

From the motion group of the trivial link to the homology of the hypertree poset

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Institut Camille Jordan (Lyon)

Friday January, 10th 2014
Seminar of the ANR HOGT project

From the motion group of the *a* *n*-component trivial link
to the homology of the *a* hypertree poset *on n vertices*

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Motivation : $P\Sigma_n$

- F_n generated by $(x_i)_{i=1}^n$
- $P\Sigma_n$, pure symmetric automorphism group
 - ▶ group of automorphisms of F_n which send each x_i to a conjugate of itself,
 - ▶ group of motions of a collection of n coloured unknotted, unlinked circles in 3-space.
- It seems that their cohomology groups are not Koszul (A. Conner and P. Goetz).

- Use of the **hypertree poset** for the computation of the l^2 -Betti numbers of $P\Sigma_n$ by C. Jensen, J. McCammond and J. Meier.
- Action of $P\Sigma_n$ on a contractible complex MM_n defined by McCullough and Miller in 1996 in terms of marking of hypertrees, whose fundamental domain is the hypertree poset on n vertices,
- $P\Sigma_n \triangleright Inn(F_n) \Rightarrow OP\Sigma_n = P\Sigma_n / Inn(F_n)$
- $OP\Sigma_n$ acts cocompactly on MM_n
- Use of a theorem by Davis, Januszkiewicz and Leary to obtain the expression of l^2 -cohomology of the group in term of the cohomology of the fundamental domain of the complex.

Summary

- 1 The hypertree poset
 - Hypertrees
 - Hypertree poset
 - Homology of the hypertree poset
- 2 Computation of the homology of the hypertree poset
 - Species
 - Counting strict chains using large chains
 - Pointed hypertrees
 - Relations between chains of hypertrees
 - Dimension of the homology
- 3 From the hypertree poset to rooted trees
 - PreLie species
 - Character for the action of the symmetric group on the homology of the poset

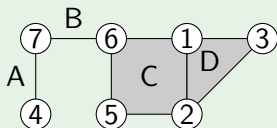
Hypergraphs and hypertrees

Definition ([Ber89])

A *hypergraph* (on a set V) is an ordered pair (V, E) where:

- V is a finite set (*vertices*)
- E is a collection of subsets of cardinality at least two of elements of V (*edges*).

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

Definition

Let $H = (V, E)$ be a hypergraph.

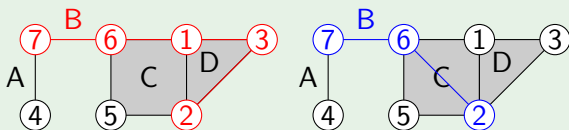
A *walk* from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f :

$$(d, \dots, e_i, v_i, e_{i+1}, \dots, f)$$

where for all i , $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.

The *length* of a walk is the number of edges and vertices in the walk.

Examples of walks



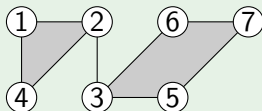
Hypertrees

Definition

A *hypertree* is a non-empty hypergraph H such that, given any distinct vertices v and w in H ,

- there exists a walk from v to w in H with distinct edges e_i , (H is *connected*),
- and this walk is unique, (H has *no cycles*).

Example of a hypertree



The hypertree poset

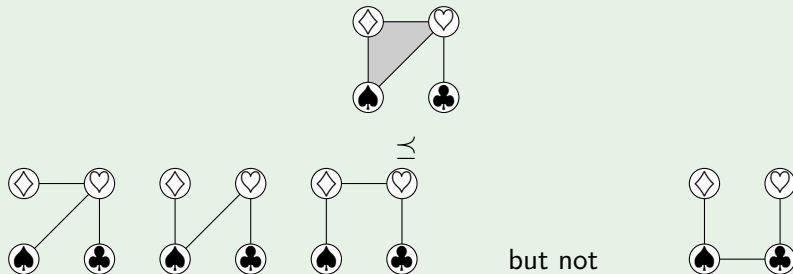
Definition

Let I be a finite set of cardinality n , S and T be two hypertrees on I .

