set of all epimorphisms from $\mathrm{G}_K$ to $G$ and the superscript

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[1] All profinite group homomorphisms are tacitly assumed to be continuous.

$\equiv$ in $\mathrm{Hom}(\mathrm{G}_{K_v}, G)^{\equiv}$ means that homomorphisms are considered up to conjugation by an element of $G$ (depending on $v$).

*Definition* 1.1. Elements $\underline{\varphi} = (\varphi_v : \mathrm{G}_{K_v} \to G)_{v\in S}$ of $\prod_{v\in S} \mathrm{Hom}(\mathrm{G}_{K_v}, G)^{\equiv}$ are called *Grunwald problems*. Given a finite Galois extension $L/K$ totally split in $K_v$ for each $v \in S$, we say that $\varphi \in \mathrm{Epi}(\mathrm{G}_L, G)$ is *an $L$-solution to the Grunwald problem $\underline{\varphi}$* if $\mathrm{Gr}_{L,S}(\varphi) = \underline{\varphi}$. The Grunwald problem $\underline{\varphi}$ is said to be *unramified* if $\mathrm{Gal}(\overline{K_v}/K_v^{\mathrm{ur}}) \subset \ker(\varphi_v)$ $(v \in S)$.

$K$-solutions are of primary interest. Scalar extension to fields $L$ as in definition 1.1 is however a natural operation; it does not change the Grunwald problem to solve.

The general question we address is whether solutions to some Grunwald problem can be found among the specializations of some Galois G-cover $f : X \to \mathbb{P}^1$. For every point $t_0 \in \mathbb{P}^1(K)$ not a branch point, what we call the *specialization* of $f$ at $t_0$ is the residue field, denoted by $K(X)_{t_0}$, of some point in $X$ above $t_0$; viewed as an homomorphism $\mathrm{G}_K \to G$, it is the action of $\mathrm{G}_K$ on the fiber $f^{-1}(t_0)$ (see §2.1).

1.2. **Main theorems.** As above, let $K$ be a field, $S$ be a set of independent non-trivial discrete valuations of $K$ and $G$ be a finite group. For each place $v$, denote the valuation ring of $K_v$ by $\mathcal{O}_v$, the valuation ideal by $\mathfrak{p}_v$, the order of the residue field of $K_v$ by $q_v$ and its characteristic by $p_v$.

The constant $c(|G|, r)$ that appears in theorem 1.2 below only depends on the order of $G$ and the branch point number $r$ of the cover involved; it is explicitly defined in §3.1.

**Theorem 1.2.** *Assume that $K$ is a number field, that $p_v \not| \, |G|$ and $q_v \geq c(|G|, r)$ $(v \in S)$. Let $f : X \to \mathbb{P}^1$ be a G-cover of group $G$ and $r$ branch points, defined over $K$ and satisfying the following* good reduction *condition:*

(good-red) *for each $v \in S$, the branch divisor $\mathbf{t} = \{t_1, \ldots, t_r\}$ is étale and there is no vertical ramification in the cover $f$ at $v$.*[2]

*Then $f$ has the following* Hilbert-Grunwald specialization *property:*

[2]Specifically, $\mathbf{t} = \{t_1, \ldots, t_r\}$ étale means that no two $\overline{K}$-points $t_i, t_j \in \overline{K} \cup \{\infty\}$ *coalesce* at $v$, and coalescing at $v$ that $|t_i|_{\overline{v}} \leq 1$, $|t_j|_{\overline{v}} \leq 1$ and $|t_i - t_j|_{\overline{v}} < 1$, or else $|t_i|_{\overline{v}} \geq 1$, $|t_j|_{\overline{v}} \geq 1$ and $|t_i^{-1} - t_j^{-1}|_{\overline{v}} < 1$, where $\overline{v}$ is any prolongation of $v$ to $\overline{K}$. For more on non vertical ramification, see addendum 1.4 (c) and §2.3. (good-red) is indeed a good reduction criterion: if $\mathbf{t}$ is étale and $p_v \not| \, |G|$, $f$ acquires good reduction at $v$ after some finite scalar extension $L/K$ [Ful69]; under the extra non-vertical ramification assumption, one can take $L = K$; see §2.4.3 and §2.4.4.

(HGr-spec) *For each unramified Grunwald problem* $(\varphi_v : \mathrm{G}_{K_v} \to G)_{v\in S}$, *there exist specializations of* $f$ *at points* $t_0 \in \mathbb{A}^1(K) \setminus \mathbf{t}$ *that are* $K$*-solution to it. More precisely the set of all such* $t_0$ *contains a set* $\mathbb{A}^1(K) \cap \prod_{v\in S\cup S_0} U_v$ *where each* $U_v \subset \mathbb{A}^1(\mathcal{O}_v)$ *is a coset of* $\mathcal{O}_v$ *modulo* $\mathfrak{p}_v$ *and* $S_0$ *is a finite set of finite places* $v \notin S$ *which can be chosen depending only on* $f$.

The constant $c(G)$ that appears in theorem 1.3 below depends only on the group $G$; it is explicitly defined in §5.

**Theorem 1.3.** *Assume that* $K$ *is a number field,* $p_v \not| \, 6|G|$ *and* $q_v \geq c(G)$ $(v \in S)$*. Then there exist a Galois extension* $L/K$ *totally split in* $K_v$ $(v \in S)$ *and a* $G$*-cover* $f : X \to \mathbb{P}^1$ *of group* $G$*, defined over* $L$ *that satisfies the good reduction condition* (good-red) *and the Hilbert-Grunwald specialization property* (HGr-spec) *with* $K$ *replaced by* $L$.

*Addendum* 1.4. (a) We will prove a more general version of theorem 1.2 with $\mathbb{P}^1$ replaced by a higher dimensional variety $B$ and $K$ by the quotient field of some Dedekind domain (theorem 3.2). Further applications, notably to the situations $K$ is PAC or is finite or is a function field $\kappa(x)$, are discussed in [DGar].
(b) The G-cover $f : X \to \mathbb{P}^1$ of theorem 1.3 depends on the set $S$. We will however be able to fix in our construction the branch point number $r$ and the ramification type $\mathbf{C}$ [3]. That is, the corresponding points on the moduli spaces will lie on the same Hurwitz space $\mathsf{H}_r(G, \mathbf{C})$; they will more precisely be on some Harbater-Mumford component $\mathsf{HM} \subset \mathsf{H}_r(G, \mathbf{C})$ defined over $K$ [Fri95] [DE06][4].
(c) Non-vertical ramification (precisely defined in §2.3) is automatic in (good-red) if the group $G$ is of trivial center (under assumption "$\mathbf{t}$ étale and $p_v \not| |G|$"). This is shown in [Bec91] and will be used in the proof of theorem 1.3 (§5). Another practical test for non-vertical ramification is this: for each $v \in S$, if an affine equation $P(t, y) = 0$ of $X$ is given with $t$ corresponding to $f$ and $P$ monic in $y$ with integral coefficients (relative to $v$), then $v$ is unramified in $f$ if the discriminant $\Delta(t)$ of $P$ with respect to $y$ is non-zero modulo the valuation ideal of $v$.

[3]Recall that the ramification type, also called inertia canonical invariant, is the collection of conjugacy classes in $G$ of the *distinguished* generators of the inertia groups above the branch points, *i.e.*, those which map to $\exp(2i\pi/e)$ through the canonical isomorphism between inertia groups and groups of $e$-th roots of 1; the integer $e$ is the corresponding ramification index. See *e.g.* [Dèb01] or [Dèb09].

[4]Deformation or patching techniques used in that paper then show that $\mathsf{HM}(K_v) \neq \emptyset$ for all places $v$ of $K$ (as in [Des95]). This yields realizations over all corresponding $K_v$s of the group $G$ and the ramification type $\mathbf{C}$. These however have bad reduction and cannot be guaranteed to satisfy condition (HGr-spec).

Theorems 1.2 and 1.3 relate to several classical topics: *Hilbert's irreducibility theorem* — Galois covers of $\mathbb{P}^1$ over $\mathbb{Q}$ (or some number field $K$) have specializations that preserve the Galois group — *the Grunwald-Wang theorem* — these specializations can also have some local unramified behaviour prescribed at any finitely many suitably large primes — and *the Regular Inverse Galois Problem* — a cover with these properties does exist, maybe not defined over $\mathbb{Q}$ but over some number field $L$ totally split in $\mathbb{Q}_p$ for each of the same primes $p$. We elaborate below on this triple aspect.

1.3. **The Grunwald-Wang theorem and the RIGP.** As an immediate consequence of theorem 1.3, we obtain this Grunwald-Wang type conclusion: with notation as in §1.2, if $p_v \not| 6|G|$ and $q_v \geq c(G)$ ($v \in S$), then every unramified Grunwald problem $\underline{\varphi} = (\varphi_v : \mathrm{G}_{K_v} \to G)_{v\in S}$ has a solution over some Galois extension $L/K$ totally split in $K_v$ ($v \in S$).

Furthermore one can take $L = K$ for an interesting class of groups. Assume indeed that $G$ is a *regular Galois group over* $K$, that is, can be realized as the Galois group of some $K$-G-cover $f : X \to \mathbb{P}^1$; this is conjecturally true for all groups (the RIGP) and known for many. Taking $p_v \gg 1$ guarantees condition (good-red). Thus, from theorem 1.2, the one cover $f$ can be used to solve over $K$ all unramified Grunwald problems $\underline{\varphi}$ with $p_v \gg 1$. We obtain the following, where $\mathrm{Hom}_{\mathrm{ur}}(\mathrm{G}_{K_v}, G)$ denotes the subset of $\mathrm{Hom}(\mathrm{G}_{K_v}, G)$ of all unramified homomorphisms.

