Gerbes and Covers

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Abstract. We use the theory of gerbes to provide a more conceptual approach to questions about models of a cover and their fields of definition.

Résumé. Nous utilisons la théorie des gerbes pour proposer une approche plus conceptuelle à des questions liées aux modèles d'un revêtement et leurs corps de définition.

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Introduction

Let K be a field and B be a regular projective geometrically irreducible K-variety. In this paper we attach a gerbe $\mathcal{G}(f)$ to each finite branched cover $f : X \to B$ defined over the separable closure K_s of K. This gerbe contains all the information about the fields of definition and the associated models of the cover f. For example the gerbe $\mathcal{G}(f)$ has a section if and only K is a field of definition of the cover f.

Previous papers of ours were devoted to fields of definition of covers ([DeDo1], [DeDo2]). Our main tool was classical Galois cohomology. Here we use the theory of gerbes, which was developed by Giraud [Gi], to propose a more conceptual approach and suggest that the notion of gerbe is the right theoretical tool to describe the models of a cover and their fields of definition. This had been noted before by M. Fried ([Fr;p.58]) but, to our knowledge, has not been developed since then.

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1. Preliminaries

We briefly recall some basics regarding covers (§1.1) and gerbes (§1.2).

1.1. Covers (see [DeDo1;§2] for more details).

1.1.1. Mere covers and G-covers. Let K be a field and B be a regular projective geometrically irreducible variety defined over K. A mere cover of B over K is a finite and generically unramified morphism $f : X \to B$ defined over K with X a normal and geometrically irreducible variety. A G-cover of B of group G over K is a Galois cover $f : X \to B$ over K given together with an isomorphism $h : G \to \mathrm{G}(K(X)/K(B))$ of G with the Galois group of the associated function field extension (over K). An isomorphism between two mere covers $f : X \to B$ and $f : X' \to B$ over K is an algebraic isomorphism $\chi : X \to X'$, defined over K and such that $\chi \circ f' = f$. An isomorphism of G-covers of group G over K is an isomorphism of mere covers that commutes with the given actions of G.

A mere cover $f : X \to B$ over a separably closed field k has two basic geometric invariants, which only depend on the isomorphism class of the cover. First the *group* G of the cover, *i.e.*, the Galois group $\mathrm{G}(\widehat{k(X)}/k(B))$ of the Galois closure $\widehat{k(X)}/k(B)$ of the field extension $k(X)/k(B)$. Second, the *ramification locus* D of the cover; from the "Purity of Branch Locus" it is a divisor of B with simple components. By invariants of a cover over a non algebraically closed field K, we always mean the invariants of the cover over the separable closure K_s of K obtained by extension of scalars.

The ramification locus D of a cover over K is defined over K. The K-affine variety $B-D$ is denoted by B^*. If F is any field containing K, the F-arithmetic fundamendal group of B^* is denoted by $\Pi_F(B^*)$ or simply by Π_F when the context is clear. Degree d mere covers of B over F with ramification locus in D correspond to transitive representations $\Psi : \Pi_F(B^*) \to S_d$ such that the restriction to $\Pi_{F_s}(B^*)$ is transitive. G-covers of B of group G over F correspond to surjective homomorphisms $\Phi : \Pi_F(B^*) \to G$ such that $\Phi(\Pi_{F_s}(B^*)) = G$.

1.1.2. Descent of the field of definition of (G-)covers. As in [DeDo1], we use the word "(G-)cover" for the phrase "mere cover [resp. G-cover]". Suppose given a (G-)cover $f : X \to B$ *a priori* defined over K_s with ramification locus D defined over K. Descent of the field of definition of the (G-)cover f can be handled simultaneously for both the mere cover and G-cover situations.

In both cases let G denote the group of the cover. Then set

$$N = \begin{cases} G & \text{in the G-cover case} \\ Nor_{S_d} G & \text{in the mere cover case} \end{cases}$$

$$C = Cen_N G = \begin{cases} Z(G) & \text{in the G-cover case} \\ Cen_{S_d} G & \text{in the mere cover case} \end{cases}$$

where $Z(G)$ is the center of G and $Nor_{S_d}G$ and $Cen_{S_d}G$ are respectively the normalizer and the centralizer of G in S_d. Finally regard N as a subgroup of S_d where d is the degree of f: in the mere cover case, an embedding $N \hookrightarrow S_d$ is given by definition; in the G-cover case, embed $N = G$ in S_d by the regular representation of G.

Then, in both the mere cover and G-cover situations, we have the following:

(1) *(a) the (G-)cover $f : X \to B$ corresponds to an homomorphism (or representation)*

$$\phi : \Pi_{K_s} \twoheadrightarrow G \subset N$$

(b) the (G-)cover f is defined over the field K if and only if the homomorphism $\phi : \Pi_{K_s} \to G \subset N$ extends to an homomorphism $\Pi_K \to N$,

(c) Two (G-)covers over K_s are isomorphic if and only if the corresponding representations ϕ and ϕ' are conjugate by an element φ in the group N, i.e., [1]

$$\phi'(x) = \phi(x)^{\varphi} \text{ for all } x \in \Pi_{K_s}$$

1.1.3. Field of moduli. As above, let $f : X \to B$ be a mere cover [resp. G-cover] *a priori* defined over K_s. For each τ in the absolute Galois group $\mathrm{G}(K)$ of K, let $f^{\tau} : X^{\tau} \to B^{\tau}$ denote the corresponding conjugate (G-)cover. Consider the subgroup $M(f)$ [resp. $M_G(f)$] of $\mathrm{G}(K)$ consisting of all the elements $\tau \in \mathrm{G}(K)$ such that the covers [resp., the G-covers] f and f^{τ} are isomorphic over K_s. Then the *field of moduli* of the cover f [resp., the G-cover f] is defined to be the fixed field $K_s^{M(f)}$ [resp. $K_s^{M_G(f)}$] of $M(f)$ [resp. $M_G(f)$] in K_s. The field of moduli of a (G-)cover is a finite extension of K contained in each field of definition containing K. So it is the smallest field of definition containing K provided that it is a field of definition. The ramification locus D of f is automatically invariant under $M(f)$ [resp. $M_G(f)$].

Assume $\phi : \Pi_{K_s} \to G \subset N$ is the homomorphism corresponding to the (G-)cover $f : X \to B$ over K_s. Then K is the field of moduli of the (G-)cover f if and only if the ramification locus D is $\mathrm{G}(K)$-invariant and the following condition — called the *field of moduli condition* —, holds.

(FMod) For each $\mathrm{U} \in \Pi_K$, there exists $\varphi_{\mathrm{U}} \in N$ such that

$$\phi(x^{\mathrm{U}}) = \phi(x)^{\varphi_{\mathrm{U}}} \qquad \text{(for all } x \in \Pi_{K_s})$$

[1] We use the exponent notation $a^b = bab^{-1}$ for actions by conjugation in a group G.

For each $\mathrm{u} \in \Pi_K$, the element $\varphi_{\mathrm{u}} \in N$ is well-defined modulo an element of $C = Cen_N G$. The well-defined map $\overline{\varphi} : \Pi_K \to N/C$ that maps each $\mathrm{u} \in \Pi_K$ on the coset of $\varphi_{\mathrm{u}} \in N$ modulo C is a group homomorphism, which is called *the representation of* Π_K *modulo* C *given by the field of moduli condition* [DeDo1;§2.7]. If f_K is any K-model of the (G-)cover f, then the associated representation $\phi_K : \Pi_K \to N$ induces the map $\overline{\varphi}$ modulo C.

