Finiteness Results in Descent Theory

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Abstract. We show that a $\overline{\mathbb{Q}}$-curve of genus g and with stable reduction (in some generalized sense) at every finite place outside a finite set S can be defined over a finite extension L of its field of moduli K depending only on g, S and K. Furthermore, there exist L-models that inherit all places of good and stable reduction of the original curve (except possibly for finitely many exceptional places depending on g, K and S). This descent result yields this moduli form of the Shafarevich conjecture: given g, K and S as above, only finitely many K-points on the moduli space $\mathcal{M}_g$ correspond to $\overline{\mathbb{Q}}$-curves of genus g and with good reduction outside S. Other applications to arithmetic geometry, like a modular generalization of the Mordell conjecture, are given.

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Introduction

This paper has two main themes: descent theory and arithmetic geometry. For both we are concerned with algebraic curves over $\overline{\mathbb{Q}}$ of genus $g \geq 2$.

For investigating fields of definition and the corresponding models of a curve, the field of moduli, which is the field of definition of the representing point on the moduli space, is a natural landmark. The field of moduli is not a field of definition in general though it is in some circumstances (see [De2]). Our main descent result (theorem 1.3) shows that given an integer $g \geq 2$, a number field K and a finite set S of places of K, there exists a finite extension L/K that is a *common* field of definition for *all* $\overline{\mathbb{Q}}$-curves X of field of moduli K and with ur-stable reduction at finite places $v \notin S$; we refer to §1 for precise definitions, in particular, for the notion of ur-good and ur-stable reduction. Furthermore, each of these curves X has a L-model $\widetilde{X}$ that has stable reduction [resp. good reduction] at every finite place where the original $\overline{\mathbb{Q}}$-curve X does (except for finitely many places depending only on g, K and S).

Conjoined with the Shafarevich conjecture proved by Faltings [Fa], our descent result leads to the following *moduli form* of the Shafarevich conjecture (theorem 3.1). Denote the moduli space of genus g curves by $\mathcal{M}_g$ (with $g \geq 2$). Then, given a number field K and a finite set S of places of K, the subset of $\mathcal{M}_g(K)$ corresponding to $\overline{K}$-curves of field of moduli K and with ur-good reduction outside S is finite. Similarly we obtain modular versions of other classical finiteness results in arithmetic geometry (§3). These variants have K-curves replaced by $\overline{K}$-curves with field of moduli K, which can be regarded as K-points on moduli spaces, thus giving the results a modular tenor. Such a variant is given for the Mordell conjecture (theorem 3.4), which involves a new notion of rational points, on $\overline{K}$-curves with field of moduli K (definition 3.3).

The main descent theorem is proved in §2. The proof rests on the theory of stable curves of Deligne and Mumford [DelMu] and on previous work in descent theory, including the local-global principle for G-covers [DeDo1] and the descent *canonical model* construction [DeEm]. Applications to arithmetic geometry (§3) use as a major extra ingredient Falting's finiteness results in arithmetic geometry [Fa].

Our conclusions can be investigated for other categories than curves (covers, G-covers, (polarized) abelian varieties, etc.) and for more general base fields. Even more general versions should result from using the language of stacks and gerbes. A subsequent work will be devoted to these generalizations.

1. The main descent result

1.1. Notation and main data. Fix a base field K. For simplicity assume K is of characteristic 0; in our applications, K is a number field.

1.1.1. Models, fields of definition and field of moduli. Given a K-integral scheme S (and an integer $g \geq 0$), by S-curve (of genus g) we mean a proper flat morphism $C \to S$ of finite type whose generic fiber is smooth and whose geometric fibers are connected curves (of genus g). For K-integral domains A (a morphism $K \to A$ is given) we say "A-curve" instead of "Spec(A)-curve".

For curves over fields, there are notions of fields of definition, models and field of moduli. Suppose given an overfield k of K and a field extension F/k; a typical situation is $F = \overline{k}$. Given a F-curve X, a *k-model* of X is a k-curve $\widetilde{X}$ such that $\widetilde{X} \otimes_k F$ and X are isomorphic; X is then said to be *defined* over k. If the field extension F/k is Galois, the *field of moduli* of X relative to the extension F/k is defined to be the fixed field in F of the subgroup of $\mathrm{Gal}(F/k)$ of all τ such that X and X^τ are isomorphic over F. In the sequel, when we say that a $\overline{k}$-curve is of field of moduli k, *i.e.*, when we omit the reference to the extension F/k, we always mean relative to the extension $\overline{k}/k$. In this *absolute* situation where $F = \overline{k}$, the Galois group $\mathrm{Gal}(\overline{k}/k)$ (the absolute Galois group of k) is denoted by G_k.

The notion of model extends to that of model over a ring (and in fact over an arbitrary K-scheme [DeDoMo]). Given a K-integral domain A, with quotient field k, an A-model of a F-curve X is an A-curve $\mathcal{X}$ such that the *generic fiber* $\mathcal{X} \otimes_A k$ becomes isomorphic to X after extending the scalars to F, *i.e.*, $(\mathcal{X} \otimes_A k) \otimes_k F \simeq X$.

In these definitions, curves can be replaced by objects from other categories. In particular we will use the following categories of *covers*. A subgroup $G \subset S_d$ (with $d \geq 1$) and a smooth projective K-curve B being fixed, we will work with the (fibered) category of covers $f : X \to B$ over K-schemes whose geometric fibers are degree d covers of monodromy group $G \subset S_d$, and with the category of G-covers of B of group G, *i.e.*, the (fibered) category of Galois covers $f : X \to B$ (over K-schemes) given with an isomorphism $\mathrm{Aut}(f) \to G$. We refer to [DeDo1] for definitions relative to covers. Recall however that to distinguish them from G-covers, objects from the former category are sometimes called *mere* covers: they are non necessarily Galois and they are given without their automorphisms.

The field of moduli need not be a field of definition. Investigating the obstruction is part of descent theory for fields of definition, which many works have been devoted to, notably by Weil, Shimura and Grothendieck (see [De2] for a survey and references).

In this paper we restrict our attention to curves of genus $g \geq 2$: curves of genus 0 and genus 1 are known to be defined over their field of moduli.

1.1.2. Good and stable reduction. Assume A is a discrete valuation ring with fraction field k and residue field κ. Recall an A-curve $\mathcal{X}$ is said to be *stable* if the special fiber is geometrically connected and reduced, with only ordinary double points and if every smooth rational component meets the other components in more than 2 points [DelMu]. A k-curve X is said to have *good reduction* [resp. *stable reduction*] if X has a geometrically integral smooth A-model (a good model for short) $\mathcal{X}$ [resp. if X has a stable A-model $\mathcal{X}$].

