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Tchebotarev theorems for function fields

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ABSTRACT

The central theme of the paper is the specialization of algebraic function field extensions. Our main results are Tchebotarev type theorems for Galois function field extensions, finite or infinite, over various base fields: under some conditions, we extend the classical finite field case to number fields, p-adic fields, PAC fields, function fields $\kappa(x)$, etc. We also compare the Tchebotarev conclusion – existence of unramified local specializations with Galois group any cyclic subgroup of the generic Galois group (up to conjugation) – to the Hilbert specialization property. For a function field extension with the Tchebotarev property, the exponent of the Galois group is bounded by the l.c.m. of the local specialization degrees. Local–global questions arise for which we provide answers, examples and counter-examples.

1. Introduction

Let K be a field, B a smooth projective and geometrically integral K-variety and $F/K(B)$ a Galois extension of group G, finite or infinite. For every overfield $k \supset K$ and each point $t_0 \in B(k)$, there is a notion of k-specialization of $F/K(B)$ at t_0. For t_0

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not in the branch locus of the extension $F/K(B)$, it is a Galois extension $(Fk)_{t_0}/k$ and the Galois group $\mathrm{Gal}((Fk)_{t_0}/k)$ identifies to some subgroup of G (well-defined up to conjugation by elements of G).

The leading question in this context consists in comparing the Galois groups of the specializations with the "generic" Galois group G. A classical tool is the *Hilbert specialization property*: essentially, a finite extension $F/K(B)$ has the Hilbert specialization property if the "special" groups equal G for "many" specializations over $k = K$.

In this paper, we introduce another specialization property, the *Tchebotarev existence property*, which is, for finite extensions, a function field analog of the existence part in the Tchebotarev density theorem for number fields. We say that $F/K(B)$ has the *Tchebotarev existence property* if for every cyclic subgroup H of G, there exists a local field k over K and a point $t_0 \in B(k)$ such that the specialization $(Fk)_{t_0}/k$ is cyclic, unramified and its Galois group is conjugate to H in G (see Definition 2.5 for more details).

A first motivation to consider this property is the following: for an extension with the Tchebotarev existence property certain local behaviors encode informations on the structure of the Galois group of the extension and vice-versa, as explained later. This provides a function field analog to some results originally obtained over number fields by S. Checcoli and U. Zannier [6].

Compared to the Hilbert property, our property allows more general base fields and base varieties and it is also defined for infinite extensions. Moreover, even if it only preserves the "local" structures, it still encapsulates a good part of the Hilbert property: for example, over $\mathbb{Q}$, Hilbert essentially follows from the Tchebotarev property merely conjoined with the Artin–Whaples theorem; actually, the Hilbert property is somehow squeezed between two variants of the Tchebotarev property (see Proposition 4.6).

The main results of this paper are some "Tchebotarev theorems for function fields" (Theorem 3.2 and Corollary 3.7), which provide concrete situations where the property holds. As a special case, we have the following:

Theorem 1.1. *A Galois extension $F/K(T)$ with F/K regular*[1] *has the Tchebotarev existence property if K is a number field, or a finite field, or a PAC field*[2] *with cyclic extensions of any degree, or a rational function field $\kappa(x)$ with κ a finite field of prime-to-$|G|$ order.*

With some extra good reduction condition on $F/K(T)$, the property is also shown to hold if K is a p-adic field or a formal Laurent series field with coefficients in a finite field, etc. Fields of the form $K = k((\theta))(x)$ are also considered in Theorem 3.10. To our knowledge only the finite field case was covered in the literature. The main ingredients in our proofs are the *twisting lemma* and the *local specialization result* of P. Dèbes

[1] *I.e.* $F \cap \overline{K} = K$ as recalled in Definition 3.1.
[2] Pseudo Algebraically Closed; the definition is recalled in Section 3.1.2 (a).

and N. Ghazi [12], concerning whether a given Galois extension E/k comes from a specialization of a Galois extension $F/K(B)$ with F/K regular.

As remarked above, it is natural to compare our property to the classical Hilbert specialization property. Recall that a field K is *Hilbertian* if the Hilbert specialization property holds for every finite Galois extension $F/K(T)$. In analogy, we say that a field K is *Tchebotarev* if the Tchebotarev existence property holds for every finite Galois extension $F/K(T)$ with F/K regular.

If K is a PAC field and G_K denotes its absolute Galois group, then being Hilbertian corresponds to a well-identified property of G_K. More precisely, basing on classical results of M.D. Fried, M. Jarden, H. Völklein [19,22] and to subsequent work of F. Pop [28] and P. Dèbes [9], it is known that K is hilbertian if and only if G_K is ω-free (definition recalled in Section 4.1). Moreover, the fact that every finite group occurs as a quotient of G_K is equivalent to a weak variant of the Hilbert property, called *RG-Hilbertian* (see Definition 4.1). In analogy to these results and as a consequence of Theorem 1.1, we prove that the property of being Tchebotarev can also be read on G_K:

Corollary 1.2. *Let K be a countable PAC field with absolute Galois group G_K. Then K is Tchebotarev if and only if every cyclic group is a quotient of some open subgroup of G_K.*

If K is not PAC, the situation is more complex: some of the PAC conclusions still hold, others do not, and some are unclear. Still, as mentioned above, we are able to show that in general, the Hilbert property is comprised between a strong and a weak variant of the Tchebotarev property (see Proposition 4.6).

We finally investigate some local–global results for infinite extensions implied by the Tchebotarev existence property. Let again $F/K(B)$ be a Galois extension of Galois group G. A nice feature is that if $F/K(B)$ has the Tchebotarev property, then it has local specializations with any prescribed cyclic subgroup of G as Galois group. We have the following consequence, where by "local specialization degrees" we mean the degrees of all the cyclic unramified local specializations of $F/K(B)$ (see Proposition 5.5 and Corollary 5.12 for more precise statements):

Theorem 1.3. *Under the Tchebotarev existence property, if the local specialization degrees of $F/K(B)$ are uniformly bounded, then the exponent of $G = \mathrm{Gal}(F/K(B))$ is finite. The converse holds too under some standard assumptions on K. Moreover, if G is abelian, the uniform boundedness of the local specialization degrees of $F/K(B)$ also implies that there exists an integer $d \geq 1$ such that F is in the compositum $K(B)^{(d)}$ of all finite extensions of $K(B)$ of degree $\leq d$.*

These results were established by the first author and U. Zannier in the situation $F/K(B)$ is a number field extension F/K [6,5]. It turns out that the core of their arguments is the Tchebotarev property that we have identified.

Checcoli and Zannier proved, in the number field case, that the hypothesis on G being abelian cannot be removed, providing counter-examples. We show that the same conclusion holds in the function field context in the situation where $\dim(B) > 0$ by providing counter-examples based on several group-theoretical constructions. Some of them are obtained combining results from the theory of extra-special groups and their modules, as in [6], with Abhyankar's Conjecture on Galois groups of function field extensions of characteristic p, proved by works of M. Raynaud [29] and D. Harbater [23]. Other constructions are given using pro-dihedral groups and generalized Frattini covers. One of them is re-used in a remark on a geometric analog of the Bogomolov property (definition recalled in Section 5.4.1).

The plan of the paper is the following.

In Section 2, after some preliminaries, we introduce the Tchebotarev existence property for finite and infinite extensions. We establish a formal set-up around this property: this enables us, in particular, to approach the above-mentioned local–global problems by dealing with both the original number field and the new function field situations.

In Section 3 we prove Theorem 1.1, which is a consequence of the more general Theorem 3.2, providing many fields having the Tchebotarev existence property. More specific examples are discussed in Corollary 3.7. We also study a strict variant of the property in Section 3.1.3.

In Section 4, we compare the Tchebotarev existence property to the classical Hilbert specialization property: in Section 4.1 we analyze the PAC case, proving Corollary 1.2; in Section 4.2 and Section 4.3 we study the general situation.

Section 5 is devoted to study some implications of the Tchebotarev existence property for infinite extensions and to the proof of Theorem 1.3 and some related questions. The paper ends with some remarks, in particular on the geometric Bogomolov property.

2. The Tchebotarev existence property

In this section we define the Tchebotarev existence property for finite and infinite extensions. We start with some preliminaries on local fields, local specializations and Frobenius subgroups.

2.1. Preliminaries

Given a field k, we fix an algebraic closure $\overline{k}$ and denote the separable closure of k in $\overline{k}$ by k^{sep} and its absolute Galois group by G_k.

Definition 2.1. Given a field K, what we call a *local field* over K is a finite extension k_v of some completion K_v of K for some discrete valuation v on K. The field k_v is complete with respect to the unique prolongation of v to k_v, which we still denote by v.

A *localization set* of K is a set $\mathcal{M}$ of finite *places* of K (*i.e.* of equivalence classes of discrete valuations on K). An $\mathcal{M}$-local field over K is a local field k_v over K with $v \in \mathcal{M}$. When the context is clear, we will drop the reference to $\mathcal{M}$.

Here are some typical examples.

Example 2.2. (a) A complete valued field K_v for a non trivial discrete valuation v will be implicitly given with the localization set $\mathcal{M} = \{v\}$. The $\mathcal{M}$-local fields over K_v are K_v and its finite extensions.

(b) A number field K will be implicitly given with the localization set $\mathcal{M}$ consisting of all the finite places of K. The $\mathcal{M}$-local fields over K are the non-archimedean completions of K and its finite extensions.

