Some Arithmetic Properties of Algebraic Covers

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ABSTRACT. Consider a Galois extension F/K and an algebraic cover $f: X \to B$ a priori defined over F. The cover f may have several models (and possibly none) over each given subfield E of F. How do these models compare to each other? Are there better models than others? We establish here a structure result for the set of all various models which can be used to investigate these questions. The structure, which is of cohomological nature, yields an interesting arithmetical tool: K-covers can be 'twisted' to provide other K-models with possibly better properties. One application is concerned with the Beckmann-Black problem. E. Black conjectures that each Galois extension E/K is the specialization of a Galois branched cover of $\mathbb{P}^1$ defined over K with the same Galois group G. We show the conjecture holds when G is abelian and K is an arbitrary field; this was known for number fields from results of S. Beckmann (1992) and E. Black (1995). Other applications include discussions of existence for a given cover, of a "good" model, a stable model, a model with a totally rational fiber, etc. Also we clarify an inacurracy in a result of Fried about field of moduli and extensions of constants. Finally, we continue our study of local-global principles for covers: if a cover is defined over each $\mathbb{Q}_p$, does it follow it is defined over $\mathbb{Q}$? Here we consider the case of Galois covers of a general base space B.

1. Introduction

This paper is organized as follows. §2 is devoted to the structure result mentioned in the abstract for the set of all K-models of a given cover $f : X \to B$ *a priori* defined over a Galois extension of K. There are two versions. In the first one (§2.4) we assume that the base space B of the cover satisfies a certain condition introduced in [DeDo1] and called the (Seq/Split) condition. That condition, which is recalled in §2.3, holds for example if the base space has unramified K-rational points. The general case of the structure result is given in §2.5. §2.1-§2.3 review some basics relative to the arithmetic of covers.

§2 is used in §3 to study two questions about the realization of covers with some arithmetical constraint on some fiber. The first one (§3.1) is known as the Beckmann-Black problem: given a field K, is every finite Galois extension E/K the specialization of a Galois branched cover of $\mathbb{P}^1$ which is defined over K and has the same Galois group? Beckmann and Black answered positively when G is abelian and K is a number field. We extend this result to arbitrary fields K. The second question (§3.2) is about the existence for a given K-cover of an unramified totally K-rational fiber. We comment on this property: in particular we recapitulate what is known about it and review several situations where it revealed helpful.

§4 is about field of moduli (§4.2) and extension of constants (§2.5.1). Given a K-cover f_K, consider, on one hand, the Galois closure $\widehat{f_K}$ of f_K over K and, on the other hand, the Galois closure $\widehat{f}$ of $f_K \otimes_K K_s$ over the separable closure K_s of K. Th.4.1 (§4.3) relates the extension of constants of f_K in $\widehat{f_K}$ and the field of moduli of $\widehat{f}$ as G-cover. Two consequences are given (§4.4). The first one is a criterion for the field of moduli of $\widehat{f}$ to be a field of definition (as G-cover). The second one clarifies an inacurracy in a result of Fried about extensions of constants in a cover with centerless group. A counter-example to Fried's original statement is given in §4.1.

The theme of §5 is the local-to-global principle for covers: if a cover is defined over each $\mathbb{Q}_p$, does it follow it is defined over $\mathbb{Q}$? The state of the question is recalled in §5.1. Essentially the local-to-global principle holds for G-covers; and for mere covers, it holds under some additional hypotheses on the monodromy group and is conjectured not to hold in general. We consider the case of mere covers that are Galois over $\overline{\mathbb{Q}}$. The local-to-global principle is known to hold then if the base space B satisfies the (Seq/Split) condition. Here we consider the general case, that is, we do not assume the (Seq/Split) condition. Our main result appears in §5.3. Our approach uses the notion of Galois covers given with the action of a subgroup of the automorphism group. These objects, called SG-covers, and which generalize both mere covers and G-covers, are introduced in §5.2.

Notation — Given a Galois extension E/k, its Galois group is denoted by $\mathrm{G}(E/k)$. Given a field k, we denote by k_s a separable closure of k and by $\mathrm{G}(k)$ the absolute Galois group $\mathrm{G}(k_s/k)$.

2. Structure result for models of a cover

2.1. Mere covers and G-covers. The main topic of this paper is the arithmetic of covers. There are classically two situations. One is concerned with not necessarily Galois covers — traditionally called "mere covers" — while the other one considers "G-covers", *i.e.*, Galois covers given with the Galois action.

Given a field K, a regular projective geometrically irreducible variety B defined over K, mere covers $f : X \to B$ over K correspond (*via* an equivalence of categories) to finite separable regular field extensions $K(X)/K(B)$ while G-covers of B of group G over K correspond to regular Galois extensions $K(X)/K(B)$ given with an isomorphism of the Galois group $\mathrm{G}(K(X)/K(B))$ with G.

We denote the variety B with the reduced ramification divisor D removed by B^* and the K-arithmetic fundamendal group of B^* by $\Pi_K(B^*)$ or simply by Π_K when the context is clear. Degree d mere covers of B over F, unramified over B^* correspond to transitive representations $\Psi : \Pi_F(B^*) \to S_d$ such that the restriction to $\Pi_{F_s}(B^*)$ is transitive. G-covers of B of group G over F correspond to surjective homomorphisms $\Phi : \Pi_F(B^*) \to G$ such that $\Phi(\Pi_{F_s}(B^*)) = G$.

We freely use these notions in the sequel; see [DeDo1;§2] for more details. In the rest of this section we fix a field K and a variety B as above.

2.2. Descent of the field of definition of [G-]covers. As in [DeDo1], we frequently use the word "[G-]cover" for the phrase "mere cover [resp. G-cover]". Suppose given a Galois extension F/K and a [G-]cover $f : X \to B$ *a priori* defined over F and such that the ramification divisor D is defined over K. In the mere cover situation, we will always assume that the Galois closure over F of the mere cover is, as G-cover, defined over F. This insures that the group of the cover (*i.e.*, the Galois group of the Galois closure) is the same over F as over F_s. This is of course not restrictive in the absolute situation, *i.e.*, when F is separably closed.

A K-model of the [G-]cover f is a [G-]cover $f_K : X_K \to B$ over K such that the [G-]cover over F obtained from f_K by extension of scalars from K to F, which we denote by $f_K \otimes_K F$, is isomorphic to f over F. A [G-]cover f is said to be defined over K if it has a K-model f_K. A significant problem is to study the descent of the field of definition of the [G-]cover f, and, more generally, to find its k-models for $K \subset k \subset F$. We introduce some notation that makes it possible to handle these questions simultaneously in both the mere cover and G-cover situations.