**Corollary 1.5.** *Every regular Galois group $G$ over $K$ has the following* unramified Grunwald property*: for all finite sets $S$ of finite places $v$ of $K$ with $p_v \gg_G 1$ ($v \in S$),*

(Gr-ur) *the set* $\prod_{v\in S} \mathrm{Hom}_{\mathrm{ur}}(\mathrm{G}_{K_v}, G)^{\equiv}$ *is in the image of the Grunwald map* $\mathrm{Gr}_{K,S} : \mathrm{Epi}(\mathrm{G}_K, G) \to \prod_{v\in S} \mathrm{Hom}(\mathrm{G}_{K_v}, G)^{\equiv}$.

Equivalently, condition (Gr-ur), which only depends on $G$ and $K$, is a necessary condition (possibly vacuous) for $G$ to be a regular Galois group over $K$. Corollary 1.5 can be compared to Saltmann's result that existence of some generic extension for $G$ over $K$ (which is stronger than being a regular Galois group over $K$) implies the full surjectivity of the Grunwald map $\mathrm{Gr}_{K,S}$ [Sal82, theorem 5.8]. Recall Saltmann used Wang's counter-example to Grunwald's theorem — condition (Gr-ur) does not hold for $K = \mathbb{Q}$, $S = \{2\}$, $G = \mathbb{Z}/8\mathbb{Z}$ — to show that there can be no generic extension for $\mathbb{Z}/8\mathbb{Z}$ over $\mathbb{Q}$ and consequently that the Noether program does not work in general [Sal82, theorem 5.11].

Corollary 1.6 below and the more general corollary 4.1 (both proven in §4) provide even stronger obstructions (though still possibly vacuous) to the Regular Inverse Galois Problem. Given a Galois extension $E/\mathbb{Q}$, let $\pi^E_{\rm nts}(x)$ denote the number of primes $p \leq x$ that are not totally split or are ramified in $E/\mathbb{Q}$. From the Čebotarev density theorem we have

$$\pi^E_{\rm nts}(x) \sim_E (1 - \frac{1}{|G|}) \frac{x}{\log x} \quad \text{(when } x \to \infty)$$

**Corollary 1.6.** *Let $G$ be a finite group. Assume there exist two functions $\ell(x)$ and $m(x)$ tending to $\infty$ with $x$ such that the following holds: if $E/\mathbb{Q}$ is a Galois extension of group $G$ and discriminant $d_E$,*

$$(*) \qquad \pi^E_{\rm nts}(x) \geq m(x) \quad \text{if } \log|d_E| \leq x\,\ell(x)$$

*Then $G$ is not a regular Galois group over $\mathbb{Q}$.*

The Čebotarev theorem has the following effective version, proved by Lagarias and Odlyzko ([LO77]; see also [Ser81, §2.2]): if $\pi(x)$ denotes the number of primes $\leq x$, then

$$(**) \qquad \pi^E_{\rm nts}(x) \geq \pi(x) - \frac{2}{|G|}\frac{x}{\log x} \quad \text{if } \beta|G|\log^2|d_E| \leq \log x$$

for some absolute constant $\beta$. Thus condition (*) holds *for all* finite groups $G$ if $\log|d_E| \leq x\,\ell(x)$ is replaced by $\beta|G|\log^2|d_E| \leq \log x$ (or by $\beta|G|\log|d_E| \leq \sqrt{x}/\log x$ under GRH). Producing a *single* group satisfying the exact condition (*) would disprove the RIGP.

Classical methods for establishing such analytic estimates as (**) depend on the possiblity of finding appropriate zero-free regions for Hecke L-functions. It would be interesting to investigate to what extent the Galois structure can be taken advantage of to improve these estimates for some specific groups. The difference between (*) and (**) is essentially a "log" in the condition on $d_E$. We note that a "log" can be gained in a related problem: concerning the least prime ideal in the Čebotarev density theorem (instead of the number of primes), Linnick's theorem shows that difference between the general estimate and that of the specific situation of Dirichlet's theorem (see [LMO79]).

*Remark* 1.7. We have considered the totally split behaviour for simplicity of exposition. Similar conclusions hold for any possible local behaviour. See corollary 4.1 which is a more general and fully effective version of the above results. Also note that for groups $G$ that are known to be regular Galois groups over $\mathbb{Q}$, corollary 4.1 can be used positively to produce Galois extensions $E_x/\mathbb{Q}$ of groups $G$ that are totally split at almost all primes $\leq x$ and for which an upper bound for $d_{E_x}$ that compares to the Lagarias-Odlyzko bound can be given.

1.4. **The RIGP over $p$-adic fields.** Conclusion of theorem 1.3 implies that the G-cover $f : X \to \mathbb{P}^1$ is defined over the field $K^{\mathrm{tot}S}$ of all totally $S$-adic algebraic numbers (that is, all numbers $x \in \overline{K}$ such that $K$-conjugates of $x$ are in $K_v$ for each $v \in S$). Existence of a realizing $K^{\mathrm{tot}S}$-G-cover, for any group $G$, is a classical application of the so-called patching methods [Har87], [Dèb95], [Pop96]. However these methods lead by essence to covers with bad reduction. Here, as the cover $f : X \to \mathbb{P}^1$ also satisfies (good-red) and $p_v \not| |G|$, we obtain:

**Corollary 1.8.** *Given $K$, $S$, $G$ as before, assume as in theorem 1.3 that $p_v \not| 6|G|$ and $q_v \geq c(G)$ ($v \in S$). Then there exists a G-cover $f : X \to \mathbb{P}^1$ of group $G$ defined over $K^{\mathrm{tot}S}$ and with good reduction at every place $v \in S$ (including no vertical ramification).*

Consequently, for the same places $v$, $G$ is a regular Galois group over the finite fields $\mathbb{F}_{q_v}$. That each finite group is a regular Galois group over all big enough fields $\mathbb{F}_q$ was proved in [FV91] and [Pop96]; our results and approach also relate to these works.

1.5. **The Hilbert aspect.** Conclusion (HGr-spec) includes the Hilbert property: the specializations $K(X)_{t_0}/K$ have a Galois group equal to the generic Galois group $G$. Our method leads to some bound for the least integer $t_0 > 0$ for which the specialization $K(X)_{t_0}/K$ has group $G$ that depends only on $|G|$, $r$ and the number $\mathrm{bb}(\mathbf{t})$ of primes for which the (good-red) condition from §1.2 does not hold (see §4)[5]. Existing bounds usually involve the height $H$ of some affine equation of $f$. Recall that conjecturally — notably under Lang's conjecture on rational points on higher dimensional varieties — neither $\mathrm{bb}(\mathbf{t})$ nor $H(f)$ are necessary; a bound should exist depending only on $|G|$ and $r$ [DW08].

1.6. **Earlier works.** Similar Hilbert-Grunwald-RIGP questions are addressed in a paper of Plans and Vila [PV05], for a few groups and for specific G-covers $X \to \mathbb{P}^1$, generally derived from the rigidity method. Here, $G$ can be any group in theorem 1.3 and theorem 1.2 *a priori* applies to all $K$-G-covers $X \to \mathbb{P}^1$. We have however a big enough condition on $q_v$ and $p_v$ for $v \in S$. This condition can in fact not be removed: as Wang's counter-example to Grunwald's theorem or other examples in [PV05] show, there are situations where some local unramified behaviours cannot occur. These counter-examples however all involve the prime $p = 2$ and it seems unknown whether counter-examples exist with other primes.

[5]Note however that these bounds are for *regular* Galois extensions. A classical argument to deduce the full situation of Hilbert's irreducibility theorem uses Čebotarev's theorem, which may lead to constants involving other parameters.

From earlier works on the ramification of specializations of covers, it also seems difficult to remove the unramified assumption on the $\varphi_v$ (though it remains an interesting question in general). The specializations $t_0$ from conclusion (HGr-spec) will indeed be constructed so not to reduce to branch points of the cover modulo the valuation ideal of $v$. From [Bec91] or more general results of Grothendieck [Gro71], the specializations $K(X)_{t_0}/K$ are then necessarily unramified at $v$ unless $v$ is one from a finite list of bad places.

Theorem 1.2 is in the line of a series of works, of Eichler [Eic39], Fried [Fri74], Ekedahl [Eke90], Colliot-Thélène [Ser92, §3] on the Hilbert specialization property. Our method rests on the same basic idea — go to finite fields — and is in fact rather similar to that of Ekedahl. Our contribution is the Grunwald aspect: we realize given *local extensions* at some given places while it suffices, for Hilbert's irreducibility theorem, to realize suitable *decomposition groups*, at any possible places. This has led us to closely investigate the local situation and prevented from using asymptotic arguments. Our results are totally explicit and effective (see corollary 4.1). With the starting twisting lemma 2.1 we also offer a new approach that unifies earlier works over various fields like PAC and finite fields. For example Fried's Čebotarev theorem for rational function fields $\kappa(x)$ over a finite field $\kappa$ and Colliot-Thélène's result that varieties over a number field with the "weak weak approximation property" have the Hilbert property can be obtained as special cases of our approach; this is detailed in [DGar].