1.2. Gerbes (see [Gi] for a detailed treatment).

1.2.1. Basic definitions [Gi;Ch.II §1]. Let S be an étale site (*e.g.* [Ta]). A fibered category $\mathcal{G}$ [2] of groupoids over S (*i.e.*, for each open subset $U \subset S$, the U-fiber $\mathcal{G}(U)$ is a groupoid [3]) is called

• a *prestack* if, for each open subset $U \subset S$, and all objects x, y in the category $\mathcal{G}(U)$, $\mathrm{Hom}(x, y)$ is a sheaf,

• a *stack* if in addition the following similar local-to-global condition holds for objects: given any open subset $U \subset S$, any open covering $(U_i)_{i\in I}$ of U and any family $(x_i)_{i\in I} \in \prod_i \mathcal{G}(U_i)$, if for all $i, j \in I$, there exists an isomorphism φ_{ij} between the restrictions of x_i and x_j to $U_i \cap U_j$ and that $\varphi_{ij} = \varphi_{ik}\varphi_{kj}$ over $U_i \cap U_j \cap U_k$ ($k \in I$), then there exists $x \in \mathcal{G}(U)$ such that the restriction of x to each U_i is isomorphic to x_i ($i \in I$).

Given an open subset $U \subset S$, we call U-objects [resp. U-morphisms] the objects [resp. morphisms] of the category $\mathcal{G}(U)$; U-objects are also traditionally called sections of $\mathcal{G}$ above U.

A *S-gerbe* is a stack $\mathcal{G}$ over S satisfying the following additional conditions [Gi;p.129]:

(i) any two sections above an open subset U are locally isomorphic, *i.e.*, there exists an open subset V of U such that the restrictions to V of the two given sections are isomorphic in the category $\mathcal{G}(V)$.

(ii) locally each fiber is nonempty, *i.e.*, each open subset U of S admits an open subset V such that the fiber above V is nonempty [4].

A S-gerbe $\mathcal{G}$ is said to have a *section* if the fiber $\mathcal{G}(S)$ above S is nonempty. The gerbe $\mathcal{G}$ is then also said to be *neutral*. Here we will consider gerbes on the étale site $S = \mathrm{Spec}(K)_{\mathrm{et}}$ of a field K, K-gerbes for short. Recall that $\mathrm{Spec}(K)_{\mathrm{et}}$ is the site with associated open covers all the morphisms $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ where L/K is any étale algebra, *i.e.*, any product of separable field extensions (see [Ta;p.86] or [Sh;p.204] for a formal definition).

2 It is part of our definition that in a fibered category, pullbacks exist and are unique [Gi;p.18].

3 A groupoid is a category such that all morphisms are invertible.

4 Of course there may be some empty fibers.

1.2.2. The embedding problem example. To every embedding problem for the absolute Galois group $\mathrm{G}(K)$

$$\begin{array}{ccccccccc} & & & & & & \mathrm{G}(K) & & \\ & & & & & & \downarrow p & & \\ 1 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 1 \end{array} \tag{2}$$

can be attached the following K-gerbe: for each object $U = (\mathrm{Spec}(E) \rightarrow \mathrm{Spec}(K))$ of the étale site $\mathrm{Spec}(K)_{\mathrm{et}}$, with E/K a finite Galois extension,
- the *U-objects* of the gerbe are the weak[5] solutions $f_E : \mathrm{G}(E) \rightarrow B$ of the embedding problem for $\mathrm{G}(E)$ obtained by composing p with the inclusion map $\mathrm{G}(E) \hookrightarrow \mathrm{G}(K)$
- the *U-morphisms* between two such objects f_E and f'_E are the elements $\varphi \in A$, such that $f'_E = \varphi f_E \varphi^{-1}$.

This indeeds defines a gerbe: note that p becomes trivial after extension of scalars to the fixed field E_o of $\ker(p)$ in $\overline{K}$ (*i.e.*, p restricts to 1 on the open subset $\mathrm{Spec}(E_o) \rightarrow \mathrm{Spec}(K)$); conditions (i) and (ii) defining gerbes clearly follow. For a formal proof see [Gi;Ch.VIII §7.4]; is considered there the situation of extensions of $\mathrm{G}(K)$ (*i.e.*, with notation above, the case $C = \mathrm{G}(K)$ and $p = \mathrm{Id}$) but one classically reduces to that case by noting that weak solutions to embedding problems as above exactly correspond to sections of the following exact sequence, obtained by pulling back along p the exact sequence in (2),

$$1 \rightarrow A \rightarrow p^*(B) \rightarrow \mathrm{G}(K) \rightarrow 1$$

The gerbe attached to an embedding problem is neutral if and only if the embedding problem has a weak solution over K.

1.2.3. Band [Gi;Ch.IV §2.2]. For each open subset $U \subset S$, denote the category of U-sheaves of groups by $\mathrm{FAGR}(U)$. Then define FAGR to be the stack over S with the category $\mathrm{FAGR}(U)$ above any given open subset $U \subset S$. Consider then the prestack over S defined as follows: for each open subset $U \subset S$, the category above U has the same objects as $\mathrm{FAGR}(U)$ but the morphisms between two objects F and G are the cosets of $\mathrm{Hom}(F, G)$ modulo the action of inner automorphisms F and G (see [Gi;Ch.IV §1.1]). Through a standard procedure [Gi; Ch.II Th.2.1.3 and §2.1.3.2], a stack can be attached to that prestack. We will denote by Li the resulting stack over S. A S-band $\mathcal{L}$ is a section

[5] By "weak" we mean that the solution f_E need not be surjective.

of Li above S, *i.e.*, an object of the category $\mathrm{Li}(S)$. We denote the natural stack morphism $\mathrm{FAGR} \to \mathrm{Li}$ by $\mathcal{B}$.

If $S = \mathrm{Spec}(K)_{\mathrm{et}}$ then a S-band $\mathcal{L}$ corresponds to a group A endowed with a homomorphism $\mathrm{G}(K) \to Out(A)$ [Gi; Ch.VIII Prop.6.1.2]. Here $\mathrm{Out}(A)$ denotes the outer automorphism group of A, *i.e.*, $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$.

To each S-gerbe $\mathcal{G}$ can be attached a *band* $\mathcal{L}(\mathcal{G})$ in the following way. For each open subset U of S, define a functor from the category $\mathcal{G}(U)$ in the category $\mathrm{FAGR}(U)$ by mapping each U-object x of $\mathcal{G}(U)$ to the U-sheaf $\mathrm{Aut}(x)$ of automorphisms of x; the functor maps each U-morphism $i : x \to y$ to the U-morphism $\mathrm{inn}(i) : \mathrm{Aut}(x) \to \mathrm{Aut}(y)$ sending $a \in \mathrm{Aut}(x)$ to $iai^{-1} \in \mathrm{Aut}(y)$. Compose it with the functor $\mathcal{B}(U) : \mathrm{FAGR}(U) \to \mathrm{Li}(U)$ to obtain a functor $\mathcal{G}(U) \to \mathrm{Li}(U)$. Denote by $\mathcal{L}_x$ the image of x by this functor. It follows from property (i) of gerbes and the construction of Li that, given two objects x and y of the category $\mathcal{G}(U)$, there exists an open subset V of U such that the restrictions of $\mathcal{L}_x$ and $\mathcal{L}_y$ are equal in the category $\mathrm{Li}(V)$ [Gi; Ch.IV §2.2.2.2]. The fact that Li is a stack makes it possible to patch the objects $\mathcal{L}_x$ (where x ranges over the objects of $\mathcal{G}(U)$ and U runs over the open subsets of S) to yield a global section of Li above S, *i.e.*, a S-band $\mathcal{L}(\mathcal{G})$ called the band of the gerbe $\mathcal{G}$; the gerbe $\mathcal{G}$ is also said to be bound by $\mathcal{L}(\mathcal{G})$.

For example, the band of the gerbe $\mathcal{G}$ attached to an embedding problem for $\mathrm{G}(K)$ (§1.2.2) can be viewed as the homomorphism $\mathrm{G}(K) \to \mathrm{Out}(A)$ obtained by composing the homomorphism $C \to \mathrm{Out}(A)$ attached to the exact sequence in (2) with p [Gi;Ch.VIII §7.3].