Let $\Psi : \Pi_F(B^*) \to S_d$ [resp. $\Phi : \Pi_F(B^*) \to G$] the representation of Π_F corresponding to the mere cover [resp. G-cover] $f : X \to B$. In both cases let G denote the group of the cover. Then set

$$N = \begin{cases} G & \text{in the G-cover case} \\ \mathrm{Nor}_{S_d} G & \text{in the mere cover case} \end{cases}$$

$$C = \mathrm{Cen}_N G = \begin{cases} Z(G) & \text{in the G-cover case} \\ \mathrm{Cen}_{S_d} G & \text{in the mere cover case} \end{cases}$$

where $Z(G)$ is the center of G and $\mathrm{Nor}_{S_d} G$ and $\mathrm{Cen}_{S_d} G$ are respectively the normalizer and the centralizer of G in S_d. Finally regard N as a subgroup of S_d where d is the degree of f: in the mere cover case, an embedding $N \hookrightarrow S_d$ is given by definition; in the G-cover case, embed $N = G$ in S_d by the regular representation of G.

Then, in both the mere cover and G-cover situations, the [G-]cover $f : X \to B$ corresponds to the representation $\phi : \Pi_F(B^*) \twoheadrightarrow G \subset N$ and the following holds [DeDo1].

Proposition 2.1 — *(a) The K-models of f correspond to the homomorphisms $\Pi_K(B^*) \to N$ that extend the homomorphism $\phi : \Pi_F(B^*) \to G \subset N$. In particular, the [G-]cover f can be defined over the field K if and only if the homomorphism $\phi : \Pi_F(B^*) \to G \subset N$ extends to an homomorphism $\Pi_K(B^*) \to N$,*

(b) Two [G-]covers over F are isomorphic if and only if the corresponding representations ϕ and ϕ' of Π_F are conjugate by an element φ in the group N, that is, $\phi'(x) = \varphi\phi(x)\varphi^{-1}$ for all $x \in \Pi_F(B^*)$

A representation $\Pi_K(B^*) \to N$ extending $\phi : \Pi_F(B^*) \to G \subset N$ will be called a K-model of ϕ. From (a), K-models of the representation ϕ correspond to K-models of the [G-]cover f.

2.3. Fibers of a cover.

2.3.1. Condition (Seq/Split). Fix a Galois extension F/K, a divisor D of B with simple components defined over K and assume that the exact sequence of fundamental groups

$$1 \to \Pi_F(B^*) \to \Pi_K(B^*) \to \mathrm{G}(F/K) \to 1$$

splits. This condition was introduced in [DeDo1] where it is called (Seq/Split).

Consider the special case $F = K_s$ and the base space B has K-rational points off the branch point set D. Then condition (Seq/Split) classically holds: indeed each unramified K-rational point t_o provides a section $\mathrm{s}_{t_o} : \mathrm{G}(K) \to \Pi_K$ [1].

2.3.2. Arithmetic action of $\mathrm{G}(F/K)$ on a fiber. Suppose given a degree d mere cover $f_K : X_K \to B$ over K unramified over B^* and let $\phi_K : \Pi_K(B^*) \to \mathrm{Nor}_{S_d}G$ be the associated representation. Given an unramified K-rational point t_o on B, denote the compositum of all fields of definition over K of points in the fiber $f_K^{-1}(t_o)$ by K_{f_K,t_o}; equivalently, K_{f_K,t_o} is the compositum of all residue fields at t_o of the Galois closure of the extension $K(X)/K(B)$. We call the field K_{f_K,t_o} the *splitting field of f_K at t_o*.

[1] On the other hand, condition (Seq/Split) does not always hold: an example in which it does not is given in [DeEm].

Proposition 2.2 [De1;Prop.2.1] — *For each $\tau \in \mathrm{G}(K)$, the element $(\phi_K \mathrm{s}_{t_o})(\tau)$ is conjugate in S_d to the action of τ on the fiber $f_K^{-1}(t_o)$. Furthermore, the splitting field K_{f_K,t_o} of f_K at t_o corresponds via Galois theory to the homomorphism $\phi_K \mathrm{s}_{\mathrm{T_o}} : \mathrm{G}(K) \to N$; that is, it is the fixed field in K_s of $Ker(\phi_K \mathrm{s}_{t_o})$ and the Galois group of the extension $K_{f_K,t_o}/K$ is the image group of $\phi_K \mathrm{s}_{t_o}$.*

Return to the general case: let $\mathrm{s} : \mathrm{G}(F/K) \to \Pi_K$ denote a section to the map $\Pi_K \to \to \mathrm{G}(F/K)$ (not necessarily of the form s_{t_o}). Each element of Π_K induces a permutation of the different embeddings of the function field $K(X_K)$ in a separable closure $(K(B))_s$ of $K(B)$. This set of embeddings $K(X_K) \hookrightarrow (K(B))_s$ can be viewed as the geometric generic fiber of the cover. By analogy with the case $\mathrm{s} = \mathrm{s}_{t_o}$, for each $\tau \in \mathrm{G}(F/K)$, the element $(\phi_K \mathrm{s})(\tau)$ is called the *arithmetic action of τ on the generic fiber associated with the section* s [DeDo1;§2.9]. Furthermore, the fixed field in F of $Ker(\phi_K \mathrm{s})$ is denoted by $K_{f_K,\mathrm{s}}$ and called the *splitting field* of f at s; the Galois group of the extension $K_{f_K,\mathrm{s}}/K$ is the image group of $\phi_K \mathrm{s}$.

2.3.3. Remarks on K-points versus K-sections.

(a) Assume $F = K_s$ and call K-sections on B^* the sections to the map $\Pi_K(B^*) \to \mathrm{G}(K)$. For simplicity, assume $\mathrm{char}(K) = 0$. It is unclear how big is the set of K-sections on B^* that do not come from K-rational points on B. This set need not be empty. Take for K a non PAC field of cohomological dimension ≤ 1 (*e.g.* $K = \mathbb{Q}^{ab}$); there is a smooth projective K-curve B with no K-rational points but from condition $\mathrm{cd}(K) \leq 1$ there are K-sections on B^*.

(b) For some purposes, K-sections can be as useful as K-rational points. For example, assume a finite group G can be realized as the Galois group of a regular Galois extension $E/K(B)$; let $\phi_K : \Pi_K \twoheadrightarrow G$ be the associated representation. Assume B is a K-rational variety and K is hilbertian. Using Prop.2.2, the hilbertian property can be rephrased to assert that there exists an unramified K-rational point $t_o \in \mathbb{P}^1$ such that the composed map $\phi_K \mathrm{s}_{t_o} : \mathrm{G}(K) \to G$ is onto, thus yielding a realization of G as Galois group over K. In fact any K-section such that $\phi_K \mathrm{s} : \mathrm{G}(K) \to G$ is onto would be just as good. So the question arises whether there is a weaker assumption on B than "B is K-rational" that guarantees that if K is hilbertian, there exists a K-section s such that $\phi_K \mathrm{s} : \mathrm{G}(K) \to G$ is onto.