$S \preceq T \iff$ Each edge of S is the union of edges of T

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices



- Graded poset by the number of edges [McCullough and Miller 1996],
- There is a unique minimum $\hat{0}$,
- $HT(I)$ = hypertree poset on I ,
- HT_n = hypertree poset on n vertices.
- Möbius number : $(n - 1)^{n-2}$ [McCammond and Meier 2004]

Goal:

- New computation of the homology dimension
- Computation of the action of the symmetric group on the homology (Conjecture of Chapoton 2007)

Homology of the poset

To every poset P , one can associate a **simplicial complex** (nerve of the poset seen as a category) whose

- vertices are elements of P ,
- faces are the chains of P .

Definition

A *strict k -chain* of hypertrees on I is a k -tuple (a_1, \dots, a_k) , where a_i are hypertrees on I different from the minimum $\hat{0}$ and $a_i \prec a_{i+1}$.

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Let C_k be the vector space generated by strict k -chains. We define $C_{-1} = \mathbb{C}.e$. We define the following linear map on the set $(C_k)_{k \geq -1}$:

$$\partial_k(a_1 \prec \dots \prec a_{k+1}) = \sum_{i=1}^k (-1)^i (a_1 \prec \dots \prec \hat{a}_i \prec \dots \prec a_k),$$

$$(a_1 \prec \dots \prec a_{k+1}) \in C_k.$$

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The homology is defined by:

$$\tilde{H}_j = \ker \partial_j / \text{im} \partial_{j+1}.$$

Theorem ([MM04])

The homology of \widehat{HT}_n is concentrated in maximal degree $(n - 3)$.

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Corollary

The character for the action of the symmetric group on \tilde{H}_{n-3} is given in terms of characters for the action of the symmetric group on C_k by:

$$\chi_{\tilde{H}_{n-3}} = (-1)^{n-3} \sum_{k=-1}^{n-3} (-1)^k \chi_{C_k}, \text{ where } n = \#I.$$

What are species?

Definition

A *species* F is a functor from the category of finite sets and bijections to itself. To a finite set I , the species F associates a finite set $F(I)$ independent from the nature of I .

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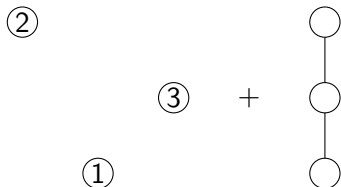
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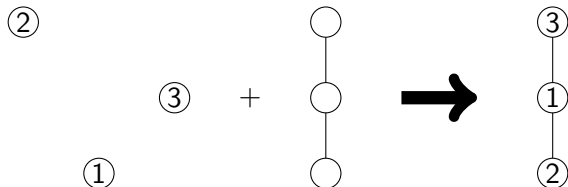


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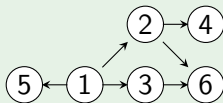
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Counterexamples

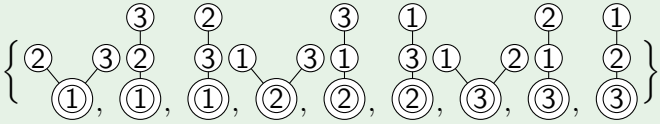
The following sets are not obtained using species:

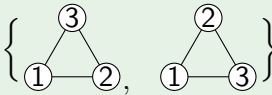
- $\{(1, \mathbf{3}, 2), (2, 1, \mathbf{3}), (2, \mathbf{3}, 1), (3, 1, \mathbf{2})\}$ (set of permutations of $\{1, 2, 3\}$ with exactly 1 descent)
- (graph of divisibility of $\{1, 2, 3, 4, 5, 6\}$)



Examples of species

- $\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$ (Species of lists Assoc on $\{1, 2, 3\}$)
- $\{\{1, 2, 3\}\}$ (Species of non-empty sets Comm)
- $\{\{1\}, \{2\}, \{3\}\}$ (Species of pointed sets Perm)

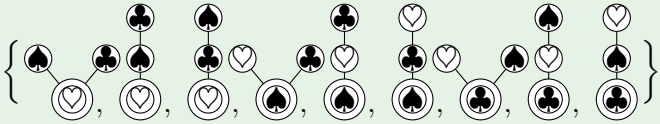
-  (Species of rooted trees PreLie)


-  (Species of cycles)

These sets are the image by species of the set $\{1, 2, 3\}$.

