Recursive Computation of Padé–Legendre Approximants and Some Acceleration Properties

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Abstract

In this paper, after recalling the two definitions of the generalizations of the Padé approximants to orthogonal series, we will define the Padé–Legendre approximants of a Legendre series. We will propose two algorithms for the recursive computation of some sequences of these approximants. We will also estimate the speed of convergence of the columns of the Padé–Legendre table from the asymptotic behaviour of the coefficients of the Legendre series. Finally we will illustrate these results with some numerical examples.

Keywords: Padé approximants, Legendre series, orthogonal polynomials, orthogonal expansions, rational approximation

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1 Introduction

Let us consider a function f given by an orthogonal series in a domain $\mathcal{D}$

$$f(z) = \sum_{k=0}^{\infty} f_k P_k(z) \quad z \in \mathcal{D}, \tag{1}$$

where $\{P_k\}$ is a family of orthogonal polynomials in the interval $[a, b]$ with respect to the weight function $w(x) \geq 0$ and

$$f_k = c_k(f) = \frac{1}{\| P_k \|_2^2} \int_a^b f(x) P_k(x) w(x) dx,$$

that is, (1) is the Fourier expansion of f with respect to the system $\{P_k\}$.

The generalization of the concept of Padé approximant to an orthogonal series leads to the construction of two essentially different classes of rational approximants:

1) The *nonlinear Padé approximant* $\Phi_{L,M}$ of order (L, M) of the series (1) is a rational function (numerator of degree L, denominator of degree M) whose Fourier expansion with respect to the system $\{P_k\}$,

$$\Phi_{L,M}(z) = \phi_0 P_0(z) + \phi_1 P_1(z) + \cdots + \phi_n P_n(z) + \cdots$$

satisfies

$$\phi_i = f_i \quad i = 0, \cdots, L + M.$$

In other words, the approximant $\Phi_{L,M}$ is determined by the system of nonlinear equations

$$c_k(\Phi_{L,M}) = c_k(f) \quad k = 0, 1, \cdots, L + M. \tag{2}$$

This system does not have always a solution. In the case of the Legendre polynomials, a method for solving (2) is given in [4].

2) The *Frobenius–Padé approximant* $[L/M]_f^P = \frac{N}{D}$ (or linear Padé approximant) of order (L, M) of the series (1) is a rational function of type (L, M), where the numerator and denominator polynomials satisfy

$$c_k(Df - N) = 0 \quad k = 0, \cdots, L + M,$$

which gives a linear system of equations for computing the coefficients of the polynomials. The case of the Tchebyshev polynomials is studied in [1] and [2].

The nonlinear approximants have some advantages over the Frobenius–Padé approximants with respect to the degree of approximation. In fact, it is easy to see that to achieve the order of approximation of $L+M$, we need to know the first $L+2M+1$ coefficients of the series to construct $[L/M]_f^P$ and only $L+M+1$ coefficients to construct $\Phi_{L,M}$. But they have some drawbacks: they are much more difficult to compute and it is more difficult to show their existence.

Many results on the convergence of these approximants have been obtain. In [6], the authors have compared and obtained bounds on the speed of convergence of diagonal sequences of these approximants , $(\Phi_{n+j,n})_n$ and $([n + j/n]_j^P)_n$ for functions of Markov type. The problems of convergence of the columns of the generalized Padé table have been studied in [8], [9] and [10], where an analog of the Montessus de Ballore's theorem is proved for the nonlinear approximants.

In this paper, we will concentrate on the Frobenius–Padé approximants for the Legendre series. We will develop two recursive algorithms for their computation (which are easily generalized to other orthogonal series), and will estimate, from the asymptotic behaviour of the coefficient sequence $(f_k)_k$, the speed of convergence of the columns of the Padé–Legendre table.

Let us begin by some definitions.

2 Generalizations of the Padé approximants to orthogonal series.

Let f be given in a domain $\mathcal{D}$ by its Fourier expansion (1). We want to compute a rational approximation of f of the form

$$[p/q]_f^P(z) = \frac{N^{[p/q]}(z)}{D^{[p/q]}(z)}, \text{ with}$$

$$\left\{ \begin{array}{lcl} N^{[p/q]}(z) & = & \sum_{i=0}^{p} a_i P_i(z) \\ D^{[p/q]}(z) & = & \sum_{j=0}^{q} b_j P_j(z) \end{array} \right.$$

where $N^{[p/q]}$ and $D^{[p/q]}$ satisfy the following property

$$D^{[p/q]}(z) f(z) - N^{[p/q]}(z) = \sum_{k=p+q+1}^{\infty} e_k P_k(z) \quad \text{for} \quad z \in \mathcal{D}. \tag{3}$$

$[p/q]_f^P$ is called the Padé approximant of order (p, q) to the orthogonal series f.

