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Arithmetic variation of fibers in families of curves Part I: Hurwitz monodromy criteria for rational points on all members of the family

By *Pierre Debes*[1]) at Paris and *Mike Fried*[2]) at Irvine

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Highly structured families of curves have played a large role in the service of such well known problems as the production of high rank elliptic curves and the realization of groups as Galois groups over $\mathbb{Q}$. Arithmetically the essential concern arising from the consideration of families of curves is the variation of the set of rational (or integral points) on each member (or fiber) of the family: e.g., the cardinality of these sets or the rank of the rational points on the Jacobian, in a genus $\geqq 1$ situation. For example, Silverman [Si] investigates the possibility of finding an upper bound for the cardinality of the set of (quasi) integral points on each fiber of an algebraic family of curves. If K is a field of definition for the family, for each fiber defined over K this bound is expected to be an explicit function of the rank and the cardinality of the torsion points of the Jacobian (over K). In two cases, Silverman has achieved a version of this goal [Si]: when $g=1$; and when the family is geometrically constant (i.e., over the algebraic closure $\bar{K}$ of K, all fibers are isomorphic).

From the "conceptual" theorems of diophantine geometry (Siegel, Mordell-Weil, Faltings), the genus of the fibers is a controlling factor for these problems. This paper uses a more precise invariant of a cover of $\mathbb{P}^1$: *the Nielsen class*. This data depends on the ramification of the covering map: the monodromy group G of the cover, the number of branch points and the conjugacy classes in G of the associated branch cycles are part of this data; algebraically, the group G is identified with the Galois group of the Galois closure of the associated function field extension, the branch cycles then correspond to the generators of the inertia groups. The precise definitions are given in § 1. 1 and § 1. 2.

Under certain assumptions, there exists an algebraic family defined over a number field K, called *the universal Hurwitz family*, among whose fibers are unique representatives of all the covers in a given Nielsen class (§ 1. 3, Theorem 1. 7). The algebraic structure can be taken advantage of to study the fibers of this family; all covers in the Nielsen class then inherit properties of these fiber covers. For example, under suitable assumptions, the existence of a rational point on the covering space is true on all the

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covers in a Nielsen class, if it is true generically on the associated Hurwitz family (§ 1. 3, Prop. 1. 8 and § 3. 1, Prop. 3. 3). The Lewis-Schinzel [LSc] criterion for families of conics to have sections shows furthermore that the converse holds as well in the genus 0 case (§ 3. 1, Prop. 3. 4).

The point of Hurwitz families is their somewhat abstract group theoretic production: no equations are involved. The main emphasis/innovation of this paper: We display a method essentially based on group theory that leads to arithmetic results. There is a natural group action attached to the definition of the Hurwitz family: the Hurwitz monodromy action. In § 1 we recall from [Fr2] geometric interpretations of the Hurwitz monodromy action. Then we define extensions of this action on *pointed Nielsen classes* (§ 3. 3). Via identifications, these have a representative collection of points on each fiber of the family. The Main Theorem (§ 3. 4, Theorem 3. 14) relates the monodromy action on pointed Nielsen classes to the action of conjugation, over the field of definition of the fiber, on these points. Specifically, the rational divisors with support contained in the set of points above the branch points correspond, on the generic fiber, to orbits of these group actions. Thus, explicit group actions provide information on the arithmetic of the fibers of Hurwitz families. As a consequence of Theorem 3. 14 we obtain a criterion, based on ramification data, for the existence of a rational point on a cover of $\mathbb{P}^1$; this criterion is particularly efficient in the genus 0 or 1 cases. We conjecture, under suitable assumptions (cf. § 3. 1), that the satisfaction of this criterion is the only obstruction to the existence of a rational point on *all* covers of $\mathbb{P}^1$ in the Nielsen class.

Several examples, which originate in an exceptional case of the Hilbert-Siegel problem [Fr3], illustrate the theory (§ 2). In § 3. 6, we apply the criterion of Th. 3. 14 to our main example and show that no rational points are produced. We may then check our conjecture in this special case. § 4 then shows that the Hurwitz family in question is a family of conics defined over $\mathbb{Q}$ some of which have rational points and others of which don't: the family is *arithmetically nonconstant* (§ 4. 4, Th. 4. 2). In the genus 1 case, we intend to continue this work to consider the production of families of high rank elliptic curves over $\mathbb{Q}$; an introductory example is given in § 3. 7.

§ 1. Nielsen classes and Hurwitz families

This section sets up the notations and foundational definitions for this paper. We start by recalling Riemann's existence theorem, Nielsen classes, Hurwitz families and the role of the Hurwitz monodromy group.

1. 1. Riemann's existence theorem. Let $\varphi: X \to \mathbb{P}_z^1$ be a finite cover of the projective line by an irreducible projective nonsingular curve. This cover is ramified over a finite set of points $z_1, \dots, z_r$ called *the branch points of the cover*. For $z_0 \notin \{z_1, \dots, z_r\}$ consider a labeling of the points $\mathbf{p}_1, \dots, \mathbf{p}_n$ of $\varphi^{-1}(z_0)$. There is a natural transitive action $T: \pi_1 \to S_n$, called *the monodromy action* of the fundamental group π_1 of $\mathbb{P}_z^1 \setminus \{z_1, \dots, z_r\}$ on $\{1, 2, \dots, n\}$: for $[\Gamma]$ the homotopy class of a path Γ based at z_0, $T([\Gamma])$ is the element of S_n that maps i to j where $\mathbf{p}_j$ is the terminal point of the unique lift of Γ (through f) which has initial point $\mathbf{p}_i$, $i \in \{1, 2, \dots, n\}$. Up to conjugation by an element of S_n, this action is independent of the choices of z_0, the representative of $[\Gamma]$, and the labeling of the $\mathbf{p}$'s.

The group $G = T(\pi_1)$ is called the monodromy group of the cover. It is defined up to conjugation by an element of S_n. Consider π_1. It is isomorphic to the free group on r generators $g_1, \dots, g_r$ with the one relation $g_1 \cdots g_r = 1$. For $i = 1, \dots, r$, map g_i to the homotopy class $[\Gamma_i]$ of a path Γ_i homotopic to $\lambda_i \delta_i \lambda_i^{-1}$ where δ_i is a clockwise boundary of a small disc Δ_i about z_i and λ_i is any (piecewise smooth) path from z_0 to a point on δ_i with these additional properties: no two of the $\lambda \cup \Delta$'s intersect anywhere (or go through z_0) besides at the already indicated endpoints of the λ's; and the first points of intersection of the λ's with a small disc Δ_0 about z_0 appear in order clockwise around Δ_0. The only complete proof we know that these conditions alone yield an isomorphism is that of [Fr4], Theorem 2.10, where the paths $\Gamma_1, \dots, \Gamma_r$ are called a *"sample bouquet"*. Of course, it is well known when the paths are placed in some kind of standard position relative to a simple triangulation of the sphere.

Then set $\sigma_i = T([\Gamma_i]) \in S_n$; the σ_i's generate the monodromy group of the cover and satisfy $\sigma_1 \cdots \sigma_r = 1$. The r-tuple $(\sigma_1, \dots, \sigma_r)$ is called a *branch cycle description* of the cover. From Riemann's existence theorem [Gro], this branch cycle information essentially determines the cover:

Theorem 1.1. *Fix r points $z_1, \dots, z_r \in \mathbb{P}_z^1$, a base point $z_0 \notin \{z_1, \dots, z_r\}$ and a sample bouquet $\Gamma_1, \dots, \Gamma_r$. Then the association* cover $\to$ branch cycle description *associated to this data produces a one-one correspondence between the following sets*:

— *degree n covers $\varphi: X \to \mathbb{P}_z^1$ ramified over $\{z_1, \dots, z_r\}$ (up to equivalence of covers), and*

— *r-tuples $(\sigma_1, \dots, \sigma_r) \in S_n^r$ such that $\sigma_1 \cdots \sigma_r = 1$ and the group $G(\sigma)$ generated by the σ_i's is transitive on the set $\{1, 2, \dots, n\}$ (up to conjugation, componentwise, by an element of S_n).*

Furthermore, the disjoint cycles β in σ_i correspond to the points $x \in \varphi^{-1}(z_i)$, the ramification index of x over z_i equals the length of β.

Through function fields, degree n covers $\varphi: X \to \mathbb{P}_z^1$ correspond to degree n extensions of $\mathbb{C}(z)$. It is standard to identify the monodromy group of the cover with the Galois group of the Galois closure of the function field extension and the entries of a branch cycle description of the cover with generators of inertia groups over the branch points. The specific identification is this: Let $(\sigma_1, \dots, \sigma_r)$ be any branch cycle description of $\varphi: X \to \mathbb{P}_z^1$. Denote the Galois group of the Galois closure of the function field extension $\mathbb{C}(X)/\mathbb{C}(z)$ by $G(\hat{X}/\mathbb{P}_z^1)$. Then in any identification of $G(\hat{X}/\mathbb{P}_z^1)$ with a subgroup of S_n given by its action on the n conjugates of a primitive element of the extension $\mathbb{C}(X)/\mathbb{C}(z)$, there exists $\gamma \in S_n$ such that

(1.1) $\gamma \sigma_i \gamma^{-1}$ is a generator of the inertia group of some prime of the Galois closure of $\mathbb{C}(X)$, $i = 1, \dots, r$.

In particular, the monodromy group $G(\boldsymbol{\sigma})$ and the Galois group $G(\hat{X}/\mathbb{P}_z^1)$ are conjugate subgroups of S_n.

Also, for any subgroup H of $G(\hat{X}/\mathbb{P}_z^1)$, of index $n(H)$, consider the cover

$$\varphi_H: \hat{X}_H \to \mathbb{P}_z^1 \tag{1.2}$$

associated to the function field extension $\mathbb{C}(\hat{X})^H/\mathbb{C}(z)$ of $\mathbb{C}(z)$ by the fixed field of H in $\mathbb{C}(\hat{X})$. On the other hand, order the right cosets of H in $G(\boldsymbol{\sigma})$ to obtain a permutation representation $T_H: G(\boldsymbol{\sigma}) \to S_{n(H)}$. Then $T_H(\boldsymbol{\sigma}) = (T_H(\sigma_1), \dots, T_H(\sigma_r))$ are branch cycles for $(\hat{X}_H, \varphi_H)$.

The Riemann-Hurwitz formula gives the genus, $g(X) = g$, of X by

$$(1.3) \qquad 2(n + g(X) - 1) = \sum_{i=1}^{r} \operatorname{ind}(\sigma_i)$$

where $\operatorname{ind}(\sigma)$ denotes n minus the number of disjoint cycles of σ.

Branch cycle descriptions provide a great deal of information but they depend on the choice of much data: a base point z_0; a labeling of the points in $\varphi^{-1}(z_0)$; an ordering of the branch points $z_1, \dots, z_r$ and a sample bouquet $\Gamma_1, \dots, \Gamma_r$. The dependence on these data has special significance for this paper since most of the complications in dealing with *families of covers*, rather than one cover at a time, arise from careful consideration of the deformation of these data. There exists a more intrinsic notion. Lemma 1 of [Fr2] shows that for any branch cycle description $(\sigma_1, \dots, \sigma_r)$ of a cover, the conjugacy class of σ_i in the monodromy group $G(\boldsymbol{\sigma})$ does not depend on the sample bouquet $\Gamma_1, \dots, \Gamma_r$, $i = 1, \dots, r$. Denote by C_i this conjugacy class. Up to conjugation by elements of S_n, the data consisting of the monodromy group $G(\boldsymbol{\sigma})$ and the set of conjugacy classes $\{C_1, \dots, C_r\}$ of $G(\boldsymbol{\sigma})$ is an invariant of the cover. This is the key to the definition of Nielsen classes.

1.2. Nielsen Classes. Let G be a subgroup of S_n and let $\mathbf{C} = (C_1, \dots, C_r)$ be an r-tuple of nontrivial (not necessarily distinct) conjugacy classes of G.

Definition 1.2. The *Nielsen Class* of $(\mathbf{C}, G)$ is the collection (assumed nonempty)

$$\operatorname{ni}(\mathbf{C}) = \{\boldsymbol{\sigma} \in G^r \mid G(\boldsymbol{\sigma}) = G,\ \sigma_1 \cdots \sigma_r = 1 \quad \text{and for some} \quad \alpha \in S_r,\ \sigma_{(i)\alpha} \in C_i,\ i = 1, \dots, r\}.$$

A cover $X \to \mathbb{P}^1$ is said to be in $\operatorname{ni}(\mathbf{C})$ if, up to conjugation by elements of S_n, any branch cycle description $\boldsymbol{\sigma}$ of the cover is in $\operatorname{ni}(\mathbf{C})$ (i.e., there exists $\gamma \in S_n$ such that $\gamma \boldsymbol{\sigma} \gamma^{-1} = (\gamma \sigma_1 \gamma^{-1}, \dots, \gamma \sigma_r \gamma^{-1}) \in \operatorname{ni}(\mathbf{C})$). Note that it makes no difference in what order we list the conjugacy classes.

The *straight Nielsen class* of $(\mathbf{C}, G)$ is

$$\operatorname{sni}(\mathbf{C}) = \{\tau \in \operatorname{ni}(\mathbf{C}) \mid \tau_i \in C_i,\ i = 1, \dots, r\}.$$

A cover $X \to \mathbb{P}^1$ given with an ordering of its branch points is said to be in $\operatorname{sni}(\mathbf{C})$ if, up to conjugation by elements of S_n, any branch cycle description of the cover computed with this ordering is in $\operatorname{sni}(\mathbf{C})$.

The *normalizer* (resp. the *straight normalizer*) of the Nielsen class is

$$\mathrm{N}(\mathbf{C}) = \{\gamma \in S_n \mid \text{conjugation by } \gamma \text{ permutes } C_1, \dots, C_r\},$$

$$\mathrm{SN}(\mathbf{C}) = \{\gamma \in S_n \mid \text{conjugation by } \gamma \text{ fixes } C_1, \dots, C_r\}.$$

Note that $\mathrm{N}(\mathbf{C})$ acts on the Nielsen class $\operatorname{ni}(\mathbf{C})$ by conjugation: $\gamma \in \mathrm{N}(\mathbf{C})$ maps $\boldsymbol{\sigma} \in \operatorname{ni}(\mathbf{C})$ to $\gamma^{-1} \boldsymbol{\sigma} \gamma \in \operatorname{ni}(\mathbf{C})$. Similarly, $\mathrm{SN}(\mathbf{C})$ acts on the straight Nielsen class $\operatorname{sni}(\mathbf{C})$. Denote the quotients of these actions by $\operatorname{ni}(\mathbf{C})^{\mathrm{ab}}$, $\operatorname{sni}(\mathbf{C})^{\mathrm{ab}}$, the *absolute Nielsen classes.*

Example 1.3. A Nielsen class representing genus 0 covers.

Let $r=4$, $n=5$, $G=S_5$. Consider the 4-tuple $\mathbf{C}=(C_1, \dots, C_4)$ of conjugacy classes in S_5 represented, respectively, by $\sigma_1=(23)(45)$, $\sigma_2=(12)$, $\sigma_3=(14)$, $\sigma_4=(54321)$. The normalizer $\mathrm{N}(\mathbf{C})$ of the Nielsen class is S_5. Here is a listing of representatives of $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$ where in each case $\tau_4=(54321)$ [Fr3], (1.10):

(1.4)

a) $\tau_1=(2\ 3)(4\ 5)$, $\tau_2=(1\ 2)$, $\tau_3=(1\ 4)$;

b) $\tau_1=(2\ 3)(4\ 5)$, $\tau_2=(1\ 4)$, $\tau_3=(2\ 4)$;

c) $\tau_1=(2\ 3)(4\ 5)$, $\tau_2=(2\ 4)$, $\tau_3=(1\ 2)$;

d) $\tau_1=(2\ 5)(3\ 4)$, $\tau_2=(1\ 2)$, $\tau_3=(3\ 5)$;

e) $\tau_1=(2\ 5)(3\ 4)$, $\tau_2=(3\ 5)$, $\tau_3=(1\ 2)$.

Remark 1.6 of § 1.3 derives $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ from $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$. □

1.3. Algebraic families, Hurwitz families and Hurwitz monodromy. Throughout this paper an algebraic family $\mathcal{F}$ (of covers of $\mathbb{P}_z^1$ over a field $K \subset \mathbb{C}$) is defined by a triple $\mathcal{F}=(\mathcal{T}, \mathcal{H}, \Phi)$ where $\Phi: \mathcal{T} \to \mathcal{H} \times \mathbb{P}_z^1$ is a finite morphism between quasiprojective varieties $\mathcal{T}$ and $\mathcal{H}$ defined over K; $\mathcal{H}$ is irreducible; Φ is a finite morphism; and the generic fiber of $\mathrm{pr}_1 \circ \Phi$ is irreducible where pr_1 is projection onto the first factor of $\mathcal{H} \times \mathbb{P}_z^1$. We refer, respectively, to $\mathcal{T}$, $\mathcal{H}$ and Φ as the *total space*, the *parameter space* and the *covering map*.

For each $\mathbf{x} \in \mathcal{H}$ denote the fiber (cover) of $\mathcal{F}$ over $\mathbf{x}$ by $\mathcal{F}_{\mathbf{x}}$:

$$\Phi_{\mathbf{x}}: \mathcal{T}_{\mathbf{x}} \to \mathbb{P}_z^1 \quad \text{with} \quad \mathcal{T}_{\mathbf{x}} = \mathcal{T} \times_{\mathcal{H}} \mathrm{Spec}(K(\mathbf{x}))$$

obtained from $\mathcal{F}$ by taking the fiber product over the map $\mathrm{Spec}(K(\mathbf{x})) \to \mathcal{H}$ that has image $\mathbf{x}$. When $\mathbf{x}$ is the generic point we use the subscript "gen". The cover $\mathcal{F}_{\mathbf{x}}$ is defined over $K(\mathbf{x})$; throughout this paper, the phrase "rational on $\mathcal{T}_{\mathbf{x}}$ (or $\mathbb{P}_{K(\mathbf{x})}^1 = \mathbb{P}_z^1$)" means rational on $\mathcal{T}_{\mathbf{x}}$ (or $\mathbb{P}_{K(\mathbf{x})}^1$) over the field $K(\mathbf{x})$. For $\mathbf{x} \in \mathcal{H}(\mathbb{C})$, the cover $\mathcal{F}_{\mathbf{x}} \otimes_K \mathbb{C}$ obtained from $\mathcal{F}_{\mathbf{x}}$ by extension of scalars is a cover of $\mathbb{P}_z^1$ in the sense of § 1.1. In the sequal we drop the expression $\otimes_K \mathbb{C}$ and any other field notation that is understood from the context.

Let G be a transitive subgroup of S_n and let $\mathbf{C}$ be an r-tuple of nontrivial conjugacy classes of G.

Definition 1.4. An algebraic family $\mathcal{F}$ is said to be a *Hurwitz family associated to the Nielsen class* $\mathrm{ni}(\mathbf{C})$ if $\mathrm{pr}_1 \circ \Phi$ is smooth and projective [H], p. 268, and the covers $\mathcal{F}_{\mathbf{x}}$ are in the Nielsen class $\mathrm{ni}(\mathbf{C})$ for all $\mathbf{x} \in \mathcal{H}(\mathbb{C})$.

Note that all of the fibers $\mathcal{F}_{\mathbf{x}}$ in a Hurwitz family have the same genus. We call this the *genus* of $\mathcal{F}$. Also, the fibers are covers with exactly r branch points. Lemma 1.5 below is a converse to this.

