Simultaneous Frobenius-Padé approximants

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Abstract : *In this paper, we extend to simultaneous approximation the notion of Frobenius-Padé approximants; we then construct rational approximants for vector functions given by their expansion in an orthogonal series. After giving the definitions and notations for simultaneous Frobenius-Padé approximants and table, we develop recursive relations for computing different sequences in the table of approximants. We then propose algorithms to compute, in the two dimensional case, antidiagonal (Kronecker type algorithm) and diagonal sequences.*

Keywords : rational approximation, orthogonal polynomials, orthogonal series, Frobenius-Padé approximants, vector Padé approximants.

1 Introduction

Rational approximants for a function given as an orthogonal series (expansion with respect to a family of orthogonal polynomials) has been defined and studied for a rather long time ([11], [12], [13]). Different definitions can be given, known as linear or non linear (see for instance [1], [2], [6]), and we restrict our study to the so-called linear case, classically denoted Frobenius-Padé approximants.

The effective computation of these approximants was considered by [8], and continued in [9] where the author looks for all possible recurrence relations in the Frobenius -Padé array.

In this paper, we try to extend to simultaneous approximation the notion of Frobenius -Padé approximants. The ideas concerning simultaneous rational approximation of functions given by their expansion in power series can be found at their beginning in [4]. Practical computational methods for this problem have been developpped by De Bruin [3, 5] and in [7] analogues of Frobenius identities have been derived to develop recursive algorithms to compute "anti-diagonal" sequences as in the Padé table (the so called Kronecker algorithm).

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These ideas have been precised, then extended by vector-orthogonal polynomials, and matrix-orthogonal polynomials for example in [10], [14], where the set of free parameters is limited (regular indices) and so a well-structured table is obtained.

Here, following the ideas of these two origins, we study the array of simultaneous Frobenius-Padé approximants. In section 2, we give the most possible canonical definition for approximating d functions given as orthogonal expansions. Then, we limit the number of functions to $d = 2$, to keep the concepts clear, we are then ready to look for recursive algorithms to compute any of these approximants. In section 3, we look for the shortest possible relations, i.e. three term relations. In section 4, we define algorithms linking approximants having the same accuracy, similar to the antidiagonal in the scalar Padé case. To increase the accuracy, diagonal sequences are considered in section 5. In section 6, the number of free parameters is reduced : the numerators have the same degree, and accuracy is regularly distributed on the different functions, as done in "vector approximation" with "regular indices". The aim is to have a well-organized table (2 parameters instead of 4) and a quick increase of accuracy.

The algorithms which are developed in the different sections are independent, in the sense that the involved sequences are different. None can be reduced to a combination of the others.

2 Definitions and notations

The notations are extensions of what was taken in [8]. In the sequel, a 'double' capital letter will denote a vector. We consider a vector function

$$\mathbb{F}(z) = (f^1(z), f^2(z), \cdots, f^d(z)) \in \mathbb{C}^d[[z]] \ .$$

$(P_k)_{k\geq 0}$ being the system of orthogonal polynomials with respect to the weight function $w(x)$ on the interval $[a,b]$, each f^j is supposed to be in $\mathcal{L}_2((a,b),w)$ and given as an orthogonal series

$$f^j(z) = \sum_{k=0}^{\infty} f^j_k P_k(z), \quad f^j_k = \frac{1}{\| P_k \|_2^2} \int_a^b f^j(x) P_k(x) w(x) dx \ , \quad j = 1, \ldots, d.$$

We are going to construct a rational approximant to $\mathbb{F}(z)$ in the following way:

- we define two multi-indices $\mathbf{p} = (p_1, p_2, \cdots, p_d)$ and $\mathbf{q} = (q_1, q_2, \cdots, q_d)$,
- the numerator polynomials are denoted by $\mathbb{N}\mathrm{um}(z) = (N^1(z), \cdots, N^d(z))$ and their degrees are specified by
$$\deg(N^j(z)) \leq p_j, \quad j = 1, \cdots, d,$$
- the denominator polynomial $D(z)$ has degree at most $q = q_1 + \cdots + q_d$,
- these polynomials satisfy the accuracy to order conditions, i.e. the remainder term
$$\mathbb{R}\mathrm{em}(z) = \big(R^1(z), R^2(z), \cdots, R^d(z)\big)$$

satisfies

$$R^j(z) = D(z)f^j(z) - N^j(z) = e^j_{p_j+q_j+1}P_{p_j+q_j+1}(z) + \cdots, \quad j = 1, \cdots, d, \tag{1}$$

which means that for each component of $\mathbb{R}\mathrm{em}(z)$, the first p_j+q_j coefficients of the expansion with respect to the orthogonal system $(P_k)_{k\geq 0}$ are zero. We have omitted the indexes $(\mathbf{p}, \mathbf{q})$ in $\mathbb{R}\mathrm{em}(z), \mathbb{N}\mathrm{um}(z), D(z), e^j_k$ in order to simplify notations.

These conditions lead to a homogeneous linear system which has always a non trivial solution, and the approximant is called a **simultaneous Frobenius-Padé approximant**

$$[\mathbf{p}, \mathbf{q}] \ or \ [p_1, \cdots, p_d; q_1, \cdots, q_d] = \mathbb{N}\mathrm{um}(z)/D(z). \tag{2}$$

We set

$$N^j(z) = \sum_{i=0}^{p_j} a^j_i P_i(z), \ j = 1, \cdots, d, \quad D(z) = \sum_{i=0}^{q} b_i P_i(z) \ ,$$

and, in the same way as in [8], we define the quantities $(h^j_{ki}), j = 1, \cdots, d$, by

$$f^j(z)P_i(z) = \sum_{k=0}^{\infty} h^j_{ki} P_k(z), \ \text{where}$$

$$h^j_{ki} = \frac{1}{\| P_k \|^2{}_2} \int_a^b f^j(x)P_i(x)P_k(x)w(x)dx \ . \tag{3}$$

Classically, i.e. as for any kind of Padé approximation, equations (1) can be written

$$\sum_{k=0}^{\infty} \left(\sum_{l=0}^{q} b_l h^j_{kl} \right) P_k(z) - \sum_{l=0}^{p_j} a^j_l P_l(z) = \mathcal{O}(P_{p_j+q_j+1}), \quad j = 1, \cdots, d,$$

where the notation $\mathcal{O}(P_l)$ means that all the coefficients in the polynomial series expansion are zero up to $l-1$. We then obtain the two following systems

$$\sum_{l=0}^{q} b_l h^j_{kl} \ = \ a^j_k, \ k = 0, \cdots, p_j, \ j = 1, \cdots, d, \tag{4}$$

$$\sum_{l=0}^{q} b_l h^j_{kl} \ = \ 0 \quad k = p_j + 1, \cdots, p_j + q_j, \ j = 1, \cdots, d. \tag{5}$$

The system (5) of q equations enables us to compute the $q+1$ coefficients of $D(z)$. Once the (b_l) are computed, the numerator coefficients are given immediately from (4). The (b_l) are the solution of a homogeneous system of q equations in $q+1$ unknowns and so there is always a non-trivial solution. Proceeding like in [8], it is easy to see that, for computing all the coefficients of the previous systems, we need to know the quantities $f^j_k, \quad k = 0, \cdots, p_j + 2q_j, j = 1, \cdots, d$, that is, the first $p_j + 2q_j + 1$ coefficients of each series $f^j(z), j = 1, \cdots, d$.

Apart from a multiplicative factor, the denominator $D(z)$ has the following determinantal representation (we define for all i, $n_i = p_i + q_i$)

$$D(z) = \begin{vmatrix} h^1_{p_1+1,0} & h^1_{p_1+1,1} & \cdots & h^1_{p_1+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h^1_{n_1,0} & h^1_{n_1,1} & \cdots & h^1_{n_1,q} \\ \vdots & \vdots & \vdots & \vdots \\ h^d_{p_d+1,0} & h^d_{p_d+1,1} & \cdots & h^d_{p_d+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h^d_{n_d,0} & h^d_{n_d,1} & \cdots & h^d_{n_d,q} \\ P_0(z) & P_1(z) & \cdots & P_q(z) \end{vmatrix}. \quad (6)$$

and the numerators can then be written

$$N^j(z) = \begin{vmatrix} h^1_{p_1+1,0} & h^1_{p_1+1,1} & \cdots & h^1_{p_1+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h^1_{n_1,0} & h^1_{n_1,1} & \cdots & h^1_{n_1,q} \\ \vdots & \vdots & \vdots & \vdots \\ h^d_{p_d+1,0} & h^d_{p_d+1,1} & \cdots & h^d_{p_d+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h^d_{n_d,0} & h^d_{n_d,1} & \cdots & h^d_{n_d,q} \\ \sum_{i=0}^{p_j} h^j_{i,0} P_i(z) & \sum_{i=0}^{p_j} h^j_{i,1} P_i(z) & \cdots & \sum_{i=0}^{p_j} h^j_{i,q} P_i(z) \end{vmatrix} \quad j = 1, \cdots, d. \quad (7)$$

For the multi-indices $\mathbf{p}, \mathbf{q}$, let us define the determinants (generalization of the classical Hankel determinants) $H_{\mathbf{p},\mathbf{q}}$, $q \times q$ for $q = \sum_1^d q_k$, by:

$$H_{\mathbf{p},\mathbf{q}} = \begin{vmatrix} h^1_{p_1+1,0} & h^1_{p_1+1,1} & \cdots & h^1_{p_1+1,q-1} \\ \cdots & \cdots & \cdots & \cdots \\ h^1_{n_1,0} & h^1_{n_1,1} & \cdots & h^1_{n_1,q-1} \\ \vdots & \vdots & \vdots & \vdots \\ h^d_{p_d+1,0} & h^d_{p_d+1,1} & \cdots & h^d_{p_d+1,q-1} \\ \cdots & \cdots & \cdots & \cdots \\ h^d_{n_d,0} & h^d_{n_d,1} & \cdots & h^d_{n_d,q-1} \end{vmatrix}.$$

If $H_{\mathbf{p},\mathbf{q}} \neq 0$ then the system (5) has full rank q and has a one-parameter family of solutions that can be obtained by Cramer's rule. This is equivalent to the fact that the approximant is uniquely defined, and also equivalent to the property that the denominators and numerators are of exact degree $(p_1, \ldots, p_d, q)$. The table of approximants is, in this case, called regular or normal. In the sequel, we will restrict ourselves to this regular case, i.e.

$$H_{\mathbf{p},\mathbf{q}} \neq 0 \text{ for all } \mathbf{p}, \mathbf{q}.$$

As in [8] for the Legendre polynomials, the quantities $h^j_{k,i}$ defined by (3) can be computed recursively from the data $f^j_i, i \geq 0, j = 1, \cdots, d$ using the three term recurrence relation for the family $(P_k)_{k\geq 0}$ of orthogonal polynomials

$$xP_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + C_k P_{k-1}(x),\ k \geq 1\ .$$

We obtain the relation

$$h^j_{k,i+1} = r_i h^j_{k,i-1} + s_{ki} h^j_{k-1,i} + t_{ki} h^j_{k,i} + u_{ki} h^j_{k+1,i}, \quad j = 1, \cdots, d\ , \tag{8}$$

with the initializations ($P_0 = 1,\ \mu_k = \|P_k\|_2^2$)

$$h^j_{k,0} = f^j_k,\ h^j_{0,i} = \frac{\mu_i}{\mu_0} f^j_i,\ h^j_{i,-1} = 0,\ j = 1, \cdots, d\ .$$

The quantities $r_i, s_{ki}, t_{ki}, u_{ki}$ are defined from the coefficients of the recurrence relation in the following way

$$r_i = -\frac{C_i}{A_i}, s_{ki} = \frac{C_k}{A_i}\frac{\mu_{k-1}}{\mu_k}, t_{ki} = \frac{B_k - B_i}{A_i}, u_{ki} = \frac{A_k}{A_i}\frac{\mu_{k+1}}{\mu_k},$$

for all values of i and $k \geq 1$.