Examples of species

- $\{(\heartsuit, \spadesuit, \clubsuit), (\heartsuit, \clubsuit, \spadesuit), (\spadesuit, \heartsuit, \clubsuit), (\spadesuit, \clubsuit, \heartsuit), (\clubsuit, \heartsuit, \spadesuit), (\clubsuit, \spadesuit, \heartsuit)\}$
(Species of lists Assoc on $\{\clubsuit, \heartsuit, \spadesuit\}$)
- $\{\{\heartsuit, \spadesuit, \clubsuit\}\}$ (Species of non-empty sets Comm)
- $\{\{\heartsuit\}, \{\spadesuit\}, \{\clubsuit\}\}$ (Species of pointed sets Perm)

-  (Species of rooted trees PreLie)

-  (Species of cycles)

These sets are the image by species of the set $\{\clubsuit, \heartsuit, \spadesuit\}$.

Operations on species and associated series

Proposition

Let F and G be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)

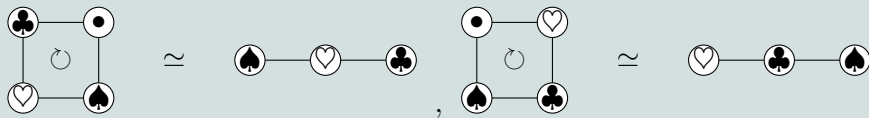
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Example: Derivative of the species of cycles on $I = \{\heartsuit, \spadesuit, \clubsuit\}$



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- $(F \circ G)(I) = \bigsqcup_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J)$, (substitution) where $\mathcal{P}(I)$ runs on the set of partitions of I .

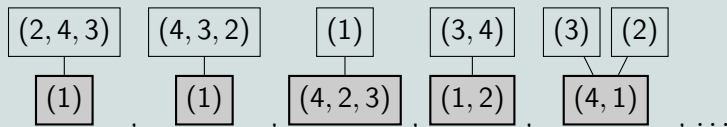
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Example of substitution: Rooted trees of lists on $I = \{1, 2, 3, 4\}$



Definition

To a species F , we associate its *generating series*:

$$C_F(x) = \sum_{n \geq 0} \#F(\{1, \dots, n\}) \frac{x^n}{n!}.$$

Examples of generating series:

- The generating series of the species of lists is $C_{\text{Assoc}} = \frac{1}{1-x}$.
- The generating series of the species of non-empty sets is $C_{\text{Comm}} = \exp(x) - 1$.
- The generating series of the species of pointed sets is $C_{\text{Perm}} = x \cdot \exp(x)$.
- The generating series of the species of rooted trees is $C_{\text{PreLie}} = \sum_{n \geq 0} n^{n-1} \frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{\text{Cycles}} = -\ln(1-x)$.

Definition

The *cycle index series* of a species F is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \dots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \dots \right),$$

- with F^σ , is the set of F -structures fixed under the action of σ ,
- and σ_i , the number of cycles of length i in the decomposition of σ into disjoint cycles.

Examples

- The cycle index series of the species of lists is $Z_{\text{Assoc}} = \frac{1}{1-p_1}$.
- The cycle index series of the species of non empty sets is $Z_{\text{Comm}} = \exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right) - 1$.

Operations on cycle index series

Operations on species give operations on their cycle index series:

Proposition

Let F and G be two species. Their cycle index series satisfy:

$$\begin{aligned}Z_{F+G} &= Z_F + Z_G, & Z_{F \times G} &= Z_F \times Z_G, \\Z_{F \circ G} &= Z_F \circ Z_G, & Z_{F'} &= \frac{\partial Z_F}{\partial p_1}.\end{aligned}$$

Definition

The *suspension* Σ of a cycle index series $f(p_1, p_2, p_3, \dots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \dots).$$

Counting strict chains using large chains

Let I be a finite set of cardinality n .

Definition

A *large k -chain* of hypertrees on I is a k -tuple (a_1, \dots, a_k) , where a_i are hypertrees on I and $a_i \preceq a_{i+1}$.

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Let $M_{k,s}$ be the set of words on $\{0, 1\}$ of length k , with s letters "1". The species $\mathcal{M}_{k,s}$ is defined by:

$$\begin{cases} \emptyset & \mapsto M_{k,s}, \\ V \neq \emptyset & \mapsto \emptyset. \end{cases}$$

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Proposition

The species \mathcal{H}_k of large k -chains and \mathcal{HS}_i of strict i -chains are related by:

$$\mathcal{H}_k \cong \sum_{i \geq 0} \mathcal{HS}_i \times \mathcal{M}_{k,i}.$$

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Proof.