Suppose $\mathcal{F}$ is an algebraic family such that each fiber $\mathcal{F}_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{H}(\mathbb{C})$ is a cover with exactly r branch points, for r a given positive integer. There is a map naturally attached to $\mathcal{F}$ that sends each cover $\mathcal{F}_{\mathbf{x}}$ to its branch point set: it is called the *branch point reference map*. Formally this would be said in terms of the variety $(\mathbb{P}^1)^r \setminus \Delta_r$, denoted $\mathcal{U}_r$, with Δ_r the subset of $(\mathbb{P}^1)^r$ with 2 or more coordinates equal. The process of taking the quotient of the action of S_r on $\mathcal{U}_r$ gives the Noether map [Fr1], § 0,

$$(1.5) \qquad \Psi_r: (\mathbb{P}^1)^r \backslash \Delta_r \to \mathbb{P}^r \backslash D_r \quad (\text{denoted } \mathcal{U}^r),$$

that takes an ordered r-tuple of distinct points of $\mathbb{P}^1$ to the symmetric functions in these coordinates. Here D_r is the discriminant locus of $\mathbb{P}^r$. Then the branch point reference allusion would correspond to the map

$$(1.6) \qquad \Psi: \mathcal{H} \to \mathcal{U}^r$$

that takes each $\mathbf{x} \in \mathcal{H}$ to the unordered branch point set of the cover $\mathcal{F}_{\mathbf{x}}$.

Lemma 1.5. *Let $\mathcal{F}$ as above be an algebraic family such that each fiber $\mathcal{F}_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{H}(\mathbb{C})$ is a cover with exactly r branch points. Suppose* $\mathrm{pr}_1 \circ \Phi$ *is smooth. Then*

a) *the branch point reference map $\Psi: \mathcal{H} \to \mathcal{U}^r$ is a morphism;*

b) *the family $\mathcal{F}$ is a Hurwitz family.*

Proof. a) The map Ψ is defined everywhere so we just need to show that Ψ is locally given by $(r+1)$-tuples of rational functions of $\mathcal{H}$. Consider the ramification locus of the finite morphism $\mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$; it is a positive divisor $D = \sum m_i H_i$ on $\mathcal{H} \times \mathbb{P}^1$ (where each H_i is a closed irreducible subscheme of codimension 1 of $\mathcal{H} \times \mathbb{P}^1$). Then the divisor $\hat{D} = \sum H_i$ has no multiplicities. Cover $\mathcal{H}$ by open subsets U such that $\hat{D}$ is given on each $U \times \mathbb{P}^1$ by a polynomial $R \in \mathcal{O}(U)$ $[z, t]$ homogeneous in (z, t) and with coefficients in $\mathcal{O}(U)$ ($\mathcal{O}$ denotes the structure sheaf of the variety H). For $\mathbf{x} \in U$, the divisor $\hat{D}_{\mathbf{x}}$ of $\mathbb{P}^1$, obtained from $\hat{D}$ by specialization to $\mathbf{x}$, can be written as $\hat{D}_{\mathbf{x}} = \sum n_i z_i$ where the z_i are the branch points of the cover $\mathcal{F}_{\mathbf{x}}$ and the multiplicities n_i are positive integers. Furthermore $n_i = 1$, $i = 1, \ldots, r$, for all $\mathbf{x} \in U$ excluding a proper Zariski closed subset of U. But since we assume the support of $\hat{D}_{\mathbf{x}}$ is of constant cardinality, this actually holds for all $\mathbf{x} \in U$. Thus, for all $\mathbf{x} \in U$, $\hat{D}_{\mathbf{x}} = \sum z_i$ (i.e. the branch points $z_1, \ldots, z_r$ are the r simple roots of the specialized polynomial $R_{\mathbf{x}}$). Therefore, the map $\Psi: \mathcal{H} \to \mathcal{U}^r$ is given on U by the $(r+1)$-tuple of the coefficients on $\mathcal{O}(U)$ of the polynomial R. □

b) Since $\mathcal{H}$ is irreducible (hence connected), we only need to prove that if a fiber $\mathcal{F} \backslash S(, \mathcal{F}_{\mathbf{x}^0})$ with $\mathbf{x}^0 \in \mathcal{H}(\mathbb{C})$ is in a specific Nielsen class ni(**C**), then all fibers $\mathcal{F}_{\mathbf{x}}$ are in this same Nielsen class, for all $\mathbf{x}$ in a neighbourhood $U \subset \mathcal{H}(\mathbb{C})$ of $\mathbf{x}^0$. Denote the set of branch points of $\mathcal{F}_{\mathbf{x}^0}$ by $\mathbf{z}^0$, and fix $z_0 \notin \mathbf{z}^0$ and a sample bouquet $\Gamma_1, \ldots, \Gamma_r$ on $\mathbb{P}_z^1 \backslash \mathbf{z}^0$. It follows from the definition of "sample bouquet" (§ 1.1) and the continuity of $\Psi: \mathcal{H} \to \mathcal{U}^r$ that we may choose a connected neighbourhood U of $\mathbf{x}^0$ on $\mathcal{H}(\mathbb{C})$ with the following property: for $\mathbf{x} \in U$, if $\mathbf{z}$ is the set of branch points of the cover $\mathcal{F}_{\mathbf{x}}$, then $\Gamma_1, \ldots, \Gamma_r$ is a sample bouquet on $\mathbb{P}_z^1 \backslash \mathbf{z}$. Next use local sections of the map $\mathcal{T} \to \mathcal{H} \times \mathbb{P}_z^1$ to label the points of $\varphi_{\mathbf{x}}^{-1}(z_0)$ for $\mathbf{x} \in U$ (restrict U if needed). For $\mathbf{x} \in U$ compute a branch cycle description of $\mathcal{F}_{\mathbf{x}}$ from this data; it is a continuous (well defined) function of $\mathbf{x}$, therefore it is a constant function on U. Since the Nielsen class of a cover is independent of the choice of a sample bouquet, we are done. □

Under certain assumptions (below) we can assure the existence of a universal Hurwitz family $\mathcal{F}(\mathbf{C})$ associated to the Nielsen class ni(**C**) in the following sense (cf. condition (1.9) b)): any other Hurwitz family associated to the Nielsen class ni(**C**) is obtained by pullback of the parameter space of $\mathscr{F}(\mathbf{C})$ over the reference variety $\mathscr{U}^r$. The key to these assumptions is a fundamental tool for this paper: Hurwitz monodromy action.

Consider the free group on generators, Q_i, $i=1, \dots, r-1$, with these relations:

(1. 7) a) $Q_i Q_{i+1} Q_i = Q_{i+1} Q_i Q_{i+1}$, $i=1, \dots, r-2$;

b) $Q_i Q_j = Q_j Q_i$ for $|i-j|>1$; and

c) $Q_1 Q_2 \cdots Q_{r-1} Q_{r-1} \cdots Q_1 = 1$.

This group, a quotient of the Artin braid group [Bo], is called the *Hurwitz monodromy group* of degree r. We denote it by $\mathrm{H}(r)$. The Q_i's act on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ by this formula: for $\boldsymbol{\sigma} \in \mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$

(1. 8) $(\boldsymbol{\sigma}) Q_i = (\sigma_1, \dots, \sigma_{i-1}, \sigma_i \sigma_{i+1} \sigma_i^{-1}, \sigma_i, \sigma_{i+2}, \dots, \sigma_r)$, $i=1, \dots, r-1$.

Thus they induce a permutation representation of $\mathrm{H}(r)$ on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$: the Hurwitz monodromy action on the Nielsen class $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$.

Remark 1. 6. The natural permutation representation α_r: $\mathrm{H}(r) \to S_r$ by $Q_i \to (i\ i+1)$, $i=1, \dots, r-1$, is surjective. Thus, each orbit of $\mathrm{H}(r)$ on $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$ contains an element of $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$. □

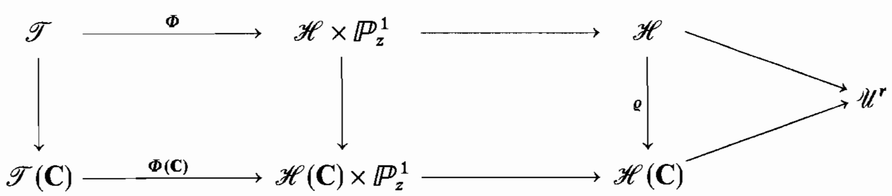
Theorem 1. 7. *Assume that the centralizer* $\mathrm{Cen}_{S_n}(G)$ *of* G *in* S_n *is trivial and that the Hurwitz monodromy action is transitive on* $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$. *Then there is an algebraic family* $\mathscr{F}(\mathbf{C})$ (*defined over* $\mathbb{C}$)

$$\Phi(\mathbf{C})\colon \mathscr{T}(\mathbf{C}) \to \mathscr{H}(\mathbf{C}) \times \mathbb{P}_z^1,$$

the universal Hurwitz family associated to $\mathrm{ni}(\mathbf{C})$, *with these properties*:

(1. 9) a) $\mathscr{F}(\mathbf{C})$ *is a Hurwitz family associated to* $\mathrm{ni}(\mathbf{C})$;

b) *If* $\Phi\colon \mathscr{T} \to \mathscr{H} \times \mathbb{P}_z^1$ *is any Hurwitz family* $\mathscr{F}$ *associated to* $\mathrm{ni}(\mathbf{C})$, *then there is a unique morphism of families of covers* $\mathscr{F} \to \mathscr{F}(\mathbf{C})$, *i.e., a unique commutative diagram of morphisms* (*defined over* $\mathbb{C}$)



that preserves the equivalence of covers (*i.e., the fiber covers* $\mathscr{F}_{\mathbf{x}}$ *and* $\mathscr{F}(\mathbf{C})_{\varrho(\mathbf{x})}$ *are equivalent covers, for all* $\mathbf{x} \in \mathscr{H}(\mathbb{C})$). *Furthermore*:

c) *The family* $\mathscr{F}(\mathbf{C})$ *can be defined over a number field* K, *which is minimal in that for any* $\sigma \in G(\bar{\mathbb{Q}}/\mathbb{Q})$, *if* $\mathscr{F}(\mathbf{C})^\sigma$ *is isomorphic to* $\mathscr{F}(\mathbf{C})$ (*i.e., if* $\mathscr{F}(\mathbf{C})^\sigma$ *satisfies* (1. 9) a) *and* (1. 9) b)), *then* σ *is fixed on* K;

d) $\Psi(\mathbf{C})\colon \mathscr{H}(\mathbf{C}) \to \mathscr{U}^r$ *is defined over* K *and étale of degree* $|\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}|$.

Comments. **a)** Theorem 1.7 appears as a special case of the results of [Fr2], § 4 and 5. We recall that the fundamental group of $\mathcal{U}^r$ is identified with the Hurwitz monodromy group $\mathrm{H}(r)$ and that the cover $\mathcal{H}(\mathbf{C}) \to \mathcal{U}^r$ is just the analytic cover associated to the action of $\mathrm{H}(r)$ on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$. Transitivity of this action ensures connectedness of $\mathcal{H}(\mathbf{C})$ and a result of Grauert and Remmert [GrR] to the effect that normal analytic covers of quasiprojective varieties are themselves quasiprojective certifies that all varieties in question are, indeed, quasiprojective (as in [Fr2]). A more algebraic approach, due to Thompson [Th] can be used to replace [GrR].

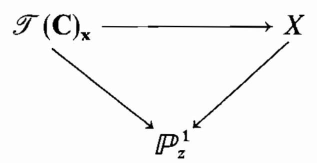
b) Theorem 1.7 demonstrates a solution to a moduli problem: equivalence classes of covers in $\mathrm{ni}(\mathbf{C})$ can be represented by a universal pair $(\mathcal{F}(\mathbf{C}) \to \mathcal{H} \times \mathbb{P}^1, \mathcal{H}(\mathbf{C}))$. That is, each cover $\varphi: X \to \mathbb{P}^1_z$ in $\mathrm{ni}(\mathbf{C})$ is equivalent to a unique fiber cover $\mathcal{F}(\mathbf{C})_{\mathbf{x}}: \mathcal{T}(\mathbf{C})_{\mathbf{x}} \to \mathbb{P}^1_z$ $(\mathbf{x} \in \mathcal{H}(\mathbf{C})(\mathbb{C}))$, which has no automorphisms. Indeed, apply (1.9) b) to the family consisting of the one cover $\varphi: X \to \mathbb{P}^1_z$ to get a fiber cover $\mathcal{F}(\mathbf{C})_{\mathbf{x}}$ equivalent to $\varphi: X \to \mathbb{P}^1_z$; uniqueness of $\mathbf{x} \in \mathcal{H}(\mathbf{C})(\mathbb{C})$ follows from (1.9) d) and uniqueness in (1.9) b) implies that the automorphism group of the cover $\mathrm{Aut}(\mathcal{T}(\mathbf{C})_{\mathbf{x}}/\mathbb{P}^1_z)$ is trivial. This, with the identification $\mathrm{Aut}(\mathcal{T}(\mathbf{C})_{\mathbf{x}}/\mathbb{P}^1_z) \approx \mathrm{Cen}_{S_n}(G)$, explains the role of the assumption $\mathrm{Cen}_{S_n}(G) = \{1\}$.

c) In addition, from this centralizer condition, a universal Hurwitz family is necessarily unique, up to a unique isomorphism of families of covers. It is standard that this guarantees condition (1.9) c). That is, there exists a minimal field of definition $K \subset \bar{\mathbb{Q}}$: K is the fixed field in $\bar{\mathbb{Q}}$ of the subgroup of $G(\bar{\mathbb{Q}}/\mathbb{Q})$ consisting of all σ such that $\mathcal{F}(\mathbf{C})^\sigma$ is isomorphic to $\mathcal{F}(\mathbf{C})$. This field K is called the field of moduli of the pair $(\mathcal{T}(\mathbf{C}) \to H \times \mathbb{P}^1, \ \mathcal{H}(\mathbf{C}))$. This minimal field of definition can be computed under certain circumstances. For example, [Fr2], Theorem 5.1, and [Fr3], Proposition 1.5, show that if the kernel $\mathrm{SH}(r)$ of the homomorphism of α_r in Remark 1.6 is transitive on $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$, then K is cyclotomic and easily computed. In particular these results show that $K = \mathbb{Q}$ if in addition $C_1, \ldots, C_r$ are rational conjugacy classes of G (see § 1.4 for a definition of "rational"). Also [Fr3], Prop. 1.7, provides a criteria for the parameter space of $\mathcal{F}(\mathbf{C})$ to be a rational variety (i.e., a variety with a purely transcendental extension of K as its function field). Note: Since $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ is the union of the $\mathrm{sni}(\mathbf{D})^{\mathrm{ab}}$s with $\mathbf{D}$ running over the r-tuples obtained by permuting the conjugacy classes of $\mathbf{C}$, and since each $\mathbf{D}$ is in the orbit of $\mathrm{H}(r)/\mathrm{SH}(r)$ on $\mathbf{C}$, it is clear that transitivity of $\mathrm{SH}(r)$ on $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$ immediately implies transitivity of $\mathrm{H}(r)$ on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$.

The following result allows us to concentrate on the universal Hurwitz family from Theorem 1.7 for most of the concerns of this paper.

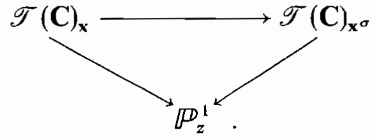
Proposition 1.8. *Let $\mathbf{x} \in \mathcal{H}(\mathbf{C})$ and $X \to \mathbb{P}^1_z$ be a cover defined over a field F. Assume that the covers $X \to \mathbb{P}^1_z$ and $\mathcal{F}(\mathbf{C})_{\mathbf{x}}$ are equivalent. Then $\mathcal{F}(\mathbf{C})_{\mathbf{x}}$ and the isomorphism $\mathcal{T}(\mathbf{C})_{\mathbf{x}} \approx X$ are defined over KF, with K the field of definition of $\mathcal{F}(\mathbf{C})$.*

Proof. Since $\mathcal{F}(\mathbf{C})_{\mathbf{x}}$ and $X \to \mathbb{P}^1_z$ are equivalent, there is a commutative diagram



where the right and left covers are defined, respectively, over $K(\mathbf{x}) \subset \bar{K}\bar{F}$ and F, and the top arrow is an isomorphism defined over $\mathbb{C}$. We descend from $\mathbb{C}$ to $\bar{K}\bar{F} = \bar{F}$ by the following argument. The top arrow is defined over a finite type extension of $\bar{F}$ which can be written in the form $\bar{F}(w_1, \dots, w_s, y)$ with $w_1, \dots, w_s$ algebraically independent over $\bar{F}$ and $[\bar{F}(w_1, \dots, w_s, y): \bar{F}(w_1, \dots, w_s)] < \infty$. Specialize $w_1, \dots, w_s, y$ to have values in $\bar{F}$ so as to get an isomorphism defined over $\bar{F}$ between $\mathscr{T}(\mathbf{C})_{\mathbf{x}}$ and X (since these are defined over $\bar{F}$). As $\mathrm{Aut}(\mathscr{T}(\mathbf{C})_{\mathbf{x}}/\mathbb{P}_z^1) = \{1\}$, this last isomorphism is the same as the original.

Consider $\sigma \in G(\bar{F}/KF)$; σ fixes the cover $X \to \mathbb{P}_z^1$, and the transform $\mathscr{F}(\mathbf{C})^\sigma$ is just $\mathscr{F}(\mathbf{C})$. Thus σ transforms the cover $\mathscr{F}(\mathbf{C})_{\mathbf{x}}$ to $\mathscr{F}(\mathbf{C})_{\mathbf{x}^\sigma}$, the fiber corresponding to $\mathbf{x}^\sigma \in \mathscr{H}(\mathbf{C})$. This gives another diagram



Necessarily $\mathbf{x} = \mathbf{x}^\sigma$ (from Comment b), two distinct fibers of $\mathscr{F}(\mathbf{C})$ are nonequivalent); then again apply $\mathrm{Aut}(\mathscr{T}(\mathbf{C})_{\mathbf{x}}/\mathbb{P}_z^1) = \{1\}$ to see that the horizontal map is necessarily the identity. Therefore $\mathscr{F}(\mathbf{C})_{\mathbf{x}}$ and the isomorphism $\mathscr{T}(\mathbf{C})_{\mathbf{x}} \to X$ are transformed into themselves by σ. It is well known, as a consequence of Hilbert's Theorem 90 and $\mathrm{char}(F) = 0$, that a variety transformed into itself by each element of $G(\bar{F}/KF)$ is defined over KF. Thus the Proposition follows. □

Thus, for example, such arithmetical results as the existence of a rational point on the generic cover, once established for the universal Hurwitz family $\mathscr{F}(\mathbf{C})$, hold for any Hurwitz family $\mathscr{F}$ associated with $\mathrm{ni}(\mathbf{C})$ defined over K: apply Prop. 1.8 to the generic cover of the family $\mathscr{F}$. From now on we consider *universal* Hurwitz families only.

1.4. Specializing z_r to ∞. We introduce an additional hypothesis for this section. For $i = 1, \dots, r$ denote the order of the elements of C_i by e_i, l.c.m. $(e_1, \dots, e_r)$ by N and $\bigcup_{(a,N)=1} C_i^a$ by $\bar{C}_i$, *the rational closure of the conjugacy class* C_i. Consider the group

$$\mathrm{N}_{S_n}^{\mathrm{gal}}(\mathbf{C}) = \{\gamma \in \mathrm{N}_{S_n}(G) \mid \text{there exists } \beta \in S_r \text{ and an integer } b \text{ prime to } N \text{ such that } \gamma C_i \gamma^{-1} = C_{(i)\beta}^b,\ i = 1, \dots, r\}.$$

Note that we have the following chain of containments:

$$\mathrm{N}_{S_n}(G) \supset \mathrm{N}_{S_n}^{\mathrm{gal}}(\mathbf{C}) \supset \mathrm{N}(\mathbf{C}) \supset \mathrm{SN}(\mathbf{C}) \supset G.$$

As John Thompson has pointed out, there are examples of $(G, \mathbf{C})$ such that $G \subset S_n$, G is transitive, $\mathrm{Cen}_{S_n}(G) = \{1\}$, $\mathrm{ni}(\mathbf{C}) \neq \emptyset$, and such that in the above chain, each containment is proper. In addition to the two hypotheses of Theorem 1.7 we assume that one conjugacy class, say C_r, has this property:

(1.10) for each $\gamma \in \mathrm{N}_{S_n}^{\mathrm{gal}}(\mathbf{C})$ and $i = 1, \dots, r-1$, $\bar{C}_r$ is distinct from $\gamma \bar{C}_i \gamma^{-1}$.