(c) Consider the Fried-Völklein/Pop theorem: a field that is countable, PAC and hilbertian necessarily has a pro-free absolute Galois group of countable rank. At some point, the proof uses the existence of K-rational points on some K-variety. If K-rational points could be replaced in the proof by K-sections, then the assumption PAC could be replaced by "$\mathrm{cd}(K) \leq 1$". Thus the Fried-Völklein conjecture — $\mathrm{cd}(K) \leq 1$ and K hilbertian implies $\mathrm{G}(K)$ pro-free of countable rank, for countable fields – would follow. And the Shafarevich conjecture — $\mathbb{Q}^{ab}$ pro-free —, which is a special case, would follow as well.

2.4. Structure result under (Seq/Split). Assume condition (Seq/Split) holds and let $\mathrm{s} : \mathrm{G}(F/K) \to \Pi_K$ be a section to the map $\Pi_K \twoheadrightarrow \mathrm{G}(F/K)$. Let $f : X \to B$ be a [G-]cover defined over F, unramified over B^* and $\phi : \Pi_F(B^*) \to G \subset N$ be the associated representation. The following notation is used below. Given two maps α, β, the composed map is denoted by $\alpha\beta$ (when it exists); for maps from a set S to a group G, the product map, sending each $s \in S$ to $\alpha(s)\beta(s)$, is denoted below by $\alpha \cdot \beta$.

Proposition 2.3 — *Assume f is defined over K. Let $\phi_K^o : \Pi_K \to N$ be the representation associated with some K-model f_K^o of f and set $\varphi^o = \phi_K^o \mathrm{s}$. Then the set of all K-models ϕ_K of ϕ is in one-one correspondence with the 1-cochain set $Z^1(\mathrm{G}(F/K), C, \varphi^o)$ (precisely defined in the proof).*

More precisely, the K-models of ϕ are those maps $\phi_K : \Pi_K \to N$ which equal ϕ on Π_F and equal $\theta \cdot \varphi^o$ on $\mathrm{G}(F/K)$ (via s), for some $\theta \in Z^1(\mathrm{G}(F/K), C, \varphi^o)$.

Furthermore, if s' is a section to the map $\Pi_K \to \mathrm{G}(F/K)$, then $\mathrm{s}' = \sigma \cdot \mathrm{s}$ for some $\sigma \in Z^1(\mathrm{G}(F/K), \Pi_F, \mathrm{s})$ and the arithmetic action of $\mathrm{G}(F/K)$ on the generic fiber of ϕ_K associated with the section s' is given by

$$\varphi = \phi\sigma \cdot \theta \cdot \varphi^o$$

Proof. The field K is the field of moduli of the [G-]cover f relative to the extension F/K (since f is defined over K). From [DeDo1] (see Main Theorem (III)), we have the following. Let $\overline{\varphi} : \Pi_K \to N/C$ be the representation of Π_K modulo C given by the field of moduli condition [DeDo1;§2.7]. The K-models of f correspond in a one-one way to the liftings $\varphi : \mathrm{G}(F/K) \to N$ of the map $\overline{\varphi}\mathrm{s}$. More precisely, to each given lifting $\varphi : \mathrm{G}(F/K) \to N$ corresponds a K-model of the representation ϕ, namely the one that equals ϕ on Π_F and equals φ on $\mathrm{G}(F/K)$ (*via* s). This K-model has the further property that the action $\varphi : \mathrm{G}(F/K) \to N \subset S_d$ is the arithmetic action of $\mathrm{G}(F/K)$ on the generic fiber associated with the section s.

The map φ^o is a lifting of $\overline{\varphi}\mathrm{s}$. Therefore the set of all liftings exactly consists of those maps $\varphi : \mathrm{G}(F/K) \to N$ of the form $\varphi = \theta \cdot \varphi^o$ where θ is any element of $Z^1(\mathrm{G}(F/K), C, \varphi^o)$, *i.e.*, is any map $\mathrm{G}(F/K) \to C$ satisfying the cocycle condition

$$\theta(uv) = \theta(u)\ \theta(v)^{\varphi^o(u)}$$

Finally, let s' be a section to the map $\Pi_K \to \mathrm{G}(F/K)$. Then $\sigma = \mathrm{s}'\mathrm{s}^{-1}$ satisfies the cocycle condition

$$\sigma(uv) = \sigma(u)\ \sigma(v)^{\mathrm{s}(u)}$$

thus, lies in $Z^1(K, \Pi_{K_s}, \mathrm{s})$. The arithmetic action on the generic fiber of ϕ_K associated with the section s' is given by

$$\phi_K\, \mathrm{s}' = \phi_K\, (\sigma \cdot \mathrm{s}) = \phi\sigma \cdot \theta \cdot \varphi^o \qquad \square$$

2.5. Structure result (General Case). We fix a Galois extension F/K and a divisor D of B with simple components defined over K. We no longer assume that condition (Seq/Split) holds.

2.5.1. Extension of constants in the Galois closure ([DeDo1;§2.8]. Let $f_K : X_K \to B$ be a [G-]cover over K and let $\phi_K : \Pi_K(B^*) \to N$ be the associated representation of $\Pi_K(B^*)$. Consider the function field extension $K(X_K)/K(B)$ associated to f_K. Denote the Galois closure of the extension $K(X_K)/K(B)$ by $\widehat{K(X_K)}/K(B)$; its Galois group is the *arithmetic* Galois group of f_K; denote it by $\widehat{G}$. Consider then the field $\widehat{K} = \widehat{K(X_K)} \cap F$. The extension $\widehat{K}/K$ is called the *extension of constants in the Galois closure* of f_K of f.

Denote by Λ the unique homomorphism $\mathrm{G}(K) \to N/G$ that makes the following diagram commute.

$$\begin{array}{ccc}
\Pi_K(B^*) & \longrightarrow & \mathrm{G}(K) \\
{\scriptstyle \phi_K}\downarrow & & \downarrow{\scriptstyle \Lambda} \\
N & \longrightarrow & N/G
\end{array}$$

Proposition 2.4 [DeDo1;Prop.2.3] — *The homomorphism* Λ *corresponds to the extension of constants* $\widehat{K}/K$ *in the Galois closure of the model* f_K *of* f_F*. That is,* $\mathrm{G}(F/\widehat{K}) = Ker(\Lambda)$*. The field* $\widehat{K}$ *can also be described as the smallest extension* k *of* K *such that* $\phi_K(\Pi_k) \subset G$*, or, equivalently, such that* $k\widehat{K(X_K)}/k$ *is a regular extension.*

The homomorphism $\Lambda : \mathrm{G}(K) \to N/G$ is called the *constant extension map (in Galois closure)* of f_K. For G-covers, $N/G = \{1\}$, Λ is trivial and $\widehat{K} = K$: by definition, G-covers over K do not have any extension of constants in their Galois closure.