Deletion of repetitions

(a_1, \dots, a_k)

$(a_{j_1}, \dots, a_{j_i})$

$u_j = 0$ if $a_j = a_{j-1}$, 1 otherwise

(u_1, \dots, u_k)

with $a_0 = \hat{0}$. □

The previous proposition gives, for all integer $k > 0$:

$$\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i.$$

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$$\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s.$$

χ_k is a polynomial $P(k)$ in k which gives, once evaluated in -1 , the character:

Corollary

$$\chi_{\tilde{H}_{n-3}} = (-1)^n P(-1) =: (-1)^n \chi_{-1}$$

The hypertrees will now be on n vertices.

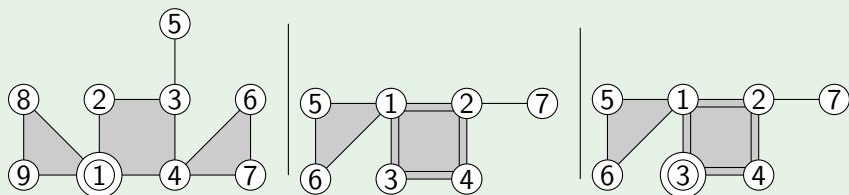
Pointed hypertrees

Definition

Let H be a hypertree on I . H is:

- *rooted* in a vertex s if the vertex s of H is distinguished,
- *edge-pointed* in an edge a if the edge a of H is distinguished,
- *rooted edge-pointed* in a vertex s in an edge a if the edge a of H and a vertex s of a are distinguished.

Example of pointed hypertrees



Proposition: Dissymmetry principle

The species of hypertrees and of rooted hypertrees are related by:

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^p + \mathcal{H}^a.$$

We write:

- \mathcal{H}_k , the species of large k -chains of hypertrees,
- \mathcal{H}_k^{pa} , the species of large k -chains of hypertrees whose minimum is rooted edge-pointed,
- \mathcal{H}_k^p , the species of large k -chains of hypertrees whose minimum is rooted,
- \mathcal{H}_k^a , the species of large k -chains of hypertrees whose minimum is edge-pointed.

Corollary ([Oge13])

The species of large k -chains of hypertrees are related by:

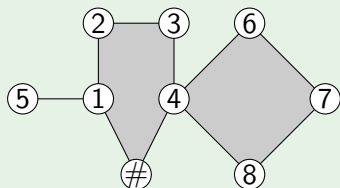
$$\mathcal{H}_k + \mathcal{H}_k^{pa} = \mathcal{H}_k^p + \mathcal{H}_k^a.$$

Last but not least type of hypertrees

Definition

A *hollow hypertree* on n vertices ($n \geq 2$) is a hypertree on the set $\{\#, 1, \dots, n\}$, such that the vertex labelled by $\#$, called the *gap*, belongs to one and only one edge.

Example of a hollow hypertree



We denote by \mathcal{H}_k^c , the species of large k -chains of hypertrees whose minimum is a hollow hypertree.

Relations between species of hypertrees

Theorem

The species \mathcal{H}_k , \mathcal{H}_k^p and \mathcal{H}_k^c satisfy:

$$\mathcal{H}_k^p = X \times \mathcal{H}'_k \quad (1)$$

$$\mathcal{H}_k^p = X \times \text{Comm} \circ \mathcal{H}_k^c + X, \quad (2)$$

$$\mathcal{H}_k^c = \text{Comm} \circ \mathcal{H}_{k-1}^c \circ \mathcal{H}_k^p, \quad (3)$$

$$\mathcal{H}_k^a = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p, \quad (4)$$

$$\mathcal{H}_k^{pa} = (\mathcal{H}_{k-1}^p - x) \circ \mathcal{H}_k^p. \quad (5)$$