Given a field extension k/K and a valuation v on k, denote the completion of k at v by k_v, the valuation ring of k_v by A_{k_v} and the residue field by κ_{k_v}. We still denote by v the unique extension of the valuation to some fixed algebraic closure $\overline{k_v}$ of k_v. Also denote the maximal unramified extension of k_v by k_v^{ur}. Since k_v^{ur} has cohomological dimension ≤ 1, any $\overline{k_v}$-curve (or $\overline{k_v}$-object from one of the categories above) with field of moduli k_v has a (and possibly several) k_v^{ur}-model(s) ([DeDo1] corollary 3.3 and [DeEm] corollary 4.3).

A $\overline{k_v}$-curve X is said to have *k_v^{ur}-good reduction* [resp. *k_v^{ur}-stable reduction*] with respect to k_v if k_v is the field of moduli of X and if X has a good $A_{k_v^{\mathrm{ur}}}$-model $\mathcal{X}_v^{\mathrm{ur}}$ [resp. if X has a stable $A_{k_v^{\mathrm{ur}}}$-model $\mathcal{X}_v^{\mathrm{ur}}$]. In that case the generic fiber X_v^{ur} of $\mathcal{X}_v^{\mathrm{ur}}$ is a k_v^{ur}-model of X. From [DelMu] theorem 1.11, *such* a k_v^{ur}-model of X with a stable $A_{k_v^{\mathrm{ur}}}$-model $\mathcal{X}_v^{\mathrm{ur}}$ is unique, up to k_v^{ur}-isomorphism. We call it *the* k_v^{ur}-model of X with stable reduction.

Remark 1.1. Let F_v/k_v be an unramified field extension.
(a) A k_v-curve X_v has good reduction [resp. stable reduction] if and only if the F_v-curve $X_v \otimes_{k_v} F_v$ has good reduction [resp. stable reduction]. The direct part is obvious (and does not need the hypothesis "F_v/k_v unramified"). The converse follows from the two following points. First, existence of a model with good reduction [resp. stable reduction] is a geometric property of the special fiber of the minimal model (*e.g* [Li] proposition 3). The second point is that the minimal model commutes with every *unramified* base change.

(b) If a k_v-curve X_v has good reduction [resp. stable reduction], then the $\overline{k_v}$-curve $X = X_v \otimes_{k_v} \overline{k_v}$ has k_v^{ur}*-good reduction* [resp. k_v^{ur}*-stable reduction*] with respect to k_v. But here the converse does not hold in general since the $\overline{k_v}$-curve X may have several k_v^{ur}-models and that the k_v-curve X_v may not induce *the* k_v^{ur}-model of X with stable reduction.

Finally recall that a k-curve $\widetilde{X}$ is said to have good reduction [resp. stable reduction] at some valuation v of k if the k_v-curve $\widetilde{X} \otimes_k k_v$ does. Similarly we say a $\overline{k}$-curve X of field of moduli k has ur-good reduction [resp. ur-stable reduction] at some valuation v of k if the $\overline{k_v}$-curve $X \otimes_{\overline{k}} \overline{k_v}$ has k_v^{ur}-good reduction [resp. k_v^{ur}-stable reduction] with respect to k_v for every k-embedding $\overline{k} \hookrightarrow \overline{k_v}$. Classically if k is a number field, then "bad reduction" may occur only at finitely many places of k.

Remark 1.2. (a) In definition above, given a place v of k, it suffices to check the reduction condition for only one k-embedding $\overline{k} \hookrightarrow \overline{k_v}$: indeed, as k is the field of moduli of the $\overline{k}$-curve X, two $\overline{k_v}$-curves $X \otimes_{\overline{k}} \overline{k_v}$ corresponding to two k-embeddings $\overline{k} \hookrightarrow \overline{k_v}$ are isomorphic. Note also that the k_v^{ur}-model of X with stable reduction, when it exists, does not depend either on the k-embedding $\overline{k} \hookrightarrow \overline{k_v}$.

(b) If ℓ/k is a finite sub-extension of $\overline{k}$ and X is a $\overline{k}$-curve of field of moduli k with ur-good reduction [resp. ur-stable reduction] at some place v of k, then the field of moduli of X relative to the extension $\overline{k}/\ell$ is ℓ and X has ur-good reduction [resp. ur-stable reduction] at every place of ℓ above v.

1.2. Statement of the main result.

Theorem 1.3 — *Let $g \geq 2$ be an integer, let K be a number field and let S be a finite set of places of K.*

There exists a finite extension L/K and a finite set $S_o = S_o(L)$ of places of K such that if X is any $\overline{K}$-curve of genus g, of field of moduli K and with ur*-stable reduction at all finite places $v \notin S$, then X has a L-model $\widetilde{X}$, which has these additional properties:*

Given a finite place v of K with $v \notin S_o$, if the $\overline{K}$-curve X has ur*-good reduction [resp.* ur*-stable reduction] at v, then the L-model $\widetilde{X}$ has good reduction [resp. stable reduction] at every extension w of v to L.*

Furthermore, $\mathrm{Aut}(\widetilde{X}) \simeq \mathrm{Aut}(X)$*, that is, the automorphisms of X are defined over L.*

Remarks 1.4. (a) Conclusion of theorem 1.3 fails if the reduction condition is removed. More specifically, there exist families of curves of genus $g \geq 2$ and field of moduli K such that no finite extension L/K is a field of definition for all curves of the family. Examples with $K = \mathbb{Q}$ and $g = 2$ can be obtained from a paper of J-F. Mestre [Me]. On the other hand, if K is a local field (finite extension of $\mathbb{Q}_p$), then it is true that all $\overline{K}$-curves of genus g with field of moduli K can be defined over a common finite field extension of K (without any reduction assumption).

(b) An explicit description of the extension L/K (which is not unique) can be obtained from the proof. Set $c_1(g) = 84(g-1)$. The extension L/K can be taken to be any finite extension L^2/K of some extension L^1/K such that the following holds:

- L^1/K is the compositum of all extensions of $K((\zeta_n)_{n \leq c_1(g)})$ of degree $\leq c_1(g)!$ and unramified above every place $v \notin S$ [1],

- for each place w of L_2 above some $v \in S$, the degree $[L^2_w : L^1_w]$ is a multiple of all exponents of abelian groups of order $\leq c_1(g)$.