We recall that a conjugacy class C is said to be *rational* if it is closed under putting elements to power relatively prime to the order of elements in the class; in that case, $\bar{C} = C$. In all the examples given in this paper, the group G is a symmetric group S_m, for some integer m. Therefore all the conjugacy classes are rational. Condition (1.10) then only requires that $C_r \neq C_1, \ldots, C_{r-1}$, since in our cases we will also have $N_{S_n}(G)=G$ with $G=S_m$ (*m* not necessarily equal to *n*). This kind of assumption is natural in those problems that arise from the consideration, for example, of maps

$$\varphi(f)\colon \mathbb{P}^1_w \longrightarrow \mathbb{P}^1_z$$

with f a polynomial in w, since such a map is totally ramified over ∞.

With hypothesis (1.10), the branch point associated to C_r in a cover $\mathscr{F}(\mathbf{C})_{\mathbf{x}}$ is automatically rational over $K(\mathbf{x})$: it is consequence of the following result called *the branch cycle argument* in [Fr2]. Set $\zeta_N = \exp(2i\pi/N)$.

Proposition 1.9. *Let $\varphi\colon X \longrightarrow \mathbb{P}^1_z$ be a cover defined over a field F and $\sigma \in G(\bar{F}/F)$. Let $z_1, \ldots, z_r$ be an ordering of the branch points for which the cover is in* sni(**C**). *Let $a \in (\mathbb{Z}/N)^*$ such that $\zeta_N^\sigma = \zeta_N^a$. Then σ acts on the branch point set $\{z_1, \ldots, z_r\}$ with this property: if $\bar{\sigma}$ denotes the permutation of $\{1, \ldots, r\}$ induced by σ then there exists $\gamma \in N_{S_n}(G)$ such that $\gamma C_i^a \gamma^{-1} = C_{(i)\bar{\sigma}}$ (in particular, $\gamma \in N^{\mathrm{gal}}_{S_n}(\mathbf{C})$).*

Now we wish to specialize the branch point corresponding to C_r to ∞. This may be done as follows. For each cover, we may assume that the branch point corresponding to C_r is z_r. But that requires us to somewhat modify the Nielsen class data. This imposes that the r-th branch cycle itself is in C_r. We may then also assume a fixed representative σ_r of the r-th branch cycle for the absolute Nielsen class. Thus we are led to consider

$$\mathrm{ni}(\mathbf{C}_\infty) = \{\tau \in \mathrm{ni}(\mathbf{C}) \mid \tau_r = \sigma_r\} \quad \text{and} \quad \mathrm{N}(\mathbf{C}_\infty) = \{\gamma \in \mathrm{N}(\mathbf{C}) \mid \gamma \text{ centralizes } \sigma_r\},$$

with the latter replacing $\mathrm{N}(\mathbf{C})$ in order to form the quotient $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$.

This requires some adjustments to $\mathscr{F}(\mathbf{C})$. Consider first the fiber product

$$\begin{array}{ccccc} & & (\mathscr{U}^{r-1} \times \mathbb{P}^1_z)' & & \\ & \nearrow & & \searrow & \\ \mathscr{H}(\mathbf{C}) \times_{\mathscr{U}^r} (\mathscr{U}^{r-1} \times \mathbb{P}^1_z)' & & & & \mathscr{U}^r \\ & \searrow & & \nearrow & \\ & & \mathscr{H}(\mathbf{C}) & & \end{array}$$

where $'$ indicates that the last coordinate z of a point (D, z), with D a divisor of degree $r-1$ on $\mathbb{P}^1_z$ is not in the support of D. The south-east arrow maps (D, z) to the divisor $D+z$ on $\mathbb{P}^1_z$. The fiber product on the left of the diagram has a component consisting of the points $(\mathbf{x}, (D, z_r))$ for which the cover $\mathscr{F}(\mathbf{C})_{\mathbf{x}}$ has C_r corresponding to the branch point z_r. Call this $\mathscr{H}(\mathbf{C}; r)$. From assumption (1.10), it is isomorphic to $\mathscr{H}(\mathbf{C})$. Therefore it is irreducible.

Next, consider the subset $\mathscr{H}(\mathbf{C}_\infty)$ of $\mathscr{H}(\mathbf{C}; r)$ that lies over the collection of points $(D, z_r) \in \mathscr{U}^{r-1} \times \mathbb{P}^1_z$ for which $z_r = \infty$. Denote this last set by $\mathscr{U}^{r-1}_\infty$. It is clearly $\mathbb{A}^{r-1} \backslash D_{r-1}$, affine $(r-1)$-space minus its discriminant locus. A further hypothesis is needed to insure irreducibility of $\mathscr{H}(\mathbf{C}_\infty)$. In § 1.3 (Comment 1), we recall that the cover $\mathscr{H}(\mathbf{C}) \longrightarrow \mathscr{U}^r$ is just the analytic cover associated to the action of $\mathrm{H}(r)$ (identified with the fundamental group of $\mathscr{U}^r$) on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ given by the action of the generators Q_i, $i=1, \ldots, r-1$, in (1.8). Denote the subgroup of $\mathrm{H}(r)$ generated by $Q_1, \ldots, Q_{r-2}$ by $\mathrm{H}(r-1)^*$. This is most naturally identified with the Artin Braid group of degree $r-1$ as the fundamental group of $\mathscr{U}^{r-1}_\infty$ (e.g., [BFr], § 1).

Proposition 1. 10. *In addition to* (1. 10) *assume that*

(1. 11) *the action of* $\mathrm{H}(r-1)^*$ *on* $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ *is transitive.*

Then the variety $\mathscr{H}(\mathbf{C}_\infty)$ *is irreducible.*

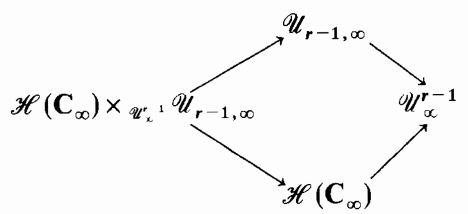
Proof. Clearly $\mathscr{H}(\mathbf{C}_\infty)$ is identified with the fiber product of $\mathscr{H}(\mathbf{C}; r)$ and $\mathscr{U}_\infty^{r-1}$ over $(\mathscr{U}^{r-1} \times \mathbb{P}_z^1)'$. The action of $\mathrm{H}(r)$ on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ induces an action of the fundamental group of $\mathscr{U}_\infty^{r-1}$ on $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ through its embedding in $(\mathscr{U}^{r-1} \times \mathbb{P}_z^1)'$: and this action is transitive if and only if $\mathscr{H}(\mathbf{C}_\infty)$ is irreducible. Thus Prop. 1. 10 follows with the identification of $\mathrm{H}(r-1)^*$ with the fundamental group of $\mathscr{U}_\infty^{r-1}$. □

Assume (1. 11) holds. Then *the universal Hurwitz family associated to* $\mathrm{ni}(\mathbf{C})$ *with the additional stipulation that* z_r *has been specialized to* ∞ is the family $\mathscr{F}(\mathbf{C}_\infty)$ given by $\mathscr{T}(\mathbf{C}_\infty) \to \mathscr{H}(\mathbf{C}_\infty) \times \mathbb{P}_z^1$ where $\mathscr{T}(\mathbf{C}_\infty)$ is defined to be the fiber product $\mathscr{T}(\mathbf{C}) \times_{\mathscr{H}(\mathbf{C})} \mathscr{H}(\mathbf{C}_\infty)$. Since all fibers of the family $\mathscr{F}(\mathbf{C})$ are connected (Def. 1. 4), $\mathscr{T}(\mathbf{C}_\infty)$ is irreducible. Compatible with the discussion of (1. 5) and (1. 6) of § 1. 3 the branch point reference for $\mathscr{H}(\mathbf{C}_\infty)$ corresponds to the map

(1. 12) $$\Psi(\mathbf{C}_\infty)\colon \mathscr{H}(\mathbf{C}_\infty) \to \mathscr{U}_\infty^{r-1}$$

that takes an equivalence class of a fiber cover of $\mathscr{F}(\mathbf{C}_\infty)$ to the unordered set of its finite branch points. This is an étale morphism of degree $|\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}|$.

1. 5. Adjunction of the branch points. In our use of the unordered branch point set as reference data for the family $\mathscr{F}(\mathbf{C}_\infty)$ we sometimes have need to use a less symmetrical situation by putting an ordering on the finite branch point set. First consider the fiber product



where $\mathscr{U}_{r-1,\infty} = \mathbb{A}^{r-1} \setminus \Delta_{r-1}$ is affine $(r-1)$-space minus the subset of points with two or more coordinates equal. The map $\mathscr{U}_{r-1,\infty} \to \mathscr{U}_\infty^{r-1}$ is the Noether map induced by the elementary symmetric functions in $r-1$ variables.

The fiber product isn't usually connected. For example there are $(r-1)!$ connected components corresponding to the different orderings of the branch points in the case that, in addition to (1. 10), the other conjugacy classes $C_1, \ldots, C_{r-1}$ are also distinct modulo $\mathrm{N}(\mathbf{C}_\infty)$. More generally, consider $\mathrm{SH}(r-1)^*$, the kernel of the morphism $\mathrm{H}(r-1)^* \to S_{r-1}$ that maps Q_i to $(i\ i+1)$, $i = 1, \ldots, r-2$. Similarly, consider $\mathrm{sni}(\mathbf{C}_\infty)^{\mathrm{ab}}$, the analogue of $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$ of § 1. 2. The following may be proved in the manner of Prop. 1. 10.

Proposition 1. 11. *In addition to* (1. 10) *assume that*

(1. 13) *the action of* $\mathrm{SH}(r-1)^*$ *on* $\mathrm{sni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ *is transitive.*

Then the number of connected components of $\mathscr{H}(\mathbf{C}_\infty)\times_{\mathscr{U}_\infty^{r-1}}\mathscr{U}_{r-1,\infty}$ *is the number of distinct orderings of the conjugacy classes* $C_1, \ldots, C_{r-1}$ *modulo* $N(\mathbf{C}_\infty)$.

Since $\mathscr{U}_{r-1,\infty}\to\mathscr{U}_\infty^{r-1}$ is a Galois cover, all of the connected components of the fiber product are isomorphic. Select one of them, $\mathscr{H}(\mathbf{C}_\infty)'$. Under assumption (1. 13), *the universal Hurwitz family associated with* $\mathrm{ni}(\mathbf{C}_\infty)$ *with the branch points adjoined* is the family $\mathscr{F}(\mathbf{C}_\infty)'$ given by $\mathscr{T}(\mathbf{C}_\infty)'\to\mathscr{H}(\mathbf{C}_\infty)'\times\mathbb{P}_z^1$, where $\mathscr{T}(\mathbf{C}_\infty)'$ is $\mathscr{T}(\mathbf{C}_\infty)\times_{\mathscr{H}(\mathbf{C}_\infty)}\mathscr{H}(\mathbf{C}_\infty)'$. The branch point reference corresponds to the map $\Psi(\mathbf{C}_\infty)'\colon\mathscr{H}(\mathbf{C}_\infty)'\to\mathscr{U}_{r-1,\infty}$ that takes the equivalence class of a fiber cover of $\mathscr{F}(\mathbf{C}_\infty)'$ to its ordered branch point set. It is an étale morphism of degree $|\mathrm{sni}(\mathbf{C}_\infty)^{\mathrm{ab}}|$. Note that the fiber covers $\mathscr{F}(\mathbf{C}_\infty)'_{\mathbf{x}}$, of the family $\mathscr{F}(\mathbf{C}_\infty)'$ derive from the corresponding covers $\mathscr{F}(\mathbf{C}_\infty)_{\mathbf{x}}$ (i.e., $\mathbf{x}$ lies below $\mathbf{x}'$ in the natural map $\mathscr{H}(\mathbf{C}_\infty)'\to\mathscr{H}(\mathbf{C}_\infty)$) by extension of scalars: the field of definition of $\mathscr{F}(\mathbf{C}_\infty)'_{\mathbf{x}'}$ is the field $K(\mathbf{x})$ with the finite branch points of the cover adjoined.

§ 2. Two introductory examples

This section uses two examples to acquaint the reader with the general theory of Nielsen classes and Hurwitz families developed in § 1 and to introduce the problems considered in the subsequent sections.

2. 1. The Hurwitz families $\mathscr{F}(o)$ and $\mathscr{F}(a)$. Throughout most of this section we take $r=4$, $G=S_5$ and $\mathbf{C}=(C_1,\ldots,C_4)$ as the respective conjugacy classes in S_5 of $\sigma_1=(2\ 3)(4\ 5)$, $\sigma_2=(1\ 2)$, $\sigma_3=(1\ 4)$ and $\sigma_4=(5\ 4\ 3\ 2\ 1)$. Our first example extends Ex. 1. 3 (of § 1. 2) for $n=5$. The 2nd example, however, considers S_5 as a subgroup of S_{10} ($n=10$) through the representation $T_2\colon S_5\to S_{10}$ given by action of S_5 on the 10 unordered pairs from $\{1, 2, 3, 4, 5\}$. For example, label the pairs $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$, $\{4,5\}$ respectively as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Then, $T_2(\boldsymbol{\sigma})$ is this 4-tuple of elements of S_{10}:

(2. 1) $$T_2(\boldsymbol{\sigma})=((1\,2)(3\,4)(7\,8)(6\,9),\ (2\,5)(3\,6)(4\,7),\ (1\,6)(2\,8)(4\,10),\ (1\,4\,10\,8\,5)(2\,7\,3\,9\,6)).$$

For the sake of computing absolute Nielsen classes these two copies of S_5 must be regarded as a priori distinct, and so we decorate them, respectively, as $\mathrm{ni}(\mathbf{C})_5^{\mathrm{ab}}$ and $\mathrm{ni}(\mathbf{C})_{10}^{\mathrm{ab}}$. But the act of applying T_2 to the coordinates of elements of the former induces a natural map $T_2^*\colon\mathrm{ni}(\mathbf{C})_5^{\mathrm{ab}}\to\mathrm{ni}(\mathbf{C})_{10}^{\mathrm{ab}}$. The following lemma demonstrates that this map identifies the two sets.

Lemma 2. 1. *The normalizer* $\mathrm{N}_{S_{10}}(S_5)$ *of* S_5 *in* S_{10} *is* S_5. *Thus,* T_2^* *is one-one.*

Proof. Every automorphism of S_5 is inner [Bu], p. 209. Thus, $\mathrm{N}_{S_5}(S_5)=S_5\times C$, where $C=\mathrm{Cen}_{S_{10}}(S_5)$. The stabilizer in S_5 of a point is isomorphic to $S_3\times S_2$, which is its own normalizer in S_5. Recall that if G is a transitive subgroup of S_{10} and $G(1)$ is its stabilizer, then $\mathrm{N}_G(G(1))/G(1)$ is isomorphic to $\mathrm{Cen}_{S_{10}}(G)$. Therefore $C=1$; S_5 is selfnormalizing in S_{10}. Thus, T_2^* is a bijection. □

Lemma 2. 1 is a useful point. Quite a few properties of Hurwitz families depend only on the absolute Nielsen classes, without regard to the particular embedding of G in S_n. Thus this group theory check has established considerable sharing of properties for our two examples: for example, the second assumption of Theorem 1. 7 i.e., transitivity of H(4) on $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ (c.f., (2. 2) below). As for the first assumption of Theorem 1. 7, $\mathrm{Cen}_{S_5}(S_5)$ is clearly trivial and the triviality of $\mathrm{Cen}_{S_{10}}(S_5)$ follows from Lemma 2. 1.

Thus Theorem 1. 7 ensures the existence of universal Hurwitz families $\mathscr{F}_5(\mathbf{C})$ and $\mathscr{F}_{10}(\mathbf{C})$ associated with the considered Nielsen classes. For simplicity, denote, respectively, the total space, parameter space and covering map of $\mathscr{F}_5(\mathbf{C})$ by $\mathscr{T}_5$, $\mathscr{H}_5$ and Φ_5 and of $\mathscr{F}_{10}(\mathbf{C})$ by $\mathscr{T}_{10}$, $\mathscr{H}_{10}$ and Φ_{10}. The next result shows the effect of the representation T_2 at the level of the covers in the families.

Proposition 2. 2. *There is a natural isomorphism between the respective parameter spaces $\mathscr{H}_5$ and $\mathscr{H}_{10}$ of the two families. Let $\varphi: X \to \mathbb{P}_z^1$ be any cover of the family $\mathscr{F}_5(\mathbf{C})$. Then the corresponding cover of the family $\mathscr{F}_{10}(\mathbf{C})$ is equivalent to the cover $\varphi_H: \hat{X}_H \to \mathbb{P}_z^1$ (as in* (1. 2)) *where H is the stabilizer of* $\{1, 2\}$ *in S_5.*

Proof. First note that the representation $T_2: S_5 \to S_{10}$ is equivalent to the representation T_H of S_5 on the cosets of the subgroup stabilizing $\{1, 2\}$. Therefore from (2. 1), if τ is a description of the branch cycles of $\varphi: X \to \mathbb{P}_z^1$, then $T_2(\tau)$ is a description of the branch cycles of $\varphi_H: \hat{X}_H \to \mathbb{P}_z^1$. That is, this last cover is a fiber of the family $\mathscr{F}_{10}(\mathbf{C})$ (i.e., lies in the Nielsen class $\mathrm{ni}(\mathbf{C})_{10}$).

Geometrically we construct this cover from $\varphi: X \to \mathbb{P}_z^1$ as follows. Form the fiber product $X \times_{\mathbb{P}_z^1} X = \{(x_1, x_2) \in X \times X \mid \varphi(x_1) = \varphi(x_2)\}$; remove the diagonal $\Delta = \{(x, x) \mid x \in X\}$; quotient out by the action of $\mathbb{Z}/2$ by switching the coordinates in $X \times X$; and take the normalization of the resulting complex analytic set. (This last is necessary because whenever both coordinates of $y = (x_1, x_2)$ consist of points that are ramified over $\mathbb{P}_z^1$, then y will be a singular point of $X \times_{\mathbb{P}_z^1} X$.) This leads to geometric construction of the family $\mathscr{F}_{10}(\mathbf{C})$ by following the same rules applied to the fiber product $\mathscr{T}_5 \times_{\mathscr{H}_5} \mathscr{T}_5$. The result is a family of covers in the Nielsen class $\mathrm{ni}(\mathbf{C})_{10}$ over the parameter space $\mathscr{H}_5$ with total space denoted $\mathscr{T}_H$. In addition, consider the natural map of (1. 9) of Theorem 1. 7, $\Psi_H: \mathscr{H}_5 \to \mathscr{H}_{10}$, given by mapping $\mathbf{x}$ to the equivalence class of the cover $\mathscr{T}_{H,\mathbf{x}} \to \mathbb{P}_z^1$. Identify $\mathscr{H}_5$ (resp., $\mathscr{H}_{10}$) with the cover of $\mathscr{U}^4$ given by the action of H(4) on $\mathrm{ni}(\mathbf{C})_5^{\mathrm{ab}}$ (resp., $\mathrm{ni}(\mathbf{C})_{10}^{\mathrm{ab}}$). We then identify Ψ_H with the natural map from the theory of fundamental groups induced by the identification via Lemma 2. 1 of $\mathrm{ni}(\mathbf{C})_5^{\mathrm{ab}}$ and $\mathrm{ni}(\mathbf{C})_{10}^{\mathrm{ab}}$. Thus Ψ_H is an isomorphism and the result follows. □

Now consider the absolute straight Nielsen class whose elements are listed in Ex. 1. 3: denote these elements, respectively, by $0_a, 0_b, 0_c, 0_d, 0_e$. Check easily (for details see [Fr3], Part I, Ex. 1. 2 cont.) that

(2. 2) $\qquad Q_1^{-2} = (0_a 0_d 0_b)(0_c 0_e)$ and $Q_1 Q_2^{-2} Q_1^{-1} = (0_a 0_c 0_e)(0_b 0_d)$.

It follows that the hypothesis (1. 13) and (a fortiori) (1. 11) of § 1 hold. Therefore we may consider the families $\mathscr{F}_5(\mathbf{C}_\infty)$, $\mathscr{F}_{10}(\mathbf{C}_\infty)$ of § 1. 4 and $\mathscr{F}_5(\mathbf{C}_\infty)'$, $\mathscr{F}_{10}(\mathbf{C}_\infty)'$ of § 1. 5. For simplicity denote these respectively by $\mathscr{F}(o)$, $\mathscr{F}(a)$, $\mathscr{F}(o)'$ and $\mathscr{F}(a)'$. These will be used as illustrations throughout the remainder of this paper.