There are many ways of choosing sequences of simultaneous Frobenius-Padé approximants (SFPA) because we have many free parameters

- the denominator degree $q = q_1 + \cdots + q_d$,
- the numerator degrees $\mathbf{p} = (p_1, p_2, \cdots p_d)$,
- the order of approximation $(p_1 + q_1, \cdots, p_d + q_d)$.

From the point of view of approximating the d functions f^j, it seems natural and interesting to consider the following cases:

- We fix the denominator degree $q = md+k$ and impose the same accuracy to order $p_i+q_i = n$ to the d functions. More precisely, the numerator polynomials will have degrees $\mathbf{p_n} = (p_1, \ldots, p_d)$ with $p_j = n - m - 1, (q_j = m + 1)$ for $j = 1, \cdots, k$, $p_j = n - m, (q_j = m)$ for $j = k + 1, \cdots, d$ and we consider the "vertical sequences" (q fixed) $([\mathbf{p_n}, \mathbf{q}])_{n\in\mathbb{N}}$.
- we impose the same degree for all numerator polynomials: $p_j = p$ and then if $q = md + k$, we define the "regular multi-index" $\mathbf{q} = (q_1, \ldots, q_d)$ by

  $$\begin{array}{lll} q_j = m+1, & n_j = p+m+1, & j = 1, \cdots, k, \\ q_j = m & n_j = p+m, & j = k+1, \cdots, d, \end{array}$$

  and then consider the "diagonal-regular" sequences $([\mathbf{p}, \mathbf{q}])_{q\in\mathbb{N}}$.
- We fix the order of accuracy $\mathbf{n} = (n_1, \cdots, n_d)$ from the number of available coefficients for the expansion of each series f^j and construct a sequence of approximants increasing the degree of the denominator from 0 to $\sum_{j=1}^d n_j$ and decreasing the numerator degrees from $\mathbf{p} = \mathbf{n}$ to $(0, \cdots, 0)$.

From now on we restrict ourselves to the case $\mathbf{d = 2}$ and will develop recursive algorithms to compute sequences of simultaneous Frobenius -Padé approximants.

3 Three term recurrence relations

Based on the determinantal formulas (6) and (7) for the representation of numerators and denominator of the simultaneous Frobenius-Padé approximant and on the Jacobi identity (see formula (9) further on) we can obtain several three term recurrence relations involving two approximants of the same column and one approximant of the previous one. Combining these relations we propose an algorithm that, starting with an approximant of a given column q, enables the computation of any approximant in that column q from suitable approximants of the column $q-1$. We can then compute all of the Frobenius-Padé table.

3.1 Three-term recurrence relations

Let $D[p_1,p_2;q_1,q_2](z)$, $N^j[p_1,p_2;q_1,q_2](z), j=1,2$ be respectively the denominator and numerators of $[p_1,p_2;q_1,q_2]$. By the determinantal representation (6) and (7) in the general case we obtain $(n_j=p_j+q_j, j=1,2; q=q_1+q_2)$

$$D[p_1,p_2;q_1,q_2](z)=\begin{vmatrix} h^1_{p_1+1,0} & h^1_{p_1+1,1} & \cdots & h^1_{p_1+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h^1_{n_1,0} & h^1_{n_1,1} & \cdots & h^1_{n_1,q} \\ h^2_{p_2+1,0} & h^2_{p_2+1,1} & \cdots & h^2_{p_2+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h^2_{n_2,0} & h^2_{n_2,1} & \cdots & h^2_{n_2,q} \\ P_0(z) & P_1(z) & \cdots & P_q(z) \end{vmatrix} = (-1)^{q_1}\begin{vmatrix} 0 & h^1_{p_1+1,0} & \cdots & h^1_{p_1+1,q} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & h^1_{n_1,0} & \cdots & h^1_{n_1,q} \\ 1 & h^1_{n_1+1,0} & \cdots & h^1_{n_1+1,q} \\ 0 & h^2_{p_2+1,0} & \cdots & h^2_{p_2+1,q} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & h^2_{n_2,0} & \cdots & h^2_{n_2,q} \\ 0 & P_0(z)) & \cdots & P_q(z) \end{vmatrix},$$

and the same for the numerators replacing the last row by $(0, \sum_{i=0}^{p_j} h^j_{i,0}P_i(z), \cdots, \sum_{i=0}^{p_j} h^j_{i,q}P_i(z))$, $j=1,2$.

As previously, H is the generalized Hankel determinant, $q\times q$, $q=q_1+q_2$

$$H[p_1,p_2;q_1,q_2]=\begin{vmatrix} h^1_{p_1+1,0} & h^1_{p_1+1,1} & \cdots & h^1_{p_1+1,q-1} \\ \vdots & \vdots & \vdots & \vdots \\ h^1_{n_1,0} & h^1_{n_1,1} & \cdots & h^1_{n_1,q-1} \\ h^2_{p_2+1,0} & h^2_{p_2+1,1} & \cdots & h^2_{p_2+1,q-1} \\ \vdots & \vdots & \vdots & \vdots \\ h^2_{n_2,0} & h^2_{n_2,1} & \cdots & h^2_{n_2,q-1} \end{vmatrix}.$$

Let us recall the **Jacobi's identity** applied to a $(n\times n)$ determinant D:

$$D_{i_1,i_2;j_1,j_2}D = D_{i_1;j_1}D_{i_2;j_2} - D_{i_1;j_2}D_{i_2;j_1} \text{ with } 1\le i_1,i_2,j_1,j_2\le n, \quad (9)$$

where $D_{m;l}$ represents the determinant obtained from D eliminating the row m and the column l.

- $(i_1, i_2; j_1, j_2) = (q+1, q+2; 1, q+2)$

$$\begin{array}{l} D[p_1,p_2;q_1,q_2](z)H[p_1,p_2;q_1+1,q_2-1] = \\ D[p_1,p_2;q_1+1,q_2-1](z)H[p_1,p_2;q_1,q_2] - D[p_1,p_2;q_1,q_2-1](z)H[p_1,p_2;q_1+1,q_2] \end{array}.$$

Interchanging the two functions $f^1 \leftrightarrow f^2$ we get the similar identity

$$\begin{array}{l} D[p_1,p_2;q_1,q_2](z)H[p_1,p_2;q_1+1,q_2-1] = \\ D[p_1,p_2;q_1-1,q_2+1](z)H[p_1,p_2;q_1,q_2] - D[p_1,p_2;q_1-1,q_2](z)H[p_1,p_2;q_1-1,q_2] \end{array}.$$

- $(i_1, i_2; j_1, j_2) = (1, q+2; 1, q+2)$

$$\begin{array}{l} D[p_1,p_2;q_1,q_2](z)H[p_1+1,p_2;q_1,q_2] = \\ D[p_1+1,p_2;q_1,q_2](z)H[p_1,p_2;q_1,q_2] - D[p_1+1,p_2;q_1-1,q_2](z)H[p_1,p_2;q_1+1,q_2] \end{array}.$$

Interchanging the two functions $f^1 \leftrightarrow f^2$ we get the similar identity

$$\begin{array}{l} D[p_1,p_2;q_1,q_2](z)H[p_1,p_2+1;q_1,q_2] = \\ D[p_1,p_2+1;q_1,q_2](z)H[p_1,p_2;q_1,q_2] - D[p_1,p_2+1;q_1,q_2-1](z)H[p_1,p_2;q_1,q_2+1] \end{array}.$$

These identities are also satisfied by the corresponding numerators and so, if as usual we denote by $S[p_1,p_2;q_1,q_2]$ either the denominators or the numerators, the previous relations can be writen:

$$\begin{array}{lll} S[p_1,p_2;q_1,q_2] & = \mu_A^0 S[p_1,p_2;q_1+1,q_2-1] + \mu_A^1 S[p_1,p_2;q_1,q_2-1], & (\mathbf{A}) \\ S[p_1,p_2;q_1,q_2] & = \mu_B^0 S[p_1,p_2;q_1-1,q_2+1] + \mu_B^1 S[p_1,p_2;q_1-1,q_2], & (\mathbf{B}) \\ S[p_1+1,p_2;q_1,q_2] & = \mu_C^0 S[p_1,p_2;q_1,q_2] + \mu_C^1 S[p_1+1,p_2;q_1-1,q_2], & (\mathbf{C}) \\ S[p_1,p_2+1;q_1,q_2] & = \mu_D^0 S[p_1,p_2;q_1,q_2] + \mu_D^1 S[p_1,p_2+1;q_1,q_2-1]. & (\mathbf{D}) \end{array}$$

The quantities $\mu_A^i, \mu_B^i, \mu_C^i, \mu_D^i, \quad i = 0,1$ are immediately obtained from the previous relations as a quotient of generalized Hankel determinants: for instance

$$\begin{array}{lll} \mu_A^0 & = & H[p_1,p_2;q_1,q_2]/H[p_1,p_2;q_1+1,q_2-1] \\ \mu_A^1 & = & -H[p_1,p_2;q_1+1,q_2]/H[p_1,p_2;q_1+1,q_2-1]. \end{array}$$

Let us now see how we can use the 4 relations **(A)** – **(D)** to compute the Frobenius-Padé table.

3.2 Recursive algorithms in the Frobenius-Padé table

Using the same notations as in the previous section let us denote by

$$[p_1 p_2; q_1, q_2] \text{ with } q_1 + q_2 = q, \;\; q \in \mathbb{N}, \tag{10}$$

the qth column of the Frobenius-Padé table.

Suppose we have the expansion of a vector function $\mathbb{F} = (f^1, f^2)$ in the orthogonal polynomial system $(P_k)_{k\geq 0}$, i.e., we know the approximants $[p_1, p_2; 0, 0], p_1, p_2 \in \mathbb{N}$ (the partial sums). Let us now show that the previous relations allow to compute the table of approximants.

- Column 1: in this column we have two types of sequences:
  - $(q_1, q_2) = (1, 0)$ i.e. $([p_1, p_2; 1, 0], p_1, p_2 \in \mathbb{N}$,
  - $(q_1, q_2) = (0, 1)$ i.e. $([p_1, p_2; 0, 1], p_1, p_2 \in \mathbb{N}$.