(c) If κ is a field and x an indeterminate, the rational function field $\kappa(x)$ will be implicitly given with the localization set $\mathcal{M}$ consisting of all the $(x - x_0)$-adic valuations where x_0 ranges over $\mathbb{P}^1(\kappa)$ (with the usual convention that $x - \infty = 1/x$). The $\mathcal{M}$-local fields over $\kappa(x)$ are the fields $\kappa((x-x_0))$ of formal Laurent series in $x-x_0$ with coefficients in κ and their finite extensions ($x_0 \in \mathbb{P}^1(\kappa)$).

A variant of this example includes the π-adic valuations corresponding to all irreducible polynomials $\pi \in \kappa[x]$. However, we will mostly consider the situation above with only degree 1 polynomials π: indeed, it is stronger for an extension to have the Tchebotarev existence property in this situation than for the variant.

(d) A field K, without any specification, will be implicitly given with the localization set $\mathcal{M}$ consisting of the sole trivial discrete valuation, denoted 0. The $\mathcal{M}$-local fields over K are K and its finite extensions.

2.1.1. Local specializations and Frobenius subgroups

Suppose given a base field K, a smooth projective and geometrically integral[3] K-variety B and a Galois extension $F/K(B)$ of Galois group G.

The following notions are classical when the extension $F/K(B)$ is finite and extend naturally to infinite extensions by writing $F/K(B)$ as the union of an increasing sequence of finite Galois extensions.

Given a point $t_0 \in B(K)$, we denote by F_{t_0}/K the *specialization of* $F/K(B)$ *at* t_0: if $\mathrm{Spec}(A) \subset B$ is some affine neighborhood of t_0, A_{t_0} the localized ring of A at the maximal ideal $\mathfrak{p} \subset A$ corresponding to t_0, $(A_{t_0})'_F$ the integral closure of A_{t_0} in F, then F_{t_0}/K is the residue extension of the integral extension $(A_{t_0})'_F/A_{t_0}$ at some prime ideal above $\mathfrak{p}$. In particular, it is a normal extension.

Definition 2.3. Given an overfield k of K and $t_0 \in B(k)$, the extension $(Fk)_{t_0}/k$ is called a *k-specialization of* $F/K(B)$ (notice that $Fk/k(B)$ is well-defined as $F/K(B)$ is Galois).

If k_v is a local field over K, points $t_0 \in B(k_v)$ are called *local points* of B, the associated k_v-specializations $(Fk_v)_{t_0}/k_v$ *local specializations* and the degrees $[(Fk_v)_{t_0} : k_v]$ *local specialization degrees* of $F/K(B)$ (these are to be understood as supernatural numbers [19, §22.8] if $F/K(B)$ is infinite).

[3] In particular the function field extension $K(B)/K$ is regular (as defined in Definition 3.1).

Denote the branch locus of $F/K(B)$ by D, *i.e.*, the formal sum of all hypersurfaces of $B \times_K K^{\text{sep}}$ such that the associated discrete valuations are ramified in the field extension $FK^{\text{sep}}/K^{\text{sep}}(B)$. If the extension $F/K(B)$ is finite, D is an effective divisor; in general D is an inductive limit of effective divisors.

Definition 2.4. Given a local field k_v over K and a local point $t_0 \in B(k_v) \setminus D$, the extension $(Fk)_{t_0}/k$ is Galois and the Galois group $\text{Gal}((Fk_v)_{t_0}/k_v)$ is called the *Frobenius subgroup* of $F/K(B)$ at t_0 over k_v. The local point $t_0 \in B(k_v) \setminus D$ is said to be k_v*-unramified* for the extension $F/K(B)$ if the associated k_v-specialization $(Fk_v)_{t_0}/k_v$ is unramified.[4]

The Frobenius subgroup is a subgroup of G whose order is the local specialization degree $[(Fk_v)_{t_0} : k_v]$. We use the phrase *unramified local specialization degree* for this degree when t_0 is k_v-unramified for $F/K(B)$.

2.2. The Tchebotarev existence property

2.2.1. Finite extensions

Definition 2.5. Let K be a field, given with a localization set $\mathcal{M}$, and let $F/K(B)$ be a finite Galois extension of group G.

$F/K(B)$ is said to have the *Tchebotarev existence property with respect to* $\mathcal{M}$ if for every element $g \in G$, there exists an $\mathcal{M}$-local field k_v over K and a local point $t_0 \in B(k_v) \setminus D$, k_v-unramified for $F/K(B)$, such that the Frobenius subgroup of $F/K(B)$ at t_0 over k_v is cyclic and conjugate to the subgroup $\langle g \rangle \subset G$.

We say further that $F/K(B)$ has the *strict Tchebotarev existence property* if in addition to the above, the $\mathcal{M}$-local fields k_v can be taken to be completions K_v of K (*i.e.*, no finite extension is necessary).

Remark 2.6. If K is a number field or if $K = \kappa(x)$ with G_κ pro-cyclic, the Frobenius subgroups of $F/K(B)$ at local points $t_0 \in B(k_v) \setminus D$, k_v-unramified for $F/K(B)$, are automatically cyclic as quotients of the pro-cyclic group $\text{Gal}(k_v^{\text{ur}}/k_v)$ (with k_v^{ur} the unramified closure of k_v).

Definition 2.5 is modeled upon the situation of number field extensions F/K. It is in fact a generalization: take $B = \text{Spec}(K)$; for every finite place of K, there is only one point in $B(K_v) = \text{Spec}(K_v)$ and the corresponding local specialization of F/K is the v-completion of F/K. From the classical Tchebotarev density theorem, Galois extensions of number fields indeed have the strict Tchebotarev existence property. As pointed out by M. Jarden, the weaker density property proved by Frobenius (*e.g.* [24, p. 134]), where is given a cyclic subgroup instead of a specific element of the Galois group, is sufficient to prove our property for number field extensions.

[4] When v is the trivial valuation, this condition is vacuous as all finite extensions of k_v are unramified.

In this paper we will be more interested in function field extensions $F/K(B)$ with $\dim(B) > 0$. Concrete situations where the Tchebotarev existence property is satisfied are given in Section 3.

2.2.2. Infinite extensions

Definition 2.5 extends in a natural way to infinite extensions.

Definition 2.7. A Galois extension $F/K(B)$ (possibly infinite) is said to have the *Tchebotarev existence property* (w.r.t. a localization set $\mathcal{M}$ of K) if $F/K(B)$ is the union of an increasing sequence of finite Galois extensions $F_n/K(B)$ that all have the Tchebotarev existence property (w.r.t. $\mathcal{M}$); and similarly for the *strict Tchebotarev existence property.*

This definition does not depend on the choice of the increasing sequence $\{F_n\}_{n\geq 1}$ such that $\bigcup_{n\geq 1} F_n = F$. This follows from the fact (left as an exercise) that given two finite Galois extensions $E/K(B)$ and $E'/K(B)$ such that $E' \supset E$, if $E'/K(B)$ has the Tchebotarev existence property (strict or not), then so does $E/K(B)$.

3. Situations with the Tchebotarev property

This section is devoted to state and prove our Tchebotarev theorems for function fields. In Section 3.1 we state the results, which are proved in Section 3.2. More specific examples where the Tchebotarev property holds are given in Section 3.1.2 and Section 3.3.

Unless otherwise specified, we assume $\dim(B) > 0$ in this section. In this function field context, we will mostly consider extensions $F/K(B)$ satisfying the following standard regularity condition.

Definition 3.1. A separable field extension F/K is said to be *regular* if $F \cap \overline{K} = K$. A separable extension $F/K(B)$ is said to be *K-regular* if F/K is regular.[5]

3.1. Main results

Theorem 3.2 below is a central result of this paper: it provides various situations where the Tchebotarev existence property is satisfied. Theorem 1.1 follows as an easy corollary from it (see Remark 3.4 below). The proof of Theorem 3.2 is given in Section 3.2.

Theorem 3.2. *Let K be a field given with a localization set $\mathcal{M}$. A finite K-regular Galois extension $F/K(B)$ has the Tchebotarev existence property in each of the three following situations:*

[5] Note that the assumption that the extension $K(B)/K$ is separable (Section 2.1.1) guarantees that F/K is separable if $F/K(B)$ is.

(a) *K is a field that is PAC and has cyclic extensions of any degree (with $\mathcal{M} = \{0\}$),*
(b) *K is a finite field (with $\mathcal{M} = \{0\}$),*
(c) *there exists a non trivial discrete valuation $v \in \mathcal{M}$ that is good for the extension $F/K(B)$ and such that the residue field κ_v is finite, or is PAC, perfect and has cyclic extensions of any degree.*

Remark 3.3. The precise definition of *good place* for $F/K(B)$ that appears in (c) requires some additional notation and we prefer to postpone it below in Section 3.1.1; it essentially corresponds to a set of conditions which classically guarantee *good reduction* of $F/K(B)$.

Remark 3.4. If $B = \mathbb{P}^1$, condition (c), and consequently the Tchebotarev property, clearly hold in each of the concrete situations of Theorem 1.1 of the introduction (see also Corollary 3.7 for some higher dimensional generalization).

In this situation and in cases (a), (b), which only depend on the field K and not on the extension $F/K(B)$, the conclusion that $F/K(B)$ has the Tchebotarev existence property also holds if $F/K(B)$ is infinite.

On the other hand, there are examples for which the Tchebotarev existence property does not hold in general: for instance if K is algebraically closed or if $K = \mathbb{R}$, as then, for any Galois extension $F/K(B)$, all specializations are of degree 1 or 2.

3.1.1. Good places

We now give the definition of *good places* appearing in the statement of Theorem 3.2.