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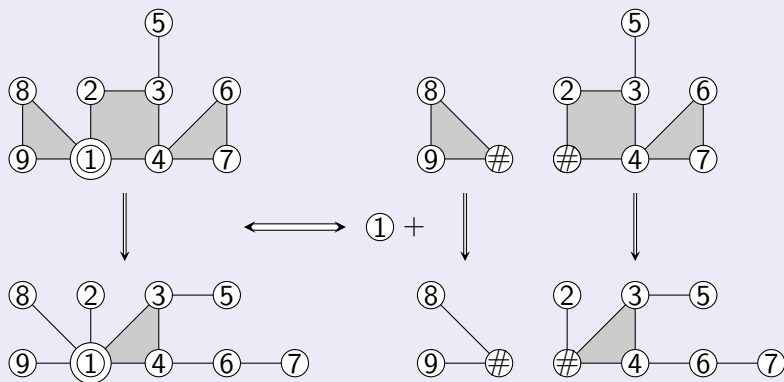
Proof.

- 1 Rooting a species F is the same as multiplying the singleton species X by the derivative of F ,

Second part of the proof.

We separate the root and every edge containing it, putting gaps where the root was,

$$\mathcal{H}_k^p = X \times \text{Comm} \circ \mathcal{H}_k^c + X,$$

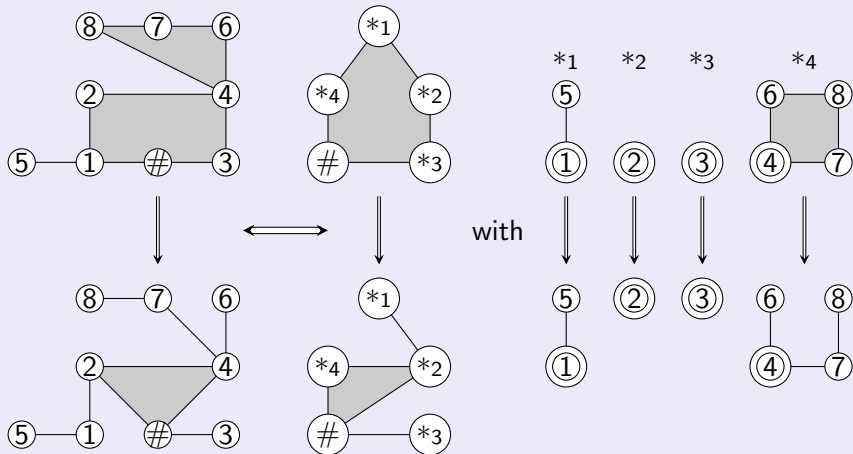


and End!

3 Hollow case:

$$\mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \quad (6)$$

$$\mathcal{H}_k^{cm} = \text{Comm} \circ \mathcal{H}_{k-1}^c. \quad (7)$$



Dimension of the homology

Proposition

The generating series of the species \mathcal{H}_k , \mathcal{H}_k^p and \mathcal{H}_k^c satisfy:

$$\mathcal{C}_k^p = x \cdot \exp \left(\frac{\mathcal{C}_{k-1}^p \circ \mathcal{C}_k^p}{\mathcal{C}_k^p} - 1 \right), \quad (8)$$

$$\mathcal{C}_k^a = (\mathcal{C}_{k-1} - x)(\mathcal{C}_k^p), \quad (9)$$

$$\mathcal{C}_k^{pa} = (\mathcal{C}_{k-1}^p - x)(\mathcal{C}_k^p), \quad (10)$$

$$x \cdot \mathcal{C}_k' = \mathcal{C}_k^p, \quad (11)$$

$$\mathcal{C}_k + \mathcal{C}_k^{pa} = \mathcal{C}_k^p + \mathcal{C}_k^a. \quad (12)$$

Lemma

The generating series of \mathcal{H}_0 and \mathcal{H}_0^P are given by:

$$\mathcal{C}_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$\mathcal{C}_0^P = xe^x.$$

Lemma

The generating series of \mathcal{H}_0 and \mathcal{H}_0^P are given by:

$$C_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$C_0^P = xe^x.$$

This implies with the previous theorem:

Theorem ([MM04])

The dimension of the only homology group of the hypertree poset is $(n - 1)^{n-2}$.

This dimension is the dimension of the vector space $\text{PreLie}(n-1)$ whose basis is the set of rooted trees on $n - 1$ vertices.

From the hypertree poset to rooted trees

- 1 This dimension is the dimension of the vector space $\text{PreLie}(n-1)$ whose basis is the set of rooted trees on $n - 1$ vertices.
The operad (a species with more properties on substitution) whose vector space are $\text{PreLie}(n)$ is PreLie .
- 2 This operad is **anticyclic** ([Cha05]): There is an action of the symmetric group \mathfrak{S}_n on $\text{PreLie}(n - 1)$ which extends the one of \mathfrak{S}_{n-1} .
- 3 Moreover, there is an **action** of \mathfrak{S}_n on the homology of the poset of hypertrees on n vertices.

Question

Is the action of \mathfrak{S}_n on $\text{PreLie}(n-1)$ the same as the action on the homology of the poset of hypertrees on n vertices?

Character for the action of the symmetric group on the homology of the poset

Using relations on species established previously, we obtain:

Proposition

The series Z_k , Z_k^p , Z_k^a and Z_k^{pa} satisfy the following relations:

$$Z_k + Z_k^{pa} = Z_k^p + Z_k^a, \quad (13)$$

$$Z_k^p = p_1 + p_1 \times \text{Comm} \circ \left(\frac{Z_{k-1}^p \circ Z_k^p - Z_k^p}{Z_k^p} \right), \quad (14)$$

$$Z_k^a + Z_k^p = Z_{k-1} \circ Z_k^p, \quad (15)$$

$$Z_k^{pa} + Z_k^p = Z_{k-1}^p \circ Z_k^p, \text{ and } p_1 \frac{\partial Z_k}{\partial p_1} = Z_k^p. \quad (16)$$

Theorem ([Oge13], conjecture of [Cha07])

The cycle index series Z_{-1} , which gives the character for the action of \mathfrak{S}_n on \tilde{H}_{n-3} , is linked with the cycle index series M associated with the anticyclic structure of PreLie by:

$$Z_{-1} = p_1 - \Sigma M = \text{Comm} \circ \Sigma \text{PreLie} + p_1 (\Sigma \text{PreLie} + 1). \quad (17)$$

The cycle index series Z_{-1}^p is given by:

$$Z_{-1}^p = p_1 (\Sigma \text{PreLie} + 1). \quad (18)$$

Theorem ([Oge13], conjecture of [Cha07])

The cycle index series Z_{-1} , which gives the character for the action of \mathfrak{S}_n on \tilde{H}_{n-3} , is linked with the cycle index series M associated with the anticyclic structure of PreLie by:

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The cycle index series Z_{-1}^P is given by:

$$Z_{-1}^P = p_1 (\Sigma \text{PreLie} + 1). \quad (18)$$

Proof.

Sketch of the proof

- 1 Computation of $Z_0 = \text{Comm}$ and $Z_0^P = \text{Perm} = p_1 + p_1 \times \text{Comm}$
- 2 Replaced in the formula giving Z_0^P in terms of itself and Z_{-1}^P

$$Z_0^P = p_1 + p_1 \times \text{Comm} \circ \left(\frac{Z_{-1}^P \circ Z_0^P - Z_0^P}{Z_0^P} \right),$$

Second part of the proof.

- ③ As $\Sigma \text{PreLie} \circ \text{Perm} = \text{Perm} \circ \Sigma \text{PreLie} = p_1$, according to [Cha07], we get:

$$Z_{-1}^p = p_1 (\Sigma \text{PreLie} + 1).$$

- ④ The dissymetry principle associated with the expressions gives:

$$\text{Comm} + Z_{-1}^p \circ \text{Perm} - \text{Perm} = \text{Perm} + Z_{-1} \circ \text{Perm} - \text{Perm}.$$

- ⑤ Thanks to equation [Cha05, equation 50], we conclude:

$$\Sigma M - 1 = -p_1 \left(-1 + \Sigma \text{PreLie} + \frac{1}{\Sigma \text{PreLie}} \right).$$



Thank you for your attention !

[Oge13] [Bérénice Oger](#) Action of the symmetric groups on the homology of the hypertree posets. [Journal of Algebraic Combinatorics](#), february 2013.

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Eccentricity

Definition

The *eccentricity* of a vertex or an edge is the maximal number of vertices on a walk without repetition to another vertex.

The *center* of a hypertree is the vertex or the edge with minimal eccentricity.

Example of eccentricity