1.2.4. Representable band. A S-band $\mathcal{L}$ is said to be *representable* if there exists a S-sheaf of groups H such that $\mathcal{L} = \mathcal{B}(H)$ [Gi;Ch.IV Def.1.2.1]. A S-band $\mathcal{L}$ is always *locally representable* by a S-sheaf of groups H, *i.e.*, for each open subset $U \subset S$, there exists an open subset $V \subset U$ such that $\mathcal{L}_V = \mathcal{B}(H_V)$ (where the subscripts V indicate we take the restriction to V). A S-band $\mathcal{L}$ locally representable by a S-sheaf of groups H induces an element of $H^1(S, \mathrm{Out}(H)))$ [Gi;Ch.IV Cor.1.1.7.3].

A gerbe $\mathcal{G}$ is said to be *locally bound* by a S-sheaf of groups H if its band $\mathcal{L}(\mathcal{G})$ is locally representable by H. Equivalently, for each open subset U of S and each U-object x, there exists an open subset $V \subset U$ such that $\mathrm{Aut}(x_V) \simeq H_V$ (in other words, the sheaf of groups $\mathrm{Aut}(x)$ is locally isomorphic to H).

For example the band of the gerbe $\mathcal{G}$ attached to an embedding problem for $\mathrm{G}(K)$ (§1.2.2) is representable if and only if the outer homomorphism $\mathrm{G}(K) \to \mathrm{Out}(A)$ can be lifted to an actual action $\mathrm{G}(K) \to \mathrm{Aut}(A)$. Such lifts exist locally, *i.e.*, for $\mathrm{G}(K)$ replaced by $\mathrm{G}(E)$ with E a suitably large Galois extension of K.

1.2.5. Morphisms of gerbes [Gi;Ch.IV]. Morphisms of gerbes are morphisms for the underlying stack structure. If $f : S' \to S$ is a morphism between the two étale sites S and

S' and $\mathcal{G}$ is a S-gerbe, the pull-back $f^*(\mathcal{G})$ is a S'-gerbe. Direct images of gerbes are not gerbes in general.

Consider two gerbes $\mathcal{G}$ and $\mathcal{G}'$ over the same site S. Suppose also given a morphism $\rho : \mathcal{L} \to \mathcal{L}'$ (in the category $\mathrm{Li}(S)$) between their respective bands $\mathcal{L}$ and $\mathcal{L}'$. For each open subset $U \subset S$, denote the groupoid of U-morphisms from the gerbe $\mathcal{G}$ to the gerbe $\mathcal{G}'$ that induce the band morphism ρ by $\mathrm{Hom}_\rho(\mathcal{G},\mathcal{G}')(U)$. Consider then the fibered category $\mathrm{Hom}_\rho(\mathcal{G},\mathcal{G}')$ over S with fiber $\mathrm{Hom}_\rho(\mathcal{G},\mathcal{G}')(U)$ above each open subset $U \subset S$. From [Gi;Ch.IV,Th.2.3.2 (i)], $\mathrm{Hom}_\rho(\mathcal{G},\mathcal{G}')$ is a S-gerbe.

Furthermore, using [Gi;Ch.IV Th.2.3.2 (ii) and Prop.3.3.8], it can be shown that this gerbe is bound by the S-centralizer sheaf $\mathrm{Cen}_{\mathcal{L}'}(\rho(\mathcal{L}))$ (regarded in the category $\mathrm{Li}(S)$). In particular, when $\rho = \mathrm{Id}_{\mathcal{L}}$, the corresponding gerbe, denoted then by $\mathrm{Hom}_{\mathcal{L}}(\mathcal{G},\mathcal{G})$, is bound by the S-center sheaf $Z(\mathcal{L})$ (regarded in the category $\mathrm{Li}(S)$).

1.2.6. 2nd cohomology sets [Gi;Ch.IV]. Given a band $\mathcal{L}$, two gerbes $\mathcal{G}$ and $\mathcal{G}'$ are said to be $\mathcal{L}$-equivalent if $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{G}') = \mathcal{L}$ and if there exists an isomorphism of gerbes $\mathcal{G} \to \mathcal{G}'$ that induces the identity on the band $\mathcal{L}$. For example, from [Gi;Ch.III Cor.2.2.6], if a gerbe $\mathcal{G}$ has a section s above S (*i.e.*, is neutral), then it is $\mathcal{L}$-equivalent to the gerbe $\mathrm{Tors}(\mathrm{Aut}(s))$ of torsors of $\mathrm{Aut}(s)$; and its band is then representable by $\mathrm{Aut}(s)$.

Given a S-band $\mathcal{L}$, a general notion of cohomological set $H^2(S,\mathcal{L})$ can be defined. The set $H^2(S,\mathcal{L})$ consists of all the $\mathcal{L}$-equivalence classes of gerbes. The class of the gerbe $\mathcal{G}$ in $H^2(S,\mathcal{L})$ will be denoted by $[\mathcal{G}]$. The class $[\mathcal{G}]$ of a gerbe with a section over S is called a neutral class of $H^2(S,\mathcal{L})$. The subset of $H^2(S,\mathcal{L})$ consisting of all neutral classes is denoted by $H^2(S,\mathcal{L})'$. In general, the class $[\mathcal{G}] \in H^2(S,\mathcal{L})$ should be regarded as the obstruction to the existence of a section (over S) for the gerbe $\mathcal{G}$.

1.2.7. Existence of neutral classes. In contrast with the abelian situation, the set $H^2(S,\mathcal{L})'$ of neutral classes may consist of several elements and possibly none. We explain below that $H^2(S,\mathcal{L})' \neq \emptyset$ if and only if the band $\mathcal{L}$ is representable.

If $H^2(S,\mathcal{L})' \neq \emptyset$ there exists a gerbe $\mathcal{G}$ with band $\mathcal{L}$ that is neutral, *i.e.*, that has a section s. From above, the gerbe $\mathcal{G}$ is $\mathcal{L}$-equivalent to the gerbe $\mathrm{Tors}(\mathrm{Aut}(s))$ and its band $\mathcal{L}$ is representable by $\mathrm{Aut}(s)$. Conversely, assume $\mathcal{L}$ is representable, *i.e.*, $\mathcal{L} = \mathcal{B}(H)$ for some S-sheaf of groups H. Then each class $[\alpha] \in H^1(S,\mathrm{Inn}(H))$ induces a neutral class in $H^2(S,\mathcal{L})$, namely the class $[\mathrm{Tors}(H_\alpha)]$ where H_α is the inner form of H obtained by twisting H by α.

Given a band $\mathcal{L}$ representable by H, the argument above actually provides a surjective map $H^1(S,\mathrm{Inn}(H)) \twoheadrightarrow H^2(S,\mathcal{L})'$.

1.2.8. Patching data [Gi;Ch.IV §3.5.1.1]. Given a S-gerbe $\mathcal{G}$, a *patching data* $(U_i, x_i, \chi_{ij})_{i,j\in I}$ consists of the following:

- an open covering $(U_i)_{i\in I}$ of S,

• for each $i \in I$, x_i is an object of the category $\mathcal{G}(U_i)$, and

• for each $i, j \in I$, χ_{ij} is an isomorphism of the category $\mathcal{G}(U_i \cap U_j)$ between the restrictions to $U_i \cap U_j$ of x_i and x_j.

For the étale topology, $U_i \cap U_j$ should really be understood as the fiber product $U_i \times_S U_j$. Consider the situation $S = \mathrm{Spec}(K)_{\mathrm{et}}$. A single open subset $U = \mathrm{Spec}(E) \to \mathrm{Spec}(K)$ constitutes an open covering of $\mathrm{Spec}(K)$. There are two natural projections from $U \times_S U = \mathrm{Spec}(E \otimes_K E)$ to $U = \mathrm{Spec}(E)$, which correspond to the two embeddings

$$E \to E \otimes_K E \simeq \prod_{\tau \in \mathrm{G}(E/K)} E^\tau :$$

the first one maps each $\xi \in E$ on $(\xi)_{\tau \in \mathrm{G}(E/K)}$ while the second one maps ξ on $(\xi^\tau)_{\tau \in \mathrm{G}(E/K)}$. Thus (U, x) determines a patching data for the gerbe $\mathcal{G}(U)$ if and only if for each $\tau \in \mathrm{G}(E/K)$, there exists an isomorphism χ_τ between the objects x and x^τ.