  ⋆ If we apply identity **(A)** with $(q_1, q_2) = (0, 1)$ we get

  $$S[p_1, p_2; 0, 1] = \mu_A^0 S[p_1, p_2; 1, 0] + \mu_A^1 S[p_1, p_2; 0, 0],$$

  which enables to compute the sequences of type $(0, 1)$ from the sequences of type $(1, 0)$ and approximants of the previous column;

  ⋆ If we apply identity **(B)** with $(q_1, q_2) = (1, 0)$ we get

  $$S[p_1, p_2; 1, 0] = \mu_B^0 S[p_1, p_2; 0, 1] + \mu_B^1 S[p_1, p_2; 0, 0],$$

  which enables to compute the sequences of type $(1, 0)$ from the sequences of type $(0, 1)$ and approximants of the previous column;

  ⋆ If we apply identity **(C)** with $(q_1, q_2) = (1, 0)$ we get

  $$S[p_1 + 1, p_2; 1, 0] = \mu_C^0 S[p_1, p_2; 1, 0] + \mu_C^1 S[p_1 + 1, p_2; 0, 0],$$

  which enables to construct the sequence $([p_1 + l, p_2; 1, 0]), \quad l \in \mathbb{N}$ from the approximant $[p_1, p_2; 1, 0]$ and approximants of column 0;

  ⋆ If we apply identity **(D)** with $(q_1, q_2) = (0, 1)$ we get

  $$S[p_1, p_2 + 1; 0, 1] = \mu_D^0 S[p_1, p_2; 0, 1] + \mu_D^1 S[p_1, p_2 + 1; 0, 0],$$

  which enables to construct the sequence $([p_1, p_2 + l; 0, 1]), \quad l \in \mathbb{N}$ from the approximant $[p_1, p_2; 0, 1]$ and approximants of column 0;

  We conclude that from the partial sums and the knowledge of one approximant in the first column $[p_1, p_2; 1, 0]$ (or $[p_1, p_2; 0, 1]$) we can compute all the approximants

  $$([p_1 + l, p_2 + m; q_1, q_2])_{l,m \in \mathbb{N}}, \ (q_1 + q_2 = 1).$$

  In general,

- Column q: suppose we have computed the approximants in column $q - 1$ and let q_1, q_2 be such that $q_1 + q_2 = q$

  ⋆ from identity **(A)**: $[p_1, p_2; q_1 - 1, q_2 + 1]$ can be obtained from $[p_1, p_2; q_1, q_2]$ and a suitable approximant of column $q - 1$;

  ⋆ from identity **(B)**: $[p_1, p_2; q_1 + 1, q_2 - 1]$ can be obtained from $[p_1, p_2; q_1, q_2]$ and a suitable approximant of column $q - 1$;

$\star$ from identities **(C)** and **(D)**:

$$([p_1+l,p_2;q_1,q_2])_l \text{ and } ([p_1,p_2+l;q_1,q_2])_l$$

from $[p_1,p_2;q_1,q_2]$ and suitable approximants of the previous column. We conclude that is sufficient to know one approximant of column q to compute all the approximants in this column if we know the column $q-1$.

To obtain the first approximant in column q we can:

1. use one of the algorithms developed in the further sections that enables to increase the denominator degree (diagonal or antidiagonal sequences),

2. construct the approximant $[p_1,p_2;q,0]$ (or $[p_1,p_2;0,q]$) which corresponds to the Frobenius-Padé approximant $[p_1,q]=N/D$ of the scalar function f^1 (resp. $[p_2,q]$ of f^2) and the partial sum of f^2D (f^1D) by an algorithm for scalar functions (see [8, 9]).

Computation of the coefficients:
From the way we obtained the identities **(A), (B), (C)** and **(D)** we know that the coefficients $\mu_A^j,\mu_B^j,\mu_C^j,\mu_D^j, j=1,2$ are quotient of generalized Hankel determinants and so they exist if the Frobenius Padé table is normal. For their computation we may proceed in the following two ways:

1. we can develop recursive formulas to the computation of these generalized Hankel determinants using the Jacobi identity;

2. we can obtain their values by imposing the order and degree conditions on the new approximant.

Let us give an example of how to proceed in this second case. We consider the identity:

$$S[p_1,p_2;q_1,q_2]=\mu_A^0 S[p_1,p_2;q_1+1,q_2-1]+\mu_A^1 S[p_1,p_2;q_1,q_2-1], \tag{11}$$

where S is either the numerator or the denominator. We have

$$\begin{aligned} R^j[p_1,p_2;q_1,q_2](z) &= D[p_1,p_2;q_1,q_2](z)f^j(z)-N^j[p_1,p_2;q_1,q_2](z)\\ &= [\mu_A^0 D[p_1,p_2;q_1+1,q_2-1](z)+\mu_A^1 D[p_1,p_2;q_1,q_2-1](z)]f^j(z)\\ &\quad -[\mu_A^0 N^j[p_1,p_2;q_1+1,q_2-1](z)+\mu_A^1 N^j[p_1,p_2;q_1,q_2-1](z)]f^j(z)]\\ &= \mu_A^0\left[D[p_1,p_2;q_1+1,q_2-1](z)f^j(z)-N^j[p_1,p_2;q_1+1,q_2-1](z)\right]\\ &\quad +\mu_A^0\left[D[p_1,p_2;q_1,q_2-1](z)f^j(z)-N^j[p_1,p_2;q_1,q_2-1](z)\right]\\ &= \mu_A^0 R^j D[p_1,p_2;q_1+1,q_2-1](z)+\mu_A^1 R^j[p_1,p_2;q_1,q_2-1](z) \quad (j=1,2) \end{aligned}$$

As

$$R^j[p_1,p_2;q_1,q_2](z)=\sum_{i\geq p_j+q_j+1} e_i^j[p_1,p_2;q_1,q_2]P_i(z),$$

from the numerators and denominator degrees and accuracy to order conditions of the approximants involved, we can easily see that this leads to one condition only to have the right order:

$$\mu_A^0 e_{p_2+q_2}^2[p_1,p_2;q_1+1,q_2-1]+\mu_A^1 e_{p_2+q_2}^2[p_1,p_2;q_1,q_2-1]=0. \tag{12}$$

As the numerators and denominator of the approximant are defined apart from a constant factor, we can choose the normalization corresponding to a monic denominator, which is equivalent to set $\mu_A^0 = 1$. Then we obtain μ_A^1 by solving (12) if $e^2_{p_2+q_2}[p_1, p_2; q_1, q_2 - 1] \neq 0$ which is the case if the table is normal. In fact, with $n_j = p_j + q_j$ and α the normalizing constant for $D[p_1, p_2; q_1, q_2 - 1](z)$,

$$\begin{array}{rcl} R^j[p_1, p_2; q_1, q_2 - 1](z) & = & D[p_1, p_2; q_1, q_2 - 1](z) f^j(z) - N^j[p_1, p_2; q_1, q_2 - 1](z) \\ & = & \sum_{l=0}^{\infty} \sum_{i=0}^{q-1} b_i h^j_{li} P_l(z) - \sum_{l=0}^{p_j} a^j_l P_l(z) = \\ & = & \left(\sum_{i=0}^{q-1} b_i h^j_{n_j+1,i} \right) P_{n_j+1}(z) + \cdots \\ & = & \left\{ \begin{array}{ll} \alpha H(p_1, p_2; q_1 + 1, q_2) & \text{if } j = 1 \\ \alpha H(p_1, p_2; q_1, q_2 + 1) & \text{if } j = 2 \end{array} \right\} P_{n_j+1}(z) + \cdots \end{array}$$

(here b_i are the coefficients of $D[p_1, p_2; q_1, q_2 - 1]$ and a^j_i are the coefficients of $N^j[p_1, p_2; q_1, q_2 - 1]$, $\quad j = 1, 2$.

Remark: The relation (11) is also satisfied by the errors $R^j[p_1, p_2; q_1, q_2](z)$ and so enables us to compute recursively the coefficients of the expansion in $(P_k)_{k \geq 0}$ of the error of the new approximant. The number of coefficients needed depends on the required sequence of approximants.

4 A Kronecker-type algorithm

The idea is to consider sequences having a fixed order of accuracy. In classical scalar approximation, this gives rise to the "antidiagonal" sequences.

We fix $n = n_1 = n_2$ and compute the sequence of approximants

$$\mathrm{AFP}_k(z) = \mathbb{N}\mathrm{um}_k(z)/D_k(z), \quad k = 1, 2, \cdots, 2n, \tag{13}$$

satisfying the accuracy to order condition, i.e. the following formula for the remainder sequences

$$R^j_k(z) = D_k(z) f^j(z) - N^j_k(z) = \mathcal{O}(P_{n+1}), \ j = 1, 2; \quad k = 1, 2, \cdots, 2n, \tag{14}$$

where

$$\left\{ \begin{array}{rcl} \deg(N^1_k(z)) & = & n - [(k+1)/2], \\ \deg(N^2_k(z)) & = & n - [k/2] \quad k = 0, \cdots, 2n, \\ \deg(D_k(z)) & = & k. \end{array} \right.$$

This means that the sequence of degrees of the numerators and denominators is

$$(p_1, p_2, q) = (n, n, 0), (n-1, n, 1), (n-1, n-1, 2), (n-2, n-1, 3), \cdots, (0, 1, 2n-1), (0, 0, 2n)$$

We search for a recurrence relation verified by the remainder terms, and simultaneously by the numerators and denominators, and of the form

$$S_{k+1}(z) = \sum_{i=1}^{l} c_i(z) S_{k+1-i}(z), \ \text{with } c_i(z) = \alpha_i + \beta_i z, \tag{15}$$

(two unknowns for each value of i, so $2l$ unknowns). The coefficients $c_i(z)$ that we are going to compute obviously depend on k but here we dropped the index to simplify notations. Here $S(z)$ represents $N^1(z), N^2(z)$ or $D(z)$. The linear terms $c_i(z)$ are computed from the accuracy to order conditions and the conditions on the degrees of the numerators polynomials.

- Let us denote by $(e^j_{k,i})(j=1,2)$ the coefficients of the expansion of the remainder:

$$R^j_k(z) = D_k(z)f^j(z) - N^j_k(z) = \sum_{i=n+1}^{\infty} e^j_{k,i}P_i(z), \quad j=1,2. \tag{16}$$

  Then, using the recurrence relation for orthogonal polynomials, we get

$$R^j_{k+1}(z) = \sum_{i=1}^{l} c_i(z)R^j_{k+1-i}(z) = e^j_{k+1,n}P_n(z) + \mathcal{O}(P_{n+1}),\ j=1,2\ . \tag{17}$$

  with $e^j_{k+1,n} = C_{n+1}\sum_{i=1}^{l}\beta_i e^j_{k+1-i,n+1}$, for $j=1,2$. So the accuracy to order principle supplies two conditions

$$e^j_{k+1,n} = 0, \quad j=1,2 \Leftrightarrow \sum_{i=1}^{l}\beta_i e^j_{k+1-i,n+1} = 0, \quad j=1,2. \tag{18}$$

  The quantities $e^j_{k+1,h}$, $h \geq n+1$ can be computed recursively using (17) as we will see below.