Proposition 2.5 — *Let* f *be a [G-]cover over* F *and* $\phi : \Pi_F(B^*) \to G \subset N$ *be the associated representation. Assume* f *is defined over* K*. Let* $\phi_K^o : \Pi_K(B^*) \to N$ *be the representation associated with some* K*-model* f_K^o *of* f*.*

Then the set of all K*-models* ϕ_K *of* ϕ *with the same constant extension map* Λ *as* ϕ_K^o *is in one-one correspondence with the* 1*-cochain set* $Z^1(\mathrm{G}(F/K), Z(G), \Lambda)$*. More precisely, the* K*-models of* ϕ *are those maps* $\phi_K : \Pi_K \to N$ *which are of the form* $(\Theta \mathrm{P})\cdot\phi_K^o$*, for some* $\Theta \in Z^1(\mathrm{G}(F/K), Z(G), \Lambda)$ *and where* P *is the natural surjection* $\mathrm{P} : \Pi_K \twoheadrightarrow \mathrm{G}(F/K)$*.*

Proof. From [DeDo1], a K-model ϕ_K with constant extension map Λ is any homomorphism $\phi_K : \Pi_K \to N$ extending $\phi : \Pi_F \to N$ and inducing the map $\Lambda : \mathrm{G}(F/K) \to N/G$. Furthermore, ϕ_K should also necessarily induce the representation $\overline{\varphi} : \Pi_K \to N/C$ given by the field of moduli condition. Consequently $\tilde{\theta} = \phi_K \cdot (\phi_K^o)^{-1}$ has values in $C \cap G = Z(G)$ and factors through $\mathrm{G}(F/K)$. Prop.2.5 immediately follows. □

3. Covers with prescribed fibers

In this section, the base space B is the projective line $\mathbb{P}^1$ and $F = K_s$. In particular, condition (Seq/Split) holds. We retain this conclusion from Prop.2.3. If a K-model of a [G-]cover f (*a priori* defined over K_s) is known, then other K-models can be obtained by "twisting" by elements of a 1-cochain set $Z^1(\mathrm{G}(K), C, -)$. More precisely, let $t_o \in \mathbb{P}^1(K)$ be an unramified point and s_{t_o} the associated section $\mathrm{G}(K) \to \Pi_K$. The representations $\phi_K : \Pi_K \to N$ associated with K-models of f are completely determined by their restriction to Π_{K_s} (which is given) and their restriction $\varphi = \phi_K \mathrm{s}_{t_o}$ to $\mathrm{G}(K)$. If one K-model is known that has $\varphi = \varphi^o$, others are obtained by replacing φ^o by $\theta\cdot\varphi^o$, for any $\theta \in Z^1(\mathrm{G}(K), C, \varphi^o)$. This arithmetical twist is an important ingredient of the results of this section.

3.1. The Beckmann-Black problem. In [Be2], S. Beckmann asks the following question: is every finite Galois extension $E/\mathbb{Q}$ the specialization of a Galois branched cover of $\mathbb{P}^1$ which is defined over $\mathbb{Q}$ and has the same Galois group? A finite group G is said to have the *lifting property* (over $\mathbb{Q}$) when the answer is "Yes" for every Galois extension $E/\mathbb{Q}$ of group G. She shows that finite abelian groups and symmetric groups have the lifting property. This problem has also been considered by E. Black who conjectured that each finite group has the lifting property over each field K (instead of $\mathbb{Q}$) and proposed a cohomological approach. Her main result is that over a hilbertian field K, a semi-direct product of a finite cyclic group A with a group H having the lifting property also has the lifting property if $(|H|,|A|)=1$ and $(\mathrm{char(K)},|A|)=1$ [Bl2]. That includes the case of abelian groups and also gives new examples of groups with the lifting property, *e.g.* the dihedral groups D_n of order $2n$ when n is odd (see also [Bl1]). Using our terminology, E. Black's conjecture can be reformulated as follows.

Conjecture 3.1 (E. Black) — *Let K be an arbitrary field, G be a finite group and E/K be a Galois extension of group G. Then there exists a G-cover $f: X_K \to \mathbb{P}^1$ of group G defined over K and some unramified point $t_o \in \mathbb{P}^1(K)$ such that the splitting field extension $K_{f_K,t_o}/K$ of f_K at t_o (see §2.3.2) is K-isomorphic to E/K.*

The main result of this section is the following one, which improves on the initial results of Beckmann and Black in that the field K is here an arbitrary field.

Theorem 3.2 — *The Black conjecture holds if G is an abelian group and K is an arbitrary field. In particular, the conjecture holds if* $\mathrm{G}(K)$ *is abelian (e.g. K is finite).*

Proof. Here is our strategy. We realize G as the group of a [G-]cover $f: X \to \mathbb{P}^1$ defined over K. The splitting field extension $K_{f,t_o}/K$ of f at the given point t_o is some extension of K. Then we twist the K-model in such a way that the splitting field extension at t_o equals the given extension E/K.

More specifically, suppose given a finite abelian group G and an extension E/K of group G. Realize it as the Galois group of a G-cover $f_K^o: X_K \to \mathbb{P}^1$ over K with at least one unramified point $t_o \in \mathbb{P}^1(K)$. This is easy if K is infinite. Indeed, take any regular Galois extension $F/K(T)$ of group G (such extensions exist (*e.g.* [MatMa;Ch.4,Th.2.4])); there exists K-rational points on $\mathbb{P}^1$ different from the branch points of the extension $F/K(T)$. However this is more difficult if K is a finite field, especially when the order of G is divisible by the characteristic of K. Nevertheless, this is possible: see [De3] where it is proved that each finite abelian group G can be realized as the Galois group of a G-cover defined over K and unramified over each element of a finite subset $D \subset \mathbb{P}^1(\overline{K})$ given in advance.

Next set $\mathrm{s} = \mathrm{s}_{t_o}$. Let $\varphi^o : \mathrm{G}(K) \to G$ be the arithmetic action of $\mathrm{G}(K)$ on the fiber above t_o. The given extension E/K corresponds to some surjective homomorphism $\varphi : \mathrm{G}(K) \twoheadrightarrow G$. For abelian G-covers, the set $Z^1(K, C, \varphi^o)$ involved in Prop.2.3 is merely $\mathrm{Hom}(\mathrm{G}(K), G)$ (since here $C = Z(G) = G$)). Consequently, the cover $f_K^o \otimes_K K_s$ has another K-model f_K (as G-cover) such that the arithmetic action on the fiber above t_o equals φ: indeed take $\theta = \varphi(\varphi^o)^{-1}$ in Prop.2.3. This concludes the proof of the first part of Th.3.2. The second part readily follows. □

3.2. Models with a totally rational fiber. In Th.3.2, the extension E/K can be more generally any Galois extension of group $H \subset G$ (instead of $H = G$). In particular for $H = \{1\}$ we obtain that each finite abelian group is the Galois group a G-cover of $\mathbb{P}^1$ over K with a totally K-rational unramified fiber $f_K^{-1}(t_o)$. Furthermore, the point t_o can be prescribed in advance in $\mathbb{P}^1(K)$. This had been noticed in [De1] and [Des].