1.7. **Strategy and organization of the paper.** We first prove theorem 1.2. The starting point is the "twisting lemma" that gives a general answer to the question of whether a field extension $E/K$ is a specialization of some $K$-G-cover: $K$-rational points should exist on a certain twisted $K$-variety (lemma 2.1). The next step is to establish some good reduction properties of this variety (§2.4). Thanks to the Lang-Weil estimates which remain as for our predecessors a basic tool, we can then deduce a local form of our result (proposition 2.2). Conjoined with some globalization arguments, this leads to theorem 1.2 and its higher dimensional version (theorem 3.2) in §3.

In order to obtain theorem 1.3 (in §5), we first explain how to reduce to the situation $Z(G) = \{1\}$ where the vertical ramification can be better controlled. Then we show in that case how to find a G-cover $f : X \to \mathbb{P}^1$ defined over some number field $L$ as in theorem 1.3 and satisfying the (good-red) condition. For this we use the Hurwitz space theory to construct a $K$-component of some Hurwitz space of G-covers of group $G$ that has $K_v$-rational points corresponding to covers with good reduction ($v \in S$).

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## 2. The local situation

We will work with covers of a more general base $B$ than $\mathbb{P}^1$.

2.1. **Basic notation.** For more details, we refer to [DD97, §2] and [Dèb99b, §2].

Given a field $k$, we denote by $\overline{k}$ an algebraic closure, its separable closure in $\overline{k}$ by $k^{\text{sep}}$ and its absolute Galois group by $\mathrm{G}_k$. If $k'$ is an overfield of $k$, we use the notation $\otimes_k k'$ for the scalar extension from $k$ to $k'$: for example, if $X$ is a $k$-curve, $X \otimes_k k'$ is the $k'$-curve obtained by scalar extension.

Given a regular projective geometrically irreducible $k$-variety $B$, a *$k$-mere cover of $B$* is a finite and generically unramified morphism $f : X \to B$ defined over $k$ with $X$ a normal and geometrically irreducible $k$-variety. Mere covers $f : X \to B$ over $k$ correspond to finite separable field extensions $k(X)/k(B)$ that are regular over $k$ through the function field functor. The term "mere" is meant to distinguish mere covers from G-covers. By *$k$-G-cover of $B$ of group $G$*, we mean a Galois cover $f : X \to B$ over $k$ given together with an isomorphism $G \to \mathrm{Gal}(k(X)/k(B))$. G-covers of $B$ of group $G$ over $k$ correspond to regular Galois extensions $k(X)/k(B)$ given with an isomorphism of the Galois group $\mathrm{Gal}(k(X)/k(B))$ with $G$. By group and branch divisor of a $k$-cover $f$, we mean those of the $k^{\text{sep}}$-cover $f \otimes_k k^{\text{sep}}$ [6].

Given a reduced positive divisor $D \subset B$, denote the $k$-fundamental group of $B \setminus D$ by $\pi_1(B \setminus D, t)_k$ where $t \in B(\overline{k}) \setminus D$ is a base point. Mere covers of $B$ of degree $d$ (resp. G-covers of $B$ of group $G$) with branch divisor contained in $D$ correspond to homomorphisms $\pi_1(B \setminus D, t)_k \to S_d$ such that the restriction to $\pi_1(B \setminus D, t)_{k^{\text{sep}}}$ is transitive (resp. to epimorphisms $\pi_1(B \setminus D, t)_k \to G$ such that the restriction to $\pi_1(B \setminus D, t)_{k^{\text{sep}}}$ is onto).

Each $k$-rational point $t_0 \in B(k) \setminus D$ provides a section $\mathrm{s}_{t_0} : \mathrm{G}_k \to \pi_1(B \setminus D, t)_k$ of the exact sequence

[6] The group of a $k^{\text{sep}}$-cover $X \to B$ is the Galois group of the Galois closure of the extension $k^{\text{sep}}(X)/k^{\text{sep}}(B)$. The branch divisor is the formal sum of all hypersurfaces of $B$ such that the associated discrete valuations are ramified in the extension $k^{\text{sep}}(X)/k^{\text{sep}}(B)$.

$$1 \to \pi_1(B \setminus D, t)_{k^{\rm sep}} \to \pi_1(B \setminus D, t)_k \to \mathrm{G}_k \to 1$$

well-defined up to conjugation by elements in $\pi_1(B \setminus D, t)_{k^{\rm sep}}$. Given a mere cover representation $\phi : \pi_1(B \setminus D, t)_k \to S_d$, the morphism $\phi \mathsf{s}_{t_0} : \mathrm{G}_k \to S_d$ is the *arithmetic action* of $\mathrm{G}_k$ on the fiber above $t_0$. If $\phi : \pi_1(B \setminus D, t)_k \to G$ represents a G-cover $f : X \to B$, the morphism $\phi \mathsf{s}_{t_0} : \mathrm{G}_k \to G$ is the *specialization representation* of $f$ at $t_0$. The fixed field in $k^{\rm sep}$ of $\ker(\phi \mathsf{s}_{t_0})$ is the residue field of $k(X)/k(B)$ at some point above $t_0$. We denote it by $k(X)_{t_0}$ and call the extension $k(X)_{t_0}/k$ the *specialization* of $f$ at $t_0$.

2.2. **Twisting G-covers.** We will use a notion of "twisted covers" introduced in [Dèb99a, §2] for covers of $\mathbb{P}^1$. As we indicate below, their definition and main properties readily extend to $k$-covers $f : X \to B$.

Let $k$ be a field and $f : X \to B$ be a $k$-G-cover. Let $\phi : \pi_1(B \setminus D, t)_k \to G$ be the epimorphism corresponding to the G-cover $f$ and let $\varphi : \mathrm{G}_k \to G$ be an homomorphism (not necessarily onto).

Denote the right-regular (resp. left-regular) representation of $G$ by $\delta : G \to S_d$ (resp. by $\gamma : G \to S_d$) where $d = |G|$. Define $\varphi^* : \mathrm{G}_k \to G$ by $\varphi^*(g) = \varphi(g)^{-1}$. Consider the map $\widetilde{\phi}^\varphi : \pi_1(B \setminus D, t)_k \to S_d$ defined by the following formula, where $r$ is the restriction map $\pi_1(B \setminus D, t)_k \to \mathrm{G}_k$ and $\times$ is the multiplication in the symmetric group $S_d$:

$$\widetilde{\phi}^\varphi(\theta) = \gamma\phi(\theta) \times \delta\varphi^* r(\theta) \qquad (\theta \in \pi_1(B \setminus D, t)_k)$$

It is easily checked that $\widetilde{\phi}^\varphi$ is a group homomorphism with the same restriction on $\pi_1(B \setminus D, t)_{k^{\rm sep}}$ as $\phi$. The associated mere cover is a $K$-model of the mere cover $f \otimes_k k^{\rm sep}$. We denote it by $\widetilde{f}^\varphi : \widetilde{X}^\varphi \to B$ and call it the *twisted cover* of $f$ by $\varphi$. The following statement contains the main property of the twisted cover.

**Twisting lemma 2.1.** *Let $t_0 \in B(k) \setminus D$. The specialization representation $\phi \mathsf{s}_{t_0} : \mathrm{G}_k \to G$ of the G-cover $f$ at $t_0$ is conjugate in $G$ to $\varphi : \mathrm{G}_k \to G$ if and only if there exists $x_0 \in \widetilde{X}^\varphi(k)$ such that $\widetilde{f}^\varphi(x_0) = t_0$.*

Furthermore, it is readily checked that the twisting operation commutes with extension of scalars: if $k'$ is an overfield of $k$, then the twisted cover of $f \otimes_k k'$ by the restriction of $\varphi : \mathrm{G}_k \to G$ to $\mathrm{G}_{k'}$ equals the cover $\widetilde{f}^\varphi \otimes_k k'$.

*Proof of twisting lemma.* Consider the section $\mathsf{s}_{t_0} : \mathrm{G}_k \to \pi_1(B \setminus D, t)_k$ associated with $t_0$. The arithmetic action of $\mathrm{G}_k$ on the fiber $(\widetilde{f}^\varphi)^{-1}(t_0)$ is $\widetilde{\phi}^\varphi \mathsf{s}_{t_0}$. Hence for each $\tau \in \mathrm{G}_k$, the action of $\tau$ on the fiber $(\widetilde{f}^\varphi)^{-1}(t_0)$ is given by

$$\widetilde{\phi}^{\varphi}(\mathsf{s}_{t_0}(\tau)) = \gamma\phi(\mathsf{s}_{t_0}(\tau))\,\delta\varphi^*(\tau)$$

In $S_d$ the element $\gamma\phi(\mathsf{s}_{t_0}(\tau)) \in G$ is the multiplication on the left by $\phi(\mathsf{s}_{t_0}(\tau))$ in $G$ while the element $\delta\varphi^*(\tau)$ is the multiplication on the right by $\varphi(\tau)^{-1}$. If the elements $\widetilde{\phi}^{\varphi}(\mathsf{s}_{t_0}(\tau))$ ($\tau \in \mathrm{G}_k$) have a common fixed point, say $\omega \in G$, then we obtain $\phi(\mathsf{s}_{t_0}(\tau)) = \omega\,\varphi(\tau)\,\omega^{-1}$. This proves the converse part. For the direct part assume $\phi\,\mathsf{s}_{t_0} = \omega\,\varphi\,\omega^{-1}$ for some $\omega \in G$. Then it is straightforwardly checked that $\omega$ is fixed under every permutation $\widetilde{\phi}^{\varphi}(\mathsf{s}_{t_0}(\tau))$ ($\tau \in \mathrm{G}_k$). The corresponding point $x_0 \in \widetilde{X}^{\varphi}$ above $t_0$ is $k$-rational. □

For example, from Faltings' theorem, it follows from lemma 2.1 that if $k$ is a number field and $X$ is a curve of genus $\geq 2$, then a given extension $E/k$ can be a specialization of some $k$-G-cover $f : X \to \mathbb{P}^1$ at only finitely many points $t_0 \in \mathbb{P}^1(k)$. In the rest of the paper, we are interested in situations where it is possible to produce $k$-rational points on the twisted variety $\widetilde{X}^{\varphi}$.