(c) The exceptional set $S_o = S_o(L)$ can also be precisely described. It consists of places v of K of three types: those which are ramified in the extension L/K, those whose residue characteristic p is $\leq 84(g-1)$ and those from the finite list obtained in the following manner. Fix a full set of representatives $\mathcal{Q}_1, \ldots, \mathcal{Q}_h$ of the ideal class group of L and let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be all those prime ideals which appear (with a non-zero exponent) in the ideal decompositions of $\mathcal{Q}_1, \ldots, \mathcal{Q}_h$. The extra finite list is the set of all restrictions to K of valuations associated to $\mathcal{P}_1, \ldots, \mathcal{P}_r$.

2. Proof of the main descent theorem

Fix an integer $g \geq 2$, a number field K and a finite set S of places of K. Let then X be a $\overline{K}$-curve of genus g, of field of moduli K and assume that for each finite place $v \notin S$, X has ur-stable reduction at v.

[1] This compositum is a *finite* extension of K. Indeed it follows from Hermite's theorem that there are only finitely many finite extensions of a number field, unramified outside a finite set of places and with bounded degree. This classical result will be of frequent use in the paper.

2.1. 1st step: reduction to covers. Consider the $\overline{K}$-curve $B = X/\mathrm{Aut}(X)$ and the associated $\overline{K}$-cover $f : X \to B$. From [DeEm], there exists a K-model $\widetilde{B}$ of the curve B, called the *K-descent canonical model of $X/\mathrm{Aut}(X)$ over the field of moduli of X* and satisfying the following properties.

The field of moduli of the cover $f : X \to B$ (with K-base $\widetilde{B}$) is equal to K. Furthermore, a field k such that $K \subset k \subset \overline{K}$ is a field of definition of the $\overline{K}$-cover $f : X \to B$ with K-base $\widetilde{B}$ if and only if it is a field of definition of the $\overline{K}$-curve X.

More precisely, each k-model $\widetilde{X}$ of X induces a k-model $\widetilde{X} \to \widetilde{B} \otimes_K k$ of the $\overline{K}$-cover $f : X \to B$ (with fixed K-base $\widetilde{B}$), and vice-versa.

(The first part is stated in [DeEm] theorem 3.1; the second part is what is more precisely proven in [DeEm] p.47).

This reduces the problem to investigating fields of definition and models of the $\overline{K}$-cover $X \to B$, which is a $\overline{K}$-Galois cover with field of moduli K.

2.2. 2nd step: reducing to K_v^{ur}-G-covers for $v \notin S$. For each $v \notin S$, denote by X_v^{ur} the K_v^{ur}-model of $X \otimes_{\overline{K}} \overline{K_v}$ with stable reduction. It follows from the construction in [DeEm] that the K_v-descent canonical model of the $\overline{K_v}$-curve $(X \otimes_{\overline{K}} \overline{K_v})/\mathrm{Aut}(X \otimes_{\overline{K}} \overline{K_v})$ is isomorphic to $\widetilde{B} \otimes_K K_v$: the main point is that descent data on $(X \otimes_{\overline{K}} \overline{K_v})/\mathrm{Aut}(X \otimes_{\overline{K}} \overline{K_v})$ are obtained from those on $X/\mathrm{Aut}(X)$ by extension of scalars (see [DeEm]). Denote then by $f_v^{\mathrm{ur}} : X_v^{\mathrm{ur}} \to \widetilde{B} \otimes_K K_v^{\mathrm{ur}}$ the K_v^{ur}-model of $f \otimes_{\overline{K}} \overline{K_v}$ associated with the K_v^{ur}-curve X_v^{ur}.

Using [DelMu] theorem 1.3 we obtain next $\mathrm{Aut}(X_v^{\mathrm{ur}} \otimes_{K_v^{\mathrm{ur}}} \overline{K_v}) \simeq \mathrm{Aut}(X_v^{\mathrm{ur}})$. Therefore, the cover $f_v^{\mathrm{ur}} : X_v^{\mathrm{ur}} \to \widetilde{B} \otimes_K K_v^{\mathrm{ur}}$ is a K_v^{ur}-model of the Galois cover $f \otimes_{\overline{K}} \overline{K_v}$ *as G-cover* (for every K-embedding $\overline{K} \hookrightarrow \overline{K_v}$).

2.3. 3rd step: reduction to G-covers. The goal of this step is to show that the field of moduli of $f : X \to B$ *as G-cover* is a finite extension of K depending only on K, S and g. Denote this field of moduli by $K_{\mathrm{m}}^{\mathrm{G}}$. From 2nd step, we know that $K_{\mathrm{m}}^{\mathrm{G}} \subset K_v^{\mathrm{ur}}$ for all $v \notin S$ (for every K-embedding $\overline{K} \hookrightarrow \overline{K_v}$). We will now show that $[K_{\mathrm{m}}^{\mathrm{G}} : K]$ can be uniformly bounded.

Set $G = \mathrm{Aut}(X)$, $d = |G|$ and let $G \subset S_d$ be the regular representation of G. As in [DeDo1], we view the mere cover $X \to B$ as a representation $\phi : \pi_1(B^*) \to G \subset \mathrm{Nor}_{S_d}(G)$ of the $\overline{K}$-fundamental group of the variety B with the branch locus removed. Consider then the *constant extension map* $\lambda : \mathrm{G}_K \to \mathrm{Nor}_{S_d}(G)/(G\mathrm{Cen}_{S_d}(G))$ *modulo* $\mathrm{Cen}_{S_d}(G)$ *given by the field of moduli condition* [DeDo1;§3.1].