A gerbe $\mathcal{G}$ associated with an embedding problem for $\mathrm{G}(K)$ (§1.2.2) is automatically equipped with a patching data. Indeed pick a finite Galois extension E/K such that $\mathrm{G}(E) \subset Ker(p)$. The trivial function 1 is a solution to the embedding problem over E, which is invariant by the action of $\mathrm{G}(K)$. Thus $(\mathrm{Spec}(E) \to (\mathrm{Spec}(K), 1, Id)$ yields a patching data for the gerbe $\mathcal{G}$.

1.2.9. Abelian case over $\mathrm{Spec}(K)_{\mathrm{et}}$. Take $S = \mathrm{Spec}(K)_{\mathrm{et}}$ and let $\mathcal{L}$ be an abelian K-band, *i.e.*, a K-sheaf of abelian groups regarded in the category $\mathrm{Li}(S)$. Equivalently, $\mathcal{L}$ can be regarded as a K-object H in the category of sheaves of abelian groups. In particular, the band $\mathcal{L}$ is representable (by H). Since $\mathrm{Inn}(H) = \{1\}$, the set $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L})'$ consists of a unique element (§1.2.7). From [Gi;Ch.IV Cor.3.5.3], there exists an embedding

$$H^2(K, H) \hookrightarrow H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L})$$

where $H^2(K, H)$ is the usual cohomology group $H^2(\mathrm{G}(K), H))$ [6] (the action is the homomorphism $\mathrm{G}(K) \to \mathrm{Out}(H) = \mathrm{Aut}(H)$ associated with the band $\mathcal{L}$ (§1.2.3)). Furthermore the image of this embedding identifies with the subset of $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L})$ of all classes of gerbes equipped with a patching data.

Consider the example of a gerbe attached to an embedding problem (§1.2.2) with abelian kernel A. Classically, the obstruction to the existence of a weak solution to the embedding problem can be measured by a 2-cocycle in $H^2(K, A)$ (for the action given by the embedding problem). This 2-cocycle is the one that maps to the class $[\mathcal{G}] \in H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L})$ *via* the embedding above.

6 In Giraud's terminology, the group $H^2(K,H)$ corresponds to the group $\check{H}^2((Spec(K_s) \to Spec(K)), H)$.

1.2.10. Reduction to the center of the band [Gi;Ch.IV §3.3]. Given a S-band $\mathcal{L}$, consider the abelian S-center band $Z(\mathcal{L})$. Then $H^2(S, Z(\mathcal{L}))$ is a group which acts on $H^2(S, \mathcal{L})$: the action rests on the notion of "produit contracté" of gerbes introduced by Giraud; we refer to [Gi;Ch.IV] for a precise definition of this action. Furthermore Giraud shows this action is free and transitive (provided that $H^2(S, \mathcal{L}) \neq \emptyset$); in other words, the set $H^2(S, \mathcal{L})$ is a principal homogeneous space under the group $H^2(S, Z(\mathcal{L}))$. In fact, given two S-gerbes $\mathcal{G}$ and $\mathcal{G}'$ with the same band $\mathcal{L}$, the element in $H^2(S, Z(\mathcal{L}))$ which maps the class $[\mathcal{G}]$ to the class $[\mathcal{G}']$ turns out to be the class of the gerbe $\mathrm{Hom}_{\mathcal{L}}(\mathcal{G}, \mathcal{G}')$ introduced in §1.2.5 [Gi;Ch.IV,Th.3.3.3] (so somehow the class $[\mathrm{Hom}_{\mathcal{L}}(\mathcal{G}, \mathcal{G}')]$ can be interpreted as the difference $[\mathcal{G}'] - [\mathcal{G}]$).

Assume further the band is representable, *i.e.*, $\mathcal{L} = \mathcal{B}(H)$ for some S-sheaf of groups H. Denote the distinguished neutral class $[\mathrm{Tors}(H)]$ by ε. The set $H^2(S, \mathcal{L})$ identifies to the group $H^2(S, Z(\mathcal{L}))$: identify each element $g \in H^2(S, Z(\mathcal{L}))$ with $g \cdot \varepsilon \in H^2(S, \mathcal{L})$. The subset of $H^2(S, Z(\mathcal{L}))$ that corresponds to $H^2(S, \mathcal{L})'$ under this identification is the image of the coboundary operator

$$\delta^1 : H^1(S, \mathrm{Inn}(H)) \to H^2(S, Z(\mathcal{L}))$$

which can be associated [Gi;Ch.IV Prop.3.2.6] to the exact sequence

$$1 \to Z(\mathcal{L}) \to H \to \mathrm{Inn}(H) \to 1$$

Thus we obtain:

A gerbe $\mathcal{G}$ with band $\mathcal{L}$ is neutral if and only if its class $[\mathcal{G}] \in H^2(S, \mathcal{L})$ lies, via the identification $H^2(S, \mathcal{L}) \approx H^2(S, Z(\mathcal{L}))$, in the image of the coboundary operator δ^1.

Finally take $S = \mathrm{Spec}(K)_{\mathrm{et}}$ and assume the gerbe $\mathcal{G}$ can be equipped with a patching data (which is a necessary condition for the gerbe $\mathcal{G}$ to be trivial). As above regard the class $[\mathcal{G}]$ as an element of $H^2(S, Z(\mathcal{L}))$. Then from §1.2.9, this class identifies with a 2-cocycle $\Omega_{\mathcal{G}} \in H^2(K, Z(H))$. Using the natural identification $H^1(K, \mathrm{Inn}(H)) \approx H^1(\mathrm{Spec}(K)_{\mathrm{et}}, \mathrm{Inn}(H))$, one obtains that

The gerbe $\mathcal{G}$ is neutral if and only if the 2-cocycle $\Omega_{\mathcal{G}} \in H^2(K, Z(H))$ is in the image of the (classical) coboundary operator [Se]

$$\delta^1 : H^1(K, \mathrm{Inn}(H)) \to H^2(K, Z(H))$$

associated with the exact sequence

$$1 \to Z(H) \to H \to \mathrm{Inn}(H) \to 1$$

2. The gerbe $\mathcal{G}(f)$ of models of a cover f

Let $f : X \to B$ be a (G-)cover, unramified above B^* and *a priori* defined over K_s. The field K is assumed to be the field of moduli of the cover f. To f we will attach a *gerbe* $\mathcal{G}(f)$ on the étale site $\mathrm{Spec}(K)_{\mathrm{et}}$ of K. The class of $\mathcal{G}(f)$ in $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L}(\mathcal{G}(f)))$ will represent the obstruction to K being a field of definition of the (G-)cover f.

In [DeDo1] we handle a more general problem where the extension K_s/K can be replaced by any Galois extension F/K. We refer to it as the *relative* form of the problem, the one considered here being the *absolute* form. The relative form can actually also be treated using non-abelian cohomology. We briefly indicate how in Remark 2.1.