Let $F/K(B)$ be a finite K-regular extension with branch locus D. This corresponds, through the function field functor, to a K-cover $f : X \to B$ which is étale above $B \setminus D$, from the Purity of the Branch Locus (see *e.g.* [8] for more on covers and the correspondence).

Given a local field k_v over K, denote by A_v the valuation ring, by $\mathfrak{p}_v$ the valuation ideal and by κ_v the residue field, which is assumed to be perfect and of characteristic p_v.

If $\mathcal{B}_v$ is an integral smooth projective model of B over A_v, denote by $\mathcal{F}_v : \mathcal{X}_v \to \mathcal{B}_v$ the morphism corresponding to the normalization of $\mathcal{B}_v$ in $k_v(X)$, by $\mathcal{F}_{v,0} : \mathcal{X}_{v,0} \to \mathcal{B}_{v,0}$ its special fiber and by $\mathcal{D}_v$ the Zariski closure of D in $\mathcal{B}_v$.

Also recall that f is said to have *no vertical ramification at v* if $\mathcal{F}_v : \mathcal{X}_v \to \mathcal{B}_v$ is unramified above $\mathfrak{p}_v$ viewed as a prime divisor of $\mathcal{B}_v$. We can now give the following definition:

Definition 3.5. A place v of K is said to be *good* for $F/K(B)$ if

(i) B has an integral smooth projective model $\mathcal{B}_v$ over A_v,
(ii) $p_v = 0$ or p_v does not divide the order of G,
(iii) each irreducible component of $\mathcal{D}_v$ is smooth over A_v and $\mathcal{D}_v \cup \mathcal{B}_{v,0}$ is a sum of irreducible regular divisors with normal crossings over A_v,
(iv) there is no vertical ramification at v in the cover f.

The k_v-cover $f_v := f \times_K k_v : X_v \to B_v$ then has *good reduction at* v: specifically, the special fiber $\mathcal{F}_{v,0} : \mathcal{X}_{v,0} \to \mathcal{B}_{v,0}$ is a cover over the residue field κ_v with group G and branch divisor $\mathcal{D}_{v,0}$; this follows from classical results of Grothendieck as explained in [12, §§2.4.1–2.4.4].

Example 3.6. In the typical situation where $k_v = \mathbb{Q}_p$ and $\mathcal{B} = \mathbb{P}^1_{\mathbb{Z}_p}$, condition (iii) amounts to the branch divisor $\mathbf{t}$ being étale at p, and more specifically to no two branch points $t_i, t_j \in \overline{\mathbb{Q}}_p \cup \{\infty\}$ *coalescing* at v. Here coalescing at v means that $|t_i|_{\bar{v}} \leq 1$, $|t_j|_{\bar{v}} \leq 1$ and $|t_i - t_j|_{\bar{v}} < 1$, or else $|t_i|_{\bar{v}} \geq 1$, $|t_j|_{\bar{v}} \geq 1$ and $|t_i^{-1} - t_j^{-1}|_{\bar{v}} < 1$, where $\bar{v}$ is any prolongation of v to $\overline{\mathbb{Q}}_p$. As to the non-vertical ramification in condition (iv), a practical test is this: if an affine equation $P(t, y) = 0$ of X is given with t corresponding to f and $P \in \mathbb{Z}_p[t, y]$ monic in y, there is no vertical ramification if the discriminant $\Delta(t)$ of P with respect to y is non-zero modulo p.

3.1.2. Some concrete examples in situations (a)–(c) of Theorem 3.2

Situation (a) Recall that a field k is said to be PAC if every non-empty geometrically irreducible k-variety has a Zariski-dense set of k-rational points. Classical results show that, in some sense, PAC fields are "abundant" [19, Theorem 18.6.1]; a concrete example is the field $\mathbb{Q}^{\mathrm{tr}}(\sqrt{-1})$, where $\mathbb{Q}^{\mathrm{tr}}$ is the field of totally real numbers (algebraic numbers such that all conjugates are real).

There are many fields as in situation (a) of Theorem 3.2. We list some examples:

- It is a classical result [19, Corollary 23.1.3] that for every projective profinite group $\mathcal{G}$, there exists a PAC field K such that $\mathrm{G}_K \simeq \mathcal{G}$.
  For $\mathcal{G}$ chosen so that $\widehat{\mathbb{Z}}$ is a quotient, the field K satisfies condition (a) of Theorem 3.2. Any non-principal ultraproduct of distinct finite fields is a specific example of a perfect PAC field with absolute Galois group isomorphic to $\widehat{\mathbb{Z}}$ [19, Proposition 7.9.1].
- Examples of subfields of $\overline{\mathbb{Q}}$ can be given. The PAC field $\mathbb{Q}^{\mathrm{tr}}(\sqrt{-1})$ is one: indeed it is known to be hilbertian and, consequently (see Proposition 4.4), its absolute Galois group is a free profinite group of countable rank. It has been also proved that for every integer $e \geq 1$, for almost all $\sigma = (\sigma_1, \ldots, \sigma_e) \in \mathrm{G}_{\mathbb{Q}}^e$, the fixed field $\overline{\mathbb{Q}}^\sigma$ of σ in $\overline{\mathbb{Q}}$ is PAC and $\mathrm{G}_{\overline{\mathbb{Q}}^\sigma}$ is isomorphic to the free profinite group $\widehat{F}_e$ of rank e [19, Theorems 18.5.6 & 18.6.1]; here "almost all" is to be understood as "off a subset of measure 0" for the Haar measure on $\mathrm{G}_{\mathbb{Q}}^e$. We note that for such fields $\overline{\mathbb{Q}}^\sigma$, a related Tchebotarev property already appeared in [25].

Situation (b) The situation "K finite" is rather classical. There even exist quantitative forms of the property, similar to the Tchebotarev density property for number fields; see [34,31,20,17], [19, §6]. As shown in [11, §3.5] and [15, §4.2], our approach also leads to the quantitative forms. We focus here on the existence part which also applies to infinite fields.

Situation (c) The following statement provides examples. By the phrase used in (c1) and (c2) that *the branch locus D is good* (over K), we mean that it is a sum of irreducible smooth divisors with normal crossings over K. This is automatic if B is a curve, or if, as in (c3) and (c4), K has a place that is good for the extension $F/K(B)$.

Corollary 3.7. *A finite K-regular Galois extension $F/K(B)$ has the Tchebotarev existence property in each of the following situations:*

(c1) *K is a number field and the branch locus D is good,*
(c2) *$K = \kappa(x)$, $\mathrm{char}(\kappa) = p \nmid |G|$, the branch locus D is good, and κ is either a finite field, or a perfect, PAC field having cyclic extensions of any degree,*
(c3) *$K = K_v$ is the completion of a number field at some finite place v that is good for $F/K(B)$,*
(c4) *$K = \kappa((x))$, the x-adic valuation is good for $F/K(B)$ and κ is either a finite field, or a perfect, PAC field having cyclic extensions of any degree.*

Proof. (c3) and (c4) are obvious special cases of Theorem 3.2 (c). This is true too for (c1) and (c2): the main point is that in these cases, the localization set $\mathcal{M}$ contains infinitely many places and that only finitely many can be bad, which is clear from Definition 3.5 (under the assumption that the branch locus D is good over K). □

3.1.3. The strict variant

Addendum 3.8 *(to Theorem 3.2). The strict Tchebotarev existence property is satisfied in case (a) of Theorem 3.2. It also holds in case* (c1) *of Corollary 3.7 and in cases* (c2), (c4) *when κ is PAC having cyclic extensions of any degree.*

Finite fields are typical examples over which the non-strict variant holds (Theorem 3.2) but the strict variant does not: for example if p is an odd prime, the extension $F/\mathbb{F}_p(T)$ given by the polynomial $Y^2 - Y - (T^p - T)$ has trivial specializations at all points $t_0 \in \mathbb{F}_p$ and so at all unbranched points $t_0 \in \mathbb{P}^1(\mathbb{F}_p)$ (∞ *is* a branch point). A similar argument (given in Section 4.2) shows that over $\mathbb{Q}_p$ the strict variant does not hold either. However we do not know whether the non-strict variant holds over $\mathbb{Q}_p$, *i.e.* if the condition "v good" can be removed in Corollary 3.7 (c3).

3.2. Proof of Theorem 3.2 and of its Addendum 3.8

A central ingredient is [12]: we notably use two statements called there *twisting lemma* and *local specialization result*. Both are answers to the question as to whether a Galois extension E/k is a specialization of a K-regular Galois extension $F/K(B)$.

Fix a finite K-regular Galois extension $F/K(B)$ with group G and branch locus D. We use the cover viewpoint and the notation introduced in Section 3.1.1: $f : X \to B$ is the K-cover corresponding to $F/K(B)$ and we set $f_v = f \times_K k_v$ for every local field k_v over K.

Let $g \in G$. The strategy is to construct an $\mathcal{M}$-local field k_v over K such that

(i) there exists an unramified Galois extension E/k_v with Galois group isomorphic to the subgroup $\langle g\rangle \subset G$, and
(ii) the extension E/k_v is a specialization of the extension $F k_v/k_v(B)$ at some point $t_0 \in B(k_v) \setminus D$.

We will conclude that the group $\mathrm{Gal}((F k_v)_{t_0}/k_v)$, *i.e.*, the Frobenius subgroup of $F/K(B)$ at t_0 over k_v, is cyclic and conjugate to $\langle g\rangle$ in G.