We set

$$P_j(z) f(z) = \sum_{k=0}^{\infty} h_{kj} P_k(z) \text{ where } h_{kj} = \frac{1}{\| P_k \|_2^2} \int_a^b f(x) P_j(x) P_k(x) w(x) dx.$$

Inserting this in (3) we get

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{q} b_j h_{kj} \right) P_k(z) - \sum_{k=0}^{p} a_k P_k(z) = \mathcal{O}(P_{p+q+1}(z)),$$

(where the notation $\mathcal{O}(P_{p+q+1}(z))$ means that the first term in the orthogonal expansion of the function which is non–zero has index at least $p+q+1$), so the coefficients (a_i) and (b_j) must be a solution of the two following linear systems:

$$\begin{array}{ll} \sum_{j=0}^{q} b_j h_{kj} = 0 & k = p+1, \cdots, p+q; \\ \sum_{j=0}^{q} b_j h_{kj} = a_k & k = 0, \cdots, p. \end{array} \tag{4}$$

We see that once the (b_j) computed, the computation of the (a_j) follows immediately. The (b_j) are the solution of an homogeneous system of q equations in $q+1$ unknowns. So there is always a non–trivial solution. We will be interested in a solution with $b_q \neq 0$.

In order to simplify notation, let us define the following determinants

Definition 1

$$H_{l+1,p}^{n} = \left| \begin{array}{ccccc} h_{n,0} & h_{n,1} & \cdots & h_{n,l-1} & h_{n,p} \\ h_{n+1,0} & h_{n+1,1} & \cdots & h_{n+1,l-1} & h_{n+1,p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{n+l,0} & h_{n+l,1} & \cdots & h_{n+l,l-1} & h_{n+l,p} \end{array} \right| \quad \textit{with } p \geq l. \tag{5}$$

We easily see that if $H_{q+1,q}^{p+1} \neq 0$, then there is a solution of (4) with $b_q \neq 0$. We can then write the numerator and denominator of the approximant (which are defined up to a multiplicative factor) in the determinantal form

$$D^{[p/q]}(z) = \begin{vmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & h_{p+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & h_{p+q,q} \\ P_0(z) & P_1(z) & \cdots & P_q(z) \end{vmatrix} \tag{6}$$

and

$$N^{[p/q]}(z) = \begin{vmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & h_{p+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & h_{p+q,q} \\ \sum_{i=0}^{p} h_{i,0}P_i(z) & \sum_{i=0}^{p} h_{i,1}P_i(z) & \cdots & \sum_{i=0}^{p} h_{i,q}P_i(z) \end{vmatrix}. \tag{7}$$

If $H_{q+1,q}^{p+1} = 0$, there is a solution of (4) with $b_q = 0$ which means that the order condition (3) is satisfied by some polynomials of degrees $p_1 < p$, $q_1 \leq q$ and so the approximant $[p/q]$ coincides with $[p_1/q_1]$.

We will be interested in constructing the approximants $[p/q]$ for different values $p \geq 0, q \geq 0$ and place them in a table (like in the classical Padé case).

Definition 2 *The table of the Padé approximants to an orthogonal series is normal iff* $H_{q,q-1}^{p+1} \neq 0$ *for all* $p \geq 0, q \geq 1$.

This implies that

- all the approximants in the table are different because, for each $[p/q]$, the denominator is of exact degree q and the numerator is of exact degree p;in fact

$$b_q = H_{q+1,q}^{p+1} a_q = \sum_{j=0}^{q} b_j h_{p,j} = \begin{vmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & h_{p+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & h_{p+q,q} \\ h_{p,0} & h_{p,1} & \cdots & h_{p,q} \end{vmatrix} = (-1)^q H_{q+1,q}^{p};$$

- the coefficient of the first term in the error expansion (3), e_{p+q+1}, is non–zero; in fact

$$e_{p+q+1} = \sum_{j=0}^{q} b_j h_{p+q+1,j} = \begin{vmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & h_{p+1,q} \\ \cdots & \cdots & \cdots & \cdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & h_{p+q,q} \\ h_{p+q+1,0} & h_{p+q+1,1} & \cdots & h_{p+q+1,q} \end{vmatrix} = H_{q+1,q}^{p+1} \neq 0.$$

We are going to see how to compute recursively some sequences of these approximants in the case of a normal table. But let us begin by the recursive computation of the quantities h_{kj}.