2.1. Definition of the gerbe $\mathcal{G}(f)$. Let $\phi_{K_s} : \Pi_{K_s}(B^*) \to G \subset N$ denote the representation associated to the (G-)cover f. Notation is that of §1. Define a fibered category $\mathcal{G}(f)$ as follows: for each object $U = (\mathrm{Spec}(E) \to \mathrm{Spec}(K))$ of the étale site $\mathrm{Spec}(K)_{\mathrm{et}}$, with E/K a finite Galois extension,

- the *U-objects* of $\mathcal{G}(f)$ are the E-models of f, that is, the homomorphisms

$$\phi_E : \Pi_E(B^*) \to N$$

that induce on $\Pi_{K_s}(B^*)$ some conjugate $\phi_{K_s}^{\varphi}$ of ϕ_{K_s} by some element $\varphi \in N$,

- the *U-morphisms* between two such objects ϕ_E and ϕ'_E are the elements $\varphi \in N$ such that

$$\phi'_E(x) = \varphi \phi_E(x) \varphi^{-1} \text{ for all } x \in \Pi_E(B^*)$$

(1) *The fibered category $\mathcal{G}(f)$ is a K-gerbe.*

Proof. $\mathcal{G}(f)$ is a prestack: Let $U = \mathrm{Spec}(E) \to \mathrm{Spec}(K)$ be an open subset of the étale site $\mathrm{Spec}(K)_{\mathrm{et}}$ (with E/K a finite Galois extension) and ϕ_E, ϕ'_E be two representations $\Pi_E \to N$ associated to two E-models of the (G-)cover f. We have to show that $\mathrm{Hom}(\phi_E, \phi'_E)$ is a sheaf. Suppose given an étale covering $V = \mathrm{Spec}(F) \to \mathrm{Spec}(E)$ with F/E a finite Galois extension and a V-morphism whose restrictions to $V \times_U V = \mathrm{Spec}(F \otimes_E F)$ coincide (see §1.2.8 for a description of these restrictions). We have to show that this V-morphism extends to a U-morphism. In concrete terms, we are given a F-isomorphism χ between two E-models of a (G-)cover; the compatibility condition on $V \times_U V$ merely means that $\chi^\tau = \chi$ for all $\tau \in G(F/E)$. It follows indeed that χ is defined over E, *i.e.*, yields a U-morphism.

$\mathcal{G}(f)$ is a stack: Let $U = \mathrm{Spec}(E) \to \mathrm{Spec}(K)$ be an open subset of the étale site $\mathrm{Spec}(K)_{\mathrm{et}}$ (with E/K a finite Galois extension). Let $V = \mathrm{Spec}(F) \to \mathrm{Spec}(E)$ be an étale cover of U with F/E a finite Galois extension. Suppose also given a V-object of the gerbe $\mathcal{G}(f)$, *i.e.*, a F-model ϕ_F of a (G-)cover, together with isomorphisms between the two restrictions of ϕ_F to $V \times_U V = \mathrm{Spec}(F \otimes_E F)$. We assume in addition that these isomorphisms satisfy the cocycle condition involved in the definition of stack (§1.2.1). We have to show that ϕ_F extends to a U-object. In concrete terms, we are given a F-model $f_F : X_F \to B$ of a (G-)cover, (that is, in function field terms, a separable finite field extension $F(X_F)/F(B)$ regular over F), together with isomorphisms χ_τ between f_F and f_F^τ, (or, equivalently, automorphisms χ_τ^* of the extension $F(X_F)/E(B)$) ($\tau \in G(F/E)$). The cocycle condition $\varphi_{ij} = \varphi_{ik}\varphi_{kj}$ then rewrites as the classical Weil cocycle condition $\chi_{\sigma\tau} = \chi_\tau^\sigma \chi_\sigma$ (or, equivalently, $\chi_{\sigma\tau}^* = \chi_\tau^* \chi_\sigma^*$). It follows indeed from the Weil's cocycle criterion for descending the field of definition of an algebraic variety [We] that the F-model of the (G-)cover has a E-model: the function field extension of this E-model is the fixed field in $F(X_F)$ of the automorphisms χ_τ^* ($\tau \in G(F/E)$).

Finally the two additional conditions in the definition of gerbes are satisfied. Condition (ii) follows from the fact that (G-)covers can be defined over suitably large finite Galois extensions of the field of moduli K and condition (i) from the fact that two (G-)covers that are isomorphic over K_s are automatically isomorphic over some finite extension of K. □

2.2. Properties of the gerbe $\mathcal{G}(f)$. By definition, the gerbe $\mathcal{G}(f)$ has a section above $\mathrm{Spec}(K)$ if and only if there exists a morphism $\phi_K : \Pi_K \to N$ extending ϕ_{K_s}, *i.e.*, if the (G-)cover f has a model over K. Denote the band of the gerbe $\mathcal{G}(f)$ by $\mathcal{L}$. In other words we have

(2) *The gerbe $\mathcal{G}(f)$ defines a class $[\mathcal{G}(f)]$ in $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L})$) that is neutral if and only if the (G-)cover f is defined over K.*

If $U = (\mathrm{Spec}(E) \to \mathrm{Spec}(K))$ is an open subset of $(\mathrm{Spec}(K))_{\mathrm{et}}$ and $x = \phi_E$ is a U-object of $\mathcal{G}(f)$, the sheaf of groups $\mathrm{Aut}(x)$ of automorphisms of the U-object ϕ_E is locally (for the étale topology) isomorphic to the centralizer $C = \mathrm{Cen}_N G$. Indeed, both maps ϕ_{K_s} and ϕ_E coincide on Π_{K_s} (up to conjugation by some element $\varphi \in N$). Thus the U-automorphisms of ϕ_E are locally given by the elements $\varphi \in N$ such that $\phi_{K_s} = \phi_{K_s}^\varphi$, *i.e.*, by the elements of C. It follows that

(3) *The gerbe $\mathcal{G}(f)$ is locally bound by the constant sheaf C. Consequently (§1.2.4) the band $\mathcal{L}$ induces a class in $H^1((\mathrm{Spec}(K))_{\mathrm{et}}, \mathrm{Out}(C))$.*

Finally:

(4) *Since K is the field of moduli of the (G-)cover f, the gerbe $\mathcal{G}(f)$ can be equipped with a patching data.*

Proof of (4). There exists a finite Galois extension E/K satisfying these two conditions
(i) the (G-)cover f has a E-model f_E over E,

(ii) for each $\tau \in \mathrm{G}(E/K)$, the covers f_E and f_E^τ are isomorphic over E.
For example pick a finite Galois extension E_o of K over which the cover f is defined and take for E the Galois closure over K of the field generated by E_o and the coefficients of isomorphisms between the cover f and its distinct conjugates under $\mathrm{G}(E_o/K)$.

Let U be the open étale subset $U = (\mathrm{Spec}(E) \to \mathrm{Spec}(K))$. For each $\tau \in \mathrm{G}(E/K)$, denote by $\phi_E^\tau : \Pi_E \to N$ the representation associated with the E-model f_E^τ and by φ_τ an element in N satisfying $\phi_E^\tau = \phi_E^{\varphi_\tau}$. The two natural restrictions of ϕ_E to $U \times_K U = \mathrm{Spec}(E \otimes_K E)$ are the two families $(\phi_E)_{\tau \in \mathrm{G}(E/K)}$ and $(\phi_E^\tau)_{\tau \in \mathrm{G}(E/K)}$ (§1.2.8). The element $(\varphi_\tau)_{\tau \in \mathrm{G}(E/K)}$ conjugates one onto the other. Conclude that $(U, \phi_E, (\varphi_\tau)_{\tau \in \mathrm{G}(E/K)})$ is a patching data for the gerbe $\mathcal{G}(f)$. □

Remark 2.1 (Absolute vs relative situation). Consider here the more general relative situation (as in [DeDo1]): a (G-)cover $f_F : X \to B$ is given as above, unramified above B^* but f_F is *a priori* defined over an arbitrary Galois extension F of K (instead of $F = K_s$ as in the absolute situation considered above); the problem is whether this F-model is defined over K, *i.e.*, has a K-model isomorphic to f_F over F.

Assume further that K is the field of moduli of the (G-)cover f_F *relative to the extension* F/K [DeDo1;§2.7]. The obstruction to K being a field of definition of f_F can also be represented by a gerbe in this more general situation. This gerbe is very similar to the gerbe $\mathcal{G}(f)$. The only difference in the definition is that the étale site $(\mathrm{Spec}(K))_{et}$ has to be replaced by the étale sub-site relative to the extension F/K. This sub-site is the site with associated open covers all the morphisms $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ where L/K is any product of separable finite field extensions contained in F. The relative gerbe is a gerbe over this sub-site.