- Let us now denote by $r^j_{i,k}$ $(j=1,2)$ the coefficients of the numerator polynomials written in the orthogonal basis $(P_i)_{i\geq 0}$. As the table has been supposed "regular", they are of exact degree

$$N^j_k(z) = \sum_{i=0}^{d_k} r^j_{i,k}P_i(z),\ r^j_{d_k,k} \neq 0,\ j=1,2. \tag{19}$$

  Then by the recurrence relation satisfied by the orthogonal polynomials $P_k(z)$ we immediately get the coefficients $s^j_{i,k}$ in

$$zN^j_k(z) = \sum_{i=0}^{d_k+1} s^j_{i,k}P_i(z),\ s^j_{d_k+1,k} \neq 0\ j=1,2. \tag{20}$$

  Using the recurrence relation (15) we can write down the equations that impose the degrees of numerator and denominator polynomials and determine the smallest value of l (number of terms in the recurrence) for which the number of conditions is less or equal the number of parameters. This occurs for $l=5$ as we will see below. Let us explain how to proceed. We consider 2 cases:

  1. <u>$k+1$ odd</u>: We have

$$\begin{array}{rcl} \deg(N^1_{k+1}) & = & n - \left[\frac{k+2}{2}\right] = k_1, \\ \deg(N^2_{k+1}) & = & n - \left[\frac{k+1}{2}\right] = k_1+1, \end{array} \tag{21}$$

and we summarize in the next table the degrees for the other polynomials involved in the recurrence relation:

$k+1-j$	$\deg(c_j(z)N^1_{k-j+1}(z))$	$\deg(c_j(z)N^2_{k-j+1}(z))$	$\deg(D^1_{k-j+1}(z))$
k	k_1+2	k_1+2	k
$k-1$	k_1+2	k_1+3	$k-1$
$k-2$	k_1+3	k_1+3	$k-2$
$k-3$	k_1+3	k_1+4	$k-3$
$k-4$	k_1+4	k_1+4	$k-4$

.

By using relation (15) we obtain for the right hand side (r.h.s.) and for the left hand side (l.h.s.) the following degrees:

	r.h.s.	l.h.s.
deg N^1	k_1+4	k_1
deg N^2	k_1+4	k_1+1
deg D	$k+1$ if $\beta_1 \neq 0$	$k+1$

.

So to get the good polynomial degrees we have to impose 7 degree conditions. If we write them down explicitly we get:

N^1	k_1+4	$\beta_5=0$
N^2	k_1+4	$\beta_4=0$
N^1	k_1+3	$\alpha_5 r^1_{k_1+3,k-4}+\beta_3 s^1_{k_1+3,k-2}=0$
N^2	k_1+3	$\alpha_5 r^2_{k_1+3,k-4}+\alpha_4 r^2_{k_1+3,k-3}+\beta_3 s^2_{k_1+3,k-2}+\beta_2 s^2_{k_1+3,k-1}=0$
N^1	k_1+2	$\sum_{i=3}^5 \alpha_i r^1_{k_1+2,k-i+1}+\sum_{i=1}^3 \beta_i s^1_{k_1+2,k-i+1}=0$
N^2	k_1+2	$\sum_{i=2}^5 \alpha_i r^2_{k_1+2,k-i+1}+\sum_{i=1}^3 \beta_i s^2_{k_1+2,k-i+1}=0$
N^1	k_1+1	$\sum_{i=1}^5 \alpha_i r^1_{k_1+1,k-i+1}+\sum_{i=1}^3 \beta_i s^1_{k_1+1,k-i+1}=0$

. (22)

In this table, for each row the first two columns indicate that the equation is obtained by eliminating the term of order k_1+j in the right hand side of (15) applied to N^i. We have then 7 degree conditions and 2 order conditions for 10 unknowns and so, this homogeneous linear system always has a solution.

2. $k+1$ even: proceeding in the same way as in the previous case, the conditions on the numerator degrees are equivalent to the following system of equations :

N^2	k_1+4	$\beta_4=0$
N^1	k_1+3	$\beta_5=0$
N^2	k_1+3	$\alpha_5 r^2_{k_1+3,k-4}+\beta_3 s^2_{k_1+3,k-2}=0$
N^1	k_1+2	$\alpha_5 r^1_{k_1+2,k-4}+\alpha_4 r^1_{k_1+2,k-3}++\beta_3 s^1_{k_1+2,k-2}+\beta_2 s^1_{k_1+2,k-1}=0$
N^2	k_1+2	$\sum_{i=3}^5 \alpha_i r^2_{k_1+2,k-i+1}+\sum_{i=1}^3 \beta_i s^2_{k_1+2,k-i+1}=0$
N^1	k_1+1	$\sum_{i=2}^5 \alpha_i r^1_{k_1+1,k-i+1}\sum_{i=1}^3 \beta_i s^1_{k_1+1,k-i+1}=0$
N^2	k_1+1	$\sum_{i=1}^5 \alpha_i r^2_{k_1+1,k-i+1}+\sum_{i=1}^3 \beta_i s^2_{k_1+1,k-i+1}=0$

. (23)

If we regroup the order and degree conditions we obtain the homogeneous linear system of 7 equations and 8 unknowns

$$\begin{pmatrix} - & \times & \times & & & & & \\ - & \times & \times & & & & & \\ & & - & + & & & & \\ & - & - & - & + & & & \\ - & - & - & - & - & + & & \\ - & - & - & - & - & - & + & \\ - & - & - & - & - & - & - & + \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \alpha_5 \\ \alpha_4 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{pmatrix}.$$

We denote by "+" the terms which can be shown to be non zero under normality conditions, by "−" the terms that can be nonzero and by "× " the terms on which we will impose a condition in order to get a solution for the system (see below) .
The first two equations correspond to the accuracy to order conditions (18). This system has always a nontrivial solution. Let us suppose that the first k approximants of the sequence are "regular", that means they have the exact numerator and denominator degrees. We look for the $k+1$ st approximant to have a denominator of degree $k+1$ and so we need to have $\beta_1 \neq 0$. This is the case if the first 2 equations have a solution with $\beta_1 = 1$, that is, if

$$\det \begin{pmatrix} e^1_{k-1,n+1} & e^1_{k-2,n+1} \\ e^2_{k-1,n+1} & e^2_{k-2,n+1} \end{pmatrix} \neq 0. \tag{24}$$

In this case, we get β_2, β_3 by solving

$$e^j_{k-1,n+1}\beta_2 + e^j_{k-2,n+1}\beta_3 = -e^j_{k,n+1}, \quad j = 1, 2, \tag{25}$$

and we then obtain a lower triangular system for the values of $\alpha_i, i = 1, \cdots, 5$. We easily see that the diagonal coefficients are nonzero because they correspond to the coefficient of highest degree of the numerators of the previous approximants (for instance in the case of $k+1$ odd, the diagonal terms are $(r^1_{k_1+3,k-4}, r^2_{k_1+3,k-3}, r^1_{k_1+2,k-2}, r^2_{k_1+2,k-1}, r^1_{k_1+1,k})$). We summarize these results in the following theorem.

Theorem 1 *Let $\mathbb{F}(z) = (f^1(z), f^2(z))$ be a vector function given by its expansion in an orthogonal series and let us consider the anti-diagonal sequence of simultaneous Frobenious-Padé approximants $(AFP_k)_{k\geq 0}$ defined by (13). We suppose this sequence to be normal. Then if*

$$det \begin{pmatrix} e^1_{k-1,n+1} & e^1_{k-2,n+1} \\ e^2_{k-1,n+1} & e^2_{k-2,n+1} \end{pmatrix} \neq 0 \text{ for all } k,$$

(the quantities $e^j_{m,l}$ are defined in (16)) the numerators and denominators of these approximants can be computed by a six term recurrence relation of the form

$$S_{k+1}(z) = (\alpha^k_1 + \beta^k_1 z)S_k(z) + (\alpha^k_2 + \beta^k_2 z)S_{k-1}(z) + (\alpha^k_3 + \beta^k_3 z)S_{k-2}(z) + \alpha^k_4 S_{k-3}(z) + \alpha^k_5 S_{k-4}(z) \tag{26}$$

where $\beta_1^k := 1$ *and the other coefficients are computed by (22,23) and (25). The* $C_k^j(z)$ *being the* k*-th partial sum of the series expansion, the initializations are*

$$\left\{\begin{array}{ll} D_{-j}(z) = 1 & j = 1,\cdots,4 \\ N_{-j}^1 = & \left\{\begin{array}{lll} C_n^1(z) & \text{if} & j=1 \\ C_{n+1}^1(z) & \text{if} & j=2,3 \\ C_{n+2}^1(z) & \text{if} & j=4 \end{array}\right. \\ N_{-j}^2 = & \left\{\begin{array}{lll} C_{n+1}^2(z) & \text{if} & j=1,2 \\ C_{n+2}^2(z) & \text{if} & j=3,4 \end{array}\right. \end{array}\right. .$$

Recursive computation of $e_{k+1,l}^j$: From the recurrence relation (26) we obtain

$$\begin{array}{ll} D_{k+1}(z)f^j(z) - N_{k+1}^j(z) & = \sum_{l=n+1}^{\infty} e_{k+1,l}^j P_l(z) \\ & = \sum_{l=n+1}^{\infty} \left(\sum_{i=1}^{5} \alpha_i e_{k-i+1,l}^j\right) P_l(z) + \sum_{l=n+1}^{\infty} \left(\sum_{i=1}^{3} \beta_i e_{k-i+1,l}^j\right) zP_l(z) \\ & = \sum_{l=n+1}^{\infty} \theta_l^j P_l(z) + \sum_{l=n+1}^{\infty} \tau_l^j zP_l(z), \quad j=1,2. \end{array} \tag{27}$$

From the three term recurrence relation for the orthogonal system $(P_k)_{k\geq 0}$ we know that

$$zP_l(z) = A_l P_{l+1}(z) + B_l P_l(z) + C_l P_{l-1}(z),$$

and finally the recurrence relation for the quantities $(e_{k,l}^j)$ is

$$e_{k+1,l}^j = \theta_l^j + A_{l-1}\tau_{l-1}^j + B_l\tau_l^j + C_{l+1}^j\tau_{l+1}^j, \quad l \geq n+1, j=1,2. \tag{28}$$

with τ_l^j and θ_l^j defined in (27).

How many quantities do we need to compute at each step? Suppose we want to compute the approximants $\text{AFP}_k(z)$ for $k = 1,\cdots,2n$.

- we deduce from (25) that, for a fixed k, we need to know $e_{k,n+1}^j, e_{k-1,n+1}^j, e_{k-2,n+1}^j, j=1,2$ to compute the approximant $\text{AFP}_{k+1}(z)$. So, to compute the sequence $\text{AFP}_k(z)$ for $k = 1,\cdots,2n$, we will need the quantities $e_{i,n+1}^j, j=1,2;\;\; i=0,\cdots,2n-1$;

- from the recurrence relation (28) we deduce that to compute $e_{k+1,l}^j$ we need the quantities

$$\left\{\begin{array}{l} e_{k-4,l}^j \\ e_{k-3,l}^j \\ e_{k-2,l-1+i}^j, i=0,1,2 \qquad (j=1,2). \\ e_{k-1,l-1+i}^j, i=0,1,2 \\ e_{k,l-1+i}^j, i=0,1,2 \end{array}\right.$$

Then we easily obtain that we need to compute the quantities

$$e_{i,l}^j \text{ for } l=n+1,\cdots,3n-i, \quad i=1,\cdots,2n-1 \quad (j=1,2).$$

5 Computation of diagonal sequences

The idea is to increase accuracy, increasing simultaneously the degrees of the numerators and denominators. In this part, from one term to the next one of the sequence, we have one parameter more and gain one order of accuracy for one of the components.

Let us fix p_1 and p_2 and consider the following sequence of approximants

$$[p_1, p_2; 0, 0], [p_1, p_2; 1, 0], [p_1 + 1, p_2; 1, 0], [p_1 + 1, p_2; 1, 1], [p_1 + 1, p_2 + 1; 1, 1], \cdots$$

or, in general, the sequence $(\mathrm{AFP}_m)_{m \geq 0}$ defined by:

$$\begin{aligned}(\mathrm{AFP}_{4k+1}, \mathrm{AFP}_{4k+2}, \mathrm{AFP}_{4k+3}, \mathrm{AFP}_{4k+4}) = \\ = \quad & ([p_1 + k, p_2 + k; k + 1, k], \\ & [p_1 + k + 1, p_2 + k; k + 1, k], \\ & [p_1 + k + 1, p_2 + k; k + 1, k + 1], \\ & [p_1 + (k + 1), p_2 + (k + 1); k + 1, k + 1])\end{aligned} \tag{29}$$

We remark that from step k to step $k + 1$ we need to compute 4 terms and:

- we increase the denominator degree by 2
- we increase numerators degree by 1 and the accuracy to order of each function by 2;

Theorem 2 *We consider the sequence of Frobenius-Padé approximants defined by (29). This sequence can be computed by a six-term recurrence relation of the form*

$$S_{m+1}(z) = \alpha_1^m S_m(z) + (\alpha_2^m + \beta_2^m z) S_{m-1}(z) + (\alpha_3^m + \beta_3^m z) S_{m-2}(z) \tag{30}$$

$$+ (\alpha_4^m + \beta_4^m z) S_{m-3}(z) + \alpha_5^m S_{m-4}(z)$$

where α_j^m, β_j^m are constants and $S_i(z)$ represents either the numerators or the denominators of the approximants.
This relation is minimal with respect to the number of terms.

Proof: from the definition (29) of the sequence of approximants $(\mathrm{AFP}_{m \geq 0})$ we can immediately deduce that we have to consider 2 different situations
Case 1:
obtain a recurrence relation corresponding to increasing the degree of numerator and the order of convergence for the first component;
Case 2:
obtain a recurrence relation corresponding to increasing the degree of denominator and the order of convergence for the first component. With respect to the second component, the relations are obtained trivially from the first ones by interchanging the roles of f^1 and f^2. The proof is done in the following way: we try to obtain the new term $S_{m+1}(z)$ of the sequence starting from the last computed term $S_m(z)$ multiplied by a linear factor $(\alpha_1^m + \beta_1^m z)$ and add successively

previous terms of the sequence multiplied by linear factors until the number of unknowns is greater or equal to the number of accuracy to order and degree conditions. We then write the system of equations corresponding to these conditons and show that it has a solution. To simplify notations and without loss of generarity we set $k = 0$ in (29)(the general case follows immediately). We introduce some notation: let $S[p_1, p_2; q_1, q_2]$ represent either the numerators N^1, N^2, the denominator D or the error term $R^i = Df^i - N^i, i = 1, 2$ of the approximant $[p_1, p_2; q_1, q_2]$. Let us obtain the two recurrence relations mentioned before.

Case 1:

In the following table we write down the degrees of polynomials, the degrees of their product by a linear factor and orders of errors for the corresponding approximants, for the indices appearing in the recurrence relation (30) with $k = 0$.

	$\deg(N^1)$	$\mathrm{ord}(Df^1 - N^1)$	$\deg(N^2)$	$\mathrm{ord}(Df^2 - N^2)$	$\deg(D)$
$S[p_1+2, p_2+1; 2, 1]$	p_1+2	p_1+5	p_2+1	p_2+3	3
$S[p_1+1, p_2+1; 2, 1]$	p_1+1	p_1+4	p_2+1	p_2+3	3
$\times(\alpha_1+\beta_1 z)$	p_1+2	p_1+3	p_2+2	p_2+2	4
$S[p_1+1, p_2+1; 1, 1]$	p_1+1	p_1+3	p_2+1	p_2+3	2
$\times(\alpha_2+\beta_2 z)$	p_1+2	p_1+2	p_2+2	p_2+2	3
$S[p_1+1, p_2; 1, 1]$	p_1+1	p_1+3	p_2	p_2+2	2
$\times(\alpha_3+\beta_3 z)$	p_1+2	p_1+2	p_2+1	p_2+1	3
$S[p_1+1, p_2; 1, 0]$	p_1+1	p_1+3	p_2	p_2+1	1
$\times(\alpha_4+\beta_4 z)$	p_1+2	p_1+2	p_2+1	p_2	2
$S[p_1, p_2; 1, 0]$	p_1	p_1+2	p_2	p_2+1	1
$\times(\alpha_5+\beta_5 z)$	p_1+1	p_1+1	p_2+1	p_2	2

.

We now write the linear combination appearing in (30) and obtain from the previous table

S	left-hand side what we get	right-hand side what we want to get
$S = N^1$	$\deg(N^1) = p_1 + 2$	$\deg(N^1) = p_1 + 2$
$S = N^2$	$\deg(N^2) = p_2 + 2$	$\deg(N^2) = p_1 + 1$
$S = D$	$\deg(D) = 4$	$\deg(D) = 3$
$S = R^1$	$\mathrm{ord}(R^1) = p_1 + 1$	$\mathrm{ord}(R^1) = p_1 + 5$
$S = R^2$	$\mathrm{ord}(R^2) = p_2$	$\mathrm{ord}(R^2) = p_2 + 3$

.

We immediately conclude that we have to impose 9 conditions to obtain the good degrees and orders. These conditions are linear equations in the unknowns α_i, β_i with second member equal zero. In the following table we summarize each condition. Each row corresponds to one condition: in the first column we note the type of condition (order or degree), in the second column we write the order or degree of the coefficient we are eliminating and in the third one the we write down the unknowns appearing in the condition (i.e. for which the coefficient is non zero)

$\deg(D)$	4	β_1									
$\deg(N^2)$	p_2+2	β_1	β_2								
$\operatorname{ord}(R^1)$	p_1+1					β_5					
$\operatorname{ord}(R^2)$	p_2				β_4	β_5					
$\operatorname{ord}(R^1)$	p_1+2		β_2	β_3	β_4	β_5	α_5				
$\operatorname{ord}(R^2)$	p_2+1			β_3	β_4	β_5	α_5	α_4			
$\operatorname{ord}(R^2)$	p_2+2	β_1	β_2	β_3	β_4	β_5	α_5	α_4	α_3		
$\operatorname{ord}(R^1)$	p_1+3	β_1	β_2	β_3	β_4	β_5	α_5	α_4	α_3	α_2	
$\operatorname{ord}(R^1)$	p_1+4	β_1	β_2	β_3	β_4	β_5	α_5	α_4	α_3	α_2	α_1

.

This table shows us that the order and degree conditions are equivalent to an homogeneous system of 9 equations on 10 unknowns and then there is always a non-trivial solution. From the first four conditions we obtain immediately that $\beta_1 = \beta_2 = \beta_4 = \beta_5 = 0$. For the normality of the table we need to have $\beta_3 \neq 0$ and if we fix $\beta_3 = 1$ we obtain a lower triangular system giving the recurrence coefficients.

Case 2

Let us now consider the second case (that is, the new term corresponds to increasing the denominator degree) and proceeding like in Case 1 we get the following tables

	$\deg(N^1)$	$\operatorname{ord}(Df^1 - N^1)$	$\deg(N^2)$	$\operatorname{ord}(Df^2 - N^2)$	$\deg(D)$
$S[p_1+1, p_2+1, 2, 1]$	p_1+1	p_1+4	p_2+1	p_2+3	3
$S[p_1+1, p_2+1, 1, 1]$	p_1+1	p_1+3	p_2+1	p_2+3	2
$\times(\alpha_1+\beta_1 z)$	p_1+2	p_1+2	p_2+2	p_2+2	3
$S[p_1+1, p_2, 1, 1]$	p_1+1	p_1+3	p_2	p_2+2	2
$\times(\alpha_2+\beta_2 z)$	p_1+2	p_1+2	p_2+1	p_2+1	3
$S[p_1+1, p_2, 1, 0]$	p_1+1	p_1+3	p_2	p_2+1	1
$\times(\alpha_3+\beta_3 z)$	p_1+2	p_1+2	p_2+1	p_2	2
$S[p_1 p_2, 1, 0]$	p_1	p_1+2	p_2	p_2+1	1
$\times(\alpha_4+\beta_4 z)$	p_1+1	p_1+1	p_2+1	p_2	2
$S[p_1, p_2, 0, 0]$	p_1	p_1+1	p_2	p_2+1	0
$\times(\alpha_5+\beta_5 z)$	p_1+1	p_1	p_2+1	p_2	1

.

We now write the linear combination appearing in (30) and obtain from the previous table

S	left-hand side what we get	right-hand side what we want to get
$S = N^1$	$\deg(N^1) = p_1+2$	$\deg(N^1) = p_1+1$
$S = N^2$	$\deg(N^2) = p_2+2$	$\deg(N^2) = p_1+1$
$S = D$	$\deg(D) = 3$	$\deg(D) = 3$
$S = R^1$	$\operatorname{ord}(R^1) = p_1$	$\operatorname{ord}(R^1) = p_1+4$
$S = R^2$	$\operatorname{ord}(R^2) = p_2$	$\operatorname{ord}(R^2) = p_2+3$

.

The conditions are linear equations in the unknowns α_i, β_i with second member equal zero. More precisely we get from the previous table:

$\deg(N^2)$	p_2+2	β_1									
$\mathrm{ord}(R^1)$	p_1					β_5					
$\deg(D)$	3	β_1	β_2								
$\deg(N^1)$	p_1+2	β_1	β_2	β_3							
$\mathrm{ord}(R^2)$	p_2			β_3	β_4	β_5					
$\mathrm{ord}(R^1)$	p_1+1				β_4	β_5	α_5				
$\mathrm{ord}(R^1)$	p_1+2	β_1	β_2	β_3	β_4	β_5	α_5	α_4			
$\mathrm{ord}(R^2)$	p_2+1		β_2	β_3	β_4	β_5	α_5	α_4	α_3		
$\mathrm{ord}(R^2)$	p_2+2	β_1	β_2	β_3	β_4	β_5	α_5	α_4	α_3	α_2	
$\mathrm{ord}(R^1)$	p_2+3	β_1	β_2	β_3	β_4	β_5	α_5	α_4	α_3	α_2	α_1

This table shows us that the order and degree conditions are equivalent to an homogeneous system of 9 equations on 10 unknowns and then there is always a non-trivial solution. From the first two conditions we immediately get $\beta_1 = \beta_5 = 0$ and, from the normality of the table and the third equation, we get $\beta_2 \neq 0$. If we fix $\beta_2 = 1$ then the other coefficients follow from solving a triangular system.

6 Regular indices, diagonal regular approximants

6.1 Definitions and notations

We will follow ideas of "regular approximation" and regular indices as developed in ([10], [14]) for Padé and Padé-Hermite approximants. The idea is to improve quickly the accuracy and so we are only interested in Frobenius-Padé approximants that can be called diagonal-regular, i.e. the degrees are identical for the different numerators, and the approximation is regularly distributed, i.e. q_1, q_2 is the "regular" multi-index associated to q : ($[q/2]$ is the integer part of $q/2$ and $q = q_1 + q_2$.)

$$p_1 = p_2, \qquad \begin{array}{l} q_1 = [q/2] + \varepsilon_q, \;\; \varepsilon_q = 1 \text{ if } q \text{ odd }, 0 \text{ if } q \;\text{ even} \\ q_2 = [q/2] \end{array}$$

The same could be done with paradiagonal sequences, i.e. $p_1 \neq p_2$ but the difference $p_1 - p_2$ being constant. This brings no problem, and, for sake of simplicity, we stay with the same $p = p_1 = p_2$.

These approximants can be indicated using two indices only, and are displayed in a 2-dimensional array similar to the Padé table, the columns are for q constant and the rows for p constant. The notation are as follows

$$\begin{array}{l} [p,q] = \mathbb{N}\mathrm{um}_{p,q}/D_{p,q}, \\ \left\{ \begin{array}{l} \mathbb{N}\mathrm{um}_{p,q} = (N^1_{p,q}, N^2_{p,q}),\; deg(N^i_{p,q}) = p,\; i = 1,2,\; \deg(D_{p,q}) = q, \\ \mathbb{R}\mathrm{em}_{p,q} = \mathbb{F}D_{p,q} - \mathbb{N}\mathrm{um}_{p,q} = (O(P_{p+[q/2]+\varepsilon_q+1}), O(P_{p+[q/2]+1}))\;. \end{array} \right. \end{array} \tag{31}$$

Let us expand these notations: for $q = 2q'$

$$
\begin{array}{l}
\mathbb{R}\mathrm{em}_{p,q} = \sum_{k\geq p+q'+1} \mathbf{e}_{p,q;k} P_k,\ \ \mathbf{e}_{p,q;k} = (e^1_{p,q;k}, e^2_{p,q;k})^t = (\langle R^i_{p,q}, P_k\rangle)^t_{i=1,2},\\
\mathbb{R}\mathrm{em}_{p,q+1} = \sum_{k\geq p+q'+1} \mathbf{e}_{p,q+1;k} P_k,\ \ \mathbf{e}_{p,q+1;p+q'+1} = (0, e^2_{p,q+1;p+q'+1})^t,\\
z\ \mathbb{R}\mathrm{em}_{p,q} = \sum_{k\geq p+q'} \tilde{\mathbf{e}}_{p,q;k} P_k,\\
\tilde{\mathbf{e}}_{p,q;k} = A_{k-1}\mathbf{e}_{p,q;k-1} + B_k \mathbf{e}_{p,q;k} + C_{k+1}\mathbf{e}_{p,q;k+1}\ ,
\end{array}
$$

in the first formula the scalar product is the product defining the orthogonal sequence $(P_k)_{k\geq 0}$; $\mathbf{e}_{p,q;k} = 0$ if k is outside the specified range (31) and, as $q+1$ is odd, the first non zero $\mathbf{e}_{p,q+1;k}$ has the special form given in the second formula ; the A_k, B_k, C_k are the coefficients of the recurrence formula for the P_k: $zP_k = A_k P_{k+1} + B_k P_k + C_k P_{k-1}$.

Moreover, if the table is regular in the sense that all Hankel determinants are non zero, then the first terms of each expansion ($\mathbb{R}\mathrm{em}_{p,q}, z\mathbb{R}\mathrm{em}_{p,q}$) are non zero, as follows: with $q = 2q'$

$$
\left\{\begin{array}{l}
\langle R^1_{p,q}, P_k\rangle = 0,\ \ k = 0,\dots,p+q'\\
\langle R^2_{p,q}, P_k\rangle = 0,\ \ k = 0,\dots,p+q'
\end{array}\right.,\quad \langle R^1_{p,q}, P_{p+q'+1}\rangle = e^1_{p,q;p+q'+1} = H_{p,q+1}/H_{p,q}, \tag{32}
$$

$$
\left\{\begin{array}{l}
\langle R^1_{p,q-1}, P_k\rangle = 0, k = 0,\dots,p+q'\\
\langle R^2_{p,q-1}, P_k\rangle = 0, k = 0, p+q'-1
\end{array}\right.,\langle R^2_{p,q-1}, P_{p+q'}\rangle = e^2_{p,q-1;p+q'} = H_{p,q}/H_{p,q-1}, \tag{33}
$$

and so the first terms $\tilde{e}^1_{p,q;p+q'} = C_{p+q'+1} e^1_{p,q;p+q'+1}$ of $z\mathbb{R}\mathrm{em}_{p,q}$ and
$\tilde{e}^2_{p,q-1;p+q'} = C_{p+q'+1} e^2_{p,q-1;p+q'+1}$ of $z\mathbb{R}\mathrm{em}_{p,q-1}$ are also non zero.

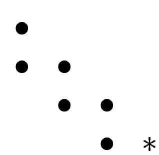
6.2 Diagonal-staircase and diagonal relations

We will look for two types of recurrence relations between the $\mathbb{R}\mathrm{em}_{p,q}$, first two descending staircase relations, second a diagonal relation. These relations will also be satisfied by the corresponding numerators and denominators so, like before, we will denote by $S_{p,q}$ either $N^1_{p,q}, N^2_{p,q}, D_{p,q}, R^1_{p,q}$ or $R^2_{p,q}$.
Let us now consider the descending staircases, there are two cases, ending by a horizontal or by a vertical line. The sequence considered is, in both cases, defined by the following indices: $(p,q), (p+1,q), (p+1,q+1), (p+2,q+1), \dots$

Theorem 3 *For all* (p,q)*, there exist polynomials* $\phi_j(z) = \alpha_j + \beta_j z, j = 1,\dots,6,$ *of degree respectively* $0,1,1,1,1,0$*, such that the following relation holds*

$$
\begin{array}{rl}
S_{p+1,q+1}(z) = & \phi_1(z)S_{p+1,q}(z) + \phi_2(z)S_{p,q}(z) + \phi_3(z)S_{p,q-1}(z) +\\
& \phi_4(z)S_{p-1,q-1}(z) + \phi_5(z)S_{p-1,q-2}(z) + \phi_6(z)S_{p-2,q-2}(z)\ .
\end{array} \tag{34}
$$

The coefficients of the polynomials $\phi_j(z)$ *are computed by a linear system, with a matrix* A *'nearly' Hessenberg, given in formula (35).*
The relation involves the following approximants



Proof: We have to prove an identity of the form $\mathbb{F}$ times a polynomial minus a polynomial is equal to zero. So if we suppose that one (at least) of the components of $\mathbb{F}$ is not a rational function, this is equivalent to say that the two polynomials are zero.

How to imagine the existence of a recurrence relation as (34)? The relation is written to avoid "degree conditions" for the denominators, i.e. the degree of the terms on the right hand side are of degree smaller than $q+1$, which means that the degrees of the polynomials $\Phi_0, \Phi_1, ...$ is at most $1, 2, ...$ Then we consider the relation for the residuals $\mathbb{R}\mathrm{em}_{p+k,q+k}$, we count the number of accuracy conditions (or orthogonality conditions) to be satistied by the right hand side (let us say m), and the number of unknowns coefficients of the polynomials $\Phi_0, \Phi_1, ...$, let us say m'. When $m = m' - 1$, we have a homogeous linear system, one unknown more than the numlber of equations, so we are sure to have a solution to the system.

Let us consider identity (34) for the denominators

$$D_{p+1,q+1}(z) = \phi_1(z) D_{p+1,q}(z) + \phi_2(z) D_{p,q}(z) + \phi_3(z) D_{p,q-1}(z) + \dots ,$$

as the degree of $D_{i,j}$ is j, identifying the degrees gives the leading coefficient of $\phi_2(z)$.

The approximation order and the respective degrees define the approximants up to multiplication by a scalar. So we consider the expression on the right hand side of (34) for the error terms $\mathbb{R}\mathrm{em}_{p,q}$ and write the conditions of orthogonality satisfied by the left hand side. If this is possible, it means that the right hand side is proportional to the left one. In short, this means that we must obtain m equations and m' unknowns, satisfying $m' = m + 1$. Then we always have a solution. The condition on the degrees, already mentioned, gives the last equation and the unicity of the solution.

Let us write the proof for the case $q = 2q'$, then $\mathbb{R}\mathrm{em}_{p+1,q+1} = (O(P_{p+q'+2+1}), O(P_{p+q'+1+1}))$. Let us consider the right hand side. We get, detailed for the second term $(\alpha_2 + \beta_2 z)\mathbb{R}\mathrm{em}_{p,q}$, with $q = 2q'$

$$\begin{aligned} (\alpha_2 + \beta_2 z)\mathbb{R}\mathrm{em}_{p,q}(z) &= (\alpha_2 + \beta_2 z) \textstyle\sum_{k \geq p+q'+1} \mathbf{e}_{p,q;k} P_k(z) \\ &= \beta_2 \tilde{\mathbf{e}}_{p,q;p+q'} P_{p+q'}(z) + \textstyle\sum_{k \geq p+q'+1} (\alpha_2 \mathbf{e}_{p,q;k} + \beta_2 \tilde{\mathbf{e}}_{p,q;k}) P_k(z) \ . \end{aligned}$$

Doing similarly for each term of the sum, we obtain for the last term the lowest degree of approximation

$$\alpha_6 \mathbb{R}\mathrm{em}_{p-2,q-2} = \alpha_6 \textstyle\sum_{k \geq p+q'-2} \mathbf{e}_{p-2,q-2;k} P_k \ .$$

So, the coefficients of the polynomials $\phi_j(z)$ must be taken so that the following orthogonality conditions are satisfied: rhs means the expression on the right hand side of (34)

$$\begin{aligned} \langle rhs^1, P_k \rangle = 0, \quad & k = p+q'-2, \dots, p+q'+2 \\ \langle rhs^2, P_k \rangle = 0, \quad & k = p+q'-2, \dots, p+q'+1 \end{aligned}$$

this gives 9 equations for 10 unknowns (the coefficients of the polynomials $\phi_j(z) = (\alpha_j + \beta_j z)$), so the system always has a solution. We write the orthogonality conditions, successively for component 1 and 2, first for $P_{p+q'-2}$, up to $P_{p+q'+1}$, and finally the first component for $P_{p+q'+2}$.