Let $v \notin S$ and $\sigma_v : \overline{K} \hookrightarrow \overline{K_v}$ be a K-embedding. As $f \otimes_{\overline{K}} \overline{K_v}$ is defined over K_v^{ur} as G-cover, the map λ becomes trivial after extending the scalars to K_v^{ur}: indeed, for these places, there is no extension of constants in the Galois closure (as $X_v^{\mathrm{ur}} \to \widetilde{B} \otimes_K K_v^{\mathrm{ur}}$ *is* Galois and regular). Equivalently the fixed field $\overline{K}^{\ker(\lambda)}$ of $\ker(\lambda)$ is contained in K_v^{ur}. This containment holds for all places $v \notin S$ and all K-embeddings $\overline{K} \hookrightarrow \overline{K_v}$. Since we have $[\overline{K}^{\ker(\lambda)} : K] \leq |\mathrm{Nor}_{S_d}(G)|/(|G||\mathrm{Cen}_{S_d}(G)|)$, conclude from Hermite's theorem that $\overline{K}^{\ker(\lambda)}$ is contained in a finite extension L^1/K depending only on K, S and G. Further this extension L^1/K can be required to depend only on K, S and g, as, in view of Hurwitz' theorem, $|G| = |\mathrm{Aut}(X)|$ can be bounded by $84(g-1)$. The next argument shows that $K_{\mathrm{m}}^{\mathrm{G}} \subset L^1$. With no loss of generality we may assume that $L^1 = K$, and so, $\lambda : \mathrm{G}_K \to \mathrm{Nor}_{S_d}(G)/(G\mathrm{Cen}_{S_d}(G))$ is trivial.

Recall the map λ is induced by the *representation* $\overline{\varphi} : \pi_1(\widetilde{B}^*) \to \mathrm{Nor}_{S_d}(G)/\mathrm{Cen}_{S_d}(G)$ *of* $\pi_1(\widetilde{B}^*)$ *given by the field of moduli condition* (of the mere cover $X \to B$) [DeDo1;§2.7]. Triviality of λ translates to this: for each element U of the K-fundamental group $\pi_1(\widetilde{B}^*)$, there exists $\varphi_{\mathrm{U}} \in G$ (whereas *a priori* $\varphi_{\mathrm{U}} \in \mathrm{Nor}_{S_d}(G)$) such that

$$\phi(x^{\mathrm{U}}) = \phi(x)^{\varphi_{\mathrm{U}}} \qquad \text{(for all } x \in \pi_1(B^*))$$

That indeed means K is the field of moduli of the Galois cover $X \to B$ *as G-cover.*

2.4. 4th step: descending to K_v for $v \notin S$. From the field of moduli condition (relative to the extension $\overline{K}/K$), for each $\tau \in \mathrm{Gal}(K_v^{\mathrm{ur}}/K_v)$, there exists a $\overline{K_v}$-isomorphism $\chi_\tau : f_v^{\mathrm{ur}} \otimes_{K_v^{\mathrm{ur}}} \overline{K_v} \to (f_v^{\mathrm{ur}})^\tau \otimes_{K_v^{\mathrm{ur}}} \overline{K_v}$ of G-covers (with fixed base $\widetilde{B} \otimes_K K_v$). From [DelMu] theorem 1.3, since both the K_v^{ur}-curves X_v^{ur} and $(X_v^{\mathrm{ur}})^\tau$ have stable $A_{K_v^{\mathrm{ur}}}$-models, we have $\mathrm{Isom}(X_v^{\mathrm{ur}} \otimes_{K_v^{\mathrm{ur}}} \overline{K_v}, (X_v^{\mathrm{ur}})^\tau \otimes_{K_v^{\mathrm{ur}}} \overline{K_v}) \simeq \mathrm{Isom}(X_v^{\mathrm{ur}}, (X_v^{\mathrm{ur}})^\tau)$. Conclude that the isomorphisms χ_τ can be defined over K_v^{ur} (with $\tau \in \mathrm{Gal}(K_v^{\mathrm{ur}}/K_v)$) and so K_v is the field of moduli of the K_v^{ur}-G-cover f_v^{ur} *relative to the extension* K_v^{ur}/K_v. As $\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v) \simeq \widehat{\mathbb{Z}}$ and hence is projective, we obtain that the mere cover f_v^{ur} has a K_v-model $f_v : X_v \to \widetilde{B} \otimes_K K_v$ ([DeDo1] corollary 3.3). Furthermore, the K_v-curve X_v, which is obviously a K_v-model of X_v^{ur} has stable reduction (Remark 1.1 (a)).

Note for later use that the content of this 4th step actually holds for all finite places v of K where the $\overline{K}$-curve X has ur-stable reduction (and not only for places $v \notin S$). Furthermore, if the $\overline{K}$-curve X has ur-*good* reduction at v, then the K_v-curve X_v has good reduction at v.

Remark 2.1. Arguments above apply to mere covers as well to yield the following:

Let (A_v, v) be a complete (or more generally, henselian) discrete valuation ring, let k_v be its fraction field and let X be a $\overline{k_v}$-curve of genus $g \geq 2$ and of field of moduli k_v. If X has k_v^{ur}-stable reduction with respect to k_v, then X has a k_v-model with stable reduction.

In this direction the following can be obtained from [DeHa] and [Em]: the same holds if the residue characteristic does not divide the order of $\mathrm{Aut}(X)$ and if the branch locus of the cover $X \to X/\mathrm{Aut}(X)$ is smooth at v. From results of Fulton [Fu], these assumptions ensure that X has k_v^{ur}-good reduction (*a fortiori* k_v^{ur}-stable reduction) with respect to k_v.

2.5. 5th step: controlling the obstruction for $v \in S$. With no loss of generality we may and will assume that K contains all n-th roots of unity with $n \leq 84(g-1)$. Fix a finite place $v \in S$ and a K-embedding $\overline{K} \hookrightarrow \overline{K_v}$. Consider the G-cover $f \otimes_{\overline{K}} \overline{K_v}$ obtained from f by extension of scalars. The field of moduli of $f \otimes_{\overline{K}} \overline{K_v}$ (relative to the extension $\overline{K_v}/K_v$) is K_v and the obstruction to K_v being a field of definition corresponds to the vanishing of a 2-cocycle $\Omega \in H^2(K_v, Z(G))$ (with trivial action) [DeDo1]. We explain below how to construct a finite extension L^2/K such that

(*) *for all places $v \in S$ and all K-embeddings $\overline{K} \hookrightarrow \overline{K_v}$, all 2-cocycles in $H^2(K_v, Z(G))$ become trivial in $H^2(L_v^2, Z(G))$, where $L_v^2 = K_v L^2$;*

in particular, L_v^2 will be a field of definition of $f \otimes_{\overline{K}} \overline{K_v}$ (as G-cover).