Recall that, given a cover $f_K : X_K \to \mathbb{P}^1$ and a point $t_o \in \mathbb{P}^1(K)$ not a branch point, the fiber $f_K^{-1}(t_o)$ is said to be *totally K-rational* if it consists only of K-rational points on X_K. Below we comment on this condition; in particular we review some situations where the existence of a totally rational fiber revealed helpful.

3.2.1. Existence results. It is known that a finite group G is the Galois group a G-cover of $\mathbb{P}^1$ over K with a totally K-rational fiber in the following situations:

- K is an arbitrary field and G is an abelian group (see above).

- K is an ample field and G is an arbitrary group: this result is essentially due to Harbater and Pop (see *e.g.* [DeDes]). Recall a field K is called *ample* if every smooth K-curve with at least one K-rational point has infinitely many K-rational points. Algebraically closed fields, separably closed fields, more generally PAC fields are ample. Local fields are ample too. The fields $\mathbb{Q}^{tr}$ [resp. $\mathbb{Q}^{tp}$] of all totally real [resp p-adic] algebraic numbers are other typical examples of ample fields.

It is unclear whether this existence result holds for any field K and any group G. That would follow from a generalization of E. Black's conjecture, where the extension E/K would be allowed to be any Galois extension of group contained in the given group G.

3.2.2. Field of definition of G-covers. A mere cover over K which is Galois over K_s and has a totally K-rational fiber is then automatically defined over K as [G-]cover [De1].

3.2.3. Patching covers. D. Harbater showed that, over a complete valued field K, G-covers with a totally K-rational fiber can be patched and glued to provide a G-cover still defined over K and of group the group generated by the groups of the initial covers. This patching and gluing result is an essential tool in the proof of the Regular Inverse Galois Problem over a complete valued field [Ha1] and also in the proof of Abhyankar's conjecture [Ha2].

3.2.4. Stable models. I proved in [De1] that a [G-]cover f_K defined over a Galois extension K of k and which has a totally K-rational unramified fiber, is then *stable* over k. That is, the field of moduli of f_K relative to the extension K/k equals the field of moduli of $f = f_K \otimes_k k_s$ relative to the extension k_s/k. D. Harbater and I combined this Stability Criterion with a Good Models result due to S. Beckmann [Be1] to prove the following result [DeHa]. A G-cover of $\mathbb{P}^1$ over $\overline{\mathbb{Q}_p}$ is defined over its field of moduli (relative to the extension $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$), except possibly if p is a *bad* prime, *i.e.*, if p divides the order of the group of the cover or if the branch points of the cover coalesce modulo p. The result actually holds in a more general situation where $\mathbb{Q}_p$ is replaced by the fraction field of a henselian discrete valuation ring, with a perfect residue field of cohomological dimension ≤ 1. M. Emsalem recently extended this result to mere covers [Em].

3.2.5. The Beckmann-Black problem. In [De2] I prove, for ample fields, a *mere* form of the conjecture where the Galois cover is required to be defined over K but only as mere cover. Existence of a G-cover defined over K with a totally K-rational unramified fiber is also a key ingredient of the proof.

4. Field of moduli and extension of constants

4.1. On a result of M. Fried. In [Fr;Prop.2], M. Fried states that if $f : X \to \mathbb{P}^1$ is a mere cover defined over K with group a centerless group G then the arithmetic Galois group $\widehat{G}$ (see §2.5.1) satisfies: $\widehat{G} \cap \mathrm{Cen}_{S_d} G = \{1\}$ (where d is the degree of f). It seems that this statement is not exactly correct. Here is an argument showing why. Here again, we use the structure result (Prop.2.3) and the derived arithmetical 'twist' on K-models of a cover.

We use the function field viewpoint. Start with a regular Galois extension $E/K(T)$ of group G and with a totally K-rational fiber above some K-rational unramified point t_o: if K is an ample field, this can be achieved for any group G. Let $\phi_K : \Pi_K \to G$ be the representation of Π_K associated with the extension $E/K(T)$ and ϕ be the restriction of ϕ_K to Π_{K_s}: for $x \in \Pi_{K_s}$ and $\tau \in \mathrm{G}(K)$, we have, using Prop.2.2,

$$\phi_K(x\mathrm{s}_{t_o}(\tau)) = \phi(x)$$

Let $C = \mathrm{Cen}_{S_d} G$ be the centralizer of G in its regular representation and $\varphi : \mathrm{G}(K) \to C$ be any homomorphism. From Prop.2.3, the representation ϕ has a K-model $\phi_K^\varphi : \Pi_K \to \mathrm{Nor}_{S_d} G$ which equals ϕ on Π_{K_s} and equals φ on $(\mathrm{G}(K)$ (*via* s) (since here $\varphi \in Z^1(\mathrm{G}(K), C, 1) = \mathrm{Hom}(\mathrm{G}(K), C))$. Let $E^\varphi/K(T)$ be the field extension associated with the "twisted" representation ϕ_K^φ: this extension is still Galois over K_s but no longer over K. The arithmetic Galois group is the group generated by G and $\varphi(\mathrm{G}(K))$. This contradicts Prop.2 of [Fr] since elements in $\varphi(\mathrm{G}(K))$ centralize G.

In the proof of Prop.2 of [Fr], it seems that an inaccuracy occurs when the author says (after display (2.6)): "Therefore $F^o(Y) \subset F^o(Y^o) \subset \widehat{F(Y)}$". Indeed $Y^o \to X$ is obtained by descent from a cover $\widehat{Y} \to X$ which is defined only up to $\widehat{F}$-isomorphism. Thus the containments above hold only up to $\widehat{F}$-isomorphism (but not up to F-isomorphism).

However it is possible to modify Fried's statement so it holds true. The rectified version is the second consequence of Th.4.1 (Cor.4.3 below).

4.2. Field of moduli [DeDo1;§2]**.** We recall below some basics about fields of moduli. Fix a Galois extension F/K. Given a mere cover [resp. G-cover] $f : X \to B$ *a priori* defined over F, for each $\tau \in \mathrm{G}(F/K)$, let $f^\tau : X^\tau \to B^\tau$ denote the corresponding conjugate [G-]cover. Consider the subgroup $M(f)$ [resp. $M_G(f)$] of $\mathrm{G}(F/K)$ consisting of all elements $\tau \in \mathrm{G}(F/K)$ such that the covers [resp., the G-covers] f and f^τ are isomorphic over F. Then the *field of moduli* of the cover f [resp., the G-cover f] (relative to the extension F/K) is defined to be the fixed field $F^{M(f)}$ [resp.$F^{M_G(f)}$] of $M(f)$ [resp. $M_G(f)$] in F. The field of moduli of a [G-]cover is contained in each field of definition k such that $K \subset k \subset F$. So it is the smallest field of definition provided that it *is* a field of definition.