3 Recursive computation of h_{kj}

In this section we will proceed as Holdeman in [5]. It is well–known (see [11] for instance) that the family $\{P_k\}$ of orthogonal polynomials satisfies a recurrence relation of the form:

$$P_{k+1}(x) = (\alpha_k x + \beta_k)P_k(x) - \gamma_k P_{k-1}(x), \quad k = 0, 1, \cdots.$$

We can then write

$$P_{k+1}(x)P_j(x) = \alpha_k x P_j(x)P_k(x) + \beta_k P_k(x)P_j(x) - \gamma_k P_{k-1}(x)P_j(x) \tag{8}$$

$$xP_j(x) = \frac{P_{j+1}(x)}{\alpha_j} - \frac{\beta_j}{\alpha_j}P_j(x) + \frac{\gamma_j}{\alpha_j}P_{j-1}(x). \tag{9}$$

Replacing (9) in (8) we get

$$\begin{aligned} P_{k+1}(x)P_j(x) &= \frac{\alpha_k}{\alpha_j}P_k(x)P_{j+1}(x) - \frac{\alpha_k}{\alpha_j}\beta_j P_k(x)P_j(x) + \\ &+ \frac{\alpha_k}{\alpha_j}\gamma_j P_k(x)P_{j-1}(x) + \beta_k P_k(x)P_j(x) - \gamma_k P_{k-1}(x)P_j(x). \end{aligned}$$

If we set $\mu_k = \| P_k \|_2^2$ for all k, multiply both sides of the equality by $f(x)w(x)$, and integrate from a to b, we get

$$\frac{\mu_{k+1}}{\mu_k}h_{k+1,j} = \frac{\alpha_k}{\alpha_j}h_{k,j+1} + \left(\beta_k - \frac{\alpha_k}{\alpha_j}\beta_j\right)h_{k,j} + \frac{\alpha_k}{\alpha_j}\gamma_j h_{k,j-1} - \gamma_k \frac{\mu_{k-1}}{\mu_k}h_{k-1,j}, \; j \geq 1, k \geq 0. \tag{10}$$

This enables us to compute recursively the quantities h_{kj} after providing the appropriate initialization. We will use this recurrence relation for the case of a Legendre series.

4 Definition of the Padé–Legendre approximants

We are going to consider the particular case of a Legendre series, that is $P_k = L_k$, where $\{L_k\}$ are the Legendre polynomials: they are orthogonal in $[-1, 1]$ with respect to the weight function $w(x) = 1$. These polynomials $\{L_k\}$ satisfy the recurrence relationship [11]

$$L_{k+1}(x) = \frac{2k+1}{k+1}xL_k(x) - \frac{k}{k+1}L_{k-1}(x), \quad k \geq 1 \tag{11}$$

$$L_0(x) = 1, \; L_1(x) = x. \tag{12}$$

So, in this case, we have

$$\alpha_k = \frac{2k+1}{k+1}, \; \beta_k = 0, \; \gamma_k = \frac{k}{k+1}, \; \mu_k = \frac{2}{2k+1}.$$

Inserting this into (10), we obtain

$$h_{k,j+1} = \frac{2j+1}{j+1}\frac{k+1}{2k+3}h_{k+1,j} - \frac{j}{j+1}h_{k,j-1} + \frac{2j+1}{j+1}\frac{k}{2k-1}h_{k-1,j}. \tag{13}$$

The quantities h_{kj} can belisted in a double-entry table and computed recursively if we initialize the recursions by providing the first two columns and the first row in the following way:

$$\begin{aligned} h_{j0} &= f_j \quad j = 0, 1, \cdots \\ h_{0j} &= \frac{1}{\mu_0} \int_{-1}^{1} f(x) L_j(x) L_0(x) dx = \frac{\mu_j}{\mu_0} f_j = \frac{1}{2j+1} f_j \quad j \geq 0. \end{aligned}$$

By using the recurrence relation for the $\{L_k\}$, we obtain

$$h_{k1} = \frac{1}{\mu_k} \int_{-1}^{1} f(x) L_1(x) L_k(x) dx = \frac{k+1}{2k+3} h_{k+1,0} + \frac{k}{2k-1} h_{k-1,0},$$

which corresponds to the relation (13) for $j = 0$ if we set $h_{k,-1} = 0$ ($\forall k \geq 1$).