The form of the system is

$$\begin{pmatrix} + & + & & & & & & & & \\ - & - & + & & & & & & & \\ - & - & - & + & & & & & & \\ - & - & - & - & + & + & & & & \\ - & - & - & - & - & - & + & & & \\ - & - & - & - & - & - & - & + & & \\ - & - & - & - & - & - & - & - & + & \\ - & - & - & - & - & - & - & - & - & \\ - & - & - & - & - & - & - & - & - & + \end{pmatrix} \begin{pmatrix} \beta_5 \\ \alpha_6 \\ \beta_4 \\ \alpha_5 \\ \beta_3 \\ \alpha_4 \\ \beta_2 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{pmatrix} = 0 \tag{35}$$

Terms denoted by a $+$ are now proven to be non zero. We use formulae (32), (33), $q = 2q'$ being taken even. The coefficients noted by a $+$ are row by row, from left to right and top to bottom:

row 1 : $\tilde{e}^1_{p-1,q-2;p+q'-1}$, $e^1_{p-2,q-2;p+q'-2}$,
row 2 : $\tilde{e}^2_{p-1,q-1;p+q'-1}$,
row 3 : $e^1_{p-1,q-2;p+q'-1}$
row 4 : $\tilde{e}^2_{p,q-1;p+q'}$, $e^2_{p-1,q-1;p+q'-1}$,
row 5 : $\tilde{e}^1_{p,q;p+q'+1}$,
row 6 : $e^2_{p,q-1;p+q'}$,
row 7 : $e^1_{p,q;p+q'+1}$,
row 9 : $e^1_{p+1,q,p+q'+2}$,

and each of these are from (32, 33) non zero because of the regularity of the Frobenius-Padé array.

Using the previous properties and the recurrence relation of the P_k, the first equation is

$$\beta_5 A_{p+q'-2} \frac{H_{p-1,q-1}}{H_{p-1,q-2}} + \alpha_6 \frac{H_{p-2,q-1}}{H_{p-2,q-2}} = 0 \ ,$$

none of α_6, β_5 can be zero, the relation (34) cannot be shortened.

The proof is similar in the case where $q = 2q' + 1$ is odd. ∎

Let us now consider the second possible staircase

Theorem 4 *For all (p, q), there exist polynomials $\phi_j(z) = \alpha_j + \beta_j z + \gamma_j z^2, j = 0, \ldots, 6$, of degree respectively $0, 1, 1, 2, 1, 0$, such that the following relation holds*

$$\begin{aligned} S_{p+1,q+1}(z) = \ & \phi_1(z)S_{p,q+1}(z) + \phi_2(z)S_{p,q}(z) + \phi_3(z)S_{p-1,q}(z) + \\ & \phi_4(z)S_{p-1,q-1}(z) + \phi_5(z)S_{p-2,q-1}(z) + \phi_6(z)S_{p-2,q-2}(z) \ . \end{aligned} \tag{36}$$

The coefficients of the polynomials $\phi_j(z)$ are computed by a linear system, with a matrix given in (37).

The relation involves the following approximants

$$
\begin{array}{cccc}
\bullet & \bullet & & \\
 & \bullet & \bullet & \\
 & & \bullet & \bullet \\
 & & & *
\end{array}
$$

Proof: The proof follows the general lines of the preceding one. As before, the recurrence relations will hold for the numerators and denominators, if $\mathbb{F}$ is supposed to have, at least, one no rational component.

The identity for the denominators gives, for monic denominators, the equation

$$1 = \alpha_1 + \beta_2 + \beta_3 + \gamma_4 \ .$$

Let us come now to the orthogonality properties. The proof is written for the case $q = 2q'$, where $\mathbb{R}\text{em}_{p+1,q+1} = (O(P_{p+q'+2+1}), O(P_{p+q'+1+1}))$. We consider the relation (36) for the error terms and its right hand side. We obtain for the last term the lowest degree of approximation the terms $\phi_4 \mathbb{R}\text{em}_{p-1,q-1}$ and $\phi_5 \mathbb{R}\text{em}_{p-2,q-1}$ which are orthogonal to P_k up to $k = p + q' - 3$ for the first component and $k = p + q' - 4$ for the second. So the orthogonality conditions to satisfy, to obtain (36), are

$$
\begin{array}{ll}
\langle rhs^1, P_k \rangle = 0, & k = p + q' - 2, \ldots, p + q' + 2, \\
\langle rhs^2, P_k \rangle = 0, & k = p + q' - 3, \ldots, p + q' + 1,
\end{array}
$$

this gives 10 equations for 11 unknowns (the coefficients of the polynomials $\phi_j(z) = \alpha_j + \beta_j z + \gamma_j z^2, j = 1, \ldots, 6$), plus the degree condition. We write the orthogonality conditions, successively for component 1 and 2, first for $\langle rhs^2, P_{p+q'-3} \rangle$, up to $\langle rhs^1, P_{p+q'+2} \rangle$. The form of the system is

$$
\left(
\begin{array}{ccccccccccc}
+ & + & & & & & & & & & \\
- & - & + & & & & & & & & \\
- & - & - & + & + & & & & & & \\
- & - & - & - & - & + & & & & & \\
- & - & - & - & - & - & + & & & & \\
- & - & - & - & - & - & - & + & + & & \\
- & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & + & \\
- & - & - & - & - & - & - & - & - & - & + \\
- & - & - & - & - & - & - & - & - & - & -
\end{array}
\right)
\left(
\begin{array}{c}
\gamma_4 \\ \beta_5 \\ \alpha_6 \\ \beta_4 \\ \alpha_5 \\ \beta_3 \\ \alpha_4 \\ \beta_2 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1
\end{array}
\right)
= 0. \qquad (37)
$$

The coefficients noted by a $+$ are non zero. This is proven as in the previous result, i.e. these terms are either $\langle R^1_{i,2j}, P_{i+j+1} \rangle$ or $\langle R^2_{i,2j+1} P_{i+j+1} \rangle$, up to the multiplication by A_{i+j} for the coefficients of the β, and by $A_{i+j-1} A_{i+j}$ for the coefficient of the γ, which are ratio of two Hankel determinants, and are supposed to be non zero for the regularity of the Frobenius-Padé table. ■

Let us consider now the diagonal problem, that is the considered sequences are, for each pair (p, q), $(\mathbb{R}\text{em}_{p+k,q+k})_{k \in \mathbb{Z}}$.

Theorem 5 *For all* (p,q)*, there exist polynomials* $\phi_j(z) = \alpha_j + \beta_j z + \gamma_j z^2 \ldots, j = 0,\ldots,6,$ *of degree respectively* $1,2,3,4,4,2,1,$ *such that the following relation holds*

$$\begin{array}{rl} S_{p+1,q+1}(z) = & \phi_0(z)S_{p,q}(z) + \phi_1(z)S_{p-1,q-1}(z) + \phi_2(z)S_{p-2,q-2}(z) + \\ & \phi_3(z)S_{p-3,q-3}(z) + \phi_4(z)S_{p-4,q-4}(z) + \phi_5(z)Sr_{p-5,q-5}(z) + \phi_6(z)S_{p-6,q-6}(z) \ . \end{array}$$

The coefficients of the polynomials $\phi_j(z)$ *are computed by a linear system, described below.*

Proof: The proof follows the general lines of the preceding ones. As before, the recurrence relations will hold for the numerator and denominator, if $\mathbb{F}$ is supposed to have, at least, one non rational component.

The polynomials $\phi_j(z)$ are written $\alpha_j + \beta_j z + \gamma_j z^2 + \delta_j z^3 + \epsilon_j z^4, j = 0,\ldots,6$ following their degree, and the identity for the degrees of the monic denominators gives the equation

$$1 = \beta_0 + \gamma_1 + \delta_2 + \epsilon_3 \ .$$

Let us come now to the orthogonality properties. Let us write the proof for the case $q = 2q'$, then $\mathbb{R}\mathrm{em}_{p+1,q+1} = (O(P_{p+q'+2+1}), O(P_{p+q'+1+1}))$. When $S_{p,q}$ denotes the error terms $R^i_{p,q}$, the right hand side is orthogonal to P_k, up to $k = p+q'-10$ for the first component and $k = p+q'-10$ for the second. So the orthogonality conditions to satisfy, to obtain (5), are

$$\begin{array}{ll} \langle rhs^1, P_k\rangle = 0, & k = p+q'-9,\ldots,p+q'+2 \\ \langle rhs^2, P_k\rangle = 0, & k = p+q'-9,\ldots,p+q'+1 \end{array}$$

this gives 23 equations for 24 unknowns. So, again, we always have a solution. We write the orthogonality conditions, successively for component 1 and 2, first for $\langle rhs^1, P_{p+q'-9}\rangle$, up to $\langle rhs^1, P_{p+q'+2}\rangle$.

The form of the system is "pseudo-Hessenberg", i.e. the equations have an increasing number of terms: this number is (n+1) for equation n= 1, 2, n+2 for equation n= 3, n+3 for equation n=4,5, n+4 for equation n=6,7,8, n+5 for equation n=9 to 16,n+4 for equation n=17-19, n+3 for equation n=20-21, and complete (24 terms) for equation n=21-23.

The unknowns are involved in the following order

$$((\beta_6,\epsilon_4,\gamma_5,\alpha_6,\delta_4),(\beta_5,\epsilon_3,\gamma_4,\alpha_5,\delta_3)(\beta_4,\gamma_3,\alpha_4,\delta_2),(\beta_3,\gamma_2,\alpha_3),(\beta_2,\gamma_1,\alpha_2),(\beta_1,\alpha_1),(\beta_0,\alpha_0)),$$

and each time a new unknown is involved, its coefficient is non zero. ∎

6.3 Computational aspects

Despite the rather large systems we have obtained, it seems not too difficult to compute any approximant in the Frobenius-Padé table, restricted to diagonal-regular approximants, with the preceding relations. The interest is to obtain approximants with a large order of accuracy, without going through less interesting approximants.

The previous relations have been obtained only with conditions concerning the accuracy, and the idea is that going to compute all the conditions, there is only the last condition(s) to compute because the other ones were computed at the preceding step; in other words, considering the matrices to compute, there is more or less only the last row(s) to compute.

More precisely, let us first use the two descending staircases to obtain simultaneously two descending diagonals in this table $([p+k,k])_k$ and $([p+k+1,k])_k$.