2.3. **Local specialization result.** Assume $k$ is the quotient field of some complete discrete valuation ring $A$. Denote the valuation ideal by $\mathfrak{p}$, the residue field $A/\mathfrak{p}$ by $\kappa$, assumed to be perfect, and its characteristic by $p$.

Let $B$ be a smooth projective and geometrically irreducible $k$-variety and assume it is given with an integral smooth projective model $\mathcal{B}$ over $A$; in particular $\mathcal{B}$ is regular [Gro67, proposition 17.5.8].

Let $f : X \to B$ be a $k$-G-cover of group $G$. Denote by $\mathcal{F} : \mathcal{X} \to \mathcal{B}$ the normalization of $\mathcal{B}$ in $k(X)$; it is a finite morphism [Mil80, proposition 1.1]. Denote its special fiber by $\mathcal{F}_0 : \mathcal{X}_0 \to \mathcal{B}_0$ and the Zariski closure of $D$ in $\mathcal{B}$ by $\mathcal{D}$.

A finite and flat morphism $\mathcal{F}' : \mathcal{X}' \to \mathcal{B}$ with $\mathcal{X}'$ normal is called an *$A$-model of* $(f \otimes_k k^{\mathrm{sep}}, \mathcal{F}_0 \otimes_\kappa \overline{\kappa})$ if $\mathcal{F}' \otimes_A k$ is a $k$-cover that is $k^{\mathrm{sep}}$-isomorphic to $f \otimes_k k^{\mathrm{sep}}$ and the special fiber $\mathcal{F}'_0 : \mathcal{X}'_0 \to \mathcal{B}_0$ is a $\kappa$-cover that is $\overline{\kappa}$-isomorphic to $\mathcal{F}_0 \otimes_\kappa \overline{\kappa}$.

The cover $f$ is said to have no vertical ramification at $\mathfrak{p}$ if $\mathcal{F} : \mathcal{X} \to \mathcal{B}$ is unramified above $\mathfrak{p}$ viewed as a prime divisor of $\mathcal{B}$, or in other words, is unramified above the special fiber $\mathcal{B}_0$. An homomorphism $\varphi : \mathrm{G}_k \to G$ is said to be *unramified* at $\mathfrak{p}$ if the inertia subgroup $I_\mathfrak{p} \subset \mathrm{G}_k$ above $\mathfrak{p}$ is contained in $\ker(\varphi)$.

**Proposition 2.2.** *Given a $k$-G-cover $f : X \to B$ of group $G$ and an unramified homomorphism $\varphi : \mathrm{G}_k \to G$, assume that $p \not| |G|$ and that* (good-red) *$\mathcal{D}$ is a smooth divisor, $\mathcal{D} \cup \mathcal{B}_0$ is regular with normal crossings over $A$ and $f$ has no vertical ramification at $\mathfrak{p}$, and*

($\kappa$-big-enough) *for every $A$-model $\mathcal{F}' : \mathcal{X}' \to \mathcal{B}$ of $(f \otimes_k k^{\mathrm{sep}}, \mathcal{F}_0 \otimes_\kappa \overline{\kappa})$, there exist $\kappa$-rational points on $\mathcal{X}'_0$ that do not lie above any point in the closed subset $\mathcal{D}_0 \otimes_\kappa \overline{\kappa}$.*

*Then there exists $t_0 \in B(k) \setminus D$ such that the specialization representation of $f$ at $t_0$ is conjugate in $G$ to the morphism $\varphi : \mathrm{G}_k \to G$. Furthermore the set of all such points $t_0$ contains the preimage via the map $\mathcal{B}(A) \to \mathcal{B}_0(\kappa)$ of a non-empty subset $F \subset \mathcal{B}_0(\kappa) \setminus \mathcal{D}_0$.*

As we will show in §2.5, condition ($\kappa$-big-enough) holds if $\kappa$ is a big enough finite field (lemma 2.4). Also, in the case $B = \mathbb{P}^1$, the condition on $\mathcal{D} \cup \mathcal{B}_0$ can be omitted in the (good-red) assumption as it follows from the two other conditions (lemma 2.6).

2.4. **Proof of proposition 2.2.** Consider the cover $\widetilde{f}^\varphi : \widetilde{X}^\varphi \to B$ obtained by twisting $f$ by $\varphi$. Denote by $\widetilde{\mathcal{F}}^\varphi : \widetilde{\mathcal{X}}^\varphi \to \mathcal{B}$ the morphism obtained by normalizing $\mathcal{B}$ in $k(\widetilde{X}^\varphi)$. By definition we have $\widetilde{\mathcal{F}}^\varphi \otimes_A k = \widetilde{f}^\varphi$; hence $\widetilde{\mathcal{F}}^\varphi \otimes_A k$ is a $k$-cover that is $k^{\mathrm{sep}}$-isomorphic to $f \otimes_k k^{\mathrm{sep}}$. Below we prove the rest of the condition that makes $\widetilde{\mathcal{F}}^\varphi$ an $A$-model of $(f \otimes_k k^{\mathrm{sep}}, \mathcal{F}_0 \otimes_\kappa \overline{\kappa})$ so we can apply assumption ($\kappa$-big-enough).

2.4.1. *$\widetilde{\mathcal{F}}^\varphi$ is flat.* In order to show this claim, we will use this criterion: a cover is flat if it is tamely ramified along a regular divisor with normal crossings [GM71]. From our assumption that $p \not| |G|$, all covers involved in the argument will be tamely ramified. As a first step, note that the $k$-cover $\widetilde{f}^\varphi : \widetilde{X}^\varphi \to B$ is flat: its branch divisor $D$ is regular with normal crossings over $k$ (as it is over $A$ from (good-red)) and $\widetilde{f}^\varphi$ is étale above $B \setminus D$ [Mil80, theorem 3.21]. Now as $B \setminus D = \mathcal{B} \setminus (\mathcal{D} \cup \mathcal{B}_0)$, we have that $\widetilde{\mathcal{F}}^\varphi$ is étale above $\mathcal{B} \setminus (\mathcal{D} \cup \mathcal{B}_0)$ and can use again the above criterion and the assumption (good-red), applied this time over the ring $A$, to conclude our claim. [7]

2.4.2. *$\widetilde{\mathcal{F}}^\varphi$ is étale above $\mathcal{B} \setminus \mathcal{D}$.* As $\widetilde{f}^\varphi : \widetilde{X}^\varphi \to B$ is unramified over $B \setminus D$, it suffices, thanks to the Purity of Branch Locus, to check that $\mathfrak{p}$ is unramified in $\widetilde{\mathcal{F}}^\varphi$. Let $E$ be the fixed field of $\ker(\varphi)$ in $k^{\mathrm{sep}}$. The homomorphism $\varphi$ being unramified at $\mathfrak{p}$ means that $\mathfrak{p}$ is unramified in the extension $E/k$ (more exactly in the integral closure $A'_E$ of $A$ in $E$). This conjoined with $f$ having no vertical ramification at $\mathfrak{p}$ implies that

[7] As pointed out by the referee, if $B$ is a curve, flatness does not require tame ramification: a normal cover of a regular surface is flat over the base.

$\mathfrak{p}$ is unramified in the normalization of $\mathcal{B}$ in $E(X)$ [Bec91, lemma 2.1]. As $k(\widetilde{X}^\varphi) \subset E(X) = E(\widetilde{X}^\varphi)$, this is stronger than what we need.

2.4.3. *Good reduction.* We can now resort to a classical good reduction criterion for covers due to Grothendieck *et al.* [Gro71] [GM71]: in the situation above where we have a smooth divisor $\mathcal{D}$ with normal crossings of the smooth proper and geometrically irreducible scheme $\mathcal{B}$ over the complete discrete valuation ring $A$, the reduction process yields an equivalence between the category of covers of $\mathcal{B}$ tamely ramified along $\mathcal{D}$ and that of those covers of the special fiber of $\mathcal{B}_0$ that are tamely ramified above $\mathcal{D}_0$. From the properties we already know about $\widetilde{\mathcal{F}}^\varphi : \widetilde{\mathcal{X}}^\varphi \to \mathcal{B}$, we deduce that $\widetilde{\mathcal{F}}_0^\varphi : \widetilde{\mathcal{X}}_0^\varphi \to \mathcal{B}_0$ is finite, flat, étale above $(\mathcal{B} \setminus \mathcal{D})_0$ and that $\widetilde{\mathcal{X}}_0^\varphi$ is normal and irreducible ($\widetilde{\mathcal{X}}^\varphi$ is irreducible as it contains $\widetilde{X}^\varphi$ which is irreducible as a dense subset). We show in §2.4.4 below that $\widetilde{\mathcal{X}}_0^\varphi$ is geometrically irreducible.