An alternate description of the obstruction can be given when the extension F/K is finite. Let f be the (G-)cover over K_s obtained by extension of scalars. In this section we have attached to f a gerbe $\mathcal{G}(f)$ whose class lies in $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \mathcal{L})$ where $\mathcal{L} = \mathcal{L}(\mathcal{G}(f))$. Denote the étale open cover $\mathrm{Spec}(F) \to \mathrm{Spec}(K)$ by U. The given F-model f_F of f is a U-object of the gerbe $\mathcal{G}(f)$. Furthermore, as in (4) above, this U-object can be equipped with a patching data *relative to the cover* U. It follows [Gi;Ch.IV Cors.3.5.2 & 3.5.3] that the class $[\mathcal{G}(f)]$ is the image of a class $[\gamma_f] \in \check{H}^2(U, \mathcal{L})$ *via* the natural map $\check{H}^2(U, \mathcal{L}) \to H^2(\mathrm{Spec}(K)_{et}, \mathcal{L})$ (where the $\check{H}$ indicates that the cohomology involved is the Cěch cohomology). The class $[\gamma_f]$ can be interpreted as the class of the relative gerbe introduced above; that is, $[\gamma_f]$ represents the obstruction to the F-model f_F being defined over K.

To simplify the exposition we will stick to the absolute situation. However all statements in this paper carry over to the more general relative situation (provided the base site is taken to be the relative sub-site above).

3. The gerbe $\mathcal{R}\mathrm{e}(\lambda)$ of extensions of constants

Fix a (G-)cover $f : X \to B$ unramified over B^* and *a priori* defined over K_s. Let $\phi_{K_s} : \Pi_{K_s}(B^*) \to G \subset N$ be the associated representation and $\mathcal{G}(f)$ be the K-gerbe associated to f in §2.

3.1. Extension of constants in the Galois closure. Let E/K be a Galois extension such that the (G-)cover f can be defined over E. Fix a E-model f_E of f and let $\phi_E : \Pi_E \to N$ be the associated representation. Consider the function field extension $E(X_E)/E(B)$ associated to f_E. Denote the Galois closure of the extension $E(X_E)/E(B)$ by $\widehat{E(X_E)}/E(B)$. Consider then the field $\widehat{E} = \widehat{E(X_E)} \cap K_s$. The extension $\widehat{E}/E$ is called the *extension of constants in the Galois closure* of the model f_E of f.

Denote by Λ_E the unique homomorphism $\mathrm{G}(E) \to N/G$ that makes the following diagram commute

$$\begin{array}{ccc} \Pi_E(B^*) & \longrightarrow & \mathrm{G}(E) \\ {\scriptstyle \phi_E}\downarrow & & \downarrow{\scriptstyle \Lambda_E} \\ N & \longrightarrow & N/G \end{array}$$

The homomorphism $\Lambda_E : \mathrm{G}(E) \to N/G$ corresponds to the extension of constants $\widehat{E}/E$ in the Galois closure of the model f_E of f_F. That is, $\mathrm{G}(K_s/\widehat{E}) = Ker(\Lambda_E)$ [DeDo1;Prop.2.3]. The homomorphism $\Lambda_E : \mathrm{G}(E) \to N/G$ is called the *constant extension map (in Galois closure)* of the E-model f_E of f.

Assume that K is the field of moduli of the (G-)cover f. Denote the representation of Π_K modulo C given by the field of moduli condition by $\overline{\varphi} : \Pi_K \to N/C$. There exists a unique homomorphism $\lambda : \mathrm{G}(K) \to N/CG$ that makes the following diagram commute

$$\begin{array}{ccc} \Pi_K(B^*) & \longrightarrow & \mathrm{G}(K) \\ {\scriptstyle \overline{\varphi}}\downarrow & & \downarrow{\scriptstyle \lambda} \\ N/G & \longrightarrow & N/CG \end{array}$$

The homomorphism $\lambda : \mathrm{G}(K) \to N/CG$ is called *the constant extension map (in Galois closure) modulo C* given by the field of moduli condition (see [DeDo1;§3.1]).

3.2. Definition of the gerbe $\mathcal{R}\mathrm{e}(\lambda)$. Assume that K is the field of moduli of the (G-)cover f. The gerbe $\mathcal{R}\mathrm{e}(\lambda)$ is defined to be the K-gerbe attached to the following embedding problem

$$\begin{array}{ccccc} & & & & \mathrm{G}(K) \\ & & & & \big\downarrow \lambda \\ CG/G & \hookrightarrow & N/G & \longrightarrow & N/CG \end{array}$$

Let f_E be any E-model of f with $K \subset E \subset K_s$ and $\phi_E : \Pi_E \to N$ be the associated representation. From §1.1.3, ϕ_E induces $\overline{\varphi}$ modulo C. Therefore the constant extension map $\Lambda_E : \mathrm{G}(E) \to N/G$ of f_E (§3.1) induces the map λ modulo C (over $\mathrm{G}(E)$). Thus we obtain a morphism of K-gerbes

$$\mathcal{G}(f) \to \mathcal{R}\mathrm{e}(\lambda)$$

which maps each object ϕ_E to the associated constant extension map $\Lambda_E : \mathrm{G}(E) \to N/G$; the definition on morphisms is immediate. In particular, the following condition

(λ/Lift) *There exists at least one lifting* $\Lambda : \mathrm{G}(K) \to N/G$ *of* $\lambda : \mathrm{G}(K) \to N/CG$.

is a necessary condition for the field of moduli K to be a field of definition of the (G-)cover f. This condition was first introduced in [DeDo1].

The G-cover case. For G-covers, $N/G = N/CG = \{1\}$, the maps λ and Λ are trivial; by definition, G-covers do not have any extension of constants in their Galois closure. The gerbe $\mathcal{R}\mathrm{e}(\lambda)$ is trivial.

3.3. Cohomological description of $\mathcal{R}\mathrm{e}(\lambda)$. Let $\overline{\kappa} : N/CG \to \mathrm{Out}(CG/G)$ be the outer action attached to the exact sequence

$$1 \to CG/G \to N/G \to N/CG \to 1$$

The band of the gerbe $\mathcal{R}\mathrm{e}(\lambda)$ is the composed map $\overline{\kappa}\lambda : \mathrm{G}(K) \to \mathrm{Out}(CG/G)$ and the gerbe $\mathcal{R}\mathrm{e}(\lambda)$ is neutral if and only if condition (λ/Lift) holds (§1.2).

Thus there is a first obstruction to condition (λ/Lift), which is that the set $H^2(K, \overline{\kappa}\lambda)'$ of all neutral classes in $H^2(K, \overline{\kappa}\lambda)$ is not empty. That condition is equivalent to the band of the gerbe $\mathcal{R}\mathrm{e}(\lambda)$ being representable (§1.2.7). More specifically, that means that the outer action $\overline{\kappa}\lambda : \mathrm{G}(K) \to \mathrm{Out}(CG/G)$ can be lifted to an actual action $\ell : \mathrm{G}(K) \to \mathrm{Aut}(CG/G)$. This condition precisely corresponds to condition (Band/Rep) of §4.3 of [DeDo1].