Let m be the h.c.m. of all exponents of $Z(G)$ where G runs through all groups of order $\leq 84(g-1)$. For every $v \in S$, pick a finite extension L^v of K_v such that m divides $[L^v : K_v]$. Using Krasner's lemma and the weak approximation theorem, one can find a Galois extension L^2/K such that $L^v \subset L_v^2$ for every $v \in S$ and every K-embedding $\overline{K} \hookrightarrow \overline{K_v}$. In order to establish the claim (*), restrict to the case $Z(G) = \mathbb{Z}/\ell\mathbb{Z}$ with $\ell | m$; note that K contains ℓ-roots of unity. Let then $\omega \in H^2(K_v, \mathbb{Z}/\ell\mathbb{Z})$; its image $\tilde{\omega}$ in

$$H^2(K_v, \mathbb{Z}/\ell\mathbb{Z}) = H^2(K_v, \mu_\ell) \subset \mathrm{Br}(K_v)$$

is a ℓ-torsion element. Conclude from [CaFr] ch.VI §1 corollary 1 that $\tilde{\omega}$ gives 0 in $\mathrm{Br}(L_v^2)$ and so induces a trivial 2-cocycle in $H^2(L_v^2, \mu_\ell) = H^2(L_v^2, \mathbb{Z}/\ell\mathbb{Z})$, for all places $v \in S$ and all K-embeddings $\overline{K} \hookrightarrow \overline{K_v}$.

2.6. 6th step: applying the local-global principle. Let L/K be the extension L^2/K constructed in 5th step from the extension L^1/K of 3rd step. We have reached a situation where we have a $\overline{K}$-G-cover $f : X \to B$ which is defined over all completions L_w of L, *i.e.*, such that $f \otimes_{\overline{K}} \overline{K_w}$ has a L_w-model, say f_w, for every place w of L (including the archimedean places). It follows then from theorem 3.8 of [DeDo1] that the G-cover f has a model over L (note that as $\sqrt{-1}$ was adjoined to L^2, the "Grunwald-Wang" special case of theorem 3.8 of [DeDo1] cannot occur).

Remark 2.2. The full strength of the local-global principle for G-covers (as in [DeDo1]) is not used here: as m-roots of unity have been adjoined to K, this reduces to injectivity of the local-global map $\mathrm{Br}(L) \to \oplus_w \mathrm{Br}(L_w)$ (see [DeDo2] proposition 3.4).

2.7. 7th step: twisting the global model. From the preceding step, the $\overline{K}$-G-cover f has L-models. In this final step, we show that any such L-model can be twisted so as to provide another L-model satisfying the additional reduction properties of theorem 1.3.

Let $\widetilde{f} : \widetilde{X} \to \widetilde{B} \otimes_K L$ be a L-model of the G-cover $f : X \to B$. Consider the set of places v of K, unramified in the extension L/K, with residue characteristic p not dividing $|Z(G)|$ and such that the $\overline{K}$-curve X has ur-stable reduction at v. Denote the complement of this set by S_1; it is a finite set.

For each $v \notin S_1$ fix a full system Σ_v of non K_v-conjugate K-embeddings $\sigma_v^L : L \hookrightarrow \overline{K_v}$ (corresponding to all extensions of v to L); and for each $\sigma_v^L \in \Sigma_v$ pick an extension $\sigma_v : \overline{K} \hookrightarrow \overline{K_v}$ of σ_v^L to $\overline{K}$. Consider the two following K_v^{ur}-models of the G-cover $f \otimes_{\overline{K}} \overline{K_v}$ (*via* the constant extension σ_v):

- on one hand, the K_v^{ur}-G-cover $f_v^{\mathrm{ur}} : X_v^{\mathrm{ur}} \to \widetilde{B} \otimes_K K_v^{\mathrm{ur}}$ where X_v^{ur} is *the* K_v^{ur}-model of $X \otimes_{\overline{K}} \overline{K_v}$ with stable reduction at v and where the constant extension is *via* σ_v; recall f_v^{ur} was proved to be a K_v^{ur}-model of the G-cover $f \otimes_{\overline{K}} \overline{K_v}$ in 2nd step,

- on the other hand, the K_v^{ur}-G-cover $\widetilde{f} \otimes_L K_v^{\mathrm{ur}} : \widetilde{X} \otimes_L K_v^{\mathrm{ur}} \to \widetilde{B} \otimes_K K_v^{\mathrm{ur}}$ where the constant extension is *via* σ_v^L.

Classically, the set of all models over a field k of a $\overline{k}$-G-cover of group G is "parametrized" by $H^1(k, Z(G))$ (with trivial action), or is empty (see [De1] proposition 2.5 for a precise statement). Thus, for $v \notin S_1$, the two K_v^{ur}-G-covers above differ by an element $\gamma_{v,\sigma_v^L} \in H^1(K_v^{\mathrm{ur}}, Z(G))$. Note that γ_{v,σ_v^L} is the trivial 1-cocycle in $H^1(K_v^{\mathrm{ur}}, Z(G))$ for all but finitely many places $v \notin S_1$. We are done if can show that the family $(\gamma_{v,\sigma_v^L})_{v,\sigma_v^L}$ is in the image of the morphism

$$H^1(L, Z(G)) \to \prod_{\substack{v \notin S_1 \\ \sigma_v^L \in \Sigma_v}} H^1(K_v^{\mathrm{ur}}, Z(G))$$

To show this we may reduce to the case $Z(G) = \mathbb{Z}/\ell\mathbb{Z}$ with ℓ dividing the exponent of $Z(G)$. The above map then identifies with the map

$$L^\times/(L^\times)^\ell \to \prod_{\substack{v \notin S_1 \\ \sigma_v^L \in \Sigma_v}} (K_v^{\mathrm{ur}})^\times/((K_v^{\mathrm{ur}})^\times)^\ell$$

For each place $v \notin S_1$, the group $(K_v^{\mathrm{ur}})^\times/((K_v^{\mathrm{ur}})^\times)^\ell$ is the cyclic group of order ℓ generated by the class of a uniformizing parameter of K_v (as $p \not| \, |Z(G)|$). This leaves us with showing that one can find elements of L with prescribed order at valuations from a finite set and that are units at all other finite places of L. This may not be true in general but it becomes true if some finitely many places of L are excluded. More precisely, let $\mathcal{Q}_1, \ldots, \mathcal{Q}_h$ be a full set of representatives of the ideal class group of L and let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be all those prime ideals which appear (with a non-zero exponent) in the decomposition of some of the $\mathcal{Q}_i$s; denote the associated valuations of L by $w_{\mathcal{P}_1}, \ldots, w_{\mathcal{P}_r}$. If $\mathcal{P}$ is the valuation ideal of some finite place, say $w_{\mathcal{P}}$, of L, then $\mathcal{P} = (\pi)\mathcal{Q}_i$ for some $i \in \{1, \ldots, h\}$ and $\pi \in L$. From the uniqueness of the ideal decomposition, if $\mathcal{P} \neq \mathcal{P}_1, \ldots, \mathcal{P}_r$, we have $w_{\mathcal{P}}(\pi) = 1$ and $w(\pi) = 0$ for all places w of L such that $w \neq w_{\mathcal{P}}, w_{\mathcal{P}_1}, \ldots, w_{\mathcal{P}_r}$.

Denote by S_o the set consisting of the restrictions to K of the exceptional places $w_{\mathcal{P}_1}, \ldots, w_{\mathcal{P}_r}$ and of the places of K that are ramified in L or have residue characteristic $p \leq 84(g-1)$. Construction of elements of L with prescribed order at finitely many places w of L (above some $v \notin S_1$) and that are units at all other finite places except possibly those lying above places in S_o readily follows from the above argument. Conclude that one can find a 1-cocycle $\gamma \in H^1(L, Z(G))$ such that the L-G-cover obtained from $\widetilde{f}$ by twisting by γ satisfies the reduction properties stated in theorem 1.3. Here we also use remark 1.1(a): to obtain the good [resp. stable] reduction of a L-model $\widetilde{X}$ at some place w of L above some $v \notin S_1$, it suffices to check the good [resp. stable] reduction of $\widetilde{X} \otimes_L K_v^{\mathrm{ur}}$.
□

Remark 2.3. Given a number field L and a prime ideal $\mathcal{P} \in \operatorname{Spec}(\mathcal{O}_L)$, one can show that there exists $f \in \mathcal{O}_L$ satisfying $f \notin \mathcal{P}$ and such that the localization $(\mathcal{O}_L)_f$ is a principal integral domain. This yields the following refinement in theorem 1.3: if w_o is a place of L unramified in the extension L/K and with residue characteristic $p > 84(g-1)$, then for any $\overline{K}$-curve as in theorem 1.3 and with ur-good reduction [resp. ur-stable reduction] at $w_o|_K$, a L-model of X can be found with good reduction [resp. stable reduction] at w_o.

3. Modular arithmetic geometry

In this section we combine our main descent theorem to classical finiteness results in arithmetic geometry to get some *modular* versions of these results.

3.1. The Shafarevich conjecture. A consequence of our main descent result (theorem 1.3) is the following *moduli form* of the Shafarevich conjecture.

Theorem 3.1 — *Let $g \geq 2$ be an integer, let K be a number field and let S be a finite set of places v of K. Denote the moduli space of genus g curves by $\mathcal{M}_g$. The subset of $\mathcal{M}_g(K)$ consisting of $\overline{K}$-curves of field of moduli K and with* ur*-good reduction at all places $v \notin S$ is finite.*

The classical form of the Shafarevich conjecture proved by Faltings [Fa] provides the same finiteness conclusion but for the set of K-curves with good reduction at places $v \notin S$. That is, the curves in question are supposed to be *defined* over K, while here the hypothesis concerns the field of definition of the corresponding points on $\mathcal{M}_g$, *i.e.*, the *field of moduli* of the curves. Theorem 3.1 readily follows from our theorem 1.3 conjoined with the classical Shafarevich conjecture.

Remark 3.2. The classical Shafarevich conjecture also straightforwardly follows from theorem 3.1, *if* one knows that given a K-curve $\widetilde{X}$ of genus $g \geq 2$ and with good reduction outside S, its $\overline{K}$-isomorphism class only contains finitely many other K-curves $\widetilde{X}'$ with good reduction outside S. This is of course a consequence of the Shafarevich conjecture but a weak one for which independent arguments can be given. For the convenience of the reader, we recall one.

Let $(\mathrm{Jac}(\widetilde{X}), \lambda(\widetilde{X}))$ [resp. $(\mathrm{Jac}(\widetilde{X}'), \lambda(\widetilde{X}'))$] be the polarized jacobian variety of $\widetilde{X}$ [resp. $\widetilde{X}'$]. Pick a $\overline{K}$-isomorphism between $X = \widetilde{X} \otimes_K \overline{K}$ and $X' = \widetilde{X}' \otimes_K \overline{K}$ and denote by ψ the $\overline{K}$-isomorphism induced between $(\mathrm{Jac}(X), \lambda(X))$ and $(\mathrm{Jac}(X'), \lambda(X'))$. Fix a prime number $\ell \geq 3$, denote the set of places of K above ℓ by S_ℓ and set $L_{\widetilde{X}'} = K(\mathrm{Jac}(\widetilde{X})[\ell](\overline{K}), \mathrm{Jac}(\widetilde{X}')[\ell](\overline{K}))$. Then, on one hand, the isomorphism ψ is defined over $L_{\widetilde{X}'}$ (see [Mi1] proposition 17.5 and [Si]), and, on the other hand, by the Neron-Ogg-Shafarevich theorem, the field $L_{\widetilde{X}'}$ is an extension of K of degree bounded by ℓ^{4g} and is unramified outside $S \cup S_\ell$ (the jacobians $\mathrm{Jac}(\widetilde{X})$ and $\mathrm{Jac}(\widetilde{X}')$ have good reduction outside S as the curves $\widetilde{X}$ and $\widetilde{X}'$ do). So, by Hermite's theorem, the compositum L of all possible $L_{\widetilde{X}'}$ is a finite extension of K depending only on K, S and g. Taking the Galois closure, one can further assume that L/K is Galois. Using Torelli's theorem, and more precisely corollary 12.2 of [Mi2], one obtains that $\widetilde{X} \otimes_K L$ and $\widetilde{X}' \otimes_K L$ are L-isomorphic. Now the set of K-curves $\widetilde{X}'$ which become L-isomorphic to $\widetilde{X} \otimes_K L$ after extending the scalars is parametrized by $H^1(\mathrm{Gal}(L/K), \mathrm{Aut}(\widetilde{X} \otimes_K L))$, which is a finite set (as $\mathrm{Gal}(L/K)$ and $\mathrm{Aut}(\widetilde{X} \otimes_K L)$ are).

3.2. The Mordell conjecture. One of Faltings' goals in proving the Shafarevich conjecture was to apply Parshin's construction to then deduce the Mordell conjecture.

In this paragraph we explain how our modular variant of the Shafarevich conjecture also implies a modular version of the Mordell conjecture.

We first define a modular notion of rational points.

Definition 3.3 — *Let X be a $\overline{K}$-curve of genus $g \geq 2$ and of field of moduli K. Let S be a finite set of places of K and $d \geq 1$ be an integer. A point $x \in X(\overline{K})$ is said to be (S,d)-rational on X if the following conditions are satisfied:*
(1) There exists a family $(f_\sigma)_{\sigma \in \mathrm{G}_K}$ of $\overline{K}$-isomorphisms $f_\sigma \in \mathrm{Isom}(X^\sigma, X)$ such that the set $\{f_\sigma(x^\sigma) \mid \sigma \in \mathrm{G}_K\}$ has at most d elements.
(2) For every place $v \notin S$ where X has ur*-stable reduction, if X_v^{ur} is the K_v^{ur}-model of $X \otimes_{\overline{K}} \overline{K_v}$ with stable reduction (for some K-embedding $\overline{K} \hookrightarrow \overline{K_v}$ [2]), then there exists a $\overline{K_v}$-isomorphism $\psi_v : X \otimes_{\overline{K}} \overline{K_v} \to X_v^{\mathrm{ur}} \otimes_{K_v^{\mathrm{ur}}} \overline{K_v}$ such that $\psi_v(x) \in X_v^{\mathrm{ur}}(K_v^{\mathrm{ur}})$.*
The set of all points $x \in X(\overline{K})$ that are (S,d)-rational on X will be denoted by $X(S,d)$.

Note that every $\overline{K}$-isomorphism $\chi : X \to X'$ between two $\overline{K}$-curves of genus $g \geq 2$ induces a one-one correspondence between $X(S,d)$ and $X'(S,d)$.

Also, if $\widetilde{X}$ is a K-curve of genus $g \geq 2$, if $X = \widetilde{X} \otimes_K \overline{K}$ and F/K is a finite extension, then we have $X(F) \subset X(S,d)$ where $d = [F:K]$ and S is the set of places of K that are ramified in F or where the K-curve $\widetilde{X}$ does not have stable reduction.

The following statement is the announced modular version of the Mordell conjecture.

Theorem 3.4 — *Let $g \geq 2$ be an integer, let K be a number field, let S and S' be two finite sets of places of K and let $d \geq 1$ be an integer. There exists a constant $c = c(g, K, S \cup S', d)$ such that for every $\overline{K}$-curve X of genus $g \geq 2$ of field of moduli K and with* ur*-good reduction for all $v \notin S'$, we have* $\mathrm{card}(X(S,d)) \leq c$.

Proof. As in the statement fix an integer $g \geq 2$, a number field K, two finite sets S and S' of places of K and an integer $d \geq 1$. We divide the proof into two stages.

1st stage: fixing the field of definition. Let L/K be a finite extension and let $\widetilde{X}$ be a L-curve of genus $g \geq 2$. Set $X = \widetilde{X} \otimes_L \overline{K}$ and assume that X is of field of moduli K. Let $x \in X(S,d)$.

Let $(f_\sigma)_{\sigma \in \mathrm{G}_K}$ be a family as in condition (1) of definition 3.3; for each $\sigma \in \mathrm{G}_L$, we have $f_\sigma \in \mathrm{Aut}(X)$ (as $X^\sigma = X$) and $x^\sigma = f_\sigma^{-1}(f_\sigma(x^\sigma))$. It follows then from $\mathrm{card}(\mathrm{Aut}(X)) \leq$

[2] It is easily checked that the condition does not depend on the K-embedding $\overline{K} \hookrightarrow \overline{K_v}$.

$84(g-1)$ and condition (1) that the set $\{x^\sigma | \sigma \in \mathrm{G}_L\}$ is finite and of cardinality $\leq 84(g-1)d$. Hence the extension $L(x)/L$ is of degree $\leq 84(g-1)d$.

Let v be a finite place of K, not in S and unramified in L. Assume further the L-curve $\widetilde{X}$ has stable reduction at every place w of L above v. Thus the corresponding K_v^{ur}-curve $\widetilde{X} \otimes_L K_v^{\mathrm{ur}}$ is the K_v^{ur}-model X_v^{ur} of $X \otimes_{\overline{K}} \overline{K_v}$ with stable reduction (for every K-embedding $\overline{K} \hookrightarrow \overline{K_v}$). From condition (2) of "$x \in X(S,d)$", we have $\psi_v(x) \in X_v^{\mathrm{ur}}(K_v^{\mathrm{ur}})$. Using again [DelMu] theorem 1.3, we obtain that the automorphism ψ_v of $X \otimes_{\overline{K}} \overline{K_v}$ comes from an automorphism of X_v^{ur}, *i.e.*, is defined over K_v^{ur}. Conclude then that $x \in \widetilde{X}(K_v^{\mathrm{ur}})$, or, equivalently, that $L(x) \subset K_v^{\mathrm{ur}}$ (for every K-embedding $\overline{K} \hookrightarrow \overline{K_v}$).

2nd stage: using theorem 3.1 and Faltings' theorems. By theorem 3.1, there exists a finite extension L/K (depending only on g, K and S') and a finite set of L-curves, say $\widetilde{X}_1, \ldots, \widetilde{X}_s$ of genus g such that any $\overline{K}$-curve X as in theorem 3.4 is $\overline{K}$-isomorphic to $X_i = \widetilde{X}_i \otimes_L \overline{K}$ for some index $i \in \{1, \ldots, s\}$. Furthermore, if S_o' is the union of the exceptional set S_o from theorem 1.3 and of the finite set of places of K which are ramified in L, then the L-curves $\widetilde{X}_1, \ldots, \widetilde{X}_s$ have good reduction outside the set of all places of L lying above places in $S_o' \cup S'$.

Let M/L be the compositum of all extensions F/L of degree $\leq 84(g-1)d$ and unramified at every finite place of L not above a place $v \in S \cup S' \cup S_o'$. By Hermite's theorem the extension M/L is finite. By 1st stage, we have $X_i(S,d) \subset \widetilde{X}_i(M)$, $i = 1, \ldots, s$. From the Mordell's conjecture proved by Faltings, the set $\widetilde{X}_i(M)$ is finite, $i = 1, \ldots, s$. Thus for $c(g, K, S \cup S', d) = \max_{1 \leq i \leq s} \mathrm{card}(\widetilde{X}_i(M))$, the proof is complete. □

3.3. Further finiteness results. The following statement is another consequence of our theorem 1.3. The special case $K_S = K$ corresponds to Faltings' finiteness theorem 1.

Theorem 3.5 — *Let $g \geq 2$ be an integer, let K be a number field and let S be a finite set of finite places of K. Let K_S be an algebraic extension of K unramified outside S. Then there exist only finitely many K_S-isomorphisms classes of K_S-curves X_S of genus g, of field of moduli K relative to the extension K_S/K and such that the jacobians* $\mathrm{Jac}(X_S)$ *belong to a given K_S-isogeny class.*

Proof. Fix a K_S-curve Y_S of genus $g \geq 2$ and of field of moduli K relative to the extension K_S/K. Let S_{Y_S} be a finite set of finite places of K containing S such that for every $v \notin S_{Y_S}$, the K_v^{ur}-curve $Y_S \otimes_{K_S} K_v^{\mathrm{ur}}$ has good reduction for every K-embedding $K_S \hookrightarrow K_v^{\mathrm{ur}}$. Fix $v \notin S_{Y_S}$. The K_v^{ur}-jacobian $\mathrm{Jac}(Y_S \otimes_{K_S} K_v^{\mathrm{ur}}) = \mathrm{Jac}(Y_S) \otimes_{K_S} K_v^{\mathrm{ur}}$ has good reduction for every K-embedding $K_S \hookrightarrow K_v^{\mathrm{ur}}$. Denote the set of K_S-curves X_S of genus g, of field of moduli K relative to K_S/K and such that $\mathrm{Jac}(X_S)$ and $\mathrm{Jac}(Y_S)$ are K_S-isogenous by $E_g(Y_S)$. Pick $X_S \in E_g(Y_S)$. From the Koizumi-Shimura theorem, the jacobian $\mathrm{Jac}(X_S \otimes_{K_S} K_v^{\mathrm{ur}}) = \mathrm{Jac}(X_S) \otimes_{K_S} K_v^{\mathrm{ur}}$ has good reduction for every K-embedding $K_S \hookrightarrow K_v^{\mathrm{ur}}$. From [DelMu] theorem 2.4, $X_S \otimes_{K_S} K_v^{\mathrm{ur}}$ has stable reduction for every K-embedding $K_S \hookrightarrow K_v^{\mathrm{ur}}$. It follows that the $\overline{K}$-curve $X = X_S \otimes_{K_S} \overline{K}$ has ur-stable reduction at v; note that K is the field of moduli of X relative to the extension $\overline{K}/K$ (as K_S is the field of moduli of X_S relative to K_S/K)).

Apply then theorem 1.3 to conclude that there exists a finite extension L/K depending only on K, g and S_{Y_S} and an exceptional set $S_o = S_o(L)$ of finite places of K such that X has a L-model $\widetilde{X}$ with stable reduction at every extension of a place $v \notin S_{Y_S} \cup S_o$. Let S' be the finite set of extensions to L of places v of K ramified in L or in $S_{Y_S} \cup S_o$. Let $w \in S'$ and v its restriction to K. The K_v^{ur}-curve $\widetilde{X} \otimes_L K_v^{\mathrm{ur}}$ is *the* K_v^{ur}-model of $X_S \otimes_{K_S} \overline{K_v}$ with stable reduction. Hence $\widetilde{X} \otimes_L K_v^{\mathrm{ur}}$ and $X_S \otimes_{K_S} K_v^{\mathrm{ur}}$ are isomorphic. So are the jacobians $\mathrm{Jac}(\widetilde{X} \otimes_L K_v^{\mathrm{ur}})$ and $\mathrm{Jac}(X_S \otimes_{K_S} K_v^{\mathrm{ur}})$. As the latter has good reduction at w, so does the former. Conclude that $\mathrm{Jac}(\widetilde{X})$ has good reduction at w (as in remark 1.1 (a)).

Fix a prime number $\ell \geq 3$. By the Neron-Ogg-Shafarevich theorem, $L(\mathrm{Jac}(\widetilde{X})[\ell](\overline{K}))$ is a finite extension of L of degree $\leq \ell^{2g}$ and is unramified outside $S' \cup S_\ell$, where S_ℓ is the set of finite places of L lying above ℓ. Therefore, from Hermite's theorem, there exists a finite extension M/L such that $L(\mathrm{Jac}(\widetilde{X})[\ell](\overline{K})) \subset M$ for every $X_S \in E_g(Y_S)$. Let Φ be a $\overline{K}$-isogeny between $\mathrm{Jac}(\widetilde{X}) \otimes_L \overline{K}$ and $\mathrm{Jac}(\widetilde{Y}) \otimes_L \overline{K}$. From a result already used in remark 3.2, Φ is in fact defined over $L(\mathrm{Jac}(\widetilde{X})[\ell](\overline{K}), \mathrm{Jac}(\widetilde{Y})[\ell](\overline{K}))$, and so over M. Hence for every $X_S \in E_g(Y_S)$, the M-jacobians $\mathrm{Jac}(\widetilde{X}) \otimes_L M$ and $\mathrm{Jac}(\widetilde{Y}) \otimes_L M$ are M-isogenous.

Use next Faltings' finiteness theorem 1 to obtain that the set of M-isomorphism classes of $\mathrm{Jac}(\widetilde{X}_M)$, where $\widetilde{X}_M = \widetilde{X} \otimes_L M$ and X_S runs through $E_g(Y_S)$, is finite. Now a given M-abelian variety A can be equipped with a M-polarization of degree bounded by a fixed constant, in only finitely many ways (up to M-isomorphism of polarized abelian varieties) ([NaNo], [Mi1] theorem 18.1). Conclude that, for the principal polarization $\lambda(\widetilde{X}_M)$ induced by $\widetilde{X}_M$ on $\mathrm{Jac}(\widetilde{X}_M)$, the set of M-isomorphism classes of $(\mathrm{Jac}(\widetilde{X}_M), \lambda(\widetilde{X}_M))$, where X_S ranges over $E_g(Y_S)$, is finite. Next use Torelli's theorem (again the form given in [Mi2]) to conclude that the corresponding set of M-isomorphism classes of $\widetilde{X}_M$ is finite. Hence the same is true for the set of MK_S-isomorphism classes of $X_S \otimes_{K_S} MK_S$, and so also for the set of K_S-isomorphism classes of X_S, where X_S ranges over $E_g(Y_S)$. □

There are other classical finiteness results in arithmetic geometry, concerning (polarized) abelian varieties in particular. Similar variants of these results can be given if the analog of theorem 1.3 is proved in the context of (polarized) abelian varieties.

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