To achieve (ii) we use the *twisting lemma* from [12], which says the following. Let $\varphi : \mathrm{G}_{k_v} \to \langle g\rangle$ be an epimorphism such that the fixed field $(k_v^{\mathrm{sep}})^{ker(\varphi)}$ is an extension E of k_v as in (i). Then there is a regular k_v-cover[6] $\widetilde{f}_v^{\varphi} : \widetilde{X}_v^{\varphi} \to B_v$ (with $B_v = B \times_K k_v$) such that

(*) *condition* (ii) *holds if and only if there exists a k_v-rational point on $\widetilde{X}_v^{\varphi}$ not lying above any point in the branch locus D.*

The cover $\widetilde{f}_v^{\varphi} : \widetilde{X}_v^{\varphi} \to B_v$ is obtained by “twisting” $Fk_v/k_v(B)$, viewed as a regular Galois k_v-cover $f_v : X_v \to B_v$, by the epimorphism φ, whence the terminology and the notation.

Proof of (a). This follows at once. Take for v the trivial valuation on K (for which $K_v = K$). From the assumption an extension E/K as in (i) exists, and by definition of PAC fields, the set $\widetilde{X}_v^{\varphi}(K)$ is Zariski-dense, and so (ii) holds as well.[7] Furthermore it is the strict Tchebotarev existence property (and so Addendum 3.8, case (a)) that has been proved.

Remark 3.9. The non-strict Tchebotarev existence property holds under a weaker condition: the argument above shows that it is sufficient that every cyclic subgroup C be the Galois group of some finite extension E_C/k_C with k_C a finite extension of K.

Proof of (b). This goes along similar principles but with the Lang–Weil estimates replacing the PAC property. More precisely assume that K is the field $\mathbb{F}_{q_0}$ with q_0 elements. Pick a suitably large integer m; more specifically $q = q_0^m$ should be bigger than the constant c from [11, Corollary 3.5], which depends only on G, B and D. Then from that result, if d is the order of g, the extension $\mathbb{F}_{q^d}/\mathbb{F}_q$ is the specialization of $F\,\mathbb{F}_q/\mathbb{F}_q(B)$ at some point $t_0 \in B(\mathbb{F}_q)\setminus D$. So the extension $F\,\mathbb{F}_q/\mathbb{F}_q(B)$ satisfies conditions (i) and (ii)

[6] As defined in [15, §2.2], a cover $X \to B$ defined over some field k is said to be a *regular k-cover* if the function field extension $k(X)/k(B)$ is k-regular (in the sense of Definition 3.1).
[7] For PAC fields, stronger results can be proved for which $\langle g\rangle$ can be replaced by any subgroup of G; see [11, Corollary 3.4].

above for v the trivial place on $K = \mathbb{F}_{q_0}$ and $k_v = \mathbb{F}_q$. We note that we have used a scalar extension (from $\mathbb{F}_{q_0}$ to $\mathbb{F}_q$) and only proved the (non-strict) Tchebotarev property.

Proof of (c). The proof of this last part relies on Proposition 2.2 from [12], which we apply to the k_v-cover $f_v \times_{K_v} k_v$ and to the unramified homomorphism $\varphi : \mathrm{G}_{k_v} \to \langle g \rangle \subset G$ defined as follows. If the residue field κ_v is PAC, take $k_v = K_v$ and if it is a finite field $\mathbb{F}_{q_0}$ with q_0 elements, take k_v equal to the unique unramified extension of K_v with residue extension $\mathbb{F}_{q_0^m}/\mathbb{F}_{q_0}$ with $q = q_0^m$ bigger than the constant c from [12, Lemma 2.4] (which is some version of the constant c used above). In both cases, denote the residue field of k_v by $\tilde{\kappa}_v$. From the hypotheses, the field $\tilde{\kappa}_v$ has a Galois extension $\varepsilon_v/\tilde{\kappa}_v$ of group $\langle g \rangle$. Let E_v/k_v be the unique unramified extension with residue extension $\varepsilon_v/\tilde{\kappa}_v$ and $\varphi : \mathrm{G}_{k_v} \to \langle g \rangle$ be an epimorphism such that the fixed field $(k_v^{\mathrm{sep}})^{ker(\varphi)}$ is E_v.

Proposition 2.2 from [12] has two assumptions which are labeled (good-red) and (κ-big-enough). The former is here covered by the assumption that v is good for $F/K(B)$. The latter holds as well: this follows from the PAC property if κ_v is PAC, and from [12, Lemma 2.4] if κ_v is finite of order $> c$. Conclude then from [12, Proposition 2.2] that there exists $t_0 \in B(k_v) \setminus D$ such that the specialization $(F\, k_v)_{t_0}/k_v$ is conjugate to E_v/k_v. In particular $\mathrm{Gal}((F\, k_v)_{t_0}/k_v)$ is cyclic and conjugate to $\langle g \rangle$ in G. Furthermore we have proved the strict Tchebotarev property in the case of a PAC residue field (and so Addendum 3.8, cases (c2) and (c4)) but only the non-strict Tchebotarev property in the case of a finite residue field.

It remains to show case (c1) of Addendum 3.8. That is, to prove the strict Tchebotarev property assuming that K is a number field and the branch locus D is good. Denote by $\mathcal{B}$ an integral projective model of B over the ring R of integers of K; $\mathcal{B}$ is smooth over the completion R_v for all finite places of K but in a finite subset S_0. Pick a place v of K that is good (in particular $v \notin S_0$) and has a residue field κ_v of order bigger than the constant $C(f, \mathcal{B})$ from [12, Lemma 3.1]. As above, assumptions (good-red) and (κ-big-enough) from [12, Proposition 2.2] are guaranteed and it can be concluded that there exists $t_0 \in B(k_v) \setminus D$ such that the specialization $(F\, K_v)_{t_0}/K_v$ is conjugate to the unique unramified extension E_v/K_v of degree the order of g. □

3.3. A further example

We illustrate our method with a last situation where the residue fields are neither PAC nor finite. A typical example we have in mind in the statement below is this: K is the field $k_0((\theta))(x)$ with x and θ two indeterminates and the localization set consists of all $(x - x_0)$-adic valuations with $x_0 \in \mathbb{P}^1(k_0((\theta)))$.

Theorem 3.10. *Assume K is given with a localization set $\mathcal{M}$ that contains a non trivial discrete valuation $v \in \mathcal{M}$ such that*

(a) *the residue field κ_v is a complete field for a non trivial discrete valuation w with a residue field $\kappa_{v,w}$ that is perfect, PAC and has cyclic extensions of any degree.*

Then a finite K-regular Galois extension $F/K(B)$ has the strict Tchebotarev existence property if $G = \mathrm{Gal}(F/K(B))$ has trivial center and B has an integral smooth projective A_v-model $\mathcal{B}_v$ such that

(b) *v is good for this model of $F/K(B)$,*
(c) *the place w is good for the extension $\kappa_v(\mathcal{X}_{v,0})/\kappa_v(\mathcal{B}_{v,0})$ (i.e., the function field extension of the special fiber of $\mathcal{F}_v : \mathcal{X}_v \to \mathcal{B}_v$).*

For $K = k_0((\theta))(x)$, condition (a) holds if k_0 is a perfect PAC field with cyclic extensions of any degree. For all but finitely many $x_0 \in \mathbb{P}^1(k_0((\theta)))$, the $(x - x_0)$-adic valuation v_{x_0} is good for $F/K(B)$, *i.e.* condition (b) holds. The special fiber is a $k_0((\theta))$-cover and condition (c) requires that the θ-adic valuation on $k_0((\theta))$ be good for it.

Proof. Fix $g \in G$. The proof follows the same strategy as in Section 3.2 and uses again [12, Proposition 2.2], applied here to the K_v-cover $f_v = f \times_K K_v$ and the unramified homomorphism $\varphi : \mathrm{G}_{K_v} \to \langle g \rangle \subset G$ defined as follows. From assumption (a), there exists a Galois extension of $\kappa_{v,w}$ of group isomorphic to $\langle g \rangle$. This extension lifts to an unramified (w.r.t. w) extension of κ_v with the same group, which in turn lifts to an unramified (w.r.t. v) extension E_v/K_v with the same group $\langle g \rangle$. Let $\varphi : \mathrm{G}_{K_v} \to \langle g \rangle \subset G$ be an associated representation of G_{K_v}, *i.e.*, the fixed field of $\ker(\varphi)$ in $\overline{K_v}$ is E_v.

The K_v-cover f_v satisfies condition (good-red) from [12, Proposition 2.2]; this is guaranteed by assumption (b).

To check condition (κ-big-enough) from [12, Proposition 2.2], we give ourselves what is called an *A_v-model of* $(f_v \times_{K_v} K_v^{\mathrm{sep}}, \mathcal{F}_{v,0} \times_{\kappa_v} \overline{\kappa_v})$ in [12], *i.e.*, a finite and flat morphism $\mathcal{F}' : \mathcal{X}' \to \mathcal{B}_v$ with $\mathcal{X}'$ normal and such that $\mathcal{F}' \times_{A_v} K_v$ is a K_v-cover that is K_v^{sep}-isomorphic to $f_v \times_{K_v} K_v^{\mathrm{sep}}$ and the special fiber $\mathcal{F}'_0 : \mathcal{X}'_0 \to \mathcal{B}_{v,0}$ is a κ_v-cover that is $\overline{\kappa_v}$-isomorphic to $\mathcal{F}_{v,0} \times_{\kappa_v} \overline{\kappa_v}$. And we have to find κ_v-rational points on $\mathcal{X}'_0$ not lying above any point in $\mathcal{D}_0 \times_{\kappa_v} \overline{\kappa_v}$.