Assume that condition (Band/Rep) holds. Let $\ell : \mathrm{G}(K) \to \mathrm{Aut}(CG/G)$ be an action that induces the band $\overline{\kappa}\lambda$. Denote the action induced by ℓ on $Z(CG/G)$ by χ_ℓ. In this specific situation, §1.2.10 reads as follows. The set $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \overline{\kappa}\lambda)$ is a principal homogeneous space under the group $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, Z(\overline{\kappa}\lambda))$. Denote the distinguished neutral class of $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \overline{\kappa}\lambda)$ associated with ℓ by $[\varepsilon]$. *Via* the identification $H^2(K, \overline{\kappa}\lambda) \approx H^2(\mathrm{Spec}(K)_{\mathrm{et}}, Z(\overline{\kappa}\lambda))$ that maps $[\varepsilon]$ to the trivial class in $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, Z(\overline{\kappa}\lambda))$, the class $[\mathcal{R}\mathrm{e}(\lambda)] \in H^2(\mathrm{Spec}(K)_{\mathrm{et}}, \overline{\kappa}\lambda)$ corresponds to some class in $H^2(\mathrm{Spec}(K)_{\mathrm{et}}, Z(\overline{\kappa}\lambda))$. In turn, this class corresponds to some 2-cocycle $\omega_\ell \in H^2(K, Z(CG/G))$ (for the action χ_ℓ) *via* the embedding $H^2(K, Z(CG/G)) \hookrightarrow H^2(\mathrm{Spec}(K)_{\mathrm{et}}, Z(\overline{\kappa}\lambda))$. The 2-cocycle ω_ℓ satisfies the following.

THEOREM 3.1 — *Assume that condition (Band/Rep) holds. Then these conditions are equivalent.*
(i) Condition (λ/Lift) holds,
(ii) The gerbe $\mathcal{R}\mathrm{e}(\lambda)$ is neutral,
(iii) The 2-cocycle $\omega_\ell \in H^2(K, Z(CG/G))$ lies in the image of the coboundary operator δ^1

$$\delta^1 : H^1(S, \mathrm{Inn}(CG/G)) \to H^2(S, Z(CG/G))$$

associated with the exact sequence

$$1 \to Z(CG/G) \to CG/G \to \mathrm{Inn}(CG/G) \to 1$$

REMARK 3.2. The cocycle $\omega_\ell \in H^2(K, Z(CG/G))$ is the cocycle ω_ℓ of Th.4.7 of [DeDo1] and the equivalence (i)⇔(iii) corresponds to the main conclusion of Th.4.7 of [DeDo1].

4. The gerbe $\mathcal{G}(f)_\Lambda$ of models with given constant extension map Λ

Fix a (G-)cover $f : X \to B$ unramified over B^* and *a priori* defined over K_s and let $\phi_{K_s} : \Pi_{K_s}(B^*) \to G \subset N$ be the associated representation. Assume K is the field of moduli. Let $\mathcal{G}(f)$ be the K-gerbe associated to f in §2 and $\mathcal{R}\mathrm{e}(\lambda)$ the K-gerbe of extensions of constants of f (§3), where λ is the constant extension map modulo C given by the field of moduli condition.

4.1. The gerbe $\mathcal{G}(f)_\Lambda$. Assume condition (λ/Lift) holds (§3.2). Fix a lifting $\Lambda : \mathrm{G}(K) \to N/G$ of λ, *i.e.*, a section $\Lambda = \Lambda_K$ over K of the gerbe $\mathcal{R}\mathrm{e}(\lambda)$. We then define the gerbe $\mathcal{G}(f)_\Lambda$ as follows. For each object $U = (\mathrm{Spec}(E) \to \mathrm{Spec}(K))$ of the étale site $\mathrm{Spec}(K)_{\mathrm{et}}$, with E/K a finite Galois extension,

- the U*-objects* of $\mathcal{G}(f)_\Lambda$ are the homomorphisms $\phi_E : \Pi_E(B^*) \to N$ that induce ϕ_{K_s} on $\Pi_{K_s}(B^*)$ and make the following diagram commute

$$\begin{array}{ccc} \Pi_E(B^*) & \longrightarrow & \mathrm{G}(E) \\ {\scriptstyle \phi_E}\Big\downarrow & & \Big\downarrow{\scriptstyle \Lambda_E} \\ N & \longrightarrow & N/G \end{array}$$

where $\Lambda_E : \mathrm{G}(E) \to N/G$ is the restriction of Λ to $\mathrm{G}(E)$. In other words, they are the U-objects of $\mathcal{G}(f)$ that are mapped on Λ_E by the gerbe morphism $\mathcal{G}(f) \to \mathcal{R}\mathrm{e}(\lambda)$ (§3.2).

- the U*-morphisms* between two objects ϕ_E and $\phi_{E'}$ of $\mathcal{G}(f)_\Lambda$ are the U-morphisms between ϕ_E and $\phi_{E'}$ regarded as objects of $\mathcal{G}(f)$. Note that the condition that objects here should restrict to ϕ_{K_s} exactly (and not only to a conjugate of it) forces the U-morphisms which are *a priori* in N, to be in G.

The gerbe $\mathcal{G}(f)_\Lambda$ should actually be understood as the fiber product $\mathcal{G}(f) \times_{\mathcal{R}\mathrm{e}(\lambda)} \mathrm{G}(K)$, or, equivalently, as the gerbe of liftings of the K-section $\Lambda = \Lambda_K$ over K of $\mathcal{R}\mathrm{e}(\lambda)$ by the morphism $\mathcal{G}(f) \to \mathcal{R}\mathrm{e}(\lambda)$ [Gi;Ch.IV §2.5.4.1].

It is readily checked that $\mathcal{G}(f)_\Lambda$ is indeed a gerbe and that it is locally bound by $C \cap G = Z(G)$. Using §1.2.9 we obtain that the class of the gerbe $\mathcal{G}(f)_\Lambda$ can be viewed as an element of $H^2(K, Z(G))$ (for the action L obtained by composing $\lambda : \mathrm{G}(K) \to N/CG$ by the action by conjugation of N/CG (*via* N) on $Z(G)$). And the vanishing of this element of $H^2(K, Z(G))$ is equivalent to the the gerbe $\mathcal{G}(f)_\Lambda$ being neutral. We have proved

Theorem 4.1 — *Let $f : X \to B$ be a (G-)cover defined over K_s with K as field of moduli. Assume that condition (λ/Lift) holds. Fix a lifting $\Lambda : \mathrm{G}(K) \to N/G$ of λ. Then the following are equivalent:*

(i) The gerbe $\mathcal{G}(f)_\Lambda$ is neutral,

(ii) The (G-)cover f can be defined over K with Λ as constant extension map,

(iii) The class $[\mathcal{G}(f)_\Lambda]$ viewed as an element of $H^2(K, Z(G))$ is trivial.

4.2. The Main Theorem of [DeDo1]. The element $[\mathcal{G}(f)_\Lambda] \in H^2(K, Z(G))$ of condition (iii) above is the 2-cocycle $\Omega_{u,v}$ of the Main Theorem of [DeDo1] and Th.4.1 corresponds to the special case $\theta = 1$ of conclusion (d) of that result. More generally, the Main Theorem of [DeDo1] gives the following criterion for the gerbe $\mathcal{G}(f)$ to be neutral.

THEOREM 4.2 ([DeDo1;Main Theorem II]) — *Let $f : X \to B$ be a (G-)cover defined over K_s with K as field of moduli. The following are equivalent:*
(i) The gerbe $\mathcal{G}(f)$ has a section,
(ii) The (G-)cover f is defined over K,
(iii) (a) Condition (λ/Lift) holds, and,
(b) if Λ is a (any) lifting of λ, the class $[\mathcal{G}(f)_\Lambda]$ viewed as an element of $H^2(K, Z(G))$, is in the image of the coboundary operator $\delta^1 : H^1(K, CG/G) \to H^2(K, Z(G))$ associated with the exact sequence

$$1 \to Z(G) \to C \to CG/G \to 1$$

In fact, the gerbe $\mathcal{G}(f)$ has a section if and only if there exists a lifting Λ of λ such that the gerbe $\mathcal{G}(f)_\Lambda$ has a section. So Th.4.2 readily follows from Th.4.1 and this lemma.

LEMMA 4.3 — *Let Λ and Λ' be two liftings of λ. As in Th.4.2, regard both classes $[\mathcal{G}(f)_\Lambda]$ and $[\mathcal{G}(f)_{\Lambda'}]$ as 2-cocycles in $H^2(K, Z(G))$. Then we have*

$$[\mathcal{G}(f)_{\Lambda'}] - [\mathcal{G}(f)_\Lambda] = \delta^1(\theta)$$

where $\theta = \Lambda'\Lambda^{-1}$ and δ^1 is the coboundary operator $\delta^1 : H^1(K, CG/G) \to H^2(K, Z(G))$.

This formula was proved in [DeDo1;Prop.4.5] through some cocycle calculations which may seem somewhat miraculous. The next paragraph interprets the gerbe $\mathcal{G}(f)_\Lambda$ and suggests what supports these calculations.

4.3. Interpretation of the gerbe $\mathcal{G}(f)_\Lambda$. Assume that condition (λ/Lift) holds and fix a lifting $\Lambda : \mathrm{G}(K) \to N/G$ of λ. Consider the following diagram, where $\Lambda^*(N)$ denotes the pull-back of N along the map $\Lambda : \mathrm{G}(K) \to N/G$.

$$
\begin{array}{ccccc}
1 & & 1 & & 1\\
\downarrow & & \downarrow & & \downarrow\\
G & = & G & \xleftarrow{\phi_{K_s}} & \Pi_{K_s}\\
\downarrow & & \downarrow & & \downarrow\\
N & \longleftarrow & \Lambda^*(N) & & \Pi_K\\
\downarrow & & \downarrow & & \downarrow\\
N/G & \xleftarrow{\Lambda} & \mathrm{G}(K) & = & \mathrm{G}(K)\\
\downarrow & & \downarrow & & \downarrow\\
1 & & 1 & & 1\\
 & & & & \\
\mathcal{G}_G & & \Lambda^*(\mathcal{G}_G) & & \mathcal{G}_\Pi
\end{array}
$$

Denote by $\mathcal{G}_G$ [resp. $\mathcal{G}_\Pi$] the gerbe corresponding to the left vertical exact sequence [resp. right vertical exact sequence]. Then denote the middle vertical exact sequence, which is obtained from $\mathcal{G}_G$ by pull-back along Λ, by $\Lambda^*(\mathcal{G}_G)$.

The homomorphism $\phi_{K_s} : \Pi_{K_s} \to G$ induces a morphism between the respective bands of the gerbes $\mathcal{G}_\Pi$ and $\Lambda^*(\mathcal{G}_G)$. Consider then the gerb $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$ (defined in §1.2.5) of morphisms between these two gerbes that induce the band morphism given by ϕ_{K_s}. Every U-object $\phi_E : \Pi_E \to N$ of the gerbe $\mathcal{G}(f)_\Lambda$ factors through $\Lambda^*(N)$ and so gives rise to an object $\Pi_E \to \Lambda^*(N)$ of $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$. Conversely each U-object $\Pi_E \to \Lambda^*(N)$ of $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$ induces by composition with $\Lambda^*(N) \to N$ an object $\phi_E : \Pi_E \to N$ of the gerbe $\mathcal{G}(f)_\Lambda$. Therefore the gerbe $\mathcal{G}(f)_\Lambda$ can be interpreted as the gerbe $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$.

The gerbe $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$ is bound by the centralizer $\mathrm{Cen}_G(\phi_{K_s}(\Pi_{K_s}))$ (§1.2.5), that is, by $Z(G)$. Consider next the gerbe $(\phi_{K_s})_*(\mathcal{G}_\Pi)$ obtained from $\mathcal{G}_\Pi$ by transport *via*

ϕ_{K_s}[7] and conclude using §1.2.10 that the class of $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$, and so also the class of the original gerbe $\mathcal{G}(f)_\Lambda$, can be interpreted as the difference (in $H^2(K, Z(G))$)

$$[\Lambda^*(\mathcal{G}_G)] - [(\phi_{K_s})_*(\mathcal{G}_\Pi)]$$

Remark 4.4. (a) The fact that $\mathrm{Hom}_{\phi_{K_s}}(\mathcal{G}_\Pi, \Lambda^*(\mathcal{G}_G))$ can be interpreted as a difference is reflected on the form of the 2-cocycle $\Omega_{u,v}$ of the Main Theorem of [DeDo1]. Indeed the 2-cocycle $\Omega_{u,v}$ is of the form

$$\Omega_{u,v} = \Phi_{u,v} \cdot \left(\phi_{K_s}(\gamma_{u,v})^{-1}\right)$$

The term $\Phi_{u,v}$ corresponds to the gerbe $\Lambda^*(\mathcal{G}_G)$ while the term $\left(\phi_{K_s}(\gamma_{u,v})^{-1}\right)$ corresponds to the gerbe $(\phi_{K_s})_*(\mathcal{G}_\Pi)$.

(b) The proof of the present Lemma 4.3 given in [DeDo1] can be rewritten as follows.

$$\begin{aligned}
[\mathcal{G}(f)_{\Lambda'}] - [\mathcal{G}(f)_\Lambda] &= ([\Lambda^*(\mathcal{G}_G)] - [(\phi_{K_s})_*(\mathcal{G}_\Pi)]) - ([(\Lambda')^*(\mathcal{G}_G)] - [(\phi_{K_s})_*(\mathcal{G}_\Pi)]) \\
&= [\Lambda^*(\mathcal{G}_G)] - [(\Lambda')^*(\mathcal{G}_G)] \\
&= \delta^1(\theta)
\end{aligned}$$

5. Final note

Stacks and gerbes have been used in recent works ([Be], [Ch], [DeDoEm], [Ek], [Wew]) in the following context; the forerunner in this area was M. Fried [Fr]. The base space S is some moduli space of covers with some prescribed ramification data (number of branch points, the group G of the cover, the conjugacy classes in G of the inertia groups, etc.). The S-gerbe $\mathcal{F}$ they consider is the following one: for each open subset U of S, $\mathcal{F}(U)$ is the category of algebraic families of covers corresponding to points in U – the so-called Hurwitz families —. Above each U there may be several such families and there may be none. The obstruction to this gerbe having a section is the obstruction to the existence of a family of covers above the whole moduli space S. There is a section for example if the covers in question have no automorphisms, that is, if S is a fine moduli space, in which case, on top of existence of a family above S, we have uniqueness of such a family, which is then called the universal family above S.

Using the theory of gerbes as in the present paper, similar conclusions can be drawn out for the gerbe $\mathcal{F}$. In particular, assume the band $\mathcal{L}(\mathcal{F})$ of the gerbe $\mathcal{F}$ is representable. Then the set $H^2(S, \mathcal{L}(\mathcal{F}))$ identifies to the group $H^2(\pi_1(S), Z(G))$ in such a way that the class $[\mathcal{F}]$ is neutral if and only it lies, *via* this identification, in the image of the coboundary operator

[7] $(\phi_{K_s})_*(\mathcal{G}_\Pi)$ is indeed a gerbe because ϕ_{K_s} is surjective.

$$\delta^1 : H^1(\pi_1(S), \mathrm{Inn}(G)) \to H^2(\pi_1(S), Z(G))$$

Furthermore, if f is a K_s-cover corresponding to some point (denoted by $[f]$) of S, the gerbe $\mathcal{G}(f)$ of the present paper can be regarded as a specialization of the gerbe $\mathcal{F}$ (with S endowed with the étale topology). Denote the band of the gerbe $\mathcal{G}(f)$ by $\mathcal{L}_f$. It would be worthwhile to precisely define and investigate the specialization morphism

$$H^2(S, \mathcal{L}(\mathcal{F})) \to \prod_{[f]\in S} H^2(\mathrm{Spec}(K(f))_{\mathrm{et}}, \mathcal{L}_f)$$

In particular, is it injective? More specifically, must there be a K-family above the whole moduli space if all covers corresponding to points in $S(K_s)$ are defined over their field of moduli? If there is no K-family above the whole moduli space, how big is the subset of $S(K_s)$ for which the corresponding covers are not defined over their field of moduli?

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