Assume $\phi : \Pi_F(B^*) \to G \subset N$ is the representation corresponding to the [G-]cover $f : X \to B$ over F. Then K is the field of moduli of the [G-]cover f relative to the extension F/K if and only if the ramification divisor D is defined over K and the following condition — called the *field of moduli condition* —, holds.

(FMod) For each $\mathrm{u} \in \Pi_K(B^*)$, there exists $\varphi_\mathrm{u} \in N$ such that

$$\phi(x^\mathrm{u}) = \varphi_\mathrm{u}\phi(x)\varphi_\mathrm{u}^{-1} \qquad \text{(for all } x \in \Pi_F(B^*))$$

4.3. Statement of the result. Let $f_K : X_K \to B$ be a mere cover over K and $\phi_K : \Pi_K(B^*) \to G \subset S_d$ be the associated representation. Set $f = f_K \otimes_K F$ and let $\widehat{f} : \widehat{X} \to B$ be the Galois closure over F of the mere cover $f : X \to B$. Recall from §2.2 that we assume that $\widehat{f}$ is defined over F as G-cover, in other words, that $FK(\widehat{X_K})$ is a regular extension of F; this assumption is not restrictive if $F = K_s$ (*i.e.*, in the absolute situation). The aim of this section is the following result.

Theorem 4.1 — *Let $\widehat{K}/K$ be the extension of constants in the Galois closure of f_K and $\widehat{G}$ be the arithmetic Galois group. Let K_G be the field of moduli of $\widehat{f}$ as G-cover (relative to the extension F/K). Then $\widehat{K}$ is an extension of K_G of degree*

$$[\widehat{K} : K_G] = [\mathrm{Cen}_{S_d} G \cap \widehat{G} : Z(G)]$$

4.4. Consequences. In this paragraph, we suppose given a mere cover $f : X \to B$ over F and its Galois closure $\widehat{f} : \widehat{X} \to B$ over F and let $\phi : \Pi_F(B^*) \to G$ be the representation corresponding to $\widehat{f}$. Let K_G be the field of moduli of $\widehat{f}$ as G-cover (relative to the extension F/K).

Corollary 4.2 — *Assume that f is defined over K and that* $\mathrm{Cen}_{S_d} G = Z(G)$. *Then $\widehat{f} : \widehat{X} \to B$ is defined over its field of moduli K_G (as G-cover).*

Proof. Let f_K be a K-model of f as mere cover. Apply Th.4.1 to get $\widehat{K} = K_G$. Now $\widehat{K}$ is a field of definition of the G-cover $\widehat{f}$ (Prop.2.4). ☐

Corollary 4.3 — *Assume that $Z(G) = \{1\}$. Then f has a unique K_G-model $f_{K_G} : X_{K_G} \to B$ with no extension of constants in its Galois closure (up to K_G-isomorphism). Furthermore the arithmetic Galois group $\widehat{G}$ of each K-model $f_K : X_K \to B$ of f_{K_G} satisfies $\widehat{G} \cap \mathrm{Cen}_{S_d} G = \{1\}$.*

§4.1 shows the last conclusion may fail if f_K is a model of f that is not a model of f_{K_G}.

Proof. The assumption $Z(G) = \{1\}$ insures that the Galois cover $\widehat{f}$ is defined over its field of moduli K_G [DeDo1;Cor.3.2]. That is, the homomorphism $\phi : \Pi_F \to G$ extends to an homomorphism $\phi_{K_G} : \Pi_{K_G} \to G$. The representation $\psi : \Pi_F \to S_d$ associated with the mere cover f is obtained by composing ϕ with some embedding $i : G \hookrightarrow S_d$. The homomorphim ψ extends to an homomorphism $\Pi_{K_G} \to S_d$, namely the homomorphism

$i\phi_{K_G}$. By construction, the associated K_G-model of f has no extension of constants in its Galois closure. The uniqueness of such a K_G-model of f follows from Prop.2.5.

Suppose given a K-model f_K of f_{K_G}, *i.e.*, an extension $\psi_K : \Pi_K \to S_d$ of ψ_{K_G}. It follows from

$$\psi_K(\Pi_{K_G}) = \psi_{K_G}(\Pi_{K_G}) \subset G$$

and the definition of $\widehat{K}$ (§2.5.1) that $\widehat{K} \subset K_G$. On the other hand, $\widehat{K}$ is a field of definition of $\widehat{f}$ as G-cover. Therefore $K_G \subset \widehat{K}$. Whence $\widehat{K} = K_G$. Thus Th.4.1 yields

$$\mathrm{Cen}_{S_d} G \cap \widehat{G} = Z(G) = \{1\} \qquad \square$$

4.5. Proof of Theorem 4.1. The field $\widehat{K}$ is a field of definition of the G-cover $\widehat{f}$. Therefore $\widehat{K}$ contains K_G which is the field of moduli of the G-cover $\widehat{f}$.

Let $\Lambda : \mathrm{G}(F/K) \to N/G$ be the constant extension map of f_K (in Galois closure) where as usual $N = \mathrm{Nor}_{S_d} G$. From Prop.2.4, $\mathrm{G}(F/\widehat{K}) = Ker(\Lambda)$ and for each field k such that $K \subset k \subset \widehat{K}$, the kernel of the restriction of Λ to $\mathrm{G}(F/k)$ is $\mathrm{G}(F/k) \cap \mathrm{G}(F/\widehat{K}) = \mathrm{G}(F/\widehat{K})$. Conclude that

$$[\widehat{K} : k] = |\Lambda\,(\mathrm{G}(F/k))| \tag{1}$$

Denote the quotient group $G \cdot (\mathrm{Cen}_{S_d} G \cap \widehat{G})/G$ by Γ; it is a subgroup of $\widehat{G}/G$. We claim that $\Lambda\,(\mathrm{G}(F/K_G)) \subset \Gamma$. Indeed, since K_G is the field of moduli of the G-cover $\widehat{f}$, for each $\mathrm{U} \in \Pi_{K_G}$, there exists some $\varphi_{\mathrm{U}} \in G$ such that

$$\phi(x^{\mathrm{U}}) = \varphi_{\mathrm{U}} \phi(x) {\varphi_{\mathrm{U}}}^{-1} \qquad (x \in \Pi_F)$$

where $\phi : \Pi_F \to G$ is the restriction of ϕ_K to Π_F. Now the above formula also holds with φ_{U} replaced by $\phi_K(\mathrm{U})$. Conclude that $\phi_K(\mathrm{U})$ lies in G, up to an element in $\mathrm{Cen}_{S_d} G$ (which also obviously lies in $\widehat{G} = \phi_K(\Pi_K)$). This proves the claim since the map Λ is the map induced modulo G by ϕ_K over $\mathrm{G}(F/K)$. Formula (1) then yields