2.4.4. *Geometric behaviour.* Denote the integral closure of $A$ in $k^{\mathrm{sep}}$ by $A^{\mathrm{sep}}$ and the normalization of $\mathcal{B}^{\mathrm{sep}} = \mathcal{B} \otimes_A A^{\mathrm{sep}}$ in $k^{\mathrm{sep}}(X)$ by $\mathcal{F}^{\mathrm{sep}} : \mathcal{X}^{\mathrm{sep}} \to \mathcal{B}^{\mathrm{sep}}$. The same argument as the one used for $\widetilde{\mathcal{F}}^\varphi$ shows that $\mathcal{F}^{\mathrm{sep}}$ is étale above $(\mathcal{B} \setminus \mathcal{D}) \otimes_A A^{\mathrm{sep}}$ and the same good reduction criterion leads to the same conclusions about the special fiber $\mathcal{F}_0^{\mathrm{sep}}$ as those deduced above for $\widetilde{\mathcal{F}}_0^\varphi : \widetilde{\mathcal{X}}_0^\varphi \to \mathcal{B}_0$. In particular $\mathcal{X}_0^{\mathrm{sep}}$ is normal and irreducible. Lemma 2.3 compares $\mathcal{F}_0^{\mathrm{sep}}$ and $\widetilde{\mathcal{F}}_0^\varphi \otimes_\kappa \overline{\kappa}$.

**Lemma 2.3.** *$\mathcal{F}^{\mathrm{sep}}$ and $\widetilde{\mathcal{F}}^\varphi \otimes_A A^{\mathrm{sep}}$ are isomorphic above a non-empty open subset $V \subset \mathcal{B}^{\mathrm{sep}}$. Consequently $\widetilde{\mathcal{X}}_0^\varphi$ is geometrically irreducible and $\widetilde{\mathcal{F}}^\varphi$ is an $A$-model of $(f \otimes_k k^{\mathrm{sep}}, \mathcal{F}_0 \otimes_\kappa \overline{\kappa})$.*

*Proof.* As $\widetilde{\mathcal{F}}^\varphi$ is étale above $\mathcal{B} \setminus \mathcal{D}$, we can find a non-empty affine open subset $U = \mathrm{Spec}(\beta) \subset \mathcal{B} \setminus \mathcal{D}$ meeting $\mathcal{B}_0$ and such that the integral closure $\beta'_{k(\widetilde{X}^\varphi)}$ of $\beta$ in $k(\widetilde{X}^\varphi)$ is a free $\beta$-module of rank $[k(\widetilde{X}^\varphi) : k(B)]$. Up to shrinking $U$, we may assume that the open subset $U \otimes_A A^{\mathrm{sep}} = \mathrm{Spec}(\beta \otimes_A A^{\mathrm{sep}})$ of $(\mathcal{B} \setminus \mathcal{D}) \otimes_A A^{\mathrm{sep}}$ has the property that the integral closure $(\beta \otimes_A A^{\mathrm{sep}})'_{k^{\mathrm{sep}}(X)}$ of $\beta \otimes_A A^{\mathrm{sep}}$ in $k^{\mathrm{sep}}(X)$ is a free $\beta \otimes_A A^{\mathrm{sep}}$-module of rank $[k^{\mathrm{sep}}(X) : k^{\mathrm{sep}}(B)]$. Furthermore if $f_1, \ldots, f_d$ is a basis of the $\beta$-module $\beta'_{k(\widetilde{X}^\varphi)}$, it is also a basis of the $\beta \otimes_A A^{\mathrm{sep}}$-module $(\beta \otimes_A A^{\mathrm{sep}})'_{k^{\mathrm{sep}}(X)}$: this follows from its discriminant being invertible in $\beta$ (so in $\beta \otimes_A A^{\mathrm{sep}}$ too) conjoined with $k^{\mathrm{sep}}(X) = k^{\mathrm{sep}}(\widetilde{X}^\varphi)$ and $[k(\widetilde{X}^\varphi) : k(B)] = [k^{\mathrm{sep}}(X) : k^{\mathrm{sep}}(B)]$. Conclude that $(\mathcal{F}^{\mathrm{sep}})^{-1}(U \otimes_A A^{\mathrm{sep}})$ and $(\widetilde{\mathcal{F}}^\varphi)^{-1}(U) \otimes_A A^{\mathrm{sep}}$ are isomorphic above $U \otimes_A A^{\mathrm{sep}}$, thus proving first claim of lemma 2.3.

It follows that $\mathcal{X}_0^{\rm sep}$ and $\widetilde{\mathcal{X}}_0^{\varphi} \otimes_\kappa \overline{\kappa}$ are birationally isomorphic. As the former is irreducible, $\widetilde{\mathcal{X}}_0^{\varphi}$ is geometrically irreducible. As $\mathcal{X}_0^{\rm sep}$ and $\widetilde{\mathcal{X}}_0^{\varphi} \otimes_\kappa \overline{\kappa}$ are also normal, the equivalence between function field extensions and covers recalled in §2.1 yields that the $\overline{\kappa}$-covers $\mathcal{F}_0^{\rm sep}$ and $\widetilde{\mathcal{F}}_0^{\varphi} \otimes_\kappa \overline{\kappa}$ are isomorphic (and not just birationally isomorphic). Using this in the special case $\varphi = 1$ yields that these two covers are also isomorphic to $\mathcal{F}_0 \otimes_\kappa \overline{\kappa}$. $\square$

2.4.5. *End of proof of proposition 2.2.* It follows from assumption ($\kappa$-big-enough) that there are $\kappa$-rational points on $\widetilde{\mathcal{X}}_0^{\varphi}$ not lying above any point in $\mathcal{D}_0$. Define $F$ to be the set $\widetilde{\mathcal{F}}_0^{\varphi}(\widetilde{\mathcal{X}}_0^{\varphi}(\kappa)) \setminus \mathcal{D}_0$. Let $\overline{t}_0 \in F$, $\overline{x} \in \widetilde{\mathcal{X}}_0^{\varphi}(\kappa)$ above $\overline{t}_0$ and $t_0 \in \mathcal{B}(A)$ be a lift of $\overline{t}_0$; such a lift exists from Hensel's lemma applied to the smooth variety $\mathcal{B}$. From Hensel's lemma applied to the morphism $\widetilde{\mathcal{F}}^{\varphi} : \widetilde{\mathcal{X}}^{\varphi} \to \mathcal{B}$ which is étale at the neighborhood of $t_0$, $\overline{x}$ can be lifted to some point in $\widetilde{\mathcal{X}}^{\varphi}(A)$; the corresponding point $x$ on the generic fiber $\widetilde{X}^{\varphi}$ is $k$-rational and lies above $t_0$ (viewed in $B(k)$). The twisting lemma 2.1 finishes the proof of proposition 2.2.

2.5. **On the hypotheses of proposition 2.2.** The following two lemmas provide practical conditions that guarantee the hypotheses ($\kappa$-big-enough) and (good-red) of proposition 2.2.

**Lemma 2.4.** *Condition* ($\kappa$-big-enough) *holds if $\kappa$ is a finite field of order bigger than a constant $c$, which is described in addendum 2.5.*

*Proof.* The proof rests on the Lang-Weil inequality and more specifically on this statement:

**Lang-Weil inequality**. *Let $V$ be a proper geometrically irreducible variety of dimension $d \geq 1$ over a finite field $\kappa$ with $q$ elements. There exists a constant $\beta$ depending only on the $\overline{\kappa}$-variety $V_{\overline{\kappa}} = V \otimes_\kappa \overline{\kappa}$ such that $|\#V(\kappa) - q^d| \leq \beta\sqrt{q^{2d-1}}$. For each prime $\ell \neq p$, the constant $\beta$ can be taken to be the largest dimension $\beta_\ell(V_{\overline{\kappa}})$ of the $\ell$-adic cohomology groups $H^i(V_{\overline{\kappa}}, \mathbb{Q}_\ell)$, $i = 0, 1, \ldots, 2d$, viewed as $\mathbb{Q}_\ell$-vector spaces.*

(This follows from the works of Grothendieck and Deligne on Weil's conjectures, for which we refer to [Del74] and [Del80]; references to [AGV73] about Grothendieck's contribution are given in [Del74, §I]. For any prime $\ell \neq p$, Grothendieck has defined some $\ell$-adic cohomology groups $H^i(V_{\overline{\kappa}}, \mathbb{Q}_\ell)$, which are finite dimensional $\mathbb{Q}_\ell$-vector spaces and relate to $\#V(\kappa)$ *via* the formula $\#V(\kappa) = \sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(F, H^i(V_{\overline{\kappa}}, \mathbb{Q}_\ell))$, where $\operatorname{Tr}(F, H^i(V_{\overline{\kappa}}, \mathbb{Q}_\ell))$ is the trace of the Frobenius automorphism induced on $H^i(V_{\overline{\kappa}}, \mathbb{Q}_\ell)$. By Deligne's theorem [Del80, théorème 3.3.1], the eigenvalues of the Frobenius automorphisms acting on $H^i(V_{\overline{\kappa}}, \mathbb{Q}_\ell)$ are of absolute value $\leq \sqrt{q^i}$. Furthermore the main term in the sum above is for $i = 2d$: $\mathrm{Tr}(F, H^{2d}(V_{\overline{\kappa}}, \mathbb{Q}_\ell)) = q^d$).