With these initializations, the quantities in this table can be computed progressing from left to right and from the top to the bottom, each quantity being as indicated:

$$\begin{array}{ccccccccc} h_{0,-1} & h_{00} & \cdots & \cdots & h_{0,j-1} & h_{0j} & & h_{0,j+1} & \cdots \\ h_{1,-1} & h_{1,0} & \cdots & \cdots & & \cdots & & & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & h_{k-1,j-1} & \boxed{h_{k-1,j}} & \searrow & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \boxed{h_{k,j-1}} & h_{k,j} & & \boxed{h_{k,j+1}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & h_{k+1,j-1} & \boxed{h_{k+1,j}} & \nearrow & \vdots & \vdots \end{array}$$

Furthermore, as

$$h_{kj} = \frac{1}{\mu_k} \int_{-1}^{1} f(x) L_j(x) L_k(x) dx = \frac{\mu_j}{\mu_k} h_{jk} = \frac{2k+1}{2j+1} h_{jk} \ (\forall j, k \geq 1),$$

we only need to compute the upper triangular part of this table.

As we have seen in the previous section, if we want to compute the coefficients of one particular Padé–Legendre approximant denoted by $[p/q]_f^L$, then the (b_i) are the solution of the linear system

$$\begin{pmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & \cdots & h_{p+1,q} \\ h_{p+2,0} & h_{p+2,1} & \cdots & \cdots & h_{p+2,q} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & \cdots & h_{p+q,q} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the coefficients (a_i) are then given by

$$a_j = h_{j,0} b_0 + h_{j,1} b_1 + \cdots + h_{j,q} b_q, \quad j = 0, 1, \cdots, p.$$

It is easy to see that for computing all the $h_{k,j}$ appearing in the previous systems, we need to know the quantities $f_j, j = 0, 1, \cdots p + 2q$, the first $p + 2q + 1$ coefficients of the series.

If we want to compute these coefficients (a_i, b_i) for a sequence of Padé–Legendre approximants, then it is better to compute them recursively. We will develop two algorithms in the next section.

5 Recursive algorithms

5.1 Frobenius type algorithms

In this section, we will suppose that the normality condition $H^{p+1}_{q,q-1} \neq 0 \ \ \forall p \geq 0, q \geq 1$ is satisfied. We will begin by giving the recurrence relations satisfied by the quantities $H^n_{l,p}$. For this, we need the following result (Sylvester's theorem)(see for instance, [1]):

Theorem 1 *Let A be a matrix, and let A_{rp} denote the matrix with row r and column p deleted. Also let $A_{rs,pq}$ denote the matrix A with rows r and s and columns p and q deleted. Provided $r < s$ and $p < q$,*

$$det(A)\,det(A_{rs,pq}) = det(A_{rp})\,det(A_{sq}) - det(A_{rq})\,det(A_{sp}).$$

Theorem 2

$$H^n_{l+1,p} = \frac{H^{n+1}_{l,p} H^n_{l,l-1} - H^{n+1}_{l,l-1} H^n_{l,p}}{H^{n+1}_{l-1,l-2}} \qquad l \geq 1, \quad p \geq l, \quad n \in \mathbb{N} \tag{14}$$

with the initializations

$$H^{p+1}_{0,-1} = 1, \quad H^{p+1}_{1,l} = h_{p+1,l}, \quad l \geq 0, \quad \forall p \geq 0.$$

Proof: Let us apply to $H^n_{l+1,p}$ the Sylvester's theorem with the values $r = 1, s = l+1$ and $p = l, q = l+1$:

$$H^n_{l+1,p} \times H^{n+1}_{l-1,l-2} = H^{n+1}_{l,p} \times H^n_{l,l-1} - H^{n+1}_{l,l-1} H^n_{l,p},$$

and the result follows. △

We will now introduce auxiliary quantities $D^{[p/q]}_l$ and $N^{[p/q]}_l$ in which the last column of $D^{[p/q]}$ and $N^{[p/q]}$, respectively, is replaced.