Denote the valuation ring of w by $A_{v,w}$. From assumption (c), the κ_v-variety $\mathcal{B}_{v,0}$ has an integral smooth projective model $\widetilde{\mathcal{B}}_0$ over $A_{v,w}$, and w is good for this model of $\kappa_v(\mathcal{X}_{v,0})/\kappa_v(\mathcal{B}_{v,0})$. It follows that w is also good for $\kappa_v(\mathcal{X}'_0)/\kappa_v(\mathcal{B}_{v,0})$. Indeed conditions (a), (b), (c) from Definition 3.5 are equivalently satisfied by the place w for either one of the two extensions. As to condition (d), we resort to a result of S. Beckmann [2] that says that non-vertical ramification is automatic under (a), (b), (c) if G has trivial center. It follows that $\widetilde{\mathcal{F}}'_0$ has good reduction (at w). As $\kappa_{v,w}$ is PAC, there exist $\kappa_{v,w}$-rational points on the reduction (at w) of $\widetilde{\mathcal{X}}'_0$ that are not in the branch locus of the reduction (at w) of $\widetilde{\mathcal{F}}'_0$. Using Hensel's lemma, these points can be lifted to κ_v-rational points on $\mathcal{X}'_0$ as desired.

Proposition 2.1 from [12] can then be applied to conclude that the unramified extension E_v/K_v, cyclic of group $\langle g \rangle$, is a K_v-specialization of the extension $F/K(B)$. □

Remark 3.11. A non-strict variant of Theorem 3.10 can be proved if the residue field $\kappa_{v,w}$ is assumed to be finite instead of PAC. The modifications are similar to those in the proof of Theorem 3.2 (for (b) *vs.* (a)): the Lang–Weil estimates replace the PAC property, a finite extension of K_v is needed to insure that the finite residue field $\kappa_{v,w}$ is big enough, etc. We leave the reader adjust the proof.

4. Tchebotarev versus Hilbert

We now compare the Tchebotarev existence property and the Hilbert specialization property. We have the following definitions:

Definition 4.1. Recall that a finite extension $F/K(T)$ is said to have the *Hilbert specialization property* if it has infinitely many specializations F_{t_0}/K at points $t_0 \in \mathbb{P}^1(K)$ of degree equal to $[F : K(T)]$.

A field K is called *hilbertian* if the Hilbert specialization property holds for every finite extension $F/K(T)$ and *RG-hilbertian* if it holds for every finite K-regular Galois extension $F/K(T)$.

We say that a field K, given with a localization set $\mathcal{M}$, is *Tchebotarev* (resp. *strict Tchebotarev*) if every finite K-regular Galois extension $F/K(T)$ has the Tchebotarev (resp. the strict Tchebotarev) existence property.

Example 4.2. From Section 3, PAC fields and number fields are strict Tchebotarev, finite fields are Tchebotarev, but not strict Tchebotarev.

4.1. The PAC situation and the proof of Corollary 1.2

The case where K is a PAC field gives a first idea of these notions hierarchy. Recall the following definition from [19, §27.1] that is used in statement (a) below:

Definition 4.3. A field K is ω-free if every embedding problem for G_K is solvable.

From a theorem of Iwasawa, if G_K is of at most countable rank, K is ω-free if and only if G_K is isomorphic to the free profinite group $\hat{F}_\omega$ with countably many generators [19, Theorem 24.8.1].

The following result shows that, for a field K, the properties of being Tchebotarev and its strict variant are equivalent to well-identified properties of the group G_K, proving in particular Corollary 1.2 from the Introduction. Notice that, as already remarked, conclusions (a) and (b) below are classical; see [19, Corollary 27.3.3] for the *if* part in (a), [22, Theorem A] for the *only if* part, and [22, Theorem B] for (b).[8] We have included them in the statement to put the new conclusions (c) and (d) in perspective.

[8] [22] assumes K of characteristic 0 and countable, but these hypotheses have been removed in subsequent works; see [28] for (a) and [9, §3.3] for (b).

Proposition 4.4. *Let K be a PAC field given with the trivial localization set $\mathcal{M} = \{0\}$.*

(a) *K is hilbertian iff K is ω-free.*
(b) *K is RG-hilbertian iff every finite group is a quotient of G_K.*
(c) *K is strict Tchebotarev iff every cyclic group is a quotient of G_K.*
(d) *K is Tchebotarev iff every cyclic group C is a quotient of some open subgroup U_C of G_K.*

In particular we have this chain of implications:

$$\text{hilbertian} \Rightarrow \text{RG-hilbertian} \Rightarrow \text{strict Tchebotarev} \Rightarrow \text{Tchebotarev}$$

Furthermore none of the reverse implications holds.

Proof. The *if* part in (c) is Theorem 3.2 (a). For the *only if* part, let G be a cyclic group. Classically every cyclic group G is the group of some K-regular Galois extension $F/K(T)$. If K is strict Tchebotarev, then a specialization F_{t_0}/K of group G does exist. Similar arguments lead to the non-strict variant (d) of (c) (use Remark 3.9 for the *if* part).

Using the classical result [19, Corollary 23.1.3] recalled in Section 3.1.2, the search of counter-examples to the reverse implications can be reduced to that of projective profinite groups $\mathcal{G}$ with appropriate properties. For a counter-example to "strict Tchebotarev $\Rightarrow$ RG-hilbertian", take $\mathcal{G} = \widehat{\mathbb{Z}}$ and a PAC field K such that $\mathrm{G}_K \simeq \mathcal{G}$. From statements (b) and (c), K is strict Tchebotarev but is not RG-hilbertian. For a counter-example to "RG-hilbertian $\Rightarrow$ hilbertian", see [22, §2]. Finally for the implication "Tchebotarev $\Rightarrow$ strict Tchebotarev", we have the following counter-example, provided to us by Bary-Soroker.

Take for $\mathcal{G}$ the universal Frattini cover [19, §22.6] of the group $\prod_{n\geq 5} A_n$ and a PAC field K such that $\mathrm{G}_K \simeq \mathcal{G}$. From [19, Lemma 22.6.3], if a cyclic subgroup C is a quotient of $\mathcal{G}$, then C is a Frattini cover of a quotient D of $\prod_{n\geq 5} A_n$. But then from [19, Lemma 25.5.3], D is a direct product of alternating groups A_n: a contradiction if C is non-trivial. Conclude *via* statement (d) that K is not strict Tchebotarev. Now as we explain below, K is Tchebotarev. For every integer $m \geq 1$, the alternating group A_{2m} is a quotient of $\mathcal{G}$. Denote by K_{2m}/K the corresponding Galois extension, of group A_{2m}. If $\sigma_m \in A_{2m}$ is the product of two m-cycles and k the fixed field of σ_m in K_{2m}, then k/K is finite and K_{2m}/k is Galois of group $\langle\sigma_m\rangle$. As m is arbitrary, this indeed shows that every cyclic subgroup is a quotient of some open subgroup of G_K and so *via* (d) that K is Tchebotarev. □

4.2. The general situation

Over non PAC fields K and for not necessarily trivial localization sets $\mathcal{M}$ the picture is more complex. We explain in this subsection what remains in general of the first four equivalences of Proposition 4.4 and in the next one how some implications between the various properties can still be obtained.

Assume K is an arbitrary field. Then:

- Implication ($\Rightarrow$) in Proposition 4.4 (a) does not hold: for example, $\mathbb{Q}$ is hilbertian but not ω-free.[9] Implication ($\Rightarrow$) in Proposition 4.4 (c) and (d) still holds: the argument is the same as for PAC fields (and this argument also shows that implication ($\Rightarrow$) in Proposition 4.4 (b) also holds if every finite group is the Galois group of some K-regular Galois extension; see also [9, §3.3.2]).
- None of the converses hold in general. For (a), see [3, Remark 2.14]. For (b) and (c), take a prime p and consider the field $\mathbb{Q}^{\mathrm{t}p}$ of all totally p-adic algebraic numbers. It is known that every finite group is a quotient of $\mathrm{G}_{\mathbb{Q}^{\mathrm{t}p}}$ [16]. But if $F/\mathbb{Q}^{\mathrm{t}p}(T)$ is the extension given by the polynomial $P(T,Y) = Y^2 - Y - (pT/T^2 - p)$, then for every $t_0 \in \mathbb{P}^1(\mathbb{Q}^{\mathrm{t}p})$, the polynomial $P(t_0,Y)$ is split in $\mathbb{Q}^{\mathrm{t}p}[Y]$ [14, Example 5.2]. Therefore $F/\mathbb{Q}^{\mathrm{t}p}(T)$ has no $\mathbb{Q}^{\mathrm{t}p}$-specialization with Galois group $\mathbb{Z}/2\mathbb{Z}$ and so $\mathbb{Q}^{\mathrm{t}p}$ is not strict Tchebotarev. This example also shows that $\mathbb{Q}_p$ is not strict Tchebotarev and so yields another counter-example to the converse in (c). One may think that $\mathbb{Q}_p$ is not even Tchebotarev; it would then also be a counter-example to ($\Leftarrow$) in (d).

4.3. Strong Tchebotarev versus Hilbert

Proposition 4.6 below shows that the Hilbert property is squeezed between a strong and a weak variant of the Tchebotarev property.

Definition 4.5. If K is given with a localization set $\mathcal{M}$, a finite Galois extension $F/K(B)$ is said to have the *strong Tchebotarev existence property with respect to* $\mathcal{M}$ if for every element $g \in G$, there exist *infinitely many places* $v \in \mathcal{M}$ with corresponding points $t_v \in B(K_v) \setminus D$ k_v-unramified for $F/K(B)$ and such that the Frobenius subgroup of $F/K(B)$ at t_0 over K_v is cyclic and conjugate to the subgroup $\langle g \rangle \subset G$.

We also say that K is *strong Tchebotarev* if every finite K-regular Galois extension $F/K(T)$ has the strong Tchebotarev existence property.

Proposition 4.6. *Let $F/K(T)$ be a finite K-regular Galois extension.*

(a) *If $F/K(T)$ has the strong Tchebotarev existence property w.r.t. a localization set $\mathcal{M}$ of K, then it has the Hilbert specialization property. In particular, if K is strong Tchebotarev, then it is RG-hilbertian.*

[9] However it is conjectured that "K hilbertian" implies that every *split* finite embedding problem over K has a solution [7] (which itself implies K ω-free if in addition G_K is projective and countable).

(b) *If K is a countable hilbertian field then $F/K(T)$ has the Tchebotarev existence property w.r.t. the trivial localization set $\mathcal{M} = \{0\}$. In particular, K is Tchebotarev w.r.t. $\mathcal{M} = \{0\}$.*

Proof. (a) Definition 4.5 makes it possible to construct a family of places $(v_g)_{g\in G}$, pairwise distinct and with the property that for each $g \in G$, there exists $t_{v_g} \in \mathbb{P}^1(K_{v_g}) \setminus D$ k_v-unramified for $F/K(T)$ and such $\mathrm{Gal}((FK_{v_g})_{t_{v_g}}/K_{v_g})$ is conjugate to $\langle g\rangle$. For each $g \in G$, the set of such points t_{v_g} is a v_g-adic subset of $\mathbb{P}^1(K_{v_g}) \setminus D$; this follows from the twisting lemma recalled in Section 3.2. Using the approximation Artin–Whaples theorem, the collection of points $(t_{v_g})_{g\in G}$ can be approximated by some point $t_0 \in \mathbb{P}^1(K) \setminus D$ such that $\mathrm{Gal}(F_{t_0}K_{v_g}/K_{v_g})$ is conjugate to $\langle g\rangle$ for each $g \in G$. As $\mathrm{Gal}(F_{t_0}K_{v_g}/K_{v_g})$ is a subgroup of $\mathrm{Gal}(F_{t_0}/K)$, we conclude that $\mathrm{Gal}(F_{t_0}/K)$ meets each conjugacy class of G. By a classical lemma of Jordan [26], $\mathrm{Gal}(F_{t_0}/K)$ is all of G.

(b) The following proof is due to L. Bary-Soroker. From [19, Theorem 18.10.2], the countable hilbertian field K can be embedded in some field E, Galois over K, PAC and ω-free. From Proposition 4.4, E is hilbertian, and consequently is strict Tchebotarev w.r.t. $\mathcal{M} = \{0\}$. It readily follows that $F/K(T)$ has the Tchebotarev existence property (and that K is Tchebotarev w.r.t. $\mathcal{M} = \{0\}$). Indeed given any $g \in \mathrm{Gal}(F/K(T)) = \mathrm{Gal}(FE/E(T))$, there exists $t_0 \in \mathbb{P}^1(E)$ such that $\langle g\rangle = \mathrm{Gal}((FE)_{t_0}/E)$ and a standard argument shows that the same is true with E replaced by some finite extension k of K. □

The proof shows that Proposition 4.6 (a) still holds if $F/K(T)$ is replaced by an extension $F/K(B)$ with B satisfying the *weak approximation property* (and even *the weak weak approximation property* [32, Définition 3.5.6]).

5. Some questions on infinite extensions

In this section we study some local–global implications of the Tchebotarev existence property for infinite extensions and some related questions, originally investigated in the number field setting in [6]. Our main results are Theorem 5.14 and Theorem 5.17 and they base on some group-theoretic constructions, in particular with families of extraspecial and dihedral groups.

5.1. A local–global conclusion for infinite extensions

An interesting feature of extensions with the Tchebotarev existence property is that certain local behaviors actually detect informations on the Galois group of the extension and vice-versa. For instance, an immediate remark is the following:

Remark 5.1. For a finite Galois extension $F/K(B)$ of Galois group G with the Tchebotarev existence property, the orders of the elements of G are exactly the unramified $\mathcal{M}$-local specialization degrees of $F/K(B)$ corresponding to cyclic Frobenius subgroups.

In particular the exponent of G is the l.c.m. of these local specialization degrees. Proposition 5.5 below shows that the above conclusion extends in some form to infinite extensions. The following definitions will be used.

Definition 5.2. A localization set $\mathcal{M}$ of a field K is said to be *standard* if the local fields k_v are perfect and the absolute Galois groups G_{k_v} are of uniformly bounded rank ($v \in \mathcal{M}$).

Example 5.3. This holds in particular in the following situations: K is a number field, a p-adic field, a perfect field with absolute Galois group of finite rank (*e.g.* a finite field), a field $K = \kappa(x)$ or $K = \kappa((x))$ with κ of characteristic 0 and with absolute Galois group G_κ of finite rank, etc.

Definition 5.4. We say that a family $(d_v)_v$ of positive integers indexed by v is *uniformly bounded* if there is a constant δ depending on $F/K(B)$ but not on v such that all integers d_v are $\leq \delta$.

We can now state our result, which is an extension to function fields of the result proved in [6] and [5] for number field extensions.

Proposition 5.5. *Let $F/K(B)$ be a Galois extension (possibly infinite) with Galois group G and with the Tchebotarev existence property. Suppose that the $\mathcal{M}$-local specialization degrees of $F/K(B)$ are uniformly bounded. Then the exponent of G,* $\exp(G)$*, is finite.*

Furthermore the converse holds too if the localization set $\mathcal{M}$ is standard (independently of the Tchebotarev property).

Proof. Write $F/K(B)$ as an increasing union of finite Galois extensions $F_n/K(B)$ ($n \geq 1$). Let $g \in G$. For each $n \geq 1$, let g_n be the projection of g onto $\mathrm{Gal}(F_n/K(B))$. From Remark 5.1, for each $n \geq 1$, the order of g_n is the unramified local specialization degree $[(F_n k_v)_{t_0} : k_v]$ for some place $v \in \mathcal{M}$ and some point $t_0 \in B(k_v) \setminus D$. In particular this order divides the local specialization degree $[(Fk_v)_{t_0} : k_v]$.

This yields to the fact, (which compares to Remark 5.1 above) that the set of orders of elements of G is a subset of the set of all $\mathcal{M}$-local specialization degrees of $F/K(B)$, proving the first part of Proposition 5.5.

For the converse, we borrow an argument from [5]. Let k_v be an $\mathcal{M}$-local field over K and $t_0 \in B(k_v)$. Fix $n \geq 1$. Assume k_v is perfect. Then $(F_n k_v)_{t_0}/k_v$ is a finite Galois extension and the local specialization degree $[(F_n k_v)_{t_0} : k_v]$ is the order of the group $\mathrm{Gal}((F_n k_v)_{t_0}/k_v)$. Assume further that there is a constant N depending only of $F/K(B)$ such that G_{k_v} is of rank $\leq N$. Then the finite group $\mathrm{Gal}((F_n k_v)_{t_0}/k_v)$, a quotient of G_{k_v}, has a generating set with at most N elements. The group $\mathrm{Gal}((F_n k_v)_{t_0}/k_v)$ is also of exponent $\leq \exp(G)$ (as a subgroup of $\mathrm{Gal}(F_n/K(B))$ which itself is a quotient of G). If $\exp(G)$ is finite, it follows from the Restricted Burnside's Problem solved by Zelmanov (see *e.g.* [33]) that the order of the group $\mathrm{Gal}((F_n k_v)_{t_0}/k_v)$ can be bounded by a constant only depending on $\exp(G)$ and N. □

Remark 5.6. If $F/K(B)$ has the strict Tchebotarev property, a strict variant of the above result holds too: namely, if G has finite exponent, then the $\mathcal{M}$-local specialization degrees corresponding to completions of K are uniformly bounded. The proof above can easily be adjusted.

5.1.1. Some concrete examples

From Proposition 5.5, the uniform boundedness of the $\mathcal{M}$-local specialization degrees is equivalent the finiteness of $\exp(G)$ if the extension $F/K(B)$ has the Tchebotarev property and the localization set $\mathcal{M}$ is standard.

The aim of this subsection is to provide some examples in which these two last conditions hold. We shall need the following definition:

Definition 5.7. For an extension (possibly infinite) $F/K(B)$ we say that the branch locus D (an inductive limit of effective divisors) is *good* if every effective divisor with support in D is good.

We have the following result:

Corollary 5.8. *Fix a K-regular Galois extension $F/K(B)$ with* $\dim(B) > 0$. *Let* $G = \mathrm{Gal}(F/K(B))$. *Then the uniform boundedness of the $\mathcal{M}$-local specialization degrees of $F/K(B)$ is equivalent the finiteness of* $\exp(G)$ *in each of the following situations:*

(1) *K is a PAC perfect field such that G_K is of finite rank and has every cyclic group as a quotient,*
(2) *K is a finite field,*
(3) *K is a number field and the branch locus D is good,*
(4) *$K = \kappa(x)$ with κ a PAC field of characteristic* 0 *such that G_κ is of finite rank and has every cyclic group as a quotient, and the branch locus D is good,*
(5) *$K = K_v$ is the completion of a number field at some finite place v that is good for $F/K(B)$,*
(6) *$K = \kappa((x))$ if the x-adic valuation is good for $F/K(B)$ and for κ a PAC field of characteristic* 0 *such that G_κ is of finite rank and has every cyclic group as a quotient.*

Proof. In all the above situations, the Tchebotarev property holds and the localization set $\mathcal{M}$ is standard: the latter claim is classical and the former holds from Theorem 3.2 and Corollary 3.7. □

Remark 5.9. It is well-known that for κ of characteristic $p > 0$, $\mathrm{G}_{\kappa((x))}$ is not of finite type: for example, if κ is algebraically closed, the Galois group of $X^{p^n} - X - (1/x)$ over $\kappa((x))$ is $(\mathbb{Z}/p\mathbb{Z})^n$ ($n \geq 1$). That is why situations (c2) and (c4) with κ finite from Corollary 3.7 do not appear here and κ is of characteristic 0 in cases (4) and (6) of Corollary 5.8.