We suppose we have obtained $[p+1,q+1]$ i.e. $\mathbb{R}\mathrm{em}_{p+1,q+1}$ $(q=2q')$ by the first staircase, let say (H), so we know the matrix of the system

$$M1 = (\beta_5, \alpha_6, \beta_4, \alpha_5, \beta_3, \alpha_4, \beta_2, \alpha_3, \alpha_2, \alpha_1)$$

where we denote the column by the name of the unknown whose coefficients are in this column (10 unknowns). It has been obtained by writing the required orthogonality conditions (9 equations)

$$\langle rhs^1, P_{p+q'-2}\rangle, \langle rhs^2, P_{p+q'-2}\rangle, \ldots, \langle rhs^1, P_{p+q'+2}\rangle \ .$$

Now we try to obtain $[p+2,q+1]$ by the second staircase, let say (V), so we look for the matrix of the system

$$M2 = (c_4, b_5, a_6, b_4, a_5, b_3, a_4, b_2, a_3, a_2, a_1)$$

where we similarly denote the column by the name of the unknown (replacing α by a...) whose coefficients are in this column (11 unknowns). The required orthogonality conditions (10 equations) are

$$\langle rhs^2, P_{p+q'-2}\rangle, \langle rhs^1, P_{p+q'-1}\rangle, \ldots, \langle rhs^1, P_{p+q'+3}\rangle \ .$$

The first row of $M1$ is no more used, and we will have to compute the last two new rows of $M2$, orthogonality of the second component w.r.t. $P_{p+q'+2}$, and orthogonality of the first component w.r.t. $P_{p+q'+3}$. Each column of $M2$ is now computed from M1 (except the 2 last terms), then completed

$$b_5 = \beta_4, b_4 = \beta_3, b_3 = \beta_2,$$
$$a_6 = \alpha_5, a_5 = \alpha_4, a_4 = \alpha_3, a_3 = \alpha_2, a_2 = \alpha_1 \ .$$

Once these columns are completed, they give rise to c_4, b_2 , for example for c_4 expressed in terms of the column b_4, (the analogue b_2, a_2)

$$\langle R^i_{p,q-1}, z^2 P_k\rangle = \langle R^i_{p,q-1}, A_k z P_{k+1}\rangle + \langle R^i_{p,q-1}, B_k z P_k\rangle + \langle R^i_{p,q-1}, C_k z P_{k-1}\rangle \ .$$

Finally, we get the new matrix with 9-1+2=10 equations, and 10-2+3=11 unknowns

Now we try to obtain $[p+2,q+2]$ by the first staircase, let say (H'), so we look for the matrix of the system

$$M1' = (\beta_5, \alpha_6, \beta_4, \alpha_5, \beta_3, \alpha_4, \beta_2, \alpha_3, \alpha_2, \alpha_1) \ .$$

The two first rows are of no more use and we have to compute the last equation, orthogonality of the second component w.r.t. $P_{p+q'+3}$. Except this last row, the columns are known as before by

$$\beta_j = b_{j+1}, j = 3,4,5, \quad , \alpha_j = a_{j+1}, j = 2, \ldots, 5 \ ,$$

then b_2 is computed from the complete column a_2, and a_1 has only one term in the last and new row. We have obtained the new matrix $M1'$ with 10-2+1=9 rows and 11-2+1=10 unknowns.

How are the new rows computed? The different scalar products $\langle R^i_{r,s}, P_k\rangle$ are the $e^i_{r,s;k}$ obtained by the expansion of the already known $\mathbb{R}\mathrm{em}_{r,s}$.

Why are the two steps non symmetric? Going from p to $p+1$ means two degrees of freedom, or an improvement of the accuracy by two (because the function is a vector of size two), while going from q to $q+1$ is only an improvement of one in the degree of accuracy.

If now we use the diagonal relation to compute, for fixed p and q, the sequence $([p+k, q+k])_k$, it is similar as we have to write the new conditions of orthogonality, 3 each time.

6.4 Other relations

The preceding relations seem a bit long, and so difficult to initialize. Here we look for relations that will involve the knowledge of the two preceding columns, i.e. to compute $[p,q]$, we use approximants of columns $q-1, q-2$. It seems that there does not exist relations involving only one column.

In all the previous results we get either conditions due to the degrees i.e. 'degrees conditions', or conditions due to the order of accuracy, i.e. 'order conditions'. We first give a relation involving only order conditions and one normalization condition.

Theorem 6 *For all (p,q), there exist polynomials $\phi_j(z) = \alpha_j + \beta_j z, j = 1, \ldots, 5,$ of degree respectively* $0, 0, 1, 1, 0$, *such that the following relation holds*

$$S_{p+1,q+1}(z) = \alpha_1 S_{p+1,q}(z) + \alpha_2 S_{p+1,q-1}(z) + \phi_3(z) S_{p,q}(z) + \phi_4(z) S_{p,q-1}(z) + \alpha_5 S_{p-1,q-1}(z)\ . \quad (38)$$

The coefficients α, β of the polynomials $\phi_j(z)$ are computed by a linear system, with a Hessenberg matrix given in (39).
The relation involves the following approximants

$$\begin{array}{ccc} \bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & * \end{array}$$

Proof: The proof follows the general lines of the preceding ones and is done for $q = 2q'$. As before, the recurrence relations will hold for the numerator and denominator, if $\mathbb{F}$ is supposed to have, at least, one non rational component.

If we write the relation for the error terms $\mathbb{R}\mathrm{em}_{p,q}$, then on the left hand side, we get $\mathbb{R}\mathrm{em}_{p+1,q+1} = O(P_{p+q'+3}, P_{p+q'+2})$. On the right side, we get $rhs = O(P_{p+q'}, P_{p+q'-1})$, so we have six equations for seven unknowns. Writing $\langle rhs^2, P_{p+q'-1}\rangle$ up to $\langle rhs^1, P_{p+q'+2}\rangle$, we get the following Hessenberg system

$$\begin{pmatrix} \langle R^2_{p,q-1}, zP_{p+q'-1}\rangle & + & & & & & \\ - & - & + & & & & \\ - & - & - & + & & & \\ - & - & - & - & + & & \\ - & - & - & - & - & + & \\ - & - & - & - & - & - & + \end{pmatrix} \begin{pmatrix} \beta_4 \\ \alpha_5 \\ \beta_3 \\ \alpha_4 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{pmatrix} = 0. \quad (39)$$

The terms of the superdiagonal are sure to be non zero (from the regularity assumption) and are given by

$$\begin{array}{ll}\langle R^2_{p-1,q-1}, P_{p+q'-1}\rangle, \langle R^1_{p,q}, zP_{p+q'}\rangle, & \langle R^2_{p,q-1}, P_{p+q'}\rangle, \\ \qquad \langle R^1_{p,q}, P_{p+q'+1}\rangle & , \langle R^2_{p+1,q-1}, P_{p+q'+1}\rangle, \langle R^1_{p+1,q}, P_{p+q'+2}\rangle \ .\end{array}$$

If all the denominators are supposed to be monic, the normalization condition gives $\beta_3 = 1$, which ensures the unicity of the solution.■

Let us now look for a relation obtained through order conditions and degrees conditions

Theorem 7 *For all (p,q), there exist polynomials $\phi_j(z) = \alpha_j + \beta_j z, j = 1,\ldots,5$, of degree respectively $0,1,1,0,0$, such that the following relation holds*

$$S_{p+1,q+1}(z) = \alpha_1 S_{p+3,q-1}(z) + \phi_2 S_{p+2,q}(z) + \phi_3(z) S_{p+2,q-1}(z) + \alpha_4(z) S_{p+1,q}(z) + \alpha_5 S_{p+1,q-1}(z) \ . \quad (40)$$

The relation involves the following approximants

$$\begin{array}{ccc} \bullet & \bullet & * \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

Proof: The numerator is of degree $p+3$ for the right hand side and $p+1$ on the left hand side, so the numerators being vectors of size 2, this gives four equations.

The order of accuracy is $\mathcal{O}(P_{p+q'+2}, P_{p+q'+1})$ for the right hand side, and $\mathcal{O}(P_{p+q'+3}, P_{p+q'+2})$ for the left hand side, which gives two equations $\langle rhs^2, P_{p+q'+1}\rangle$, $\langle rhs^1, P_{p+q'+2}\rangle$. So we get six equations for seven unknowns. We denote by $r_{p,q}, \tilde{r}_{p,q}$ the first and second (vector) coefficients of $\mathbb{N}\mathrm{um}_{p,q}$ and obtain the two subsystems

$$\left\{\begin{array}{ll} r_{p+2,q-1}\beta_3 + r_{p+2,q}\beta_2 + r_{p+3,q-1}\alpha_1 & = 0 \\ \tilde{r}_{p+2,q-1}\beta_3 + \tilde{r}_{p+2,q}\beta_2 + \tilde{r}_{p+3,q-1}\alpha_1 \quad + r_{p+2,q}\alpha_2 + r_{p+2,q-1}\alpha_3 & = 0 \end{array}\right.$$

$$\left(\begin{array}{cccc} \langle R^2_{p+2,q-1}, zP_{p+q'+1}\rangle & + & & \\ - & - & \langle R^1_{p+2,q}, zP_{p+q'+2}\rangle & + \end{array}\right) \left(\begin{array}{c} \beta_3 \\ \alpha_5 \\ \beta_2 \\ \alpha_4 \end{array}\right) = 0 \quad (41)$$

The terms $(+)$ are sure to be non zero (from the regularity assumption) and are given by

$$\langle R^2_{p+1,q-1}, P_{p+q'+1}\rangle, \langle R^1_{p+1,q}, P_{p+q'+2}\rangle \ .$$

As noted before when using these relations iteratively, we do not have to recompute everything in the order conditions, but here the remark is of less interest than before.

The initialization is done easily as the column of index zero is formed by the partial sums of the initial series $\mathbb{F}$. The column of index one can be computed from previous algorithms or directly by the general formulae. We recall that $h_{p,j} = (h^1_{p,j}, h^2_{p,j})^t$ and $h^i_{p,j} = 1/\|P_p\|^2 \langle f^i P_j, P_p\rangle$

$$D_{p,1} = \left|\begin{array}{cc} h^1_{p+1,0} & h^1_{p+1,1} \\ P_0 & P_1 \end{array}\right|, \quad \mathbb{N}\mathrm{um}_{p,1} = \left|\begin{array}{cc} h^1_{p+1,0} & h^1_{p+1,1} \\ \sum_0^p h_{k,0}P_k & \sum_0^p h_{k,1}P_k \end{array}\right| \quad (42)$$

For $j = 0$, we get from the definition

$$h^1_{k,0} = f^1_k \ ,$$

for $h^1_{k,1}$, P_1 is written as $az + b$, and then

$$\begin{aligned} h^1_{k,1} &= 1/\|P_k\|^2 \langle f^1, (az+b)P_k \rangle \\ &= b f^1_k + a(A_{k-1} f^1_{k-1} + B_k f_k + C_{k+1} f^1_{k+1}) \ , \end{aligned}$$

so the two first columns of the table are known, the last algorithms allow to compute the third one and then any of the preceding algorithms gives rise to the whole table of Frobenius-Padé approximants, restricted to diagonal-regular indices ($p_1 = p_2$ and (q_1, q_2) the regular bi-index associated to q).

7 Conclusion and perspectives

In the case of the Frobenius-Padé approximation for one function, the algorithms proposed in [8, 9] have been implemented and the numerical results showed that these approximants have good convergence and acceleration properties, specially for functions with discontinuities.

It will be interesting to program the recurrence relations developed in this paper to study their numerical convergence properties, for instance when applied to the solution of partial differential equations given by spectral methods (in this case we have the approximation of the solution as a partial sum of an orthogonal series).

It will also be important to obtain some sufficient conditions on the regularity of the table of the Frobenius-Padé approximants because this is a fundamental property to avoid breakdowns in our algorithms. This work is under progress.

Another idea is the following. In the classical (scalar) orthogonality, three terms recurrence relations are equivalent to orthogonality. Here the recurrence relations have several polynomial coefficients, so are very different. It is under study to look for the "inverse" problem, i.e., to obtain some orthogonality properties from the recurrence relations.

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