Specialization and Localization in Inverse Galois Theory

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Specialization and localization have always been at the core of Inverse Galois Theory (IGT): Hilbert's Irreducibility Theorem, the Noether Program, the Grunwald Problem, the Hurwitz moduli space approach are prominent milestones in this context.

We focus on the situation that the base field is a number field. The goal of the talk, based on the diagram in §2, was twofold. First, explain how we see the two operations, specialization and localization, and the somehow inverse ones that are parametrization and lifting still structure the area. Second, present a series of new results which are part of a joint program with J. Koenig, F. Legrand and D. Neftin. In various situations, these results roughly show that the sets of Galois extensions obtained by specialization or/and localization from natural sets of geometric Galois covers of fixed group G (singletons, families, moduli spaces) are big (in some density sense), but also cannot be too big (e.g. they generally do not contain all Galois extensions of group G).

The diagram in §2 displays a number of IGT properties for a finite group G over a given number field k. The abbreviations used for these properties refer to our two part glossary where they are fully defined: §1 for the classical ones and §3 for the more recent ones. For example:

IGP (Inverse Galois Problem): *There is a Galois extension E/k of group G.*

Left side of our diagram is more geometric than the right side; presence of indeterminates is the recognition sign of the former. Specialization connects the two. We specialize a k-regular Galois extension $F/k(T)$ or the corresponding k-cover $f : X \to \mathbb{P}^1_T$ in two ways:
- for $t_0 \in k$, F_{t_0}/k, also denoted by f_{t_0}, is the classical specialized extension of F at t_0: the residue field extension at some prime ideal above t_0 in the extension $F/k(T)$. As number fields are Hilbertian (HIT), the extension F_{t_0}/k is Galois of group G for "many" $t_0 \in k$.
- if $T_0 \in k(U)\setminus k$, $f_{T_0} : X_{T_0} \to \mathbb{P}^1_U$ is the pull-back of f along $T_0 : \mathbb{P}^1 \to \mathbb{P}^1$. As $k(U)$ is Hilbertian, for "many" $T_0 \in k(U)$, X_{T_0} is connected and the function field extension $k(X_{T_0})/k(U)$, which is equivalently obtained by specializing T to $T_0(U)$ in $k(X)/k(T)$, is Galois of group G.

1. Classical properties

RIGP (Regular IGP): *There exists a k-regular Galois extension $F/k(T)$ of group G (k-regular: $F \cap \overline{k} = k$), or, equivalently, a k-regular Galois cover $f : X \to \mathbb{P}^1_k$ of group G.*

HIT (Hilbert Irreducibility Theorem): *For every polynomial $P(T, Y)$, irreducible in $k(T)[Y]$, there exist infinitely many $t_0 \in k$ such that $P(t_0, Y)$ is irreducible in $k[Y]$.*

G **has a parametric extension** $F/k(T)$: *There is a Galois extension $F/k(T)$ of group G that is k-parametric, i.e., every Galois extension E/k of group contained in G is the specialized extension F_{t_0}/k of $F/k(T)$ at some point $\mathrm{t}_0 \in k$.* **Example:** *$k(\sqrt{T})/k(T)$ for $G = \mathbb{Z}/2\mathbb{Z}$.*

G **has a generic extension** $F/k(T)$: *There exists a Galois extension $F/k(T)$ of group G such that $FK/K(T)$ is K-parametric for every field extension K/k.*