Remark 5.10. Let K be a finite field. Corollary 5.8 ensures that the uniform boundedness of the local specialization degrees of $F/K(B)$ is equivalent to the finiteness of $\exp(G)$. But is this equivalence true if, instead of the local specialization degrees, we consider the local degrees of $F/K(B)$ i.e. the degrees of the field extensions obtained by taking the completions of F and $K(B)$ w.r.t. some valuations?

This closer function field analog of the question raised in [6] and [5] was recently negatively answered by H. Bauchère in his PhD thesis [1] when $B = \mathbb{P}^1_K$. As in the number field case, the uniform boundedness of the local degrees of $F/K(B)$ clearly implies the finiteness of $\exp(G)$. Bauchère proves that, however, the converse is not true in general if $p = \mathrm{char}(K)$ divides $\exp(G)$, providing a family of counterexamples given by infinite abelian extensions of exponent p.

5.2. A refined question

Let again $F/K(B)$ be a Galois extension of group G. A special situation where the exponent of G is finite is when, for some integer $d \geq 1$, F is contained in the compositum $K(B)^{(d)}$ of all extensions of $K(B)$ of degree at most d. Indeed in this case $\mathrm{Gal}(F/K(B))$ is a quotient of the group $\mathrm{Gal}(K(B)^{(d)}/K(B))$, which is of exponent $\leq d!$.

The following question arises:

Question 5.11. *Let $F/K(B)$ be a Galois extension with the Tchebotarev existence property. Suppose the local specialization degrees of $F/K(B)$ to be uniformly bounded. Is it true that $F \subset K(B)^{(d)}$ for some d?*

If the group G is abelian, the answer is affirmative as shown in Corollary 5.12 below. For number field extensions this was first proved in [6].

In general however, Question 5.11 has a negative answer. Counter-examples were given in [6,5] in the number field context. Constructing other counter-examples with $\dim(B) > 0$ was among the motivations for this work.

Corollary 5.12. *Let $F/K(B)$ be an abelian Galois extension with the Tchebotarev existence property. Assume that the $\mathcal{M}$-local specialization degrees of $F/K(B)$ are uniformly bounded. Then $F \subset K(B)^{(d)}$ for some d.*

Proof. From Proposition 5.5, $\exp(G)$ is finite. As noted in [6, Proposition 2.1], this implies $F \subset K(B)^{(d)}$ for some d if G is abelian. □

Remark 5.13. Notice that the assumption that $F/K(B)$ has the Tchebotarev property cannot be removed neither in Corollary 5.12 nor in the first part of Proposition 5.5. Indeed, let $F = \overline{\mathbb{Q}}(T^{1/\infty})$ be the field generated over $\overline{\mathbb{Q}}(T)$ by all d-th roots of T, with $d \in \mathbb{N}^*$, the extension $F/\overline{\mathbb{Q}}(T)$ is abelian of group $G \simeq \widehat{\mathbb{Z}}$, it has uniformly bounded local specialization degrees (as $\overline{\mathbb{Q}}$ is algebraically closed) but $F \not\subset K(B)^{(d)}$ for any d (as $\widehat{\mathbb{Z}}$ is of infinite exponent).

We now produce several examples showing that in general Question 5.11 has a negative answer.

Our examples will even satisfy a stronger local property: namely they will have the $\mathcal{M}$-local decomposition degrees of $F/K(B)$ uniformly bounded, where by *local decomposition degree* at some $\mathcal{M}$-local point $t_0 \in B(k_v)$, we mean the order of the decomposition group of $Fk_v/k_v(B)$ at t_0 (while the local specialization degree is the degree of the residue extension).

More specifically we will prove the following.

Theorem 5.14. *In the following situations, there exists an infinite Galois extension $F/K(T)$ having uniformly bounded $\mathcal{M}$-local decomposition degrees, but such that $F \not\subset K(B)^{(d)}$ for any integer d:*

(a) *The Regular Inverse Galois Problem (RIGP) holds over K and the localization set $\mathcal{M}$ is standard. Furthermore the constructed extension $F/K(T)$ is K-regular.*
(b) *K is a finite field and $B = \mathbb{P}^1$.*

Recall that the RIGP is the condition that every finite group is the Galois group of some K-regular Galois extension $F/K(T)$. The RIGP is known to hold over PAC fields and complete valued fields. So such fields with a standard localization set are examples of fields as in (a). Conjecturally the RIGP holds over every field and so all fields K with a standard localization set, *e.g.* number fields, are other examples.

To prove Theorem 5.14 we will adjust to our function field context a construction given in [6, §3] in the context of number fields which we re-sketch here.

5.2.1. Strategy

The construction uses extra-special groups. We recall their definition and refer to [13, §A.20] for more details.

Definition 5.15. Given a prime number ℓ, a finite ℓ-group E is said to be *extra-special* if its center $Z(E)$ and its commutator subgroup E' have both order ℓ (and then $Z(E) = E'$).

Fix two odd primes ℓ and q such that $\ell \mid q-1$. Then for every positive integer $m \geq 1$, is known to exist an extra-special group of order ℓ^{2m+1}, of exponent ℓ and of rank $2m$. Fix one such group E_m ($m \geq 1$). Moreover there exists an irreducible E_m-module of dimension ℓ^m over the finite field $\mathbb{F}_q$. Fix such an E_m-module W_m, and finally denote by G_m the semi-direct product $W_m \rtimes E_m$ ($m \geq 1$).

The following statement summarizes the strategy from [6, §3].

Lemma 5.16. *Assume that B is a curve and that, for each $m \geq 1$, G_m is the Galois group of a Galois extension $F_m/K(B)$. Let $F/K(B)$ be the compositum of all the extensions $F_m/K(B)$. Then F is not contained in $K(B)^{(d)}$ for any d, but the local decomposition* *degrees of $F/K(B)$ are uniformly bounded at all $\mathcal{M}$-local points $t_0 \in B(k_v)$ that are tamely branched in $Fk_v/k_v(B)$.*

Proof. The proof is given in [6] in the case $\dim(B) = 0$ and it can be used in the more general case $\dim(B) \geq 0$ with almost no changes. Proposition 3.1 and Proposition 3.3 of [6] show that F is not contained in $K(B)^{(d)}$ for any integer $d \geq 1$ and that $G = \mathrm{Gal}(F/K(B))$ is of finite exponent. From Proposition 5.5, this implies that the local specialization degrees of $F/K(B)$ are uniformly bounded. For each $t_0 \in B(k_v)$ the local specialization degree of $F/K(B)$ at t_0 is the degree of the residue field extension above the point t_0. Thus it remains to prove that the inertia subgroups at all $\mathcal{M}$-local points $t_0 \in B(k_v)$ that are tamely branched in the extension $Fk_v/k_v(B)$ are of uniformly bounded orders. By definition of "tame branching", these inertia subgroups are pro-cyclic subgroups of G, and so are of order $\leq \exp(G)$. □

Proof of Theorem 5.14. We use the construction and the notation from Section 5.2.1 with the primes ℓ, q distinct from p. Under the hypotheses of Theorem 5.14, for each $m \geq 1$, we have a Galois extension $F_m/K(T)$ of group $G_m = W_m \rtimes E_m$. This is clear in case (a) of Theorem 5.14; the extension $F_m/K(T)$ can further be taken to be K-regular. In case (b) for which K is finite, we resort to Shafarevich's theorem [27]: the group G_m is solvable, having odd order, and therefore it is the Galois group of some extension F_m of the global field $K(T)$. Note next that the groups G_m are of prime-to-p order. In particular branching is automatically tame and Lemma 5.16 concludes the proof. □

5.3. A second question: bounding the branch point set

Here we show that the uniform boundedness of the local decomposition degrees does not imply that $F \subset K(B)^{(d)}$ for some d, even if we assume further that the branch point set is finite. However the base field will be algebraically closed in our counter-examples (and so the Tchebotarev property will not hold).

Theorem 5.17. *In situation* (a) *or* (b) *below, there is an infinite Galois extension $F/K(B)$ with uniformly bounded local decomposition degrees, branched at only finitely many points, but such that $F \not\subset K(B)^{(d)}$ for any d:*

(a) *K is an algebraically closed field of characteristic $p > 0$ and B is a curve of genus ≥ 1.*
(b) *K is an algebraically closed field of characteristic* 0 *and $B = \mathbb{P}^1$.*

Proof. Case (a). Assume that K is an algebraically closed field of characteristic $p > 0$ and B is a curve of genus g. We use again the construction from Section 5.2.1; we retain the notation from there. From Lemma 5.16, we are left with realizing all groups G_m as groups of Galois extensions $F_m/K(B)$ ($m \geq 1$) with controlled branching. We will use

Abhyankar's Conjecture on Galois groups of function field extensions of characteristic p, which was proved by the work of M. Raynaud [29] and D. Harbater [23]:

The Raynaud–Harbater theorem. *A finite group G can be realized as the group of a Galois extension $F/K(B)$ unbranched outside a finite set S if and only if the minimal number of generators of the quotient $G/p(G)$ of G by the subgroup of G generated by all p-Sylow subgroups of G is at most $|S| + 2g - 1$.*

Take $\ell = p$. For each $m \geq 1$, we have the following. The group $p(G_m) = \ell(G_m)$ is a normal subgroup of G_m which properly contains the p-group E_m (since E_m is not normal in G_m). Consequently the group $p(G_m) \cap W_m$ is a non trivial normal subgroup of G_m. But as part of the theory of extraspecial groups, W_m is a minimal non trivial normal subgroup of G_m. Therefore $W_m \subset p(G_m)$ so finally $p(G_m) = G_m$. From the Raynaud–Harbater theorem, if $g \geq 1$, then G_m is the group of some Galois extension $F_m/K(B)$ unbranched everywhere.