The following is a consequence ([Fr3], Ex. 1. 2 cont.) of general principles as a result of a numerical check involving Hurwitz monodromy action (cf. Comment c) on Theorem 1. 7).

Proposition 2. 3. *The families $\mathscr{F}(o)$, $\mathscr{F}(a)$, $\mathscr{F}(o)'$ and $\mathscr{F}(a)'$ are defined over $K=\mathbb{Q}$, and each of their parameter spaces — $\mathscr{H}$ for $\mathscr{F}(o)$ and $\mathscr{F}(a)$ and $\mathscr{H}'$ for $\mathscr{F}(o)'$ and $\mathscr{F}(a)'$ — is a rational variety over $\mathbb{Q}$.*

To fix the ideas note that the function field $\mathbb{Q}(\mathscr{H})$ (resp., $\mathbb{Q}(\mathscr{H}')$) — the field of definition of the generic cover in either $\mathscr{F}(o)$ or $\mathscr{F}(a)$ (resp., $\mathscr{F}(o)'$ or $\mathscr{F}(a)'$) — is a finite extension of $\mathbb{Q}(z_1, z_2+z_3, z_2 z_3)$ (resp., $\mathbb{Q}(z_1, z_2, z_3)$) with z_1, z_2, z_3, the branch points of a generic cover, ordered so that z_i corresponds to σ_i in the labeling of Ex. 1. 3. It is actually shown that $\mathbb{Q}(\mathscr{H})$ (resp., $\mathbb{Q}(\mathscr{H})$) is a pure transcendental extension of $\mathbb{Q}(z_2+z_3, z_2 z_3)$ (resp., $\mathbb{Q}(z_2, z_3)$).

2. 2. Study of the family $\mathscr{F}(o)$. The fiber covers of the family $\mathscr{F}(o)$, as we will see, are quite simple: each is equivalent to a cover $\mathbb{P}_y^1 \to \mathbb{P}_z^1$ given by a degree 5 polynomial. In addition to preparing for the subtler study of the family $\mathscr{F}(a)$, the next two subsections will display by example some of the standard arguments and potential difficulties.

Proposition 2. 4. *For each $\mathbf{x} \in \mathscr{H}$, the fiber cover $\varphi_{\mathbf{x}}: \mathscr{T}(o)_{\mathbf{x}} \to \mathbb{P}_z^1$ is equivalent (over its field of definition $\mathbb{Q}(\mathbf{x})$) to a cover $\varphi(h)$: $\mathbb{P}_y^1 \to \mathbb{P}_z^1$ with h a degree 5 polynomial.*

Proof. With g equal to the genus of $\mathscr{T}(o)_{\mathbf{x}}$ the Riemann-Hurwitz formula applied to a description σ of the branch cycles gives $2(5+g-1)=2+1+1+4$ or $g=0$. Since σ_4 consists of just one disjoint cycle, there is a unique point of $\mathscr{T}(o)_{\mathbf{x}}$, say $\mathbf{p}_\infty$, lying over ∞ and it is automatically rational (over $\mathbb{Q}(\mathbf{x})$). The rational degree 1 divisor $\mathbf{p}_\infty$ gives an embedding $\psi: \mathscr{T}(o)_{\mathbf{x}} \to \mathbb{P}_y^1$ of degree 1, which is therefore an isomorphism. Specifically, if $\{1, f\}$ is a basis of the linear system $\mathscr{L}(\mathbf{p}_\infty)$, ψ is given by sending $\mathbf{p} \in \mathscr{T}(o)_{\mathbf{x}}$ to $f(\mathbf{p})$ if $\mathbf{p}_\infty \neq \mathbf{p}$ and to ∞ otherwise. Thus the rational function $\varphi_{\mathbf{x}} \circ \psi^{-1}$: $\mathbb{P}_y^1 \to \mathbb{P}_z^1$ of degree 5 is a polynomial. □

Proposition 2. 5. *The polynomial h in Proposition 2. 4 may be taken to have the form $\alpha((y^5/5)-s(y^4/4)+2ty^3-5st(y^2/2)+5t^2y)+\beta$ with $\alpha, \beta, s, t \in \mathbb{Q}(\mathbf{x})$, $\alpha \neq 0$ and $t \neq 0$.*

Proof. Fix a fiber cover $\mathscr{F}_{\mathbf{x}}$ with $\mathbf{x} \in \mathscr{H}$, defined over $\mathbb{Q}(\mathbf{x})$, and identify it with $\varphi(h)$: $\mathbb{P}_y^1 \to \mathbb{P}_z^1$. Denote the branch points of $\mathscr{F}_{\mathbf{x}}$ by z_i, $i=1,2,3$. Then z_1 is automatically defined over $\mathbb{Q}(\mathbf{x})$. And while it is possible (e.g., if $\mathbf{x}$ is generic) that the branch points z_2 and z_3 can be interchanged by $G(\overline{\mathbb{Q}(\mathbf{x})}/\mathbb{Q}(\mathbf{x}))$, the divisor z_2+z_3 on $\mathbb{P}_z^1$ is necessarily $\mathbb{Q}(\mathbf{x})$-rational. Let a_1 and a_2 be the points of $\mathbb{P}_y^1$ that have ramification index 2 over z_1. Similarly, let c (resp., d) be the point of $\mathbb{P}_y^1$ having ramification index 2 over z_2 (resp., z_3). Thus $\frac{d}{dy}(h(y))=m(y)(y-c)(y-d)$ with $m(y)=\alpha(y-a_1)(y-a_2)$. With a further linear change in y we may assume $a_2=-a_1=a$. Thus, we obtain:

$$(2.3)\quad \frac{d}{dy}(h(y))=\alpha(y^2-a^2)(y-c)(y-d) \quad \text{with} \quad \alpha, a^2, cd, c+d \in \mathbb{Q}(\mathbf{x}),\ \alpha \neq 0$$
$$\text{and } h(a)=h(-a)=z_1,\ h(c)=z_2 \text{ and } h(d)=z_3.$$

Expanding $\frac{d}{dy}(h(y))$ out gives

$$(2.4)\quad h(y)=\alpha((y^5/5)-(c+d)(y^4/4)+(cd-a^2)(y^3/3)+a^2(c+d)(y^2/2)-a^2cdy)+\beta$$

with $\beta \in \mathbb{Q}(\mathbf{x})$. The condition $h(a)=h(-a)$ gives $a^4/5+(cd-a^2)a^2/3-a^2cd=0$ or $a^2=-5cd$. Substituting back completes the proof with $s=c+d$ and $t=cd$; $a \neq -a$ gives $t \neq 0$. □

The last result suggests the introduction of the family of polynomials parametrized by $\mathbb{G}_m \times \mathbb{A}^2 \times \mathbb{G}_m$ (with $\mathbb{G}_m=\mathbb{A}^1-\{0\}$)

$$h_{\mathbf{x}}(y)=\alpha((y^5/5)-s(y^4/4)+2ty^3-5st(y^2/2)+5t^2y)+\beta$$

for $\mathbf{x}=(\alpha,\beta,s,t) \in \mathbb{G}_m \times \mathbb{A}^2 \times \mathbb{G}_m$. Check that

$$(2.5)\quad \frac{d}{dy}(h_{\mathbf{x}}(y))=\alpha(y^2+5t)(y^2-sy+t) \quad \text{and} \quad h(a)=h(-a)=\alpha(25st^2/4)+\beta$$

where a denotes a square root of $5t$.

Thus the morphism $\mathbf{z}^*\colon \mathbb{G}_m \times \mathbb{A}^2 \times \mathbb{G}_m \to \mathbb{A}^3$ that sends a point $\mathbf{x}$ to the unordered set of finite branch points of the cover $\varphi(h_{\mathbf{x}})\colon \mathbb{P}^1_y \to \mathbb{P}^1_z$ is defined over $\mathbb{Q}(\alpha,\beta,s,t)$.

In the notation of § 1.5 let $\mathcal{O}$ be the Zariski open subset $(\mathbf{z}^*)^{-1}(\mathcal{U}^3_\infty)$. The above discussion shows:

Proposition 2.6. *For each* $\mathbf{x} \in \mathcal{O}$, $\varphi(h_{\mathbf{x}})\colon \mathbb{P}^1_y \to \mathbb{P}^1_z$ *lies in the Nielsen class* $\mathrm{ni}(\mathbf{C}_\infty)_5$.

2.3. Čech cohomology interpretation of families of rational functions. Consider again the family of covers $\mathcal{F}$ given by $\Phi\colon \mathcal{O} \times \mathbb{P}^1_y \to \mathcal{O} \times \mathbb{P}^1_z$ with the restriction of Φ to $\mathbf{x} \times \mathbb{P}^1_y$ given by $\varphi(h_{\mathbf{x}})$ on the second factor. From Prop. 2.6 it is a Hurwitz family associated to $\mathrm{ni}(\mathbf{C})$, *and the natural morphism* $\mathbf{z}^*_{\mathcal{O}}\colon \mathcal{O} \to \mathcal{U}^3_\infty$ is surjective. But it is not the universal Hurwitz family $\mathcal{F}(o)$: since the fibers of $\mathbf{z}^*_{\mathcal{O}}$ have dimension 1, it is not a covering map. This brings up a delicate question:

Question 2.7. Is $\mathcal{F}(o)$ a family of polynomials?

A rephrasing of this considers a morphism $\Lambda\colon \mathcal{H} \to \mathbb{A}^6$ (representing the coefficients of polynomials of degree at most 5) such that $\mathbf{x} \in \mathcal{H}$ is mapped to a polynomial $\Lambda(\mathbf{x})$ which gives a cover $\varphi(\Lambda(\mathbf{x}))\colon \mathbb{P}^1_y \to \mathbb{P}^1_z$ in the equivalence class of $\mathbf{x}$.

We also know that for each $\mathbf{x} \in \mathcal{H}$ there exists a neighbourhood $U_{\mathbf{x}}$ of $\mathbf{x}$ with a morphism $\Lambda_{\mathbf{x}}\colon U_{\mathbf{x}} \to \mathbb{A}^6$ such that the polynomial $\Lambda_{\mathbf{x}}(\mathbf{x}')$ is as desired for each $\mathbf{x}' \in U_{\mathbf{x}}$. Indeed, such a neighbourhood can be taken to be a ball. We now show that $U_{\mathbf{x}}$ can be given by pullback to an étale cover of $\mathcal{H}$.

Recall that the family $\mathscr{F}(o)'$ is given by pullback over $\mathscr{H}'$. According to Prop. 1.11, this is a connected component of $\mathscr{H} \times_{\mathscr{U}^3_\infty} \mathscr{U}_{3,\infty}$. Also, the degree of $\mathscr{H}' \to \mathscr{H}$ is the number of permutations of (C_1, C_2, C_3) that leave this 3-tuple of conjugacy classes fixed. In this case this is 2 (given by the switch of C_2 and C_3). Now, there *is* a morphism $\Lambda': \mathscr{H}' \to \mathbb{A}^6$ with the desired properties: for $\mathbf{x}' \in \mathscr{H}'$ the branch points z_2 and z_3 of the corresponding cover are rational over $\mathbb{Q}(\mathbf{x}')$ and therefore so are the corresponding points c and d used in the proof of Prop. 2. 5. Thus we can normalize any polynomial that gives the cover corresponding to $\mathbf{x}'$ so that, say, $c=0$ and $d=1$. There is a *unique* polynomial with these properties and thus we define Λ' using this.

Over $\mathscr{H}$ form the sheaf of groups $\mathscr{G} \times \mathscr{A}$ (resp., $\mathscr{PGL}$) such that over each analytic open subset U (or cover of a Zariski open subset) of $\mathscr{H}$, the global sections $\Gamma(U, \mathscr{G} \times \mathscr{A})$ (resp., $\Gamma(U, \mathscr{PGL})$) are the analytic maps $U \to \mathbb{G}_m \times \mathbb{A}^1$ (resp., $U \to \mathrm{PGL}(1)$). Here is how $\mathscr{F}(o)$ produces a 1-chain with coefficients in $\mathscr{G} \times \mathscr{A}$. Let $\mathscr{V} = \{V_j | j \in J\}$ be an open cover of $\mathscr{H}$ consisting of polydiscs (we leave the étale cover description for those familiar, say, with [Ar]). In particular, the V_j's are simply connected. Thus the pullback of V_j to $\mathscr{H}'$ consists of two connected components. Select one of these and call it W_j. It maps one-one to V_j and thus turns the restriction of $\mathscr{F}(o)$ over V_j into a family of polynomial covers: $\varphi(\Lambda'(\mathbf{x}')): \mathbb{P}^1_y \to \mathbb{P}^1_z$ for $\mathbf{x} \in V_j$ where $\mathbf{x}'$ is the unique point of W_j over $\mathbf{x}$. For the discussion below denote $\mathbf{x}$ by $\mathrm{pr}(\mathbf{x}')$. This uniquely defines an element γ_{ij} of $\Gamma(V_i \cap V_j, \mathscr{G} \times \mathscr{A})$:

$$(2.6) \qquad \varphi(\Lambda'(\mathbf{x}') \circ \gamma_{ij}) = \varphi(\Lambda'(\mathbf{x}'')), \quad \mathrm{pr}(\mathbf{x}') = \mathrm{pr}(\mathbf{x}'') \in V_i \cap V_j, \quad \mathbf{x}' \in W_i \text{ and } \mathbf{x}'' \in W_j.$$

In our case γ_{ij} is either the identity map or the affine map $y \to 1-y$ that switches 0 and 1. It is standard to recognize that $\boldsymbol{\gamma} = \{\gamma_{ij}\}_{ij \in I \times J}$ defines an element of the set $H^1_{\mathrm{et}}(\mathscr{H}, \mathscr{G} \times \mathscr{A})$. It is trivial if and only if there exists $\Lambda: \mathscr{H} \to \mathbb{A}^6$ such that

$$(2.7) \qquad \text{for } \mathbf{x} \in \mathscr{H}, \quad \varphi(\Lambda(\mathbf{x})): \mathbb{P}^1_y \to \mathbb{P}^1_z \text{ is in the equivalence class of } \mathbf{x}.$$

Furthermore, the natural map $H^1_{\mathrm{et}}(\mathscr{H}, \mathscr{G} \times \mathscr{A}) \to H^1_{\mathrm{et}}(\mathscr{H}', \mathscr{G} \times \mathscr{A})$, induced by mapping a cover of $\mathscr{H}'$ to a cover of $\mathscr{H}$ maps $\boldsymbol{\gamma}$ to the trivial element. Warning! It is tempting to identify $\boldsymbol{\gamma}$ with an element $\hat{\boldsymbol{\gamma}}$ of $H^1_{\mathrm{et}}(\mathscr{H}, \mathbb{Z}/2)$ by regarding the group generated by the transformation $y \to 1-y$ as a copy of the constant sheaf $\mathbb{Z}/2$ in $\mathscr{G} \times \mathscr{A}$. But $\hat{\boldsymbol{\gamma}}$ may be nontrivial in $H^1_{\mathrm{et}}(\mathscr{H}, \mathbb{Z}/2)$ although its image $\boldsymbol{\gamma}$ in $H^1_{\mathrm{et}}(\mathscr{H}, \mathscr{G} \times \mathscr{A})$ is trivial.

This works for a general Hurwitz family $\mathscr{F}(\mathbf{C})$ of genus 0 (with $\mathscr{T}(\mathbf{C})$ the total family, $\mathscr{H}(\mathbf{C})$ the parameter space, etc.) except that we must replace $\mathscr{H}'$ (resp., $\mathbb{Z}/2$) by a more general, but explicitly computable cover $\mathscr{H}(\mathbf{C})^*$ (resp., sheaf $\mathbf{B}$ of finite groups). Denote the rational functions of degree n by $\mathscr{R}_n$, an algebraic variety through some convenient choice of coordinates.

Theorem 2. 8. *Consider a general Hurwitz family whose fibers are covers in a Nielsen class* $\mathrm{ni}(\mathbf{C})$ (*resp., with one of the branch points specialized to be* ∞, $\mathrm{ni}(\mathbf{C}_\infty)$). *There is an unramified cover* $\mathscr{H}(\mathbf{C})^* \to \mathscr{H}(\mathbf{C})$ (*resp.,* $\mathscr{H}(\mathbf{C}_\infty)^* \to \mathscr{H}(\mathbf{C}_\infty)$), *explicitly computable from a permutation representation of the Hurwitz monodromy group, with the following properties. Attached to it is an explicitly computable locally constant subsheaf* $\mathbf{B}$ (*resp.,* $\mathbf{B}_\infty$) *whose stalks are isomorphic to a finite subgroup B of* $\mathrm{PGL}(1)$ (*resp., B_∞ of* $\mathbb{G}_m \times \mathbb{A}^1$). *The Hurwitz family defines an element* $\hat{\boldsymbol{\gamma}} \in H^1_{\mathrm{et}}(\mathscr{H}(\mathbf{C}), \mathbf{B})$ (*resp.,* $H^1_{\mathrm{et}}(\mathscr{H}(\mathbf{C}_\infty), \mathbf{B}_\infty)$)

whose image γ *in* $H^1_{\text{et}}(\mathscr{H}(\mathbf{C}), \mathscr{PGL})$ *(resp.,* $H^1_{\text{et}}(\mathscr{H}(\mathbf{C}_\infty), \mathscr{G} \times \mathscr{A})$*) is trivial if and only if there exists* $\Lambda: \mathscr{H}(\mathbf{C}) \to \mathscr{R}_n$ *(resp.,* $\Lambda: \mathscr{H}(\mathbf{C}_\infty) \to \mathbb{A}_{n+1}$*) such that*

(2. 8) *for* $\mathbf{x} \in \mathscr{H}(\mathbf{C})$, $\varphi(\Lambda(x)): \mathbb{P}^1_y \to \mathbb{P}^1_z$ *is in the equivalence class of* $\mathbf{x}$.

Also the image γ^* *of* γ *in* $H^1_{\text{et}}(\mathscr{H}(\mathbf{C})^*, \mathbf{B})$ *(resp.,* $H^1_{\text{et}}(\mathscr{H}(\mathbf{C}_\infty)^*, \mathbf{B}_\infty)$ *is trivial.*

Proof. The first point that needs explanation beyond our example is the general production of $\mathscr{H}(\mathbf{C})^*$. The construction and application for $\mathscr{H}(\mathbf{C}_\infty)^*$ is similar, so we only do the former. We really need $\mathscr{H}(\mathbf{C})^*$ to have the property that pullback of the family to $\mathscr{H}(\mathbf{C})^*$ gives a family $\mathscr{F}(\mathbf{C})^*$ with 3 disjoint sections: $\psi_i: \mathscr{H}(\mathbf{C})^* \to \mathscr{T}(\mathbf{C})^*$ with $\Phi(\mathbf{C})^* \circ \psi_i$ the identity on $\mathscr{H}(\mathbf{C})^*$, $i=1,2,3$; and $\psi_i(\mathscr{H}(\mathbf{C})^*) \cap \psi_j(\mathscr{H}(\mathbf{C})^*)$ is empty for $i \neq j$. The pointed Nielsen class technique of § 3. 3 shows exactly how to do this so that the images of the ψ_i's are in the sublocus $\mathscr{T}^*_{\mathscr{R}}$ of $\mathscr{T}(\mathbf{C})^*$ lying over the branch locus of the cover $\mathscr{T}(\mathbf{C})^* \to \mathscr{H}(\mathbf{C})^* \times \mathbb{P}^1_z$ (as in § 3. 5, Part 1). Then pullback of the family to $\mathscr{H}(\mathbf{C})^*$ gives a unique isomorphism Θ from the pulled back total space $\mathscr{T}(\mathbf{C})^*$ to $\mathscr{H}(\mathbf{C})^* \times \mathbb{P}^1_y$ so that for $\mathbf{p} \in \mathscr{H}(\mathbf{C})^*$, $\psi_1(\mathbf{p})=0$, $\psi_2(\mathbf{p})=1$ and $\psi_3(\mathbf{p})=\infty$. Note that the rest of our proof shows that the simplicity of the cohomological obstruction definitely depends on which pointed Nielsen classes we use to give the three sections.