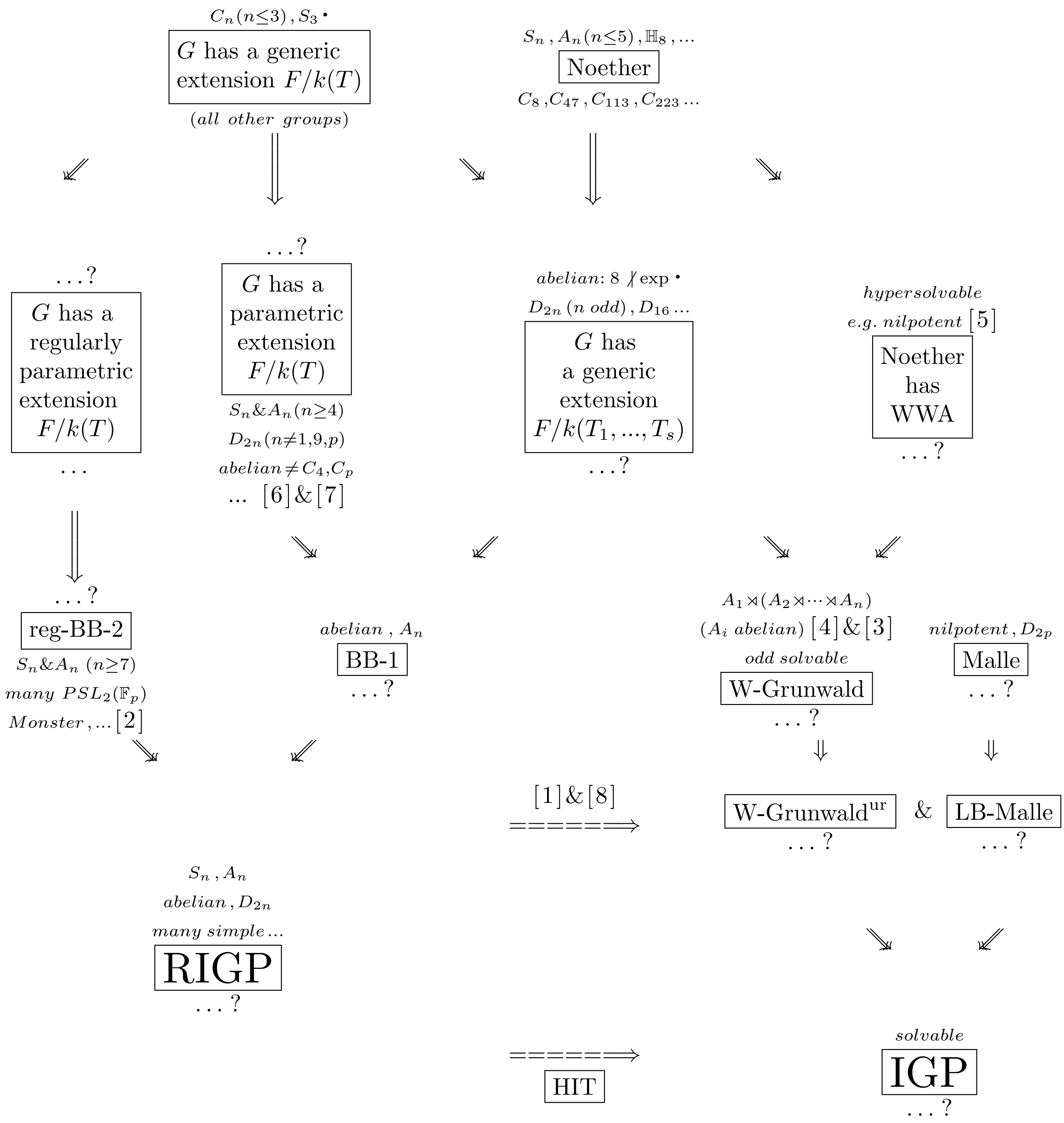
G **has a generic extension** $F/k(T_1, \ldots, T_s)$: as above with T replaced by $T_1, \ldots, T_s$ and t_0 by $\mathbf{t}_0 = (t_{01}, \ldots, t_{0s})$.

Noether: *If $\mathbf{Y} = Y_1, \ldots, Y_d$ are $d = |G|$ indeterminates, the fixed field $k(\mathbf{Y})^G$ of G in $k(\mathbf{Y})$, with $G \hookrightarrow S_d$ acting via its regular representation, is a pure transcendental extension of k. Equivalently the* Noether variety *$\mathbb{A}^d/G$ is k-rational.*

Noether has WWA: *The Noether variety $V = \mathbb{A}^d/G$ has the* Weak Weak Approximation property*: there is a finite set $\mathcal{S}_{\mathrm{exc}}$ of finite places of k such that for every finite set $\mathcal{S}$ of finite places of k, disjoint from $\mathcal{S}_{\mathrm{exc}}$, the set $V(k)$ is dense in $\prod_{v\in S} V(k_v)$.*

2. The diagram

The implication arrows show the hierarchy between the properties. Groups appearing above a given box satisfy the corresponding property, those appearing below do not, both over $k = \mathbb{Q}$. The symbol ... (resp. $\bullet$) means that the list is open (resp. closed), possibly as a question if used with a question mark. The main recent results are those assertions about groups satisfying or not satisfying a property which come with a reference. The references are given in the end.