In order to deduce the ($\kappa$-big-enough) condition, consider a $A$-model $\mathcal{F}' : \mathcal{X}' \to \mathcal{B}$ of $(f \otimes_k k^{\mathrm{sep}}, \mathcal{F}_0 \otimes_\kappa \overline{\kappa})$ and apply the Lang-Weil inequality to get a lower bound of the form $q^d - \beta_\ell(V_{\overline{\kappa}})\sqrt{q^{2d-1}}$ for the total number of $\kappa$-rational points on $V = \mathcal{X}'_0$. As $\mathcal{X}'_0 \otimes_\kappa \overline{\kappa}$ is isomorphic to $\mathcal{X}_0 \otimes_\kappa \overline{\kappa}$, we have in fact $\beta_\ell(V_{\overline{\kappa}}) = \beta_\ell(\mathcal{X}_0 \otimes_\kappa \overline{\kappa})$. Next bound from above the number of these $\kappa$-rational points that may lie above the closed subset $\mathcal{D}_0 \otimes_\kappa \overline{\kappa}$. This number, say $N$, is bounded by the degree of $f$ multiplied by the number of $\kappa$-rational points lying on some irreducible component of $\mathcal{D}_0 \otimes_\kappa \overline{\kappa}$ defined over $\kappa$. Due to the (good-red) assumption (conjoined with $p \not| |G|$), the components of $\mathcal{D}_0 \otimes_\kappa \overline{\kappa}$ correspond *via* the reduction process to the irreducible components of $D \otimes_k k^{\mathrm{sep}}$. Denote their number by $r(D)$. Using Lang-Weil again, we obtain that for any prime $\ell \neq p$, we have $N \leq |G|\, r(D)\, (1 + b_\ell(\mathcal{D}))\, q^{d-1}$ where $b_\ell(\mathcal{D})$ is the maximum of the $\beta_\ell(V_{\overline{\kappa}})$ where $V$ ranges over the components of the divisor $\mathcal{D}_0 \otimes_\kappa \overline{\kappa}$. As the main term in the lower bound is in $q^d$, we obtain the desired conclusion if $q$ is suitably large. □

*Addendum* 2.5. More precisely, the condition on $q$ is as follows. For each prime $\ell \neq p$, there is a constant $c_\ell$ depending on $r(D)$, $|G|$, $\beta_\ell(\mathcal{X}_0 \otimes_\kappa \overline{\kappa})$ and $b_\ell(\mathcal{D})$ and $q$ should be bigger than one of these $c_\ell$. For covers of $B = \mathbb{P}^1$ and with $\mathcal{B} = \mathbb{P}^1_A$, $\beta_\ell(\mathcal{X}_0 \otimes_\kappa \overline{\kappa})$ is the genus of $\mathcal{X}_0 \otimes_\kappa \overline{\kappa}$ which here equals the genus $g$ of $X$ and can be bounded in terms of $r$ and $|G|$ by the Riemann-Hurwitz formula; and $b_\ell(\mathcal{D}) = 0$. The constant $c_\ell$ depends only on $r$ and $|G|$, and no longer on the prime $\ell$. It can be made totally explicit: we have $\#\mathcal{X}_0(\kappa) - (q+1) \geq -2g\sqrt{q}$; the number $q$ should satisfy $q + 1 - 2g\sqrt{q} > |G|\, r$.

**Lemma 2.6.** *For $B = \mathbb{P}^1$, the* (good-red) *condition holds if the divisor $\mathcal{D}$ is étale over $A$, $p \not| |G|$ and $f$ has no vertical ramification at $\mathfrak{p}$.*

*Proof.* As in §1.2 where $B = \mathbb{P}^1$, denote the branch divisor $D$ by $\{t_1, \ldots, t_r\}$. We may assume that $t_1, \ldots, t_r$ are in the valuation ring $A$. The divisor $\mathcal{D}$ then corresponds to a polynomial of the form $D(T) = \delta \prod_{i=j}^r (T - t_i)$ with $\delta \in A \setminus \mathfrak{p}$. Assume that $\mathcal{D} \cup \mathcal{B}_0$ does not have normal crossings. The polynomial $D'(T)$ then vanishes at some $t_i$ modulo $\mathfrak{p}$, or, equivalently, $t_i$ is a multiple root of $D(T)$ modulo $\mathfrak{p}$. But then there is some other root $t_j \neq t_i$ of $D(T)$ equal to $t_i$ modulo $\mathfrak{p}$, contradicting that $\mathcal{D}$ is étale. □

## 3. Globalization

We explain how in the situation of theorem 1.2, local information can be obtained for each place $v \in S$ from the specialization result of §2.3 and then globalized to deduce the desired result.

3.1. **Generalization.** We will prove a more general statement. As in §1.2 we have a finite group $G$, a field $K$, a finite set $S$ of finite places of $K$ and a G-cover $f$.

The field $K$ is assumed to be the quotient field of some Dedekind domain $R$ and $S$ is a finite set of places of $K$ corresponding to some prime ideals in $R$. For every place $v$, the completion of $K$ is denoted by $K_v$, the valuation ring by $R_v$, the valuation ideal by $\mathfrak{p}_v$, the residue field $R_v/\mathfrak{p}_v$ by $\kappa_v$, the order of $\kappa_v$ by $q_v$ and its characteristic by $p_v$.

The G-cover $X \to B$ has here a more general base space than $\mathbb{P}^1$; we assume the following on $B$:

(*) *$B$ is a smooth projective and geometrically integral $K$-variety and is given with an integral model $\mathcal{B}$ over $R$ such that $\mathcal{B}_v = \mathcal{B} \otimes_R R_v$ is smooth for each $v \in S$.*

Denote the branch divisor of $f$ by $D$ and its Zariski closure in $\mathcal{B}$ and $\mathcal{B}_v$ by $\mathcal{D}$ and $\mathcal{D}_v$ respectively.

The G-cover $f$ is assumed to be defined over $K$ and we retain the good reduction assumption (good-red) from §1.2, generalized as follows for a $K$-G-cover of $B$:

(good-red) *for each $v \in S$, $p_v \not| |G|$, $\mathcal{D}_v$ is a smooth divisor, $\mathcal{D}_v \cup (\mathcal{B}_v)_0$ is regular with normal crossings (over $R_v$) and $f \otimes_K K_v$ has no vertical ramification at $\mathfrak{p}_v$.*

Finally we assume that for each $v \in S$, the residue field $\kappa_v$ is finite of order $q_v \geq C(f, \mathcal{B})$ where the constant $C(f, \mathcal{B})$ replaces the constant $c(|G|, r)$ of theorem 1.2; it depends on $f$ and $\mathcal{B}$ and is described in the proof of the following lemma 3.1.

**Lemma 3.1.** *For each $v \in S$, assumption $q_v \geq C(f, \mathcal{B})$ guarantees that condition* ($\kappa_v$-big-enough) *from proposition 2.2 holds for the $K_v$-G-cover $f \otimes_K K_v$ and $A = R_v$.*

*Proof.* It suffices to show that the constant $c$ from lemma 2.4 can be chosen depending only on $f$ and $\mathcal{B}$. Fix a prime $\ell \neq p_v$ for all $v \in S$. From addendum 2.5, for each $v \in S$, there is a constant $c_\ell$ depending on $r(D)$, $|G|$, $\beta_\ell(\mathcal{X}_0 \otimes_{\kappa_v} \overline{\kappa_v})$ and $b_\ell(\mathcal{D})$ and $q_v$ should be bigger than $c_\ell$. Here $\mathcal{X}_0$ is the special fiber of the $R_v$-scheme obtained by scalar extension from the $R$-scheme corresponding to the normalization of

$\mathcal{B}$ in $K(X)$; in other words, all $\kappa_v$-varieties $\mathcal{X}_0$ come from a global $K$-variety. In this situation we have this "standard" property of the $\ell$-adic cohomology groups:
(**) *the $\mathbb{Q}_\ell$-dimensions of $H^i(\mathcal{X}_0 \otimes_{\kappa_v} \overline{\kappa_v}, \mathbb{Q}_\ell)$ can be bounded by a constant depending only on $X$ (and independent of $v$).*

(Specifically this follows from exposé VI of [Gro73] by J.-P. Jouanolou, and in particular from proposition 1.2.6 there which shows that for every $\ell$-adic constructible sheaf $F$ on a locally noetherian prescheme $X$, there exists a constructible stratification of $X$ such that the restriction of $F$ to each stratum is a "twisted constant" $\ell$-adic sheaf).

The same remark applies to the parameter $b_\ell(\mathcal{D})$ to provide the desired conclusion. The constant $C(f, \mathcal{B})$ can be made more precise; in particular, in the case $B = \mathbb{P}^1$, it can be expressed as a constant $c(|G|, r)$ depending only on $|G|$ and $r$. □

3.2. **Argument.** Under the assumptions in §3.1, the argument below leads to theorem 3.2 which is the announced generalization of thm 1.2.

For each $v \in S$, consider the $K_v$-G-cover $f_v : X_v \to B_v$ obtained from $f$ by scalar extension from $K$ to $K_v$. Fix an unramified Grunwald problem $\underline{\varphi} = (\varphi_v : \mathrm{G}_{K_v} \to G)_{v \in S}$. From our assumptions, conditions (good-red) and ($\kappa$-big-enough) from the local specialization result (proposition 2.2) are satisfied for the cover $f_v$, and the homomorphism $\varphi_v$ is unramified, for each $v \in S$.