Denote the image of ψ_i in $\mathscr{T}^*_{\mathscr{R}}$ followed by the restriction of the projection map $\mathscr{T}(\mathbf{C})^* \to \mathscr{T}(\mathbf{C})$ by $\mathscr{T}_{\mathscr{R},i}$, $i=1,2,3$. Observe, as our example above shows, that these three sets may overlap (even be equal), but their union intersects $\mathscr{T}(\mathbf{C})_\mathbf{x}$ in a finite set of points (at least 3). Define $\mathbf{B}$ to be the sheaf of groups of automorphisms of the stalks of the sheaf $\mathscr{T}(\mathbf{C}) \to \mathscr{H}(\mathbf{C})$ that leave the sets $\mathscr{T}_{\mathscr{R},i}$, $i=1,2,3$, invariant. Using this $\mathbf{B}$ the construction works as in our example. □

A last observation about τ the generator of the cyclic group $\text{Aut}(\mathscr{H}'/\mathscr{H})$ of order 2 and $\lambda'_{\mathbf{x}'}$ the polynomial produced from $\Lambda': \mathscr{H}' \to \mathbb{A}^6$ in our example above. Assume also that $\delta: \mathscr{H}' \to \mathbb{G}_m \times \mathbb{A}^1$ is any morphism for which

$$\lambda'_{\mathbf{x}'} \circ \delta(\mathbf{x}') = \lambda'_{\tau(\mathbf{x}')} \circ \delta(\tau(\mathbf{x}')) \quad \text{for each} \quad \mathbf{x}' \in \mathscr{H}'. \tag{2. 9}$$

Then $\lambda'_{\mathbf{x}'} \circ \delta(\mathbf{x}')$ actually defines the desired $\Lambda: \mathscr{H} \to \mathbb{A}^6$ and $\mathscr{F}(o)$ is a family of polynomial maps. Indeed, (2. 9) is clearly an if and only if condition for the existence of Λ. For the general situation of Theorem 2. 8 we would let τ run over automorphisms of the Galois closure of the cover $\mathscr{H}(\mathbf{C})^* \to \mathscr{H}(\mathbf{C})$. This observation, of course, does not yet answer Question 2. 7.

§ 3. Ramification prediction

Hurwitz families are determined by giving a prescription for the ramification of their fibers. § 1 has described, based on [Fr2], § 4, how the Hurwitz monodromy action provides information on the elementary geometry properties of Hurwitz families. In this section we show that the Hurwitz monodromy action can also be used to determine arithmetic properties of the fibers. We concentrate on the investigation of rational points on the fibers of the family.

3. 1. Statement of the problem. Return to the general notation of § 1: G is a transitive subgroup of S_n with $\text{Cen}_{S_n}(G)=\{1\}$: and $\mathbf{C}=(C_1, \dots, C_r)$ is an r-tuple of nontrivial conjugacy classes of G. As in § 1.4 we make these assumptions: $\bar{C}_r$ is distinct from each of $\bar{C}_1, \ldots, \bar{C}_{r-1}$ modulo $\mathrm{N}_{S_n}^{\mathrm{gal}}(\mathbf{C})$ and $\mathrm{H}(r-1)^*$ is transitive on $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ (i.e., (1.10) and (1.11) hold). Consider the minimal Hurwitz family $\mathscr{F}(\mathbf{C}_\infty)$ of § 1.4 associated with $\mathrm{ni}(\mathbf{C})$ where z_r has been specialized to ∞. Denote this by $\mathscr{F}: \mathscr{T} \to \mathscr{H} \times \mathbb{P}^1_z$. Finally, we assume that the family $\mathscr{F}$ is defined over K, as in Theorem 1.7, where K is a number field. Our main example is a special case of this with $K = \mathbb{Q}$.

Remark 3.1. The results of this section hold equally well for the family $\mathscr{F}(\mathbf{C})$ (i.e., for the situation where z_r has not been specialized to ∞) with the following changes: $\mathrm{H}(r)$, $\mathrm{N}(\mathbf{C})$ and $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ replace $\mathrm{H}(r-1)^*$, $\mathrm{N}(\mathbf{C}_\infty)$ and $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$. The same is true for the family $\mathscr{F}(\mathbf{C}_\infty)'$ of § 1.5 with these changes: $\mathrm{SH}(r-1)^*$, $\mathrm{H}(r-1)^*$ and $\mathrm{sni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ replace $\mathrm{H}(r)$, $\mathrm{N}(\mathbf{C})$ and $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$.

The main problem: How to test for the answer to the following question.

Question 3.2. Does the fiber $\mathscr{T}_{\mathbf{x}}$ have a K-rational point for each $\mathbf{x} \in \mathscr{H}(K)$ excluding at most a proper Zariski closed subset?

If the generic cover $\mathscr{T}_{\mathrm{gen}}$ does have a rational point, then Question 3.2 has an affirmative answer. Prop. 3 3 asserts that, in addition, the answer is yes without the restriction excluding a proper Zariski closed subset.

Proposition 3.3. *Under our assumptions, if there is a rational point on the generic cover $\mathscr{T}_{\mathrm{gen}}$, then there is a rational point on each fiber cover $\mathscr{T}_{\mathbf{x}}$ with $\mathbf{x} \in \mathscr{H}(K)$.*

Proof. The variety $\mathscr{H}$ is smooth and rational. Hence every point $\mathbf{x}$ has a neighbourhood isomorphic to an open subset of $\mathbb{A}^r$. Since the property is clearly dependent only on a Zariski local neighbourhood of $\mathbf{x}$, we may assume that $\mathscr{H} = \mathbb{A}^r$ with coordinate ring $K[t_1, \ldots, t_r]$ and $\mathbf{x}$ is the origin $\mathbf{0} = (0, \ldots, 0)$. Consider a projective embedding of $\mathscr{T}$, $i: \mathscr{T} \to \mathbb{P}^m$ (for m sufficiently large). Denote the natural map $\mathrm{pr}_1 \circ \Phi: \mathscr{T} \to \mathbb{A}^r$ of the family $\mathscr{F}$ by φ. Then the embedding $\mathscr{T} \to \mathbb{A}^r \times \mathbb{P}^m$ given by the product map (φ, i) displays $\mathrm{pr}_1 \circ \Phi$ as a projective morphism. By assumption, there is a rational point $\mathbf{p}_{\mathrm{gen}}$ on $\mathscr{T}_{\mathrm{gen}}$ (i.e., a section to the map $\mathscr{T}_{\mathrm{gen}} \to \mathrm{Spec}(K(t_1, \ldots, t_r))$ given by a nonzero $(m+1)$-tuple $(g_1, \ldots, g_{m+1})$ of polynomials in $K[t_1, \ldots, t_r]$. With no loss, we may assume that the g_i's are relatively prime in the p.i.d. $K(t_1, \ldots, t_{r-1})[t_r]$. Then the polynomials $g_1(t_1, \ldots, t_{r-1}, 0), \ldots, g_{m+1}(t_1, \ldots, t_{r-1}, 0)$ cannot be all equal to 0 in $K[t_1, \ldots, t_{r-1}]$. This produces a point $\mathbf{p}_r$ on $\mathbb{A}^r \times \mathbb{P}^m$, rational over $K(t_1, \ldots, t_{r-1})$, above the point $\mathbf{x}_r \in \mathscr{H}$ obtained from the generic point of $\mathscr{H}$ by specializing t_r to 0. But the point $\mathbf{p}_r$ is necessarily in $\mathscr{T}$: since the map $\mathscr{T} \to \mathbb{A}^r$ is projective, it is proper. So there exists a point on $\mathscr{T} \subset \mathbb{A}^r \times \mathbb{P}^m$ above $\mathbf{x}_r \in \mathscr{H}$. This point must be the rational point $\mathbf{p}_r$ since the projection map $\mathbb{A}^r \times \mathbb{P}^m \to \mathbb{A}^r$ is separated. This proves the existence of a rational point on the fiber cover $\mathscr{T}_{\mathbf{x}_r}$ (i.e., on the fiber of $\mathscr{T}$ above the generic point "$(t_1, \ldots, t_{r-1}, 0)$" of the hyperplane $\{t_r = 0\}$ ($\approx \mathbb{A}^{r-1}$) of $\mathbb{A}^r$). Repeat the argument $r-1$ times to eventually get a K-rational point on the fiber $\mathscr{T}_{\mathbf{0}}$ over the point $\mathbf{0}$. □

The converse of Prop. 3.3 is of course the point of interest. We do not happen to know of any families of genus 1 curves over $\mathbb{Q}$ parametrized by a rational variety with rational points on each $\mathbb{Q}$-fiber, but none on the generic fiber. But we suspect that it can happen (cf. Selmer conjecture remark for $x^4 - (8t^2 + 5)^2 = y^2$ on p. 140 of [LSc]). For families of genus 0, however, thanks to [LSc] the converse does hold.

Proposition 3. 4. *In addition to the previous assumptions, consider the case when $\mathcal{F}$ is a Hurwitz family of genus* 0. *The following are equivalent*:

(i) *Excluding a proper Zariski closed subset of* $\mathbf{x} \in \mathcal{H}(K)$, $\mathcal{T}_{\mathbf{x}}$ *has a rational point*;

(ii) *For each* $\mathbf{x} \in \mathcal{H}(K)$ *the fiber* $\mathcal{T}_{\mathbf{x}}$ *has a rational point*;

(iii) *The generic fiber* $\mathcal{T}_{\text{gen}}$ *has a rational point.*

Proof. Let D_{gen} be the divisor of the differential dz on $\mathbb{P}^1$ regarded as a divisor on the generic cover $\mathcal{T}_{\text{gen}}$. Since $g=0$, the Riemann-Roch theorem implies that $\deg(D_{\text{gen}})=-2$. Consider the linear system $L(-D_{\text{gen}})$ of meromorphic functions on $\mathcal{T}_{\text{gen}}$ whose divisors added to $-D_{\text{gen}}$ are positive. Then $L(-D_{\text{gen}})$ has dimension 3. A basis $(f_1, f_2, f_3)=\mathbf{f}$ for $L(-D_{\text{gen}})$ gives an embedding of $\mathcal{T}_{\text{gen}}$ in $\mathbb{P}^2$ as a nonsingular conic [H], p. 296-7. For specializations of the generic point in an open subset of $\mathcal{H}$, $\mathbf{f}$ specializes to a basis embedding the corresponding fiber as a nonsingular conic in $\mathbb{P}^2$. Thus since $\mathcal{H}$ is rational, we reduce consideration of Prop. 3. 4 to the corresponding statement for a family of conics over K in $\mathbb{P}^2$ parametrized by an open subset of $\mathbb{A}^n$. For a generic point of the parameter space, the generic conic has a rational point if and only if the conics corresponding to K-specializations of the generic point in some K-open subset of the parameter space have K-points ([LSc] for the case $K=\mathbb{Q}$; [Sc], p. 211, for the general case). This gives the equivalence of (i) and (iii). Prop. 3. 3 gives the equivalence of these with (ii). □

When $\mathcal{F}$ is a Hurwitz family of genus 0, for each $\mathbf{x} \in \mathcal{H}$ the fiber $\mathcal{T}_{\mathbf{x}}$ is a projective smooth variety geometrically isomorphic to $\mathbb{P}^1$. An affirmative answer to Question 3. 2 implies that all of the fibers $\mathcal{T}_{\mathbf{x}}$ are isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}(\mathbf{x})$, the field of definition of $\mathcal{T}_{\mathbf{x}}$. In this case we say that the family $\mathcal{F}$ is *arithmetically constant.*

3. 2. Points produced by ramification. Our next example considers, for our context, "the classical numerical check provided by ramification" for producing rational points on curves.

Example 3. 5. *The fibers of* $\mathcal{F}(a)'$ (*of* § 2. 1) *have rational points.* The family $\mathcal{F}(a)$ lies in $\mathrm{ni}(\mathbf{C})_{10}$. Apply the Riemann-Hurwitz formula to any element of $\mathrm{ni}(\mathbf{C})_{10}$ (e.g., $T_2(\boldsymbol{\sigma})$ given in (2. 1)) to get the genus g of the family $\mathcal{F}(a)$: $2(10+g-1)=18$ or $g=0$. The family $\mathcal{F}(a)'$ derives from $\mathcal{F}(a)$ by "adjunction of the branch points" (§ 1. 5); we may assume that the branch points have been ordered in such a way that on each fiber cover $\mathcal{F}(a)'_{\mathbf{x}}$, the i-th branch point z_i corresponds to C_i, $i=1,\dots,4$. Then consider the divisor D_2 of *ramified* points above z_2: it is rational on $\mathcal{T}(a)'_{\mathbf{x}}$ and of degree 3. Now the canonical class on $\mathcal{T}(a)'_{\mathbf{x}}$ is represented by a degree -2 rational divisor δ. This gives a rational degree one divisor, $D_2+\delta$, on $\mathcal{T}(a)'_{\mathbf{x}}$ and from an application of the Riemann-Roch theorem, a positive degree 1 rational divisor (i.e., a rational point) on $\mathcal{T}(a)'_{\mathbf{x}}$. □

Remark 3. 6. Generalization of the argument of Ex. 3. 5 gives this well known statement. A nonsingular curve $\mathcal{C}$ of genus 0 defined over a field K has a K-rational point if and only if it has an odd degree K-rational divisor.

We explain in more generality what we mean by the phrase "ramification produces rational points". For $\mathbf{x} \in \mathcal{H}$ consider the corresponding cover $\varphi_{\mathbf{x}}: \mathcal{T}_{\mathbf{x}} \to \mathbb{P}^1$. Let $\mathcal{D}_{\mathbf{x}}$ be the collection of $K(\mathbf{x})$-rational divisors on $\mathcal{T}_{\mathbf{x}}$ with support in $\varphi_{\mathbf{x}}^{-1}(\mathbf{z})$ where $\mathbf{z}$ denotes the collection of branch points of $\varphi_{\mathbf{x}}$. Also, denote the set of rational principal divisors on $\mathcal{T}_{\mathbf{x}}$ (i.e., divisors of functions on $\mathcal{T}_{\mathbf{x}}$ defined over $K(\mathbf{x})$) by $\mathrm{Div}_{l,\mathbf{x}}$.

Definition 3.7. We say that ramification produces rational points on $\mathcal{T}_{\mathbf{x}}$ if the group generated by $\mathcal{D}_{\mathbf{x}} \cup \mathrm{Div}_{l,\mathbf{x}}$, inside the group of all rational divisors on $\mathcal{T}_{\mathbf{x}}$, contains a positive degree one divisor (i.e., a point). We say that ramification produces rational points (on the family $\mathcal{F}$) if ramification produces rational points on the generic fiber $\mathcal{T}_{\mathrm{gen}}$.

From Prop. 3.3, if ramification produces rational points, then it does on all fibers $\mathcal{T}_{\mathbf{x}}$, for $\mathbf{x} \in \mathcal{H}$. In the case of genus 0 or 1 there is a simple criterion for ramification to produce rational points (of $\mathcal{T}_{\mathbf{x}}$). Let C be a choice of one of the conjugacy classes that appears in $\mathbf{C}$ and let t be one of the lengths of the disjoint cycles of an element σ of C. From this point refer to such a pair (C, t) as a *marked conjugacy class*. Consider those of the conjugacy classes C_i that are equal to C^a modulo the normalizer $\mathrm{N}_{S_n}^{\mathrm{gal}}(\mathbf{C})$ for some integer a relatively prime to N (i.e., with the notations of § 1.4, such that $\gamma \bar{C}_i \gamma^{-1} = \bar{C}$ for some $\gamma \in \mathrm{N}_{S_n}^{\mathrm{gal}}(\mathbf{C})$). Then denote the branch points, after a possible renumbering, corresponding to these conjugacy classes by $z_1, \dots, z_v$; finally, let $\mathbf{p}_{i,j}$, $j = 1, \dots, u$, be the points of $\mathcal{T}_{\mathbf{x}}$ above z_i, $i = 1, \dots, v$, that correspond to the disjoint cycles of length t in the branch cycle description for the cover $\mathcal{F}_{\mathbf{x}}$.[3]) The set

$$P_{\mathbf{x}}(C, t) = \{\mathbf{p}_{i,j}, j = 1, \dots, u, i = 1, \dots, v\} \tag{3.1}$$

is $G(\overline{K(\mathbf{x})}/K(\mathbf{x}))$ invariant (use Prop. 1.9). Denote the collection of all $K(\mathbf{x})$-rational divisors on $\mathcal{T}_{\mathbf{x}}$ corresponding to the orbits of $G(\overline{K(\mathbf{x})}/K(\mathbf{x}))$ on the set $P_{\mathbf{x}}(C, t)$ by $\mathcal{D}_{\mathbf{x}}(C, t)$.

Proposition 3.8. a) *If the genus of the Hurwitz family is* 0, *then ramification produces rational points of* $\mathcal{T}_{\mathbf{x}}$ *if and only if for some marked conjugacy class* (C, t) *the set* $\mathcal{D}_{\mathbf{x}}(C, t)$ *contains an odd degree divisor.*

b) *If the genus of the Hurwitz family is* 1, *then ramification produces rational points of* $\mathcal{T}_{\mathbf{x}}$ *if and only if the collection of degrees of all divisors in the set* $\bigcup_{(C,t) \text{ marked conj. classes}} \mathcal{D}_{\mathbf{x}}(C, t)$ *are relatively prime.*

For $g = 0$ (resp., $g = 1$) we say that (C, t) *produces rational points on* $\mathcal{T}_{\mathbf{x}}$ if $\mathcal{D}_{\mathbf{x}}(C, t)$ contains an odd degree divisor (resp., if the degree of divisors in $\mathcal{D}_{\mathbf{x}}(C, t)$ are relatively prime). If $g = 0$, ramification produces rational points on $\mathcal{T}_{\mathbf{x}}$ if and only if some (C, t) does. This is false for $g = 1$.

Proof. From the $G(\overline{K(\mathbf{x})}/K(\mathbf{x}))$-invariance of $P_{\mathbf{x}}(C, t)$,

$$\mathcal{D}_{\mathbf{x}} \text{ is generated by } \bigcup_{(C,t) \text{ marked conj. classes}} \mathcal{D}_{\mathbf{x}}(C, t). \tag{3.2}$$

Necessity of the condition is therefore immediate since principal divisors have degree 0. The converse for a) is immediate from Remarks 3.6 and 3.9. It remains to prove the converse in case b). From (3.2) conclude that $\mathcal{D}_{\mathbf{x}}$ contains a degree one divisor. From the Riemann-Roch theorem, the linear system of a degree one divisor has dimension 1. Thus it contains a positive rational divisor. □

[3]) For the family $\mathcal{F}(\mathbf{C}_\infty)'$ — Remark 3.1 — v is always equal to 1.

Remark 3. 9. Since z_r has been placed at ∞, for any genus g there is a representative of the canonical class in $\mathscr{D}_{\mathbf{x}}$:

$$\delta_{\mathbf{x}} = \sum_{(C,t),\, C \neq C_r} (t-1)\, D_{\mathbf{x}}(C,t) - \sum_{t} (t+1)\, D_{\mathbf{x}}(C_r,t) \tag{3. 3}$$

where $D_{\mathbf{x}}(C,t)$ denotes the divisor sum of all points in $P_{\mathbf{x}}(C,t)$.

Definition 3. 7 suggests several problems:

Question 3. 10. a) Analogous to § 3. 1, Comment 2, is there an equivalence between the existence of rational points produced by ramification on all fibers $\mathscr{T}_{\mathbf{x}}$, $\mathbf{x} \in \mathscr{H}$, on one hand, and on the generic fiber $\mathscr{T}_{\mathrm{gen}}$ on the other hand?

b) Is there a simple criterion for ramification to produce rational points?

c) Does knowledge of existence of rational points produced by ramification decide the existence of rational points on all fibers $\mathscr{T}_{\mathbf{x}}$ for $\mathbf{x} \in \mathscr{H}$?

Prop. 3. 8 gives an affirmative answer to Question 3. 10 a) in the case $g=0$ or 1 (Theorem 3. 11 below), but we do not know if this is the case for each g (assuming $\mathscr{H}$ is K-rational).