Definition 3

$$D^{[p/q]}_l(z) = \begin{vmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & h_{p+1,q-1} & h_{p+1,l} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & h_{p+q,q-1} & h_{p+q,l} \\ P_0(z) & P_1(z) & \cdots & P_{q-1}(z) & P_l(z) \end{vmatrix}, \quad l \geq q, \ \ \textit{and}$$

$$N^{[p/q]}_l(z) = \begin{vmatrix} h_{p+1,0} & h_{p+1,1} & \cdots & h_{p+1,q-1} & h_{p+1,l} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{p+q,0} & h_{p+q,1} & \cdots & h_{p+q,q-1} & h_{p+q,l} \\ \sum_{i=0}^{p} h_{i,0} P_i(z) & \sum_{i=0}^{p} h_{i,1} P_i(z) & \cdots & \sum_{i=0}^{p} h_{i,q-1} P_i(z) & \sum_{i=0}^{p} h_{i,l} P_i(z) \end{vmatrix}, \quad l \geq q.$$

We remark that $D_q^{[p/q]}(z) = D^{[p/q]}(z)$ and $N_q^{[p/q]}(z) = N^{[p/q]}(z)$ for all $p, q \geq 0$. Applying again the Sylvester's theorem to the quantities $D_l^{[p/q]}(z)$ and $N_l^{[p/q]}(z)$ with the choice of the two last rows and the two last columns we get the

Frobenius-type identities:

$$D_l^{[p/q]}(z) = \frac{D_l^{[p/q-1]}(z) H_{q,q-1}^{p+1} - D_{q-1}^{[p/q-1]}(z) H_{q,l}^{p+1}}{H_{q-1,q-2}^{p+1}} \qquad p \geq 0, q \geq 1 \tag{15}$$

$$N_l^{[p/q]}(z) = \frac{N_l^{[p/q-1]}(z) H_{q,q-1}^{p+1} - N_{q-1}^{[p/q-1]}(z) H_{q,l}^{p+1}}{H_{q-1,q-2}^{p+1}} \qquad p \geq 0, q \geq 1 \tag{16}$$

with the initializations

$$D_l^{[p/0]}(z) = P_l(z), \quad N_l^{[p/0]}(z) = \sum_{i=0}^{p} h_{i,l} P_i(z).$$

We now use these identities to derive an algorithm for the computation of all the Legendre–Padé approximants that can be computed from the first $2M+1$ coefficients of the series. So let us suppose that we know $f_i, \quad i = 0, \cdots, 2M$. We can then compute the approximants $[p/q]$ with $p + 2q \leq 2M$, which are represented in the following table:

$[0/0]$	$[0/1]$	$[0/2]$	$\cdots$	$[0/M-1]$	$[0/M]$
$[1/0]$	$[1/1]$	$[1/2]$	$\cdots$	$[1/M-1]$	
$[2/0]$	$[2/1]$	$[2/2]$	$\cdots$	$[2/M-1]$	
$\vdots$	$\vdots$	$\vdots$	$\cdots$		
$\vdots$	$\vdots$	$[2M-4/2]$			
$\vdots$	$[2M-3/1]$				
$\vdots$	$[2M-2/1]$				
$[2M-1/0]$					
$[2M/0]$					

We are going to propose an algorithm to compute this table by columns. From the Frobenius type identities, we see that the recurrence relations for the computation of numerators and denominators of these approximants are the same, only the initializations are different. We denote by $S_l^{[i/j]}(z)$ either $D_l^{[i/j]}(z)$ or $N_l^{[i/j]}(z)$. It's easy to determine which auxiliary quantities we need in order to compute the previous table. For this we construct the following table where in each place,

$$\boxed{\begin{array}{c} [i/j] \\ k..l \end{array}}$$

denotes that in the position (i, j) of the table we will compute the quantities $S_n^{[i/j]}(z)$ for $n = k, k+1, \cdots, l$ with $S = D$ and $S = N$ using the recurrence relations (16) and (15).

col.0	col.1	$\cdots$	col.j	$\cdots$	col.$M-1$	col.M
$[0/0]$ $0..M$	$[0/1]$ $1..M$	$\cdots$	$\cdots$	$\cdots$	$[0/(M-1)]$ $(M-1)..M$	$[0/M]$ $M..M$
$[1/0]$ $0..(M-1)$	$[1/1]$ $1..(M-1)$	$\cdots$	$\cdots$	$\cdots$	$[1/(M-1)]$ $(M-1)..(M-1)$	
$[2/0]$ $0..(M-1)$	$[2/1]$ $1..(M-1)$	$\cdots$	$\cdots$	$\cdots$	$[2/(M-1)]$ $(M-1)..(M-1)$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		
$[2i-1/0]$ $0..(M-i)$	$[2i-1/1]$ $1..(M-i)$	$\cdots$	$[2i-1/j]$ $j..(M-i)$			
$[2i/0]$ $0..(M-i)$	$[2i/1]$ $1..(M-i)$	$\cdots$	$[2i/j]$ $j..(M-i)$			
$\vdots$	$\vdots$	$\ddots$	$\vdots$			
$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$[2M-2j/j]$ $j..j$			
$[2M-3/0]$ $0..1$	$[2M-3/1]$ $1..1$					
$[2M-2/0]$ $0..1$	$[2M-2/1]$ $1..1$					
$[2M-1/0]$ $0..0$						
$[2M/0]$ $0..0$						