Case (b). Assume that K is an algebraically closed field of characteristic 0 and $B = \mathbb{P}^1$. Fix an odd prime p. For each $m \geq 1$, take for G_m the dihedral group $\mathbb{Z}/p^m\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ of order $2p^m$. The projective limit $G = \varprojlim_{m\geq 1} G_m$ is the pro-dihedral group $\mathbb{Z}_p \rtimes \mathbb{Z}/2\mathbb{Z}$. Denote by C_m (resp. C) the conjugacy class of G_m (resp. of G) of all elements $(x, 1)$ with $x \in \mathbb{Z}/p^m\mathbb{Z}$ (resp. with $x \in \mathbb{Z}_p$). These are conjugacy classes of elements of order 2.

Pick two elements $\sigma, \tau \in C$ and denote by σ_m and τ_m their images in C_m *via* the projection map $G \to G_m$. We have $\sigma_m\sigma_m\tau_m\tau_m = 1$ and $G_m = \langle\sigma_m, \tau_m\rangle$ ($m \geq 1$). By the Riemann existence theorem, if we choose four distinct points $t_1, t_2, t_3, t_4 \in \mathbb{P}^1(K)$, there is a Galois extension $F_m/K(T)$, with group G_m, branch points t_1, t_2, t_3, t_4 and corresponding inertia groups $\langle\sigma_n\rangle$ and its conjugates for t_1, t_2 and $\langle\tau_n\rangle$ and its conjugates for t_3, t_4 ($m \geq 1$). Furthermore, by a classical compactness argument based on the fact that for each $m \geq 1$ and each 4-tuple (t_1, t_2, t_3, t_4) as above, there are only finitely many choices of the extension $F_m/K(T)$, one can perform the construction compatibly, *i.e.*, so that $F_m/K(T)$ is obtained from $F_{m+1}/K(T)$ *via* the epimorphism $G_{m+1} \to G_m$ ($m \geq 1$).

Set $F = \varinjlim_{m\geq 1} F_m$. The extension $F/K(T)$ is Galois of group G. For each $m \geq 1$, the exponent of G_m is $\geq p^m$ and so G is not of finite exponent. As already noticed (Section 5.2), this implies that F cannot be a subfield of $K(B)^{(d)}$ for any d. As K is algebraically closed, for each $t_0 \in \mathbb{P}^1(k_v)$, the local decomposition degree at t_0 is the branching index. By construction, it is 1 or 2. So the local decomposition degrees are uniformly bounded. □

Remark 5.18. The following refer to the proof of case (a) of Theorem 5.17.

- For $g = 0$, the construction leads to an extension $F/K(T)$ that is only branched at one point, say the point ∞. There is necessarily wild branching and Lemma 5.16 guarantees that the decomposition degrees at all $t_0 \in \mathbb{P}^1(K) \setminus \{\infty\}$ are uniformly bounded.

- We took $\ell = p$. If $\ell \neq p$, then if $q \neq p$, $p(G_m)$ is trivial and $G_m/p(G_m) = G_m$ and, if $q = p$, $p(G_m) = q(G_m) = W_m$. So $G_m/p(G_m)$ cannot be generated by less than $2m$ generators (E_m is of rank $2m$) and G_m cannot be realized with branch points in a fixed finite set S.

5.4. Three final remarks

The following three remarks relate to case (b) of Theorem 5.17. As in this statement assume that K is an algebraically closed field of characteristic 0.

5.4.1. On the geometric Bogomolov property

In [18], J. Ellenberg says that an infinite algebraic extension $F/K(T)$ has the *geometric Bogomolov property (GB)* if there exists some $c > 0$ such that for every non constant function x in F, $h(x) \geq c$, where h is the absolute logarithmic height. Recall that, given a non constant function $x \in \overline{K(T)}$, if $L/K(T)$ is any finite extension such that $x \in L$, $h(x)$ is the ratio $[L : K(x)]/[L : K(T)]$. Note that if C is a curve corresponding to the function field L, then $[L : K(x)]$ is the degree of x on C (equivalently, the number of zeroes – or poles – on C).

This is a geometric analog of the *Bogomolov property* of an algebraic extension $F/\overline{\mathbb{Q}}$ introduced by Bombieri and Zannier in [4], which requests that there exists some $c > 0$ such that if $x \in F$ is neither zero nor a root of unity, then $h(x) \geq c$, where $h(x)$ is the classical absolute logarithmic Weil height on $\overline{\mathbb{Q}}$.

For the Bogomolov property of algebraic extensions $F/\mathbb{Q}$, we have the following criterion proved in [4, Theorem 2].

Bombieri–Zannier criterion. *If $F/\mathbb{Q}$ is an algebraic extension with finite local degrees at some prime p, then F has the Bogomolov property.*

This result has several interesting consequences: it implies, for example, that the field $\mathbb{Q}^{tp}$ of totally p-adic numbers has the Bogomolov property, just as the field $\mathbb{Q}^{tr}$ of totally reals does (a result of Schinzel [30]).

We now provide an example showing that the geometric analog of the Bombieri–Zannier criterion does not hold, even if *all* decomposition degrees are assumed to be bounded (and not just the local specialization degrees above *one* prime).

Consider the (smooth projective) curves C_m corresponding to the function fields F_m ($m \geq 1$) from the proof of Theorem 5.17 (b). The degrees $[F_m : K(T)]$ go to infinity and the Riemann–Hurwitz formula shows that the curves C_m are all of genus 1.

Recall that the *gonality* of some K-curve C is the least degree of a non constant function $x \in K(C)$ and that the gonality of a curve is bounded above in terms of its genus. Consequently in our example above, we have that there is no real constant $c > 0$ such that the gonality of C_m is $\geq c\,[F_m : K(T)]$ ($m \geq 1$).

This implies that the compositum of all F_m's is an infinite extension of $K(T)$ without property (GB), even if it has uniformly bounded local decomposition degrees (here they are just the ramification indices).

5.4.2. A generalization using universal p-Frattini covers

The construction from the proof of Theorem 5.17 (b) extends to the following more general context; we refer to [21], [19, §22], [10], for details.

A group G_1 is given with a prime p such that $p|\ |G_1|$ and G_1 is p-perfect, *i.e.* G_1 is generated by its elements of prime-to-p order. Take for G the *p-universal Frattini cover of* G_1 (which generalizes the pro-dihedral group $\mathbb{Z}_p \rtimes \mathbb{Z}/2\mathbb{Z}$) and for $(G_m)_{m\geq 1}$ the natural collection of finite characteristic quotients of G (which generalize the dihedral groups $\mathbb{Z}/p^m\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, $m \geq 1$). Select r elements of G_1 of prime-to-p order generating G_1. The conjugacy class of each of these elements can be lifted to a conjugacy class C_i of G with the same order, $i = 1, \dots, r$ (the lifting lemma). Pick an element $\sigma_i \in C_i$, $i = 1, \dots r$ and consider the $2r$-tuple $(\sigma_1, \sigma_1^{-1} \dots, \sigma_r, \sigma_r^{-1})$; its entries generate G (the Frattini property) and are of product one.

Extensions $F_m/K(T)$ can then be constructed as in the proof of Theorem 5.17 (b) with the $2r$-tuple above replacing the 4-tuple $(\sigma, \sigma, \tau, \tau)$ and $2r$ distinct points of $\mathbb{P}^1(K)$ replacing the 4 chosen points $t_1, \dots, t_4 \in \mathbb{P}^1(K)$ in the proof of Theorem 5.17 (b). Set $F = \varinjlim_{m\geq 1} F_m$. The extension $F/K(T)$ is Galois of group G, it has uniformly bounded local decomposition degrees, but is not contained in $K(B)^{(d)}$ for any d. The main point is that G is still of infinite exponent in this more general context. Indeed the p-Sylow subgroups of G are known to be free pro-p groups and so cannot have non trivial elements of finite order.

5.4.3. One more remark on the abelian case

In the abelian situation the following can be added:

Proposition 5.19. *Let $F/K(T)$ be an abelian extension, with finitely many branch points and uniformly bounded local decomposition degrees. Then not only $F \subset K(T)^{(d)}$ but $F/K(T)$ is finite.*

Proof. Denote the branch points of $F/K(T)$ by $t_1, \dots, t_r$. Let $F_0/K(T)$ be a finite Galois sub-extension of $F/K(T)$ of group G_0. From the Riemann existence theorem, G_0 is generated by r elements $\sigma_1, \dots, \sigma_r$ such that $\sigma_1 \cdots \sigma_r = 1$; moreover σ_i is a generator of some inertia group above t_i. From the uniform boundedness of the local decomposition degrees, the order of σ_i is bounded by some constant δ, independent of i. Since G_0 is abelian we have $|G_0| \leq \delta^{r-1}$. As all finite sub-extensions of $F/K(T)$ are abelian and the argument holds for any of them, conclude that $F/K(T)$ is finite and that $[F : K(T)] \leq \delta^{r-1}$. □

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