Theorem 3. 11. *The following are equivalent for $\mathscr{F}$ a Hurwitz family of genus* 0 *or* 1:

(i) *Ramification produces rational points*;

(ii) *Ramification produces rational points on $\mathscr{T}_{\mathbf{x}}$ for each $\mathbf{x} \in \mathscr{H}(K)$; and*

(iii) *Ramification produces rational points on $\mathscr{T}_{\mathbf{x}}$ for each $\mathbf{x} \in \mathscr{H}(K)$ excluding a proper Zariski closed subset of $\mathscr{H}$.*

Proof. Hilbert's irreducibility theorem produces a Hilbert subset H of $\mathscr{H}(K)$ with the property that for each $\mathbf{x} \in H$ and each marked conjugacy class (C,t) the degrees of the divisors in $\mathscr{D}_{\mathbf{x}}(C,t)$ and $\mathscr{D}_{\mathrm{gen}}(C,t)$ are in one-one correspondence. For $g=0$ or 1 the criteria of Prop. 3. 8 for "ramification produces points" depends only on these degrees. □

All of § 3. 4 is devoted to Question 3. 10 b): Theorem 3. 14 is our main theorem. Group theoretic preliminaries appear in § 3. 3. These define an extension of the Hurwitz monodromy action. Theorem 3. 14 explains how this new action relates to the $G(\overline{K(\mathbf{x})}/K(\mathbf{x}))$ action on the set $P_{\mathbf{x}}(C,t)$. As to Question 3. 10 c), we feel that for Hurwitz families with a rational parameter space $\mathscr{H}$, the existence of rational points produced by ramification should be decisive for the existence of rational points on each K-fiber. We check this conjecture in the particular case of the family $\mathscr{F}(a)$, for which we will see that ramification does not produce rational points (§ 4).

3. 3. Pointed Nielsen classes and Hurwitz monodromy action. Let (C,t) be a marked conjugacy class (below Def. 3. 7). Consider the *pointed Nielsen class* $\mathrm{ni}(\mathbf{C}_\infty; C, t)$ whose elements are related to elements of $\mathrm{ni}(\mathbf{C}_\infty)$ as follows. For $\boldsymbol{\tau} \in \mathrm{ni}(\mathbf{C}_\infty)$ and $j=1,\dots,r$, write τ_j out as a product of disjoint cycles $\beta_1 \cdots \beta_s$; then $\mathrm{ni}(\mathbf{C}_\infty; C, t)$ is the set consisting of all $(\boldsymbol{\tau}, j, \beta_k)$ with $\boldsymbol{\tau} \in \mathrm{ni}(\mathbf{C}_\infty)$, $\tau_j \in \bar{C}$ modulo $N_{S_n}^{\mathrm{gal}}(\mathbf{C})$ (as in (1. 10)) and β_k a t-cycle in τ_j. We refer to the $(\boldsymbol{\tau}, j, \beta_k)$ as a (C,t)-pointing of $\boldsymbol{\tau}$. The following notation is convenient for calculations: we *point* τ_j by replacing β_k by a marked version β_k^1 of it; then $\boldsymbol{\tau}$ is *pointed* by replacing τ_j in $\boldsymbol{\tau}$ by the marked element.

Example 3. 12. *Pointings of elements of* $\mathrm{ni}(\mathbf{C})_{10}$ (§ 2. 1). Consider the 4-tuple $\boldsymbol{\tau}=T_2(\boldsymbol{\sigma})\in\mathrm{ni}(\mathbf{C})_{10}$ given in (2. 1). Here are three pointed versions of $\boldsymbol{\tau}$ (... indicates the remainder is as in $\boldsymbol{\tau}$):

(3. 5) a) $(\boldsymbol{\tau}, 1, (3\ 4)) = ((1\ 2)(3\ 4)^1(7\ 8)(6\ 9), \dots)$ where $(C, t)=(C_1, 2)$,

b) $(\boldsymbol{\tau}, 2, (8)) = (\dots, (2\ 5)(3\ 6)(4\ 7)(8)^1, \dots)$ where $(C, t)=(C_2, 1)$, and

c) $(\boldsymbol{\tau}, 3, (5)) = (\dots, \dots, (1\ 6)(2\ 8)(4\ 10)(5)^1, \dots)$ where $(C, t)=(C_2, 1)$.

Note that disjoint cycles of length 1 are an important part of these considerations.

There is a natural notion of absolute pointed Nielsen class $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$. The normalizer $\mathrm{N}(\mathbf{C}_\infty)$ acts on the set $\mathrm{ni}(\mathbf{C}_\infty, C, t)$ as follows: $\gamma\in\mathrm{N}(\mathbf{C}_\infty)$ sends $\boldsymbol{\tau}$ with τ_j marked on β_k to $\gamma\boldsymbol{\tau}\gamma^{-1}$ with $\gamma\tau_j\gamma^{-1}$ marked on $\gamma\beta_k\gamma^{-1}$ (i.e., $(\boldsymbol{\tau}, j, \beta_k)$ to $(\gamma\boldsymbol{\tau}\gamma^{-1}, j, \gamma\beta_k\gamma^{-1})$). The set $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$ is the quotient of $\mathrm{ni}(\mathbf{C}_\infty, C, t)$ by this action.

Hurwitz monodromy action of $\mathrm{H}(r-1)^*$ naturally extends to pointed Nielsen classes in the most natural way, too. Informally, extend the action of Q_i by applying to the selected t-cycle β_k in τ_j whatever rule that applies to $\tau_j\in\bar{C}$ in the definition of $(\boldsymbol{\tau})\,Q_i$ (cf. (1. 8)). Specifically, for $i=1, \dots, r-1$, $(\boldsymbol{\tau}, j, \beta_k)\,(Q_i)$ is defined by

(3. 6) $(\boldsymbol{\tau}, i, \beta_k)\,(Q_i) = ((\boldsymbol{\tau})\,Q_i, i+1, \beta_k)$; $(\boldsymbol{\tau}, i+1, \beta_k)\,(Q_i) = ((\boldsymbol{\tau})\,Q_i, i, \tau_i\beta_k\tau_i^{-1})$ and
$(\boldsymbol{\tau}, j, \beta_k)\,(Q_i) = ((\boldsymbol{\tau})\,Q_i, j, \beta_k)$ if $i\neq j, j-1$.

It is a simple check that this action extends from an action of generators to all of $\mathrm{H}(r-1)^*$. Since the Hurwitz monodromy action commutes with the action of the normalizer $\mathrm{N}(\mathbf{C}_\infty)$, the group $\mathrm{H}(r-1)^*$ actually acts on the absolute *pointed* Nielsen class $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$.

We now define our main tool for this paper: for any $\boldsymbol{\sigma}\in\mathrm{ni}(\mathbf{C}_\infty)$, *a natural action on the set of all the* (C, t)*-pointings of the given r-tuple* $\boldsymbol{\sigma}$ can be derived from the action described above. Consider the following subgroup of $\mathrm{H}(r-1)^*$:

$$\text{(3. 7)}\qquad \mathrm{H}_{\boldsymbol{\sigma}} = \{Q\in\mathrm{H}(r-1)^* \mid (\boldsymbol{\sigma})\,Q=\boldsymbol{\sigma} \text{ modulo } \mathrm{N}(\mathbf{C}_\infty)\}.$$

Denote the set of all possible (C, t)-pointings of $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}(C, t)$; its cardinality is $|\boldsymbol{\sigma}(C, t)|=uv$ where u and v are the integers defined in (3. 1) of § 3. 2. Now note that any element of $\mathrm{H}_{\boldsymbol{\sigma}}$, by definition, fixes the r-tuple $\boldsymbol{\sigma}$, but may move its (C, t)-markings; this is the action alluded to above. More formally, this action of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$ is defined as follows: for $Q\in\mathrm{H}_{\boldsymbol{\sigma}}$ there is a unique $\gamma\in\mathrm{N}(\mathbf{C}_\infty)$ such that $\gamma(\boldsymbol{\sigma})\,Q\gamma^{-1}=\boldsymbol{\sigma}$. (Existence of γ is part of the definition of $\mathrm{H}_{\boldsymbol{\sigma}}$, while uniqueness is a consequence of the assumption that $\mathrm{Cen}_{S_n}(G)=\{1\}$.) The action of Q as an element of $\mathrm{H}_{\boldsymbol{\sigma}}$ is then defined as the composition of the action of Q as an element of $\mathrm{H}(r-1)^*$ followed by conjugation by γ.

Remark 3. 13. a) As a consequence of the transitivity of $\mathrm{H}(r-1)^*$ on $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ this action of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$ does not depend (up to equivalence) on the choice of $\boldsymbol{\sigma}$. Indeed, given $\boldsymbol{\sigma}_i\in\mathrm{ni}(\mathbf{C}_\infty)$, $i=1, 2$, there exists Q_{12} in the subgroup of $\mathrm{Per}(\mathrm{ni}(\mathbf{C}_\infty))$ generated by $\mathrm{H}(r-1)^*$ and $\mathrm{N}(\mathbf{C}_\infty)$ such that $(\boldsymbol{\sigma}_1)\,Q_{12}=\boldsymbol{\sigma}_2$; the induced bijection $\boldsymbol{\sigma}_1(C, t)\to\boldsymbol{\sigma}_2(C, t)$ and the isomorphism $\mathrm{H}_{\boldsymbol{\sigma}_1}\to\mathrm{H}_{\boldsymbol{\sigma}_2}$ (sending $Q\in\mathrm{H}_{\boldsymbol{\sigma}_1}$ to $(Q_{12}^{-1}QQ_{12})\in\mathrm{H}_{\boldsymbol{\sigma}_2}$) identify the representations $\mathrm{H}_{\boldsymbol{\sigma}_i}\to\mathrm{Per}(\boldsymbol{\sigma}_i(C, t))$, $i=1, 2$.

b) Clearly two elements of $\boldsymbol{\sigma}(C, t)$ are in the same orbit of $\mathrm{H}_{\boldsymbol{\sigma}}$ if and only if their equivalence classes in $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$ are in the same orbit of $\mathrm{H}(r-1)^*$. That is, the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{\sigma}(C, t) & \longrightarrow & \mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}} \\ \downarrow & & \downarrow \\ \boldsymbol{\sigma}(C, t)/\mathrm{H}_{\boldsymbol{\sigma}} & \longrightarrow & \mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}/\mathrm{H}(r-1)^* \end{array} \tag{3.8}$$

where the lower horizontal arrow is injective. Again from transitivity of $\mathrm{H}(r-1)^*$ on $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ it is also bijective.

3. 4. Ramification prediction. Riemann's existence theorem identifies points over branch points with disjoint cycles in a branch cycle description of a given cover (§ 1. 1). We reformulate this to say that the correspondence identifies points over branch points and pointings of a branch cycle description. Let (C, t) be a marked conjugacy class and $\mathbf{x} \in \mathscr{H}$. Then, by definition, the elements of the set $P_{\mathbf{x}}(C, t)$ (§ 3. 2) correspond in this identification to elements of the set $\boldsymbol{\sigma}_{\mathbf{x}}(C, t)$ of all (C, t)-pointings of a branch cycle description $\boldsymbol{\sigma}_{\mathbf{x}}$ of a cover $\mathscr{F}_{\mathbf{x}}$. We now compare the action of $\mathrm{H}_{\boldsymbol{\sigma}_{\mathbf{x}}}$ on $\boldsymbol{\sigma}_{\mathbf{x}}(C, t)$ with the action of $G(\overline{\mathbb{Q}(\mathbf{x})}/\mathbb{Q}(\mathbf{x}))$ on $P_{\mathbf{x}}(C, t)$.

We state the result with $\mathbf{x}$ the generic point of $\mathscr{H}$. From Hilbert's irreducibility theorem (as in the proof of Theorem 3.11), the result still holds for all points $\mathbf{x}$ in a Zariski dense subset of $\mathscr{H}$. In the following we denote the field of definition of the generic cover $\mathscr{F}_{\mathrm{gen}}$ by F and we assume that the conjugacy class C is distinct from C_r: (C_r, t) should be studied separately. The proof of our main theorem takes up all of § 3. 5.

Theorem 3. 14. *Let $\boldsymbol{\sigma} \in \mathrm{ni}(\mathbf{C}_\infty)$. Then orbits of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$ corresponds to orbits of $G(\bar{F}/F\bar{\mathbb{Q}})$ on $P_{\mathrm{gen}}(C, t)$. Each orbit of $G(\bar{F}/F)$ on $P_{\mathrm{gen}}(C, t)$ corresponds to a disjoint union of orbits $\mathcal{O}_i$ of the same length, $i = 1, \ldots, m$, of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$. Furthermore, the induced representations of $\mathrm{H}_{\boldsymbol{\sigma}}$ in $\mathrm{Per}(\mathcal{O}_i)$, $i = 1, \ldots, m$, are isomorphic (as groups).*

In the last statement of the theorem we imply only that the images of $\mathrm{H}_{\boldsymbol{\sigma}}$ in $\mathrm{Per}(\mathcal{O}_i)$, $i = 1, \ldots, m$, are isomorphic groups. Indeed as group representations they are equivalent, but they may not be permutation equivalent representations of the image group. Below we say that $\mathrm{H}_{\boldsymbol{\sigma}}$ acts equivalently on two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ (not necessarily in the same $G(\bar{F}/F)$-orbit) if the group representations are equivalent. From Remark 3. 13 a), the conclusions of Theorem 3. 14 do not depend on the choice of $\boldsymbol{\sigma}$ (in particular, the integers m, $|\mathcal{O}_i|$ and the image of $\mathrm{H}_{\boldsymbol{\sigma}}$ in $\mathrm{Per}(\mathcal{O}_i)$ don't depend on $\boldsymbol{\sigma}$).

We list corollaries of Theorem 3. 14. To each orbit $\mathcal{O}$ of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$ associate the number lm where l is the length of the orbit and m counts the number of orbits $\mathcal{O}'$ of length l on which $\mathrm{H}_{\boldsymbol{\sigma}}$ acts equivalently to $\mathcal{O}$.

Corollary 3. 15. *Assume that the Hurwitz family $\mathscr{F}$ has genus 0 and that there is an orbit $\mathcal{O}$ of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$ with lm odd (e.g., if the number of orbits of length a given odd number l is odd). Then the marked conjugacy class (C, t) produces rational points on any cover in the Nielsen class $\mathrm{ni}(\mathbf{C})$ defined over a field containing K.*

Proof. Denote the m orbits on which $\mathrm{H}_{\boldsymbol{\sigma}}$ acts equivalently on $\boldsymbol{\sigma}(C, t)$ by $\mathcal{O}_i$, $i = 1, \ldots, m$. From Theorem 3. 14 there is a partition $(I_1, \ldots, I_s)$ of $\{1, \ldots, m\}$ so that each set $\bigcup_{i\in I_j} \mathcal{O}_i$, $j=1, \dots, s$, corresponds to an orbit of $G(\bar{F}/F)$ on $P_{\text{gen}}(C, t)$. Necessarily one of the sets $I_1, \dots, I_s$ has odd cardinality. The corresponding orbit of $G(\bar{F}/F)$ has odd length. The rest of the proof follows from Prop. 3. 8, Prop. 3. 3 and Prop. 1. 8. □

Corollary 3. 16. *Assume that the Hurwitz family $\mathcal{F}$ has genus 0 and that all orbits of H_σ on $\boldsymbol{\sigma}(C, t)$ have even length. Then (C, t) does not produce rational points.*

The proof is immediate and left to the reader. Genus 1 versions of these corollaries are not so simple to state because the quantification in Prop. 3. 8 b) is over all (C, t). In particular this requires inspection of the Galois conjugation on the points over ∞ — of necessity an ad hoc consideration — purposely excluded from the statement of Theorem 3. 14. With this included the genus 1 case is usually best treated by returning directly to Theorem 3. 14. Nevertheless one important special case is an immediate consequence of this and Prop. 3. 8 b).

Corollary 3. 17. *Assume that the Hurwitz family has genus 1 and that for each marked conjugacy class (C, t) with $C \neq C_r$, all numbers m associated to orbits of H_σ on $\boldsymbol{\sigma}(C, t)$ are equal to 1 (e.g., action of H_σ on $\boldsymbol{\sigma}(C, t)$ is transitive, in which case the group H_σ has a single orbit on $\boldsymbol{\sigma}(C, t)$ of length uv). Let $\mathcal{N}$ (resp., $\mathcal{N}_\infty$) be the collection of lengths of orbits of H_σ on $\boldsymbol{\sigma}(C, t)$ over all (C, t) with $C \neq C_r$ (resp., lengths of orbits of $G(\bar{F}/F)$ on the set of points of $\mathcal{T}_{\text{gen}}$ over ∞). Then ramification produces rational points if and only if the collection $\mathcal{N} \cup \mathcal{N}_\infty$ consists of relatively prime integers.*

We conclude this section with a few words on how we intend to apply this work in a subsequent paper to consider the production of elliptic curves with high rank over $\mathbb{Q}$. Theorem 3. 14 essentially says that, given a group G and an *r-tuple* $\mathbf{C}$ of conjugacy classes of G, inspection of the Hurwitz monodromy action may provide rational points on covers in the Nielsen class $\mathrm{ni}(\mathbf{C})$. Riemann's existence theorem asserts furthermore that there exist actual covers in the Nielsen class $\mathrm{ni}(\mathbf{C})$. So, we seem to be reduced to a pure group theory problem: finding a data $(G, \mathbf{C})$ that produces sufficiently many rational points through the inspection of the Hurwitz monodromy action. Of course, a major problem remains: can the associated covers be defined over $\mathbb{Q}$? In fact, existence of covers in $\mathrm{ni}(\mathbf{C})$ defined over $\mathbb{Q}$ is equivalent to existence of $\mathbb{Q}$-rational points on the Hurwitz space $\mathcal{H}(\mathbf{C})$. At this stage we refer back to Comment c) of Theorem 1. 7: criteria for the variety $\mathcal{H}(\mathbf{C})$ to be, for example, a $\mathbb{Q}$-rational variety (and so, with a lot of $\mathbb{Q}$-points) are available. So, the production of elliptic curves defined over $\mathbb{Q}$ with "many" rational points may already be handled through pure group theoretical calculations (cf. § 3. 7). This leaves us, of course, with the question of the rank of the group generated by these rational points produced by ramification, a subject that we expect to treat in detail in the sequel to this paper.

3. 5. Proof of Theorem 3. 14. Consider the subset $\mathcal{H}(C, t)'' \stackrel{\text{def}}{=} \bigcup_{\mathbf{x} \in \mathcal{H}} P_{\mathbf{x}}(C, t)$ (§ 3. 2) of $\mathcal{T}$. The proof is in four parts. The first three show that $\mathcal{H}(C, t)''$ is an unramified cover of $\mathcal{H}$ defined over K that corresponds, via fundamental groups, to the action of H_σ on $\boldsymbol{\sigma}(C, t)$.

Part 1: *The natural quasiprojective structure on $\mathcal{H}(C, t)''$ shows that $\mathcal{H}(C, t)'' \to \mathcal{H}$ is unramified.* Denote the branch locus of the map $\mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ by $\mathcal{R}$. Consider $\mathcal{T}_{\mathcal{R}} = \mathcal{T} \times_{\mathcal{H} \times \mathbb{P}^1} \mathcal{R}$ as a subscheme of $\mathcal{T}$ with its reduced structure. Note that the points of $\mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathbf{x}}$ are exactly the points of $\mathcal{T}_{\mathbf{x}}$ that lie over one of the branch points of the cover $\mathscr{F}_{\mathbf{x}}$. Since such points are in number independent of $\mathbf{x}$, the natural map $\mathscr{T}_{\mathscr{R}} \to \mathscr{H}$ is unramified. The multiplicity of $\mathbf{p} \in \mathscr{T}_{\mathscr{R}} \cap \mathscr{T}_{\mathbf{x}}$ on the fiber of $\mathscr{T}$ over $\Phi(\mathbf{p}) = (\mathbf{x}, z) \in \mathscr{H} \times \mathbb{P}^1$ can be identified with the order of ramification of $\mathbf{p}$ in the cover $\mathscr{F}_{\mathbf{x}}$. From the construction of Hurwitz families [Fr2], § 4, this number is locally constant, as is the conjugacy class of the canonical inertial group generator (modulo $\mathrm{N}(\mathbf{C}_\infty)$) for a point $\hat{\mathbf{p}}$ lying above $\mathbf{p}$ in the Galois closure of the cover $\mathscr{F}_{\mathbf{x}}$. Thus if $\mathscr{P}$ is a connected component of $\mathscr{T}_{\mathscr{R}}$, then $\mathscr{P}$ is either disjoint from or entirely contained in $\mathscr{H}(C, t)''$. This proves that $\mathscr{H}(C, t)''$ is a union of irreducible components of $\mathscr{T}_{\mathscr{R}}$.