Conclude from proposition 2.2 that, for each $v \in S$, the set of all $t_v \in B(K_v) \setminus D$ such that the specialization representation of $f_v$ at $t_v$ is conjugate in $G$ to $\varphi_v : \mathrm{G}_{K_v} \to G$ contains a non-empty open subset $U_v \subset B(K_v) \setminus D$. *If points $t_0$ exist in the set $B(K) \cap \prod_{v \in S} U_v$*, then for each $v \in S$, the specialization representation of $f_v$ at $t_0$ is conjugate in $G$ to $\varphi_v : \mathrm{G}_{K_v} \to G$; hence *the Galois group* $\mathrm{Gal}(K(X)_{t_0}/K)$ *contains a conjugate $H_v^{g_v}$* in $G$ of the subgroup $H_v = \varphi_v(\mathrm{G}_{K_v}) \subset G$.

3.3. **Conclusion.** The following conditions respectively guarantee that $B(K) \cap \prod_{v \in S} U_v \neq \emptyset$ and $\mathrm{Gal}(K(X)_t/K) = G$ in the argument. Theorem 3.2 is our conclusion under these additional assumptions.

(WA/$S$) *The variety $B$ has the weak approximation property with respect to $S$, i.e. $B(K)$ is dense is $\prod_{v \in S} B(K_v)$.*

(g-complete) *If $C_v$ is the conjugacy class of some generator $h_v$ of $H_v$, the set $\{C_v \,|\, v \in S\}$ is g-complete, that is, no proper subgroup of $G$ intersects each of the conjugacy classes $C_v$ ($v \in S$).*

The second condition was introduced by M. Fried [Fri95] (in another context). It does not depend on the generator $h_v$ of $H_v$.

**Theorem 3.2.** *Let $K$, $S$, $G$, $\mathcal{B}$, $f$ be as in §3.1. Assume condition* (WA/$S$) *holds. We have this Hilbert-Grunwald specialization property:*

(HGr-spec) *for each unramified Grunwald problem $(\varphi_v : \mathrm{G}_{K_v} \to G)_{v\in S}$ for which condition* (g-complete) *holds, there exist open subsets $U_v \subset B(K_v) \setminus D$ $(v \in S)$ such that $B(K) \cap \prod_{v\in S} U_v \neq \emptyset$ and each element $t_0$ in this set yields a specialization of $f$ that is a $K$-solution to the Grunwald problem $(\varphi_v : \mathrm{G}_{K_v} \to G)_{v\in S}$.*

3.4. **Final reduction.** If the group-theoretical condition ($g$-complete) does not hold, it is possible to reduce to it at the cost of throwing in more places in $S$, and assuming that the approximation condition (WA/$S$) holds for this bigger $S$. This explains the appearance of the set $S_0$ in conclusion (HGr-spec) of theorem 1.2 (whereas $S_0 = \emptyset$ in theorem 3.2). We recall below the argument, which has already been used in various versions in earlier works. This reduction ends the proof of theorem 1.2.

The main point is to construct, for each $g \in G$, a place $v_g$ of $K$ and an unramified homomorphism $\varphi_{v_g} : \mathrm{G}_{K_{v_g}} \to G$ with the following properties:

(a) *for each $g \in G$, the Galois group $H_{v_g} = \varphi_{v_g}(\mathrm{G}_{K_{v_g}})$ s conjugate to the subgroup $\langle g\rangle$ of $G$,*

(b) *for each $g \in G$, there is an open subset $U_{v_g} \subset B(K_{v_g}) \setminus D$ such that each $t \in U_{v_g}$ yields a specialization $K_{v_g}(X)_t/K_{v_g}$ of Galois group conjugate to $H_{v_g}$ in $G$,*

(c) *the set $S_0 = \{v_g \mid g \in G\}$ is disjoint from $S$.*

If then $T = S\cup S_0$ and $t_0 \in B(K)\cap(\prod_{v\in T} U_v)$, the specialization of $f$ at $t_0$ still satisfies the desired Grunwald property regarding the places $v \in S$ but has this extra property: for each $g \in G$, the conjugacy class of $g$ meets the group $\mathrm{Gal}(K(X)_{t_0}/K)$. From a classical lemma of Jordan [Jor72], the set of all conjugacy classes of a finite group is $g$-complete. So the Galois group $\mathrm{Gal}(K(X)_{t_0}/K)$ is all of $G$ and the specialization $K(X)_{t_0}/K$ is a $K$-solution to the initial Grunwald problem.

Existence of such additional places $v_g$ and associated $\varphi_{v_g}$ is guaranteed under the following conditions, which are satisfied if as in theorem 1.2, $K$ is a number field and $B = \mathbb{P}^1$:

- *$D$ is a smooth divisor with normal crossings over the field $K$,*
- *there exist infinitely many places $v$ with $\kappa_v$ finite of characteristic not dividing $|G|$ and of order $\geq C(f,\mathcal{B})$.*

These two conditions make it possible to find additional places $v_g$ satisfying the assumptions of theorem 3.2 (the (good-red) condition and

$q_v \geq C(f, \mathcal{B})$) and such that $\langle g \rangle$ is a Galois group of some unramified extension of $K_{v_g}$ ($g \in G$).

The set $S_0$ can further be chosen disjoint from any prescribed finite set of places.

## 4. Effectiveness and application to the RIGP

Our approach leads to interesting types of bounds. Assume $B = \mathbb{P}^1$ and $K = \mathbb{Q}$ (for simplicity). The following result is a fully effective version of theorem 1.2. For short, a prime $p$ is said to be *good* below (for the regular extension $F/\mathbb{Q}(T)$) if the branch divisor $\mathbf{t} = \{t_1, \ldots, t_r\}$ is étale and there is no vertical ramification at $p$, and *bad* otherwise.

**Corollary 4.1.** *Let $G$ be a finite group and $F/\mathbb{Q}(T)$ be a regular Galois extension of group $G$. There exist integers $m_0, \beta, \delta > 0$ depending only on $F/\mathbb{Q}(T)$ such that for every $x \geq m_0$, the following holds. Let $S_x$ be the set of good primes $p$ with $m_0 < p \leq x$ and $\underline{\varphi} = (\varphi_p : \mathrm{G}_{\mathbb{Q}_p} \to G)_{p \in S_x}$ be an unramified Grunwald problem. Then there exists an integer $t_0(x)$ such that*

(i) $0 \leq t_0(x) \leq \beta \prod_{p \in S_x} p$ ,

(ii) *for each integer $t \equiv t_0(x)$ modulo $(\beta \prod_{p \in S_x} p)$, $t$ is not a branch point of $F/\mathbb{Q}(T)$ and the specialization $F_t/\mathbb{Q}$ at $t$ of the extension $F/\mathbb{Q}(T)$ is a solution to the Grunwald problem $\underline{\varphi}$,*

(iii) $\log |d_{F_{t_0(x)}}| \leq \delta\, x$.

*Addendum* 4.2 (more on the constants). Denote the number of non-trivial conjugacy classes of $G$ by $\mathrm{cc}(G)$, the number of branch points of $F/\mathbb{Q}(T)$ by $r$ and the number of bad primes by $\mathrm{br}(\mathbf{t})$. One can take $m_0$ such that the interval $[4r^2|G|^2, m_0]$ contains at least $\mathrm{br}(\mathbf{t}) + \mathrm{cc}(G)$ distinct primes, and $\beta$ to be the product of $\mathrm{cc}(G)$ good primes in $[4r^2|G|^2, m_0]$. The constants $m_0$ and $\beta$ only depend on $|G|$, $r$ and $\mathrm{br}(\mathbf{t})$. The constant $\delta$ can also be made explicit but is more involved.

In particular, for $x = m_0$ for which $S_x = \emptyset$, assertion (ii) concludes that $\mathrm{Gal}(F_{t_0(x)}/\mathbb{Q}) \simeq G$. So $t_0(m_0)$ is a specialization for which the conclusion of Hilbert's irreducibility theorem holds and it is bounded only in terms of $|G|$, $r$ and $\mathrm{br}(\mathbf{t})$. Note also that this special situation corresponds to some result originally proved in [Eic39] and [Fri74]: any Hilbert subset associated with a $K$-G-cover $f : X \to \mathbb{P}^1$ contains an arithmetic progression $(am + b)_{m \in \mathbb{Z}}$.

*Proof.* The proof consists in making effective the arguments used to prove theorem 1.2. Condition $p \geq 4r^2|G|^2$ assures that $p \not| \, |G|$ and

$p \geq c(|G|, r)$; recall that in our specific situation the latter amounts to $p + 1 - 2g\sqrt{p} > |G|\, r$ (addendum 2.5) and so our claim easily follows from the Riemann-Hurwitz formula.

Fix a subset $S_0 \subset [4r^2|G|^2, m_0]$ of $\mathrm{cc}(G)$ good primes and associate in a one-one way a non trivial conjugacy class $C_p$ to each prime $p \in S_0$. For each $p \in S_0$, pick an element $g_p \in C_p$ and construct an unramified epimorphism $\varphi_p : \mathrm{G}_{\mathbb{Q}_p} \to \langle g_p \rangle$ (in other words an unramified Galois extension $E_p/\mathbb{Q}_p$ with group $\langle g_p \rangle$). Consider the Grunwald problem $\underline{\varphi} = (\varphi_p)_{p \in S_x \cup S_0}$. As all primes $p \in S_x \cup S_0$ are $\geq 4r^2|G|^2$, proposition 2.2 applies to show that one can find a coset $U_p$ of some integer $t_p(x)$ modulo $p\mathbb{Z}_p$ such that for all $t \in U_p$, $t$ is not a branch point of $F/\mathbb{Q}(T)$ and the specialization $(F\mathbb{Q}_p)_t/\mathbb{Q}_p$ corresponds to the epimorphism $\varphi_p : \mathrm{G}_{\mathbb{Q}_p} \to \langle g_p \rangle$ ($p \in S_x \cup S_0$). Use next the chinese remainder theorem to find an integer $t_0(x) \in \mathbb{Z}$ such that $t_0(x) \equiv t_p(x)$ modulo $p$ for all $p \in S_x \cup S_0$; such an integer can be chosen satisfying condition (i). Condition (ii) follows from already explained globalization arguments; the primes in the subset $S_0 \subset [4r^2|G|^2, m_0]$ and the associated morphisms $\varphi_p$ are used to guarantee that $\mathrm{Gal}(F_t/\mathbb{Q})$ meets each conjugacy class of $G$ and so equals $G$ (as explained in §3.4).