3. More recent properties

W-Grunwald: *There is a finite set* $\mathcal{S}_{\rm exc}$ *of finite places of* k *such that for every finite set* $\mathcal{E} = \{E_i/k_{v_i} | i = 1, \ldots, m\}$ *of Galois extensions* E_i/k_{v_i} *of group* $H_i \subset G$*, with* $v_i \notin \mathcal{S}_{\rm exc}$*, there is an extension* E/k *of group* G *such that* $Ek_{v_i}/k_{v_i} = E_i/k_{v_i}$, $i = 1, \ldots, m$. (The original Grunwald property, *i.e.* with $\mathcal{S}_{\rm exc} = \emptyset$, is denoted by **Grunwald**).

W-Grunwald$^{\rm ur}$: *The property* **W-Grunwald** *above but with the additional condition that the extensions* E_i/k_{v_i}, $i = 1, \ldots, N$, *are unramified.*

BB-N (Beckmann-Black Arithmetic lifting property): *For the given integer* $N \geq 1$ *and every* N *Galois extensions* $E_1/k, \ldots, E_N/k$ *of group contained in* G*, there exists a* k*-regular Galois extension* $F/k(T)$ *of group* G *that specializes to the extensions* $E_1/k, \ldots, E_N/k$.

G **has a regularly parametric extension** $F/k(T)$: *The corresponding* k*-regular Galois cover* $f : X \to \mathbb{P}^1_k$ *(of function field extension* $F/k(T)$*) has this property: every* k*-regular Galois cover* $g : Y \to \mathbb{P}^1_k$ *of group* G *is some rational pullback* $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ *of* f *(for some* T_0 *in* $k(U) \setminus k$*). Equivalently, every* k*-regular Galois extension* $L/k(U)$ *of group* G *can be obtained from the* k*-regular Galois extension* $F/k(T)$ *by specializing* $F(U)/k(U,T)$ *at some* T_0 *in* $k(U) \setminus k$.

reg-BB-N (Regular Beckmann-Black lifting property): *For the given integer* $N \geq 1$ *and every* N k*-regular Galois covers* $g_1 : Y_1 \to \mathbb{P}^1, \ldots, g_N : Y_N \to \mathbb{P}^1$ *of group* G*, there exists a* k*-regular Galois cover* f *of group* G *such that* $g_1, \ldots, g_N$ *are rational pullbacks of* f.

Malle: *The number* $N(G, y)$ *of sub-Galois extensions* E/k *of* $\overline{k}$ *of group* G *and discriminant of norm* $|N_{k/\mathbb{Q}}(d_E)| \leq y$ *satisfies*

$$c_1\, y^{a(G)} \leq N(G,y) \leq c_2\, y^{a(G)+\varepsilon} \qquad \textit{for every } y \geq y_0$$

Here $a(G) = (|G|(1 - 1/\ell))^{-1}$ *with* ℓ *the smallest prime divisor of* $|G|$ *and* c_1, c_2, $y_0 > 0$ *depend on* G *for* c_1 *and on* G, ε *for* c_2 *and* y_0.

LB-Malle (Lower Bound part of **Malle** conjecture):

$$N(G,y) \geq y^{\alpha(G)} \qquad \text{for every } y \geq y_0$$

Here $\alpha(G)$ *and* y_0 *are positive constants depending on* G.

Complement: We refer to `http://math.univ-lille1.fr/~pde/pub.html`— item 57 for the sequence of slides (converging to the diagram) used during the talk and for a more detailed description of our research project.

References

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