Part 2: *$\mathscr{H}(C, t)''$ and the natural map $\mathscr{H}(C, t)'' \to \mathscr{H}$ are defined over K.* The natural map is just the restriction of $\mathrm{pr}_1 \circ \Phi$ to $\mathscr{H}(C, t)''$. Therefore we have only to show that $\mathscr{H}(C, t)''$ (a priori defined over a finitely generated extension of $\mathbb{Q}$) is defined over K. Since $\mathscr{H}(C, t)''$ is a union of irreducible components of $\mathscr{T}_{\mathscr{R}}$, which is defined over K, $\mathscr{H}(C, t)''$ is defined over $\bar{\mathbb{Q}}$. Consider $\sigma \in G(\bar{\mathbb{Q}}/K)$ and $\mathbf{p} \in \mathscr{T}_{\mathscr{R}}(\bar{\mathbb{Q}})$. If $\mathbf{p} \in \mathscr{H}(C, t)''$ then so is $\mathbf{p}^\sigma$ (apply Prop. 1. 9). Thus $\mathscr{H}(C, t)''$ is transformed into itself by σ. It is a well known consequence of Hilbert's Theorem 90 that $\mathscr{H}(C, t)''$ is therefore defined over K.

Part 3: *Identification of the permutation group actions.* The coordinate charts of [Fr2], p. 50—51, show the effect of transporting a path, representing a branch cycle on a fiber of the Hurwitz family, around the members of the family that lie over a path in the parameter space that represents the path Q_i, $i = 1, \ldots, r$. The correspondence of a coordinate neighbourhood of a ramified point $\mathbf{p}$ in a cover $\mathscr{T}_{\mathbf{x}}$ with a disjoint cycle of the branch point z over which $\mathbf{p}$ lies extends the construction in the following way: it gives the transport of the actual coordinate neighbourhoods surrounding the ramified points (as we move over a path that represents Q_i). Restriction of this transport to just the ramified points themselves gives the data for the effect of Q_i in the permutation representation associated to the cover $\mathscr{H}(C, t)'' \to \mathscr{U}_\infty^{r-1}$ (§ 1. 4). The formula for this action [Fr2], p. 50, in terms of the coordinate neighbourhoods associated to the ramified points, is that of § 3. 3 on the pointed Nielsen classes. Since the fundamental group of $\mathscr{H}$ is identified with $\mathrm{H}_{\boldsymbol{\sigma}}$, the cover $\mathscr{H}(C, t)'' \to \mathscr{H}$ corresponds to restriction of the permutation representation of $\mathrm{H}(r-1)^*$ to $\mathrm{H}_{\boldsymbol{\sigma}}$.

Part 4: *Comparison with the Galois group action.* Let $\mathcal{O}$ be an orbit of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$; it corresponds to a connected component S of the space $\mathscr{H}(C, t)''$. Thus, the action of the fundamental group of $\mathscr{H}$ on the points of S over the generic point of $\mathscr{H}$ is equivalent to the action of $\mathrm{H}(r-1)^*$ on the orbit $\mathcal{O}$. On the other hand, since the cover $\mathscr{H}(C, t)'' \to \mathscr{H}$ is unramified, S is geometrically irreducible and the points of S over the generic point of $\mathscr{H}$ are transitively permuted by $G(\bar{F}/F\bar{\mathbb{Q}})$. That is, they form an orbit of $G(\bar{F}/F\bar{\mathbb{Q}})$ on the set $P_{\mathrm{gen}}(C, t)$. This proves the first point of Theorem 3. 14. Let now $\mathcal{O}'$ be an orbit of $G(\bar{F}/F)$ on the set $P_{\mathrm{gen}}(C, t)$. A standard group theoretical argument shows that $\mathcal{O}'$ is a disjoint union of orbits $\mathcal{O}'_1, \ldots, \mathcal{O}'_m$ of $G(\bar{F}/F\bar{\mathbb{Q}})$ on $P_{\mathrm{gen}}(C, t)$. The corresponding orbits of $\mathrm{H}_{\boldsymbol{\sigma}}$ on $\boldsymbol{\sigma}(C, t)$ are the required orbits $\mathcal{O}_1, \ldots, \mathcal{O}_m$ in Theorem 3. 14. They have the same length; indeed, the orbits $\mathcal{O}'_i$, $i = 1, \ldots, m$, are pairwise conjugate under $G(F\bar{\mathbb{Q}}/F)$. This also implies that the associated connected components S_i, $i = 1, \ldots, m$, of $\mathscr{H}(C, t)''$ are conjugate under the action of $G(\bar{\mathbb{Q}}/K)$. We conclude by noting that the monodromy groups of the covers $S_i \to \mathscr{H}$ (which are the images of $\mathrm{H}_{\boldsymbol{\sigma}}$ in $\mathrm{Per}(\mathcal{O}_i)$) are isomorphic. Indeed, the transform of S_i to S_i^σ under $\sigma \in G(\bar{\mathbb{Q}}/K)$ induces an isomorphism of the function fields $K(S_i)$ and $K(S_i^\sigma)$ that is fixed on the function field F of $\mathscr{H}$. Thus σ extends to an isomorphism of the Galois closures of the field extensions $K(S_i)/F$ and $K(S_i^\sigma)/F$. Their respective Galois groups (which are clearly isomorphic to each other) are isomorphic to the monodromy groups of the covers $S_i \to \mathscr{H}$ and $S_i^\sigma \to \mathscr{H}$. □

Final comments. As we see from the corollaries above, Theorem 3. 14 has some implied numerical checks for the cardinality of the orbits of $G(\bar{F}/F)$ on $P_{\mathrm{gen}}(C, t)$. The subtler test is to check if the action of H_σ on two distinct orbits $\mathcal{O}_i$, $i=1, 2$, gives isomorphic groups. If not, then the orbits are not $G(\bar{F}/F)$-conjugate. But if the groups are isomorphic, then the images of $\mathrm{H}_\sigma(1)$ in $\mathrm{Per}(\mathcal{O}_i)$ will also be isomorphic, $i=1, 2$. This latter test may give an easier check to distinguish between H_σ orbits that are not $G(\bar{F}/F)$-conjugate. On the other hand, as noted under the opening transitivity assumption of § 3, H_σ orbits on $\sigma(C, t)$ are in one-one correspondence with $\mathrm{H}(r-1)^*$ orbits on $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$. The corresponding transitive groups are the monodromy groups of the covers $S_i \to \mathcal{U}_\infty^{r-1}$, $i=1, 2$. Exactly as in the proof of Part 4, S_1 and S_2 cannot be conjugate under $G(\bar{\mathbb{Q}}/K)$ if these groups are not isomorphic.

3. 6. The family $\mathscr{F}(a)$. We will prove that ramification does not produce rational points on the family $\mathscr{F}(a)$. Recall the elements of $\mathrm{sni}(\mathbf{C})^{\mathrm{ab}}$ derived by applying T_2 to (1. 4) a), ..., e):

(3. 10) a) $\tau_1=(1\ 2)(3\ 4)(7\ 8)(6\ 9)$, $\tau_2=(2\ 5)(3\ 6)(4\ 7)$, $\tau_3=(1\ 6)(2\ 8)(4\ 10)$;

b) $\tau_1=(1\ 2)(3\ 4)(7\ 8)(6\ 9)$, $\tau_2=(1\ 6)(2\ 8)(4\ 10)$, $\tau_3=(5\ 8)(7\ 10)(1\ 3)$;

c) $\tau_1=(1\ 2)(3\ 4)(7\ 8)(6\ 9)$, $\tau_2=(1\ 3)(5\ 8)(7\ 10)$, $\tau_3=(2\ 5)(3\ 6)(4\ 7)$;

d) $\tau_1=(1\ 4)(2\ 3)(5\ 10)(6\ 9)$, $\tau_2=(2\ 5)(3\ 6)(4\ 7)$, $\tau_3=(2\ 4)(5\ 7)(8\ 10)$;

e) $\tau_1=(1\ 4)(2\ 3)(5\ 10)(6\ 9)$, $\tau_2=(2\ 4)(5\ 7)(8\ 10)$, $\tau_3=(2\ 5)(3\ 6)(4\ 7)$

(in each case $\tau_4=T_2((5\ 4\ 3\ 2\ 1))=((2\ 6\ 9\ 3\ 7)(4\ 1\ 5\ 8\ 10))^{-1}$).

The Hurwitz family has genus 0, so we need to show that no marked conjugacy class (C, t) produces rational points.

Case 1: $C=C_1$, $t=1$. We start with $\sigma=\tau_d$, the branch cycles of (3. 10) d). There are 2 $(=uv)$ $(C_1, 1)$-markings of τ_d. We have

$$(3.\ 11)\qquad (\tau_d, 1, (7))(Q_1 Q_2^2 Q_1^{-1})^2=(\tau_d, 1, (8)).$$

In particular H_{τ_d} acts transitively on $\tau_d(C_1, 1)$. From Corollary 3. 16, this marked conjugacy class does not produce rational points.

Case 2: $C=C_1$, $t=2$. In this case $|\tau_d(C_1, 2)|=4$ $(u=4, v=1)$ and

$$(\tau_d, 1, (1\ 4))(Q_1 Q_2^2 Q_1^{-1})=(\tau_d, 1, (2\ 3));\ (\tau_d, 1, (5\ 10))(Q_1 Q_2^2 Q_1^{-1})=(\tau_d, 1, (6\ 9)).$$

Therefore the group H_{τ_d} has either a single orbit of cardinality 4 or two orbits of cardinality 2. In either case the orbits are of even length. Conclude as in Case 1.

Case 3: $C=C_2=C_3$ (mod $\mathrm{N}(\mathbf{C}_\infty)$), $t=1, 2$. Here there are $v=2$ conjugacy classes equal modulo $\mathrm{N}(\mathbf{C}_\infty)$. Thus each orbit of H_{τ_d} on $\tau_d(C_2, t)$ will be of even length if there is $Q \in \mathrm{H}(3)^*$ that takes markings of τ_d on the second component to markings of τ_d on the third component. Check that $Q=Q_1^2(Q_1 Q_2^{-2} Q_1^{-1}) Q_1^{-2} Q_2$ does the job.

Case 4: $C=C_4$, $t=5$. This case is not covered by Theorem 3. 14. We must use an ad hoc argument. We show that the two points of $\mathscr{T}_{\mathrm{gen}}$ over ∞ are conjugate over a degree two extension of F, where F is the field of definition of the generic point. From

Prop. 2. 3 the function field $F(X)$ of $\mathcal{T}(o)_{\text{gen}}$ is (isomorphic to) the field $F(y_1)$ generated over F by a root y_1 of a polynomial $h(y)-x$ where h is a degree 5 polynomial (e.g., that of Prop. 2. 5). Also the function field $F(\hat{X}_H)$ of $\hat{X}_H = \mathcal{T}(a)_{\text{gen}}$ is the subfield of the splitting field $F(y_i)$, $i=1, \ldots, 5$, of the polynomial $h(y)-x$, fixed by the stabilizer of $\{1, 2\}$ in S_5. The function $y_1 y_2/(y_1+y_2)^2$ lies in $F(\hat{X}_H)$. Check that its Puiseux expansion at ∞ begins with $\zeta/(1+\zeta)^2$ where ζ denotes a primitive 5th root of 1, and that this number generates $F(\sqrt{5})$. The field $F(\sqrt{5})$ is therefore contained in the field of definition of each of the points over ∞. Since $\mathcal{H}(a)$ is a $\mathbb{Q}$-rational variety, it is regular over $\mathbb{Q}$. Thus $F(\sqrt{5})$ is a degree 2 extension of F. □

It is of interest to inspect further the transitivity of H_{τ_d} on $\tau_d(C, t)$, or equivalently of $\mathrm{H}(3)^*$ on $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$, in Cases 2 and 3. The history of the Hurwitz monodromy action shows it to be a common instance when it is transitive on absolute Nielsen classes (cf. [BFr1], [BFr2], [CP]). Thus it is a bit of a surprise that our simple example $\mathcal{F}(a)$ provides an *extension* of the Nielsen class action in which $\mathrm{H}(r-1)^*$ acts intransitively. We show that the action of $\mathrm{H}(3)^*$ on $\mathrm{ni}(\mathbf{C}_\infty, C, t)^{\mathrm{ab}}$ is intransitive for the marked conjugacy classes $(C, t)=(C_1, 2)$ and $(C_2, 1)$. It is transitive for $(C, t)=(C_2, 2)$.

For the last statement we are done if we can go from $(\tau_d, 3, (2\,4))$ to the other three pointings of the 3rd coordinate: check that $(\tau_d, 3, (2\,4))\,(Q_2^2)=(\tau_d, 3, (5\,7))$ and $(\tau_d, 3, (5\,7))\,(Q_1 Q_2^2 Q_1^{-1})^2=(\tau_d, 3, (8\,10))$. In particular the conjugation action on the points of $P_{\text{gen}}(C_2, 2)$ shows that

(3. 12) the points in $P_{\text{gen}}(C_2, 2)$ generate a degree 6 extension of F.

A direct by hand calculation can be used for the two intransitivity examples; we leave the details to the reader; the method is shown on a similar example in § 3. 7 (below). We concentrate on an alternative to the Hurwitz monodromy check of intransitivity in the case $(C, t)=(C_2, 1)$. Assume that H_{τ_d} is transitive on $\tau_d(C_2, 1)$. Then (3. 12) holds with $(C_2, 1)$ replacing $(C_2, 2)$ and 8 replacing 6. Thus each point on $\mathcal{T}(a)_{\text{gen}}$ over the branch point z_2 (or z_3) is defined over a degree 6 or 8 extension of F. The following shows that this is impossible.

Lemma 3. 18. *There is a point on $\mathcal{T}(a)_{\text{gen}}$ above z_2, rational over the field $F(z_2, z_3)$ (a degree 2 extension of F).*

Proof. Consider the generic cover $X=\mathcal{T}(o)_{\text{gen}}$ of the family $\mathcal{F}(o)$. The branch cycle associated to z_2 consists of a 2-cycle, so the point $\mathbf{m}$ corresponding to that 2-cycle is rational over $F(z_2, z_3)$. Now let X_{12} be the subcover (identified with the normalization of $X \times_{\mathbb{P}^1_z} X - \Delta$) of the Galois cover $\hat{X}$ of X associated with the subgroup H_{12} of S_5 fixing both 1 and 2. Since $\mathbf{m}$ has 2 as ramification index over z_2, it can be lifted to an $F(z_2, z_3)$-rational point $\mathbf{m}_{12}$ on X_{12} defined (as in Case 4 above) by $y_1(\mathbf{m}_{12})=y_2(\mathbf{m}_{12})=y_1(\mathbf{m})$. Observe that H_{12} is contained in the subgroup H of § 2. 1 that stabilizes $\{1, 2\}$. Therefore X_{12} projects onto $\hat{X}_H=\mathcal{T}(a)_{\text{gen}}$. The projection of $\mathbf{m}_{12}$ onto $\hat{X}_H$ is the required point. □

3. 7. An example of genus 1. We introduce a third example based on the data of § 2. 1: $r=4$, $G=S_5$ and $\mathbf{C}=(C_1, C_2, C_3, C_4)$ the 4-tuple of conjugacy classes in S_5 of

(3. 13) $\sigma_1=(2\,3)\,(4\,5)$, $\sigma_2=(1\,2)$, $\sigma_3=(1\,4)$, $\sigma_4=(5\,4\,3\,2\,1)$.

In this case, however, G is considered as a subgroup of S_{15} through the representation $T_3: S_5 \longrightarrow S_{15}$ given by the action of S_5 on the 15 partitions of $\{1,2,3,4,5\}$ into 2 unordered pairs plus a single element. As in § 2. 1, the representation T_3 identifies the absolute Nielsen classes $\mathrm{ni}(\mathbf{C})_5^{\mathrm{ab}}$ and $\mathrm{ni}(\mathbf{C})_{10}^{\mathrm{ab}}$ (e.g., Lemma 2. 1 holds with S_{10} replaced by S_{15}). Thus we may consider the Hurwitz family associated with $\mathrm{ni}(\mathbf{C})_{15}^{\mathrm{ab}}$ with the additional condition that z_4 has been specialized to ∞. Denote this family by $\mathscr{F}(b)$; it is defined over $\mathbb{Q}$ and its parameter space is the usual $\mathbb{Q}$-rational variety $\mathscr{H}$ of our previous examples.

Apply T_3 to the 4-tuple of (3. 13). Label the partitions of $\{1,2,3,4,5\}$, by indicating the singleton and the pair that contains the smallest integer excluding the singleton: $[1,\{2,3\}], [1,\{2,4\}], [1,\{2,5\}], [2,\{1,3\}], [2,\{1,4\}], \dots, [5,\{1,4\}]$. If these are relabeled in order as $1, 2, \dots, 15$, then in S_{15} we have

$$(3.\,14)\quad T_3(\boldsymbol{\sigma}) = ((6\,8)(5\,9)(4\,7)(12\,15)(11\,13)(10\,14),\ (1\,4)(2\,5)(3\,6)(8\,9)(12\,11)(14\,15),$$
$$(1\,12)(2\,10)(3\,11)(4\,6)(7\,9)(13\,14),\ (1\,13\,12\,7\,6)(2\,14\,11\,8\,5)(3\,15\,10\,9\,4)).$$

The genus g of the Hurwitz family is given by $2(15+g-1)=6+6+6+12$ or $g=1$. In order to show that this is a family of elliptic curves we now show that each fiber has a rational point produced by ramification: an application of Theorem 3. 14 through Hurwitz monodromy action on pointed Nielsen classes.

Consider the marked conjugacy class $(C_1, 1)$. Here $u=3$, $v=1$ (i.e., three $(C_1, 1)$-pointings of $\boldsymbol{\sigma}$). We claim that $\mathrm{H}_{\boldsymbol{\sigma}}$ has 2 orbits on $\boldsymbol{\sigma}(C_1, 1)$. Thus one of these will be of length 1 and corresponds, from Theorem 3. 14, to a rational point on the generic fiber $\mathscr{T}(b)_{\mathrm{gen}}$. For the proof we check that $\mathrm{H}(3)^*$ has two orbits on $\mathrm{ni}(\mathbf{C}_\infty, C_1, 1)^{\mathrm{ab}}$; this is equivalent (Remark 3. 13 b)) and this is easier: we know that Q_1, Q_2 generate $\mathrm{H}(3)^*$. The complete list of $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ derives from (3. 10): $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}} = \{\tau, (\tau)\,Q_1, (\tau)\,Q_1 Q_2 \mid \tau \in \mathrm{sni}(\mathbf{C}_\infty)^{\mathrm{ab}}\}$. If we denote elements of $\mathrm{sni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ by 0's, application of Q_1 to these by 1's and application of $Q_1 Q_2$ to 0's by 2's, then the action of Q_i, $i=1,2$, on $\mathrm{ni}(\mathbf{C}_\infty)^{\mathrm{ab}}$ may be summarized by ([Fr3], Ex. 2 cont.)

$$(3.\,15)\quad \text{a)}\quad Q_1 = (0_a 1_a 0_b 1_b 0_d 1_d)\,(0_c 1_c 0_e 1_e)\,(2_a 2_c 2_b)\,(2_d 2_e); \quad \text{and}$$
$$\text{b)}\quad Q_2 = (0_a 0_c 0_b)\,(0_d 0_e)\,(1_a 2_a 1_e 2_e 1_c 2_c)\,(1_b 2_b 1_d 2_d).$$

A total calculation for the effect of the Q's on $\mathrm{ni}(\mathbf{C}_\infty; C_1, 1)^{\mathrm{ab}}$ can be had by extending indication of the possible pointings on, say 1_b, by superscripts. For example, 1_b^2 indicates that, with the ordering on the 1-cycles that appear in the 2nd position of 1_b induced by its derivation from the order on the 1-cycles that appear in the 1st position of 0_b, it is the 2nd of these that has been marked. With these notations:

$$(3.\,16)\quad \text{a)}\quad Q_1 = (0_a^1 1_a^1 0_b^2 1_b^2 0_d^2 1_d^2 0_a^2 1_a^2 0_b^1 1_b^1 0_d^3 1_d^3)\,(0_a^3 1_a^3 0_b^3 1_b^3 0_d^1 1_d^1)\,(0_c^1 1_c^1 0_e^3 1_e^3)\,(0_c^2 1_c^2 0_e^2 1_e^2)$$
$$(0_c^3 1_c^3 0_e^1 1_e^1)\,(2_a^1 2_c^1 2_b^1)\,(2_d^1 2_e^1)\,(2_a^2 2_c^2 2_b^2)\,(2_d^2 2_e^2)\,(2_a^3 2_c^3 2_b^3)\,(2_d^3 2_e^3); \quad \text{and}$$
$$\text{b)}\quad Q_2 = (0_a^1 0_c^1 0_b^1)\,(0_d^1 0_e^1)\,(0_a^2 0_c^2 0_b^2)\,(0_d^2 0_e^2)\,(0_a^3 0_c^3 0_b^3)\,(0_d^3 0_e^3)\,(1_a^3 2_a^3 1_e^1 2_e^1 1_c^3 2_c^3)$$
$$(1_a^1 2_a^1 1_e^3 2_e^3 1_c^1 2_c^1 1_a^2 2_a^2 1_e^2 2_e^2 1_c^2 2_c^2)\,(1_b^1 2_b^1 1_d^3 2_d^3)\,(1_b^3 2_b^3 1_d^1 2_d^1)\,(1_b^2 2_b^2 1_d^2 2_d^2).$$

The two orbits of $\mathrm{H}(3)^*$ on $\mathrm{ni}(\mathbf{C}_\infty, C_1, 1)^{\mathrm{ab}}$ are $\{0_a^3, 1_a^3, 0_b^3, 1_b^3, 0_d^1, 1_d^1, 0_c^3, 0_e^1, 1_e^1, 1_c^3, 2_a^3, 2_c^3, 2_b^3, 2_d^1, 2_e^1\}$ and its complement. □

§ 4. Arithmetic nonconstancy of the family $\mathscr{F}(a)$

4. 1. Preliminaries. In this section we continue to investigate Question 3. 2 for the specific family $\mathscr{F}(a)$. We saw in § 3. 5 that ramification does not produce rational points, but the existence of rational points in general has still to be investigated. Here we show (Theorem 4. 2) that there are $\mathbb{Q}$-fibers of $\mathscr{F}(a)$ with no rational points. Equivalently (Prop. 3. 4), there is no rational point on the generic fiber; or the Hurwitz family is arithmetically nonconstant (end of § 3. 1). Thus, this example supports our conjecture that the production of rational points by ramification is the only obstruction to production of rational points. Philosophically: For Hurwitz families, arithmetic constancy of the family can be divined by inspection of ramification. This gives credence to the possibility for inspecting many arithmetic properties of the general fiber from pure group theory.

Here is a description of the treatment of this section. From Proposition 1. 8, a $\mathbb{Q}$-fiber of $\mathscr{F}(a)$ with no rational points comes from a cover of $\mathbb{P}_z^1$ in $\mathrm{ni}(\mathbf{C}_\infty)_{10}$ defined over $\mathbb{Q}$ having no rational points. Prop. 2. 5 of § 2. 2 provides an explicit family of polynomials $h(y)$ such that the associated covers $\varphi(h)\colon \mathbb{P}_y^1 \to \mathbb{P}_z^1$ lie in $\mathrm{ni}(\mathbf{C}_\infty)_5$. For each of them, denoted $X \to \mathbb{P}_z^1$, form the cover $\hat{X}_H \to \mathbb{P}_z^1$ associated with the subgroup H of S_5 fixing $\{1, 2\}$ (§ 1. 1 (1. 2)). From Prop. 2. 2 (§ 2. 1), this cover is in $\mathrm{ni}(\mathbf{C}_\infty)_{10}$. For suitable choice of the polynomial h we show that $\hat{X}_H$ has no rational point.

In the following fix a 4-tuple (α, β, s, t) of rational numbers in the open subset $\mathcal{O} = (\mathbf{z}^*)^{-1}(\mathscr{U}_\infty^3)$ (cf. Prop. 2. 6). In addition assume that

(4. 1) $\qquad -5\,t$ is not a square in $\mathbb{Q}$.

Then, $h(y) = \alpha((y^5/5) - s(y^4/4) + 2ty^3 - 5st(y^2/2) + 5t^2 y) + \beta$ gives the cover $\varphi(h)\colon X = \mathbb{P}_y^1 \to \mathbb{P}_z^1$ in $\mathrm{ni}(\mathbf{C}_\infty)_5$. Further recall that

(4. 2) a) $\dfrac{d}{dy}(h(y)) = \alpha(y^2 + 5\,t)(y^2 - s\,y + t)$; and

b) $h(a) = h(-a) = \alpha(25\,st^2/4) + \beta \quad$ for $\quad a = \sqrt{-5\,t}$.

Also, let z be an indeterminate over $\mathbb{Q}$ and let y_i, $i = 1, \ldots, 5$, be the roots of $h(y) - z$ in the algebraic closure of $\mathbb{Q}(z)$. Then the function field $\mathbb{Q}(X)$ of the curve X is identified with $\mathbb{Q}(y_1)$. The cover $\hat{X}_H \to \mathbb{P}_z^1$ lies in $\mathrm{ni}(\mathbf{C}_\infty)_{10}$. The function field $\mathbb{Q}(\hat{X}_H)$ is identified with $\mathbb{Q}(z, y_1 + y_2, y_1 y_2)$. We also consider $X_{12} \to \mathbb{P}_z^1$, the subcover of $\hat{X} \to \mathbb{P}_z^1$ associated with the subgroup H_{12} of S_5 fixing both 1 and 2: $\mathbb{Q}(X_{12})$ is identified with $\mathbb{Q}(y_1, y_2)$. The following diagrams summarize this:

$$\begin{array}{ccccc} & & X & & \\ & 4\nearrow & & \searrow 5 & \\ \hat{X} \to X_{12} & & \text{Covers} & & \mathbb{P}_z^1, \\ & 2\searrow & & \nearrow 10 & \\ & & \hat{X}_H & & \end{array} \qquad \begin{array}{ccccc} & & \mathbb{Q}(y_1) & & \\ & 5\nearrow & & \searrow 4 & \\ \mathbb{Q}(z) & & \text{Fields} & & \mathbb{Q}(y_1, y_2) \to \mathbb{Q}(y_1, y_2, y_3, y_4, y_5) \\ & 10\searrow & & \nearrow 2 & \\ & & \mathbb{Q}(z, y_1 + y_2, y_1 y_2) & & \end{array}$$

where the numbers are the relative degrees of the covers or field extensions.

Here is the strategy. The branch cycle of the cover $\hat{X}_H \to \mathbb{P}_z^1$ in C_4, which corresponds to the branch point at ∞, is the product in S_{10} of two 5-cycles. Let $D = \mathbf{m}_1 + \mathbf{m}_2$ be the divisor on $\hat{X}_H$ over ∞. Since D is a degree two $\mathbb{Q}$-rational divisor on the genus 0 curve $\hat{X}_H$, the linear system $\mathscr{L}(D)$ gives an embedding of $\hat{X}_H$ into $\mathbb{P}^2$ of degree 2 (as in the proof of Prop. 3. 4). We inspect if this conic has a rational point.

4. 2. A basis for $\mathscr{L}(D)$. This subsection produces a basis for $\mathscr{L}(D)$ in a three part argument.

Part 1: $y_1 + y_2 \in \mathscr{L}(D)$. Note that the polar divisor of the function z on $\hat{X}_H$ is $(z)^- = 5D$. Consider the functions $y_1, \ldots, y_5$ on $\hat{X}$. Since $h(y_i) = z$, $(y_i)^- = \frac{1}{5}(z)^-$, $i = 1, \ldots, 5$, on $\hat{X}$. Thus $\frac{1}{5}(z)^- \geqq (y_1 + y_2)^-$ on $\hat{X}$. But $y_1 + y_2$ is a function on $\hat{X}_H$ where the previous inequality is written as $(y_1 + y_2) \geqq D$.

Part 2: $(y_1 y_2 - 5t)/(y_1 + y_2) \in \mathscr{L}(D)$. The component of the branch cycle description of $X \to \mathbb{P}_z^1$ in the conjugacy class C_1 is a product of two disjoint 2-cycles. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the points of X corresponding to these two disjoint cycles. From (4. 2), they lie over the branch point $z_1 = h(a) = h(-a)$. Thus

$$y_1(\mathbf{p}_1) = a \quad \text{and} \quad y_1(\mathbf{p}_2) = -a. \tag{4. 3}$$

Denote the degree four divisor on X_{12} above $\mathbf{p}_i$ by $\mathbf{p}_{i1} + \mathbf{p}_{i2} + \mathbf{p}_{i3} + \mathbf{p}_{i4}$, $i = 1, 2$. Since a and $-a$ are roots of order 2 of the polynomial $h(y) - z_1$, two out of four of the points $\mathbf{p}_{11}, \mathbf{p}_{12}, \mathbf{p}_{13}, \mathbf{p}_{14}$ (resp., $\mathbf{p}_{21}, \mathbf{p}_{22}, \mathbf{p}_{23}, \mathbf{p}_{24}$) are sent to $-a$ (resp., a) by y_2. Indeed, these may be identified with the two points of X_{12} (regarded as the normalization of $X \times X - \Delta$, where Δ is the diagonal) that lie above $(\mathbf{p}_1, \mathbf{p}_2)$. To be explicit we assume that

$$\begin{aligned} &y_2(\mathbf{p}_{11}) = a, && y_2(\mathbf{p}_{21}) = -a, \\ &y_2(\mathbf{p}_{12}) = y_2(\mathbf{p}_{13}) = -a, && y_2(\mathbf{p}_{22}) = y_2(\mathbf{p}_{23}) = a, \\ &y_2(\mathbf{p}_{14}) = a', && y_2(\mathbf{p}_{24}) = a', \end{aligned} \tag{4. 4}$$

where a' is the root of $h(y) - z_1$ of multiplicity one.

Consider the projection onto $\hat{X}_H$ of the divisor $\mathbf{p}_{12} + \mathbf{p}_{13} + \mathbf{p}_{22} + \mathbf{p}_{23}$: it is a degree 2 divisor $\mathbf{q}_1 + \mathbf{q}_2$. Since $y_1 + y_2$ is 0 when evaluated at $\mathbf{p}_{ij}$, $i = 1, 2$ and $j = 2, 3$, conclude that $\mathbf{q}_i$, $i = 1, 2$, are zeros of $y_1 + y_2$ regarded as functions on $\hat{X}_H$. Also, the poles of this function are $\mathbf{m}_i$, $i = 1, 2$ (see Part 1). Therefore, $\deg(y_1 + y_2) = 2$ and the divisor of $y_1 + y_2$ is $\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{m}_1 - \mathbf{m}_2$. Finally, consider the divisor of zeros and of poles of the function $y_1 y_2 - a^2$ $(= y_1 y_2 - 5t)$: $(y_1 y_2 - 5t)^+ \geqq \mathbf{q}_1 + \mathbf{q}_2$ and $(y_1 y_2 - 5t)^- = 2D$. Since the zeros of $y_1 + y_2$ are zeros of $y_1 y_2 - 5t$, the polar divisor of $(y_1 y_2 - 5t)/(y_1 + y_2)$ is D.

Part 3: $\{1, y_1 + y_2, (y_1 y_2 - 5t)/(y_1 + y_2)\}$ *is a basis for* $\mathscr{L}(D)$. We have only to check that the three functions are linearly independent over $\mathbb{Q}$. Suppose that $a_1 + a_2(y_1 + y_2) + a_3((y_1 y_2 - 5t)/(y_1 + y_2)) = 0$ with $a_i \in \mathbb{Q}$, $i = 1, 2, 3$. The leading coefficient of the Puiseux expansion over $z = \infty$ gives that $a_2(1 + \zeta_5) + a_3(\zeta_5/(1 + \zeta_5)) = 0$, where ζ_5 is a primitive 5th root of 1. From § 3. 6 Case 4, $a_2 = a_3 = 0$ and the linear independence follows. □

4. 3. The conic equation. With $U=y_1+y_2$ and $V=(y_1y_2-5\,t)/(y_1+y_2)$ the previous section gives an embedding $\hat{X}_H \to \mathbb{P}^2$ by $\mathbf{p} \to (1, U(\mathbf{p}), V(\mathbf{p}))$ as a nonsingular conic defined over $\mathbb{Q}$:

$$(4.\,5) \qquad aU^2+bV^2+cUV+dU+eV+f=0, \qquad a, b, c, d, e, f \in \mathbb{Q}.$$

Inspection of the $z^{2/5}$ term of the Puiseux expansions over ∞ gives a, b, c. Indeed, the respective $z^{2/5}$ terms in U^2, V^2 and UV are $(1+\zeta_5)^2$, $\zeta_5^2/(1+\zeta_5)^2$ and ζ_5^2. With $\xi=(1+\zeta_5)^2/\zeta_5$ we get $a\xi^2+c\xi+b=0$. Since ξ is quadratic over $\mathbb{Q}$, we may choose $a=1$. Conclude that $b=N_{\mathbb{Q}(\xi)/\mathbb{Q}}(\xi)=1$ and $c=-\mathrm{Tr}_{\mathbb{Q}(\xi)/\mathbb{Q}}(\xi)=-3$. Therefore (4. 5) becomes

$$(4.\,6) \qquad U^2+V^2-3\,UV+dU+eV+f=0.$$

To determine d, e, f consider the points that are the projections onto $\hat{X}_H$ of $\mathbf{p}_{11}$ and $\mathbf{p}_{14}$. Denote these respectively by $\mathbf{p}_1^*$ and $\mathbf{p}_4^*$. Here we again use a for the parameter that appeared in § 4. 2. From (4. 3) and (4. 4) we get

$$(4.\,7) \qquad \begin{cases} (y_1+y_2)(\mathbf{p}_1^*)=2\,a, \\ (y_1y_2)(\mathbf{p}_1^*)=a^2, \end{cases} \Rightarrow \begin{cases} U(\mathbf{p}_1^*)=2\,a, \\ V(\mathbf{p}_1^*)=a, \end{cases} \text{and}$$
$$\begin{cases} (y_1+y_2)(\mathbf{p}_4^*)=a+a', \\ (y_1y_2)(\mathbf{p}_4^*)=aa', \end{cases} \Rightarrow \begin{cases} U(\mathbf{p}_4^*)=a+a', \\ V(\mathbf{p}_4^*)=a. \end{cases}$$

Substitution of $\mathbf{p}_1^*$ back in (4. 6) gives $4\,a^2+a^2-6\,a^2+2\,ad+ae+f=0$. Simplification and the similar formula for $\mathbf{p}_4^*$ gives

$$(4.\,8) \qquad (-a^2+f)+(2\,d+e)\,a=0 \quad \text{and} \quad (-a^2+(a')^2+da'+f)+(-a'+d+e)\,a=0.$$

A priori we know that a' is in $\mathbb{Q}$: it corresponds to the unique simple point on X above z_1. But we can compute it from $h(y)-z_1=\frac{\alpha}{5}(y^2-a^2)\,(y-a')$. Thus

$$a'=\frac{5\,(z_1-h(0))}{\alpha a^4}=\frac{5\,(h(a)-\beta)}{\alpha a^4}.$$

From (4. 2) and $a^2=-5\,t$ we get $a'=5\,s/4$. From assumption (4. 1), a is quadratic over $\mathbb{Q}$. This gives $-a^2+f=0$, $2\,d+e=0$, $-a^2+(a')^2+da'+f=0$ and $-a'+d+e=0$. That is, $f=a^2=-5\,t$, $d=-a'=-5\,s/4$ and $e=2\,a'=5\,s/2$. The final equation for the conic, denoted $\mathscr{C}_{\alpha,\beta,s,t}$ ($=\mathscr{C}$ when there is no possible confusion):

$$(4.\,9) \qquad U^2+V^2-3\,UV-5\,sU/4+5\,sV/2-5\,t=0.$$

4. 4. Demonstration of the arithmetic nonconstancy of $\mathscr{F}(a)$. Let $\bar{\mathscr{C}}_{\alpha,\beta,s,t}$ be the Zariski closure in $\mathbb{P}^2$ of the curve given by (4. 9) and consider the cover

$$\varphi=\varphi_{\alpha,\beta,s,t}\colon \bar{\mathscr{C}}_{\alpha,\beta,s,t} \to \mathbb{P}_z^1$$

derived from the isomorphism of $\hat{X}_H$ with $\bar{\mathscr{C}}_{\alpha,\beta,s,t}$.

Proposition 4. 1. *Assume that* (4. 1) *holds and that* $(\alpha, \beta, s, t) \in \mathcal{O}$. *Then the cover* $\varphi=\varphi_{\alpha,\beta,s,t}\colon \bar{\mathscr{C}}_{\alpha,\beta,s,t} \to \mathbb{P}_z^1$ *lies in* $\mathrm{ni}(\mathbf{C}_\infty)_{10}$ *and is defined over* $\mathbb{Q}$.

Proof. We have only to show that the covering map $\varphi_{\alpha,\beta,s,t}$ is defined over $\mathbb{Q}$. That is, we must show that the function z can be expressed as a polynomial in U and V with coefficients in $\mathbb{Q}$. From $z=h(y_i)$, $i=1, \ldots, 5$, we have $z=(h(y_1)+h(y_2))/2$. The right side is symmetric in y_1 and y_2; thus it can be written as $m(y_1+y_2, y_1y_2)$ with

$m \in \mathbb{Q}[x_1, x_2]$. But from our previous expressions for U and V, we may replace $y_1 + y_2$ by U and $y_1 y_2$ by $UV + 5t$. □

Theorem 4.2. *The Hurwitz family $\mathscr{F}(a)$ is arithmetically nonconstant.*

Proof. Classical reductions show that $\mathscr{C}_{\alpha,\beta,s,t}$ is isomorphic to the conic $U^2 - 5V^2/4 - 5(s^2 + 16t)/16 = 0$. Its Hilbert symbol is $(5/4, 5(s^2+16t)/16) = (5, 5(s^2+16t)) = (5,5)(5, s^2+16t) = (5, s^2+16t)$. Choose $\alpha = 1$, $\beta = 0$, $s = 0$, $t = 1/8$. Check easily that (4.1) holds and that $(1,0,0,1/8) \in \mathcal{O}$. Then $(5, s^2+16t) = (5,2)$. This last is -1 in the p-adic field $\mathbb{Q}_2$ as $5x^2 + 2y^2 = z^2$ has no solution mod 8. Thus $\mathscr{C}_{\alpha,\beta,s,t}$ has no $\mathbb{Q}$-rational points. Let $\mathbf{x} \in \mathscr{H}$ be the unique point such that the covers $\varphi_{\alpha,\beta,s,t}: \mathscr{C}_{\alpha,\beta,s,t} \to \mathbb{P}^1_z$ and $\mathscr{F}(a)_{\mathbf{x}}$ are equivalent. From Prop. 1.8, $\mathscr{F}(a)_{\mathbf{x}}$ and this equivalence are defined over $\mathbb{Q}$. Conclude that $\mathscr{F}(a)_{\mathbf{x}}$ is a $\mathbb{Q}$-fiber of $\mathscr{F}(a)$ with no rational points. Prop. 3.3 completes the proof. □

From Example 3.5, any point of $\mathscr{F}(a)$ that is the image of a rational point on $\mathscr{F}(a)'$ corresponds to a fiber of the family that has a rational point. Since $\mathscr{F}(a)'$ is a rational variety, that means that there is a dense set of $\mathbb{Q}$ points of $\mathscr{F}(a)$ with fibers with rational points, and a dense set of $\mathbb{Q}$ points with fibers without rational points.

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Institut Henri Poincaré, Paris, France

Department of Mathematics, University of California, Irvine CA 92717, U.S.A.

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