To prove (iii) let $\Delta(T) \in \mathbb{Z}[T]$ be the discriminant of the irreducible polynomial of some primitive element of $F/\mathbb{Q}(T)$, integral over $\mathbb{Z}[T]$. We have $|d_{F_{t_0}}| \leq |\Delta(t_0)| \leq c_1 |t_0|^{c_2}$ with $c_1, c_2$ depending on $f$. Using that $\log(\prod_{p \in S_x \cup S_0} p) \sim x$ as $x \to \infty$, we obtain $\log |d_{F_{t_0}}| \leq \delta\, x$ for some constant $\delta > 0$ depending on $f$. □

Finally we explain how corollary 1.6 follows from corollary 4.1.

*Proof of corollary 1.6.* Fix two functions $\ell(x)$ and $m(x)$ tending to $\infty$ with $x$ and a group $G$ satisfying condition (*) of corollary 1.6. Assume there exists a G-cover $f : X \to \mathbb{P}^1$ of group $G$, defined over $\mathbb{Q}$. Apply corollary 4.1 with for each prime $p \in S_x$, $\varphi_p : \mathrm{G}_{\mathbb{Q}_p} \to G$ taken to be the trivial homomorphism. Conclude that for every $x \geq m_0$, there exist specializations $F_{t_0(x)}/\mathbb{Q}$ of $F/\mathbb{Q}(T)$ at some $t_0(x) \in \mathbb{Q}$ that are unramified and totally split at every prime $p \in S_x$. Thus, with the notation of corollary 1.6, we have $\pi_{\mathrm{nts}}^{F_{t_0(x)}}(x) \leq \pi(m_0) + \mathrm{br}(\mathbf{t})$ (with $\pi(m_0)$ the number of primes $\leq m_0$). This contradicts assumption (*) from corollary 1.6 for all suitably large $x$ as from corollary 4.1 (iii) we also have $\log |d_{F_{t_0(x)}}| \leq \delta\, x$. □

## 5. Proof of theorem 1.3

As in theorem 1.3, fix a finite group $G$ and assume that the base space of the covers is $B = \mathbb{P}^1$, that $K$ is a number field and that for each $v \in S$, we have $p_v \not| 6|G|$ and $q_v \geq c(G)$. The constant $c(G)$ is defined in §5.6.

5.1. **1st step:** reduce to the situation of a group with a trivial center.

If $Z(G) \neq \{1\}$, use [FV91, lemma 2] to consider a group extension $\epsilon : \widetilde{G} \to G$ such that $Z(\widetilde{G}) = \{1\}$: one can take $\widetilde{G} = \Gamma^d \rtimes G$ with $\Gamma$ any non abelian finite simple group, $d = |G|$ and where $G$ acts on $\Gamma^d$ by permuting the factors of $\Gamma^d$ *via* the regular representation of $G$ (in other words, $\widetilde{G}$ is the wreath product of $\Gamma$ and $G$). Fix $\Gamma = \mathrm{PSL}_2(\mathbb{F}_3)$. As 2 and 3 are the only primes dividing $|\mathrm{PSL}_2(\mathbb{F}_3)|$ and no prime $p_v$ divides $6\,|G|$, no $p_v$ divides $|\widetilde{G}|$ ($v \in S$). If $Z(G) = \{1\}$, just set $\widetilde{G} = G$.

5.2. **2nd step:** construct a Hurwitz space of G-covers of group $\widetilde{G}$ with a component defined over $\mathbb{Q}$.

This can be done thanks to a construction due to Fried. Let $C_1, \ldots, C_s$ be the list of all non-trivial conjugacy classes of $\widetilde{G}$ and $\widetilde{\mathbf{C}}$ be the $r$-tuple of all pairs $(C_i, C_i^{-1})$, $i = 1, \ldots, s$, repeated twice. Denote the Hurwitz moduli space of G-covers of $\mathbb{P}^1$ of group $\widetilde{G}$ with $r$ branch points and ramification type $\widetilde{\mathbf{C}}$ by $\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})$. From [Fri95], $\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})$ has a component $\mathsf{HM}$ defined over $\mathbb{Q}$; more precisely, $\mathsf{HM}$ is in this case the unique Harbater-Mumford component of $\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})$ (see also [DE06]).

5.3. **3rd step:** rational points on the reduction of $\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})$.

From [Wew98], $\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})$ can be constructed as a scheme, smooth and of finite type over $\mathbb{Z}[1/|\widetilde{G}|]$ and the branch point assignment induces an étale morphism $\pi : \mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}}) \to \mathsf{U}_r$ over $\mathbb{Z}[1/|\widetilde{G}|]$ onto the branch point configuration space $\mathsf{U}_r$. Furthermore, there is a natural compactification $\overline{\pi} : \overline{\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})} \to \overline{\mathsf{U}_r}$ with $\overline{\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})}$ normal and proper over $\mathbb{Z}[1/|\widetilde{G}|]$ and $\overline{\pi}$ ramified only above the discriminant locus $\Delta_r = \overline{\mathsf{U}_r} \setminus \mathsf{U}_r$. As $p_v$ does not divide $|\widetilde{G}|$, the component $\mathsf{HM}$ has good reduction at each place $v \in S$; in particular the special fiber $\mathsf{HM}_{v,0}$ of $\mathsf{HM}_v = \mathsf{HM} \otimes_{\mathbb{Z}[1/|\widetilde{G}|]} R_v$ is geometrically irreducible.

As the residue fields $\kappa_v$ are finite then, from the Lang-Weil inequality, if $q_v$ is suitably large (depending on $r$ and $|\widetilde{G}|$ and so eventually only on $G$), there exist $\kappa_v$-rational points on $\mathsf{HM}_{v,0}$ that do not lie over the discriminant locus. These $\kappa_v$-rational points correspond to G-covers of group $\widetilde{G}$, of ramification type $\widetilde{\mathbf{C}}$ and with field of moduli $\kappa_v$. But these G-covers are in fact defined over $\kappa_v$: the field of moduli is a field of definition [DD97, corollary 3.3].

5.4. **4th step:** lift the $\kappa_v$-G-covers.

For each $v \in S$, use Hensel's lemma and the smoothness of the stack corresponding to the moduli space $\mathsf{H}_r(\widetilde{G}, \widetilde{\mathbf{C}})$ to lift the $\kappa_v$-G-covers from 3rd step to $K_v$-G-covers corresponding to $K_v$-points on the component $\mathsf{HM}$ (due to $Z(\widetilde{G}) = \{1\}$, the stack and the moduli space coincide, but this is not needed here). Denote by $U_v$ the $v$-adic open subset of $\mathsf{HM}(K_v)$ corresponding to $K_v$-G-covers with an étale branch divisor at $v$; by construction our lifted G-covers are in $U_v$ $(v \in S)$.

5.5. **5th step:** approximation part.

Use the local-global property of $K^{\mathrm{tot}S}$ [MB89] [MB01] to find $K^{\mathrm{tot}S}$-points on $\mathsf{HM}$ that lie in $U_v$ for each $v \in S$. From [DDMB04, corollary 1.4], such a point corresponds to some G-cover $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1$ that is defined over $K^{\mathrm{tot}S}$. By construction this G-cover $\widetilde{f}$ is defined over some Galois extension $L/K$ totally split in $K_v$ $(v \in S)$ and its branch divisor is étale at each place $v \in S$. Furthermore, as $Z(\widetilde{G}) = \{1\}$ and $p_v \not| |G|$, from [Bec91, proposition 2.3], there is no vertical ramification at each $v \in S$. Combined with lemma 2.6, this shows that the full condition (good-red) from theorem 1.3 holds.

5.6. **6th step:** the Hilbert-Grunwald property.

Fix the constant $c(G)$ in such a way the Lang-Weil inequality can be applied in §5.3 and $c(G)$ is bigger than the constant $c(|\widetilde{G}|, r)$ (defined in §3.1). By construction, the extension $\epsilon : \widetilde{G} \to G$ splits; let $s : G \to \widetilde{G}$ be a section. If $\underline{\varphi} = (\varphi_v : \mathrm{G}_{K_v} \to G)_{v \in S}$ is a given unramified Grunwald problem of group $G$, consider the unramified Grunwald problem $s\underline{\varphi} = (s\varphi_v : \mathrm{G}_{K_v} \to \widetilde{G})_{v \in S}$ of group $\widetilde{G}$. Conclude from theorem 1.2 that $\widetilde{f}$ satisfies the Hilbert-Grunwald conclusion (HGr-spec) for the triple $(\widetilde{G}, S, s\underline{\varphi})$. Consider the G-cover $f : X \to \mathbb{P}^1$ obtained from $\widetilde{f}$ by modding out by $\ker(\epsilon)$. It is readily checked that the cover $f$ is defined over $L$ and satisfies conditions (good-red) and (HGr-spec) for $(G, S, \underline{\varphi})$.

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