$$[\widehat{K} : K_G] \leq |\Gamma| \tag{2}$$

Let k be the fixed field in F of $\Lambda^{-1}(\Gamma)$. We claim that $K_G \subset k$. Indeed, let $\tau \in \mathrm{G}(F/k)$, *i.e.*, $\Lambda(\tau) \in \Gamma$. Pick an element $\mathrm{U} \in \Pi_K$ mapping to τ *via* the map $\Pi_K \to \mathrm{G}(F/K)$. The element $\phi_K(\mathrm{U})$ can be written $\phi_K(\mathrm{U}) = g \cdot c$ with $g \in G$ and $c \in \mathrm{Cen}_{S_d} G$. Thus we obtain

$$\phi(x^{\mathrm{U}}) = \phi(x)^g \qquad (x \in \Pi_F)$$

This shows that $\tau \in \mathrm{G}(K_G)$ and proves the claim. Using formula (1), we obtain

$$[\widehat{K} : K_G] \geq [\widehat{K} : k] = |\Lambda(\Lambda^{-1}(\Gamma))| = |\Gamma| \tag{3}$$

which together with (2) completes the proof (the equality $|\Lambda(\Lambda^{-1}(\Gamma))| = |\Gamma|$ holds since $\Gamma \subset \widehat{G}/G$ and that $\widehat{G}/G$ is the image group of Λ). □

5. The local-to-global principle

5.1. The problem. We retain the notation of previous sections and assume further that K is a number field and $F = \overline{\mathbb{Q}}$. Assume that the [G-]cover $f : X \to B$ can be defined over each completion K_v of K. Does it follow that the cover can be defined over K? We say that the *local-to-global* principle holds when the answer is "Yes". In his thesis, E. Dew conjectured that the local-to-global principle for G-covers of $\mathbb{P}^1$ over number fields. This was proved in [De1] except possibly for number fields that are exceptions to Grunwald's theorem (the field $\mathbb{Q}$ is not exceptional). This result was extended to G-covers of a general base space B in [DeDo1]. The case of mere covers was then considered in [DeDo2]: the local-to-global principle was shown to hold under some additional assumptions on the group G of the cover and the monodromy representation $G \hookrightarrow S_d$ (with $d = deg(f)$). Here we will consider the special case of mere covers that are Galois over $\overline{\mathbb{Q}}$.

Recall that the field of moduli of a cover embeds in each field of definition. Therefore if a cover is defined over each K_v, then its field of moduli is K (relative to the extension $\overline{K}/K$). Hence if the field of moduli is a field of definition, then the local-to-global principle holds. From [DeDo1] that is the case if the cover is Galois over $\overline{\mathbb{Q}}$ and condition (Seq/Split) (see §2.3.1) holds. The goal of this section is to investigate the problem when the cover is Galois over $\overline{\mathbb{Q}}$ but condition (Seq/Split) is not assumed.

5.2. SG-covers. Our treatment uses SG-covers which are more general objects than [G-]covers. They were originally introduced in [DeDo1;Final Note] (under a different name). Here we will only consider *Galois* SG-covers. Given a group G and a subgroup $\Gamma \subset G$, a *Galois* SG-*cover of fixed subgroup* $\Gamma \subset G$ over K is a mere cover $f : X \to B$ over K which is Galois over $\overline{K}$ and is given with an isomorphism $G \simeq \mathrm{G}(\overline{K}(X)/\overline{K}(B))$ and an embedding $\Gamma \hookrightarrow \mathrm{Aut}(K(X)/K(B))$ such that the following square diagram commutes

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{G}(\overline{K}(X)/\overline{K}(B)) \\ \uparrow & & \uparrow \\ \Gamma & \longrightarrow & \mathrm{Aut}(K(X)/K(B)) \end{array}$$

For $\Gamma = \{1\}$, SG-covers are mere covers; for $\Gamma = G$, SG-covers are G-covers. A SG-cover $f : X \to B$ over $\overline{K}$ is defined over K if the mere cover f together with the automorphisms in Γ can be defined over K.

From [DeDo1;Final Note], isomorphism classes of SG-covers of fixed subgroup $\Gamma \subset G$ $f : X \to B$ over $\overline{K}$ correspond to surjective homomorphisms $\phi : \Pi_{\overline{K}}(B^*) \to G$ regarded modulo conjugation by elements of the group

$$N = Nor_{S_d} G \cap Cen_{S_d}(\Gamma^*) \tag{1}$$

where

- the embedding $G \subset S_d$ is given by the (free transitive) action of G on the d conjugates of a primitive element of the extension $\overline{K}(X)/\overline{K}(B)$, and
- Γ^* is the image of Γ *via* the classical anti-isomorphism $^* : G \to Cen_{S_d} G$: identify each $g \in G$ with the element in S_d induced by the left-multiplication by g; then the map * send g on the right-multiplication by g.

For $\Gamma = \{1\}$ (mere cover case), we have $N = \mathrm{Nor}_{S_d}(G)$; for $\Gamma = G$ (G-cover case), we have $N = G$. Thus in general, we have $G \subset N \subset \mathrm{Nor}_{S_d}(G)$.

As in both the mere cover and G-cover situations, we have the following assertions (which generalize Prop.2.1). The group N is the one given in (1).

(2) *(a) A SG-cover $f : X \to B$ of fixed subgroup $\Gamma \subset G$ over K_s corresponds to a representation $\phi : \Pi_{K_s}(B^*) \twoheadrightarrow G \subset N$*

(b) The K-models of f (as SG-cover) correspond to the homomorphisms $\Pi_K(B^*) \to N$ that extend the homomorphism $\phi : \Pi_{K_s}(B^*) \to G$. In particular, the SG-cover f can be defined over the field K if and only if the homomorphism $\phi : \Pi_{K_s}(B^*) \to G \subset N$ extends to an homomorphism $\Pi_K(B^*) \to N$,

(c) Two [G-]covers over K_s are isomorphic if and only if the corresponding representations ϕ and ϕ' are conjugate by an element φ in the group N, that is, $\phi'(x) = \varphi\phi(x)\varphi^{-1}$ for all $x \in \Pi_{K_s}(B^*)$

The map * can be described more intrinsecally. Denote by G_+ the group

$$G_+ = G \times^s \mathrm{Aut}(G)$$

The group G_+ is isomorphic to $\mathrm{Nor}_{S_d}(G)$ (*e.g.* [DeDo1;Prop.3.1]). Next use the embedding $G \hookrightarrow G_+$ sending g to $(g,1)$ to identify G with a subgroup of G_+ (still denoted by G). For each $g \in G$, denote the conjugation by g by c_g ($c_g(h) = ghg^{-1}$). Then the anti-isomorphism * is the map

$$\begin{cases} G \to G_+ \\ g \to g^* = (g, c_g^{-1}) \end{cases}$$

Some groups play a central role in [DeDo1] and [DeDo2]: they are N (defined in (1)), $C = \mathrm{Cen}_N(G)$, N/C and $CG/G \simeq C/Z(G)$ [2].

Proposition 5.1 — *Denote the subgroup of* $\mathrm{Aut}(G)$ *of all automorphisms χ that are trivial on Γ by* $\mathrm{Aut}_\Gamma(G)$. *Then we have.*

(a) $N = \{(\gamma, \chi) \in G_+ | \gamma \in G, \chi \in \mathrm{Aut}_\Gamma(G)\}$.

(b) $N/G = \mathrm{Aut}_\Gamma(G)$.

(c) $C = (\mathrm{Cen}_G(\Gamma))^*$.

(d) CG/G is isomorphic to $\mathrm{Cen}_G(\Gamma)/Z(G)$*, which embeds in* $\mathrm{Aut}_\Gamma(G)$.

(e) $Z(G)$ is a direct summand of C if and only if $Z(G)$ is a direct summand of $\mathrm{Cen}_G(\Gamma)$.

(f) $Z(G) \subset Z(N)$ if and only if for each $\chi \in \mathrm{Aut}_\Gamma(G)$, we have $\chi|_{Z(G)} = 1$.

Proof. By definition, $N = G_+ \cap \mathrm{Cen}_{G_+}(\Gamma^*) = \mathrm{Cen}_{G_+}(\Gamma^*)$ and $C = \mathrm{Cen}_N(G)$; (a) and (c) follow straightforwardly. The projection $G_+ \to \mathrm{Aut}(G)$ mapping each $(u, \chi) \in G_+$ to $\chi \in \mathrm{Aut}(G)$ factors through G_+/G to yield the isomorphism $G_+/G \simeq \mathrm{Aut}(G)$; (b) and (d) follow immediately. The anti-isomorphism * sends $\mathrm{Cen}_G(\Gamma)$ onto $(\mathrm{Cen}_G(\Gamma))^*$ and $Z(G)$ onto itself. This provides an isomorphism $\mathrm{Cen}_G(\Gamma)/Z(G) \simeq (\mathrm{Cen}_G(\Gamma))^*/Z(G)$ and proves (e). Finally (f) readily follows from (a) and the definitions. □

[2] $Z(G)$ should *a priori* be replaced by $C \cap G$, but $C \cap G = Z(G)$ in general.

5.3. The local-to-global principle for Galois covers. [DeDo2] studies the local-to-global problem for mere covers, *i.e.*, SG-covers of subgroup $\Gamma = \{1\}$. But the paper was written in a more general setting. Namely the objects we considered were representations $\phi : \Pi_{K_s} \twoheadrightarrow G \subset N$. In [DeDo2] we were mainly interested in mere covers, which correspond to $N = \mathrm{Nor}_{S_d}(G)$, and G-covers, which correspond to $N = G$. But from (2) above, [DeDo2] applies more generally to SG-covers provided that N is understood as in (1), C as $\mathrm{Cen}_N(G)$. In this context, the second Theorem in §1 of [DeDo2] rewrites as follows (using Prop.5.1).

Theorem 5.2 — *Assume that $K = \mathbb{Q}$, or more generally, that K is a number field for which the special case of Grunwald's theorem does not occur. Then the local-to-global principle holds for SG-covers of fixed subgroup $\Gamma \subset G$ satisfying simultaneously the five conditions below.*

(i) For each $\chi \in \mathrm{Aut}_\Gamma(G)$, we have $\chi|_{Z(G)} = 1$,

(ii) $Z(G)$ is a direct summand of $\mathrm{Cen}_G(\Gamma)$

(iii/1) $Z(\mathrm{Cen}_G(\Gamma)/Z(G))$ is a direct summand of $\mathrm{Cen}_G(\Gamma)/Z(G)$.

(iii/2) $Z(\mathrm{Cen}_G(\Gamma)/Z(G)) \subset Z(\mathrm{Aut}_\Gamma(G))$.

(iii/3) $\mathrm{Inn}(\mathrm{Cen}_G(\Gamma)/Z(G))$ has a complement in $\mathrm{Aut}(\mathrm{Cen}_G(\Gamma)/Z(G))$.

These five conditions hold for example if $\mathrm{Cen}_G(\Gamma) = Z(G) \subset \Gamma$.

The local-to-global principle also holds for SG-covers satisfying simultaneously conditions (i), (ii) above and the following condition

(iii)' $\mathrm{Cen}_G(\Gamma)/Z(G)$ has a complement in $\mathrm{Aut}_\Gamma(G)$.

Originally we were interested by the local-to-global principle for Galois mere covers. Th.5.2 above can be used as follows. Suppose given a Galois mere cover defined over $\mathbb{Q}_p$ for each p. If in addition there is a subgroup $\Gamma \subset G$ such that for each p, the mere cover has a model defined over $\mathbb{Q}_p$ along with the automorphisms in Γ, then f satisfies the local assumption of the local-to-global principle not only as mere cover but also as SG-cover and Th.5.2 may be used. The bigger Γ the stronger the hypothesis. If $\Gamma = G$, *i.e.*, is as big as can be, then the hypothesis does insure that the mere cover is defined over $\mathbb{Q}$ (it is even defined over $\mathbb{Q}$ as G-cover). The weakest hypothesis is for $\Gamma = \{1\}$. Although no example is known, it is most likely that the local-to-global principle does not hold in general.

We state some intermediate special cases. In both statements below, $K = \mathbb{Q}$, or more generally, K is a number field for which the special case of Grunwald's theorem does not occur.

Corollary 5.3 — *Let* Γ *be a subgroup of a group* G *such that* $\mathrm{Cen}_G(\Gamma) \subset \Gamma \cap Z(G)$ *(e.g.* $\mathrm{Cen}_G(\Gamma) = \{1\}$*). Then the local-to-global principle holds for all SG-covers with fixed subgroup* $\Gamma \subset G$*.*

Corollary 5.4 — *Let* Γ *be a subgroup of a group* G *such that* $Z(G) \subset \Gamma$*,* $Z(G)$ *is a direct summand of* $\mathrm{Cen}_G(\Gamma)$ *and* $\mathrm{Cen}_G(\Gamma)/Z(G)$ *has a complement in* $\mathrm{Aut}_\Gamma(G)$*. Then the local-to-global principle holds for all SG-covers with fixed subgroup* $\Gamma \subset G$*.*

Indeed, assumptions in Cor.5.3 guarantee that $\mathrm{Cen}_G(\Gamma) = Z(G) \subset \Gamma$. Note that if they hold, then necessarily $Z(G) = \mathrm{Cen}_G(\Gamma) = Z(\Gamma)$. As to Cor.5.4, the assumptions guarantee here that conditions (i), (ii) and (iii)' of Th.5.2 hold. Assumptions of both Cor.5.3 and Cor.5.4 hold if $\Gamma = G$, *i.e.*, in the G-cover situation. Thus both these results generalize Th.3.8 of [DeDo1].

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