For the computation of $S_n^{[i/j]}$, the coefficients of the recurrence relations are the quantities $H_{l,k}^p$ which are also computed recursively using the relation (14). Let us put, in the following table, the quantities $H_{l,k}^p$ we need to compute:

col.1	col.2	$\cdots$	col.i	$\cdots$	col.$M-1$	col.M
$H^1_{1,j}$ $j=0..M$	$H^1_{2,j}$ $j=1..M$	$\cdots$	$H^1_{i,j}$ $j=i-1..M$	$\cdots$	$H^1_{M-1,j}$ $j=M-2..M$	$H^1_{M,j}$ $j=M-1..M$
$H^2_{1,j}$ $j=0..M$	$H^2_{2,j}$ $j=1..M$	$\cdots$	$H^2_{i,j}$ $j=i-1..M$	$\cdots$	$H^2_{M-1,j}$ $j=M-2..M$	
$H^3_{1,j}$ $j=0..M$	$H^3_{2,j}$ $j=1..M$	$\cdots$	$H^3_{i,j}$ $j=i-1..M$	$\cdots$	$H^3_{M-1,j}$ $j=M-2..M-1$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$H^M_{1,j}$ $j=0..M$	$H^M_{2,j}$ $j=1..(M-1)$	$\vdots$	$\vdots$			
$H^{M+1}_{1,j}$ $j=0..M-1$	$H^{M+1}_{2,j}$ $j=1..M-2$	$\vdots$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$H^{2M-2i}_{i,j}$ $j=i-1..i+1$			
$\vdots$	$\vdots$	$\vdots$				
$\vdots$	$\vdots$	$\vdots$	$H^{2M-2i+1}_{i,j}$ $j=i-1..i$			
$\vdots$	$\vdots$	$\vdots$				
$H^{2M-4}_{1,j}$ $j=0..4$	$H^{2M-4}_{2,j}$ $j=1..3$					
$H^{2M-3}_{1,j}$ $j=0..3$	$H^{2M-3}_{2,j}$ $j=1..2$					
$H^{2M-2}_{1,j}$ $j=0..2$						
$H^{2M-1}_{1,j}$ $j=0..1$						

Using the information of the two tables, we are now able to give the pseudo–code of the algorithm.

Algorithm

{initializations}
{recursive computations}
for $i = 1$ to M do
    {computation of column i}
    for $k = 1$ to $M - i + 1$ do
        for $j = i - 1$ to M do
            compute $H^k_{i,j}$ using relations (14)
        end do
    end do
    for $k = M - i + 2$ to $2M - 2i + 1$ do

```
          for j = i − 1 to 2M − i + 1 − k do
             compute H^k_{i,j} using relations (14)
          end do
        end do
      for k = 0 to M − i do
          for l = i to M − k do
            compute D_l^{[2i−1/j]}(z), N_l^{[2i−1/j]}(z) using (15) and (16)
          end do
        end do
end do
```

5.2 Kronecker-type algorithm

We are going to propose an algorithm for the computation of the sequence of the approximants in an anti–diagonal $[N-m/m]_f^L$, $m=0,1,\cdots,N$ (with N fixed) of the Padé–Legendre table. We let

$$[N-m/m]_f^L(z) = p_m(z)/q_m(z) \quad m=0,1,\cdots N,$$

where the polynomials p_m and q_m satisfy

$$\begin{cases} \deg(p_m) = N-m, \quad \deg(q_m) = m; \\ q_m(z)f(z) - p_m(z) = e_{m,N+1}L_{N+1}(z) + \mathcal{O}(L_{N+2}(z)). \end{cases} \tag{17}$$

We know the initial values $q_0(z)=1$, $p_0(z)=\sum_{i=0}^{N} f_i L_i(z)$. We will obtain a four–term recurrence relation

$$\begin{cases} p_{j+1}(z) &= p_{j-2}(z) + \alpha_j(z)p_{j-1}(z) + \beta_j(z)p_j(z) \\ q_{j+1}(z) &= q_{j-2}(z) + \alpha_j(z)q_{j-1}(z) + \beta_j(z)q_j(z) \end{cases} \tag{18}$$

with $\alpha_j(z)$, $\beta_j(z)$ polynomials of degree 1:

$$\begin{cases} \alpha_j(z) = a_{0,j} + a_{1,j}z \\ \beta_j(z) = b_{0,j} + b_{1,j}z \end{cases} . \tag{19}$$

The quantities $a_{0,j}, a_{1,j}, b_{0,j}, b_{1,j}$ are computed so that conditions (17) are satisfied. We set

$$p_l(z) = \sum_{i=0}^{N-l} c_{li}L_i(z) \text{ for } l=0,1,\cdots N \ (c_{0i}=f_i, \quad i=0,\cdots,N); \tag{20}$$

$$q_l(z) = \sum_{i=0}^{l} d_{li}L_i(z) \quad (d_{00}=1). \tag{21}$$

Using the recurrence relation for the Legendre polynomials and (18), (20), the recurrence (18) can be written

$$
\begin{aligned}
p_{j+1}(z) \;=\; & \left(c_{j-2,N-j+2} + a_{1,j}c_{j-1,N-j+1}\frac{N-j+2}{2N-2j+3}\right) L_{N-j+2}(z) + \\
& + \left(c_{j-2,N-j+1} + \frac{N-j+1}{2N-2j+1}b_{1,j}c_{j,N-j} + a_{0,j}c_{j-1,N-j+1} + \right. \\
& \left. a_{1,j}c_{j-1,N-j}\frac{N-j+1}{2N-2j+1}\right) L_{N-j+1}(z) + \left(c_{j-2,N-j} + c_{j,N-j}b_{0,j} + c_{j,N-j-1}\frac{N-j}{2N-2j-1}b_{1,j}\right. \\
& \left. + c_{j-1,N-j}a_{0,j} + c_{j-1,N-j+1}\frac{N-j+1}{2N-2j+3}a_{1,j} + \frac{N-j}{2N-2j-1}c_{j-1,N-j-1}a_{1,j}\right) L_{N-j}(z) \\
& + r_j(z) \text{ with } r_j(z) = \sum_{i=0}^{N-j-1} r_{j,i}L_i(z).
\end{aligned}
$$

As the polynomial p_{j+1} must have a degree less or equal to $N-j-1$, the coefficients of L_{N-j+2}, L_{N-j+1} and L_{N-j} must be equal to zero and we get three linear equations in the unknowns $a_{0,j}, a_{1,j}, b_{0,j}, b_{1,j}$. The fourth equation results from the accuracy to order condition. In fact we have:

$$
q_{j+1}f - p_{j+1} = (q_{j-2}f - p_{j-2}) + \alpha_j(q_{j-1}f - p_{j-1}) + \beta_j(q_jf - p_j). \tag{22}
$$

Setting

$$
q_l(z)f(z) - p_l(z) = e_{l,N+1}L_{N+1}(z) + \cdots \quad \text{for } l = 0, 1, \cdots, N, \tag{23}
$$

and inserting this in (22) we obtain

$$
q_{j+1}(z)f(z) - p_{j+1}(z) = \frac{N+1}{2N+3}\left(e_{j-1,N+1}a_{1,j} + e_{j,N+1}b_{1,j}\right) L_N(z) + \mathcal{O}(L_{N+1}(z)).
$$

Equating to zero the coefficient of $L_N(z)$ we get the last equation. By a simple computation we obtain the solution of the system which gives the coefficients of $\alpha_j(z)$ and $\beta_j(z)$. The results are summarized in the following theorem.

Theorem 3 *Let f be a function given by its expansion in a Legendre series $f(z) = \sum_{i=0}^{\infty} f_iL_i(z)$ for $z \in \mathcal{D}$, a domain in the complex plane. For N fixed, we want to compute the anti–diagonal of the Padé–Legendre table containing $[N-m/m]_f^L$ for $m = 0, \cdots, N$. Let us suppose the normality condition*

$$
H_{m+1,m}^{N-m} \neq 0 \quad \textit{for } m = 0, 1, \cdots, N.
$$

Then the following recursive Kronecker-type algorithm can be used to compute this sequence:

$$
\left\{
\begin{array}{lcl}
p_{j+1}(z) & = & p_{j-2}(z) + \alpha_j(z)p_{j-1}(z) + \beta_j(z)p_j(z) \\
q_{j+1}(z) & = & q_{j-2}(z) + \alpha_j(z)q_{j-1}(z) + \beta_j(z)q_j(z)
\end{array}
\right. \quad \textit{for } j = 0, \cdots N-1,
$$

where $[N-j/j]_f^L(z) = p_j(z)/q_j(z)$ *and , for* $j \ge 0$, $\alpha_j(z), \beta_j(z)$ *are defined by* $\alpha_j(z) = a_{0,j} + a_{1,j}z$, $\beta_j(z) = b_{0,j} - b_{1,j}z$ *where*

$$\left\{\begin{array}{rcl} a_{1,j} & = & -\dfrac{2N-2j+3}{N-j+2}\dfrac{c_{j-2,N-j+2}}{c_{j-1,N-j+1}} \\ b_{1,j} & = & -\dfrac{e_{j-1,N+1}}{e_{j,N+1}}a_{1,j}\ ,\ b_{1,0} = \dfrac{a_{1,0}}{c_{0,n+1}} \\ a_{0,j} & = & -\dfrac{1}{c_{j-1,N-j+1}}\left(\dfrac{N-j+1}{2N-2j+1}c_{j,N-j}b_{1,j} + \dfrac{N-j+1}{2N-2j+1}c_{j-1,N-j}a_{1,j} + c_{j-2,N-j+1}\right) \\ b_{0,j} & = & -\dfrac{1}{c_{j,N-j}}\left(c_{j-1,N-j}a_{0,j} + \dfrac{N-j}{2N-2j-1}(b_{1,j}c_{j,N-j-1} + a_{1,j}c_{j-1,N-j-1}) + \right. \\ & + & \left.\dfrac{N-j+1}{2N-2j+3}c_{j-1,N-j+1}a_{1,j} + c_{j-2,N-j}\right) \\ e_{j+1,i} & = & e_{j-2,i} + a_{0,j}e_{j-1,i} + b_{0,j}e_{j,i} + \\ & + & a_{1,j}\left[\frac{1}{2i-1}e_{j-1,i-1} + \frac{i-1}{2i+3}e_{j-1,i+1}\right] + b_{1,j}\left[\frac{1}{2i-1}e_{j,i-1} + \frac{i-1}{2i+3}e_{j,i+1}\right]\ , i \ge N+1 \end{array}\right. . \tag{24}$$

For the initialization of this algorithm we let:

$$\left\{\begin{array}{lll} p_{-2}(z) = L_{N+2}(z) & p_{-1}(z) = L_{N+1}(z) & p_0(z) = \sum_{i=0}^{N} f_i L_i(z) \\ q_{-2}(z) = 0 & q_{-1}(z) = 0 & q_0(z) = 1. \end{array}\right.$$

Remark: under the normality condition, we can see that this algorithm cannot break down. In fact, the denominators which appear in the computations of the coefficients of $\alpha_j(z)$ and $\beta_j(z)$ are

- the coefficient of the first term in the expansion of the error

$$e_{j,N+1} = \sum_{i=0}^{j} d_{j,i}h_{N-j,i} \neq 0 \Leftrightarrow H_{j+1,j}^{N-j+1} \neq 0;$$

- for each value of j, the coefficient of the term of degree $N-j$ of the numerator of $[N-j/j]_f^L$

$$c_{j,N-j} = \sum_{i=0}^{j} d_{j,i}h_{N-j,i} = (-1)^j H_{j+1,j}^{N-j} \neq 0.$$

We will next be concerned with some convergence properties of these approximants.

6 Some results on the speed of convergence of the columns of the Padé–Legendre table

Given a Legendre series, its domain of convergence is well–known (see for instance [3])in the following two cases:

- if f is given by a Legendre series $f(z) = \sum_{k=0}^{\infty} f_k L_k(z)$ with $\limsup_{k\to\infty} |f_k|^{1/k} = 1/r$, with $r > 1$, then the series converges to f in the interior of the ellipse $\mathcal{E}_r$ (foci at ± 1, semi–axes $a = \frac{1}{2}(r + r^{-1}), b = \frac{1}{2}(r - r^{-1})$), so $\mathcal{D} = \mathcal{E}_r$;

- if $f(z) = \sum_{k=0}^{\infty} f_k L_k(z)$ is analytic in $-1 \le x \le 1$ with $\limsup_{k\to\infty} |f_k|^{1/k} = 1$, then the series converges to f in $-1 < x < 1$ and so $\mathcal{D} =]-1, 1[$.

The speed of convergence of the partial sums of a Legendre series can be measured by the order of the coefficient sequence $(f_k)_k$: if $f(z) = \sum_{k=0}^{\infty} f_k L_k(z)$ for $z \in \mathcal{D}_f$, $g(z) = \sum_{k=0}^{\infty} g_k L_k(z)$ for $z \in \mathcal{D}_g$ and $\lim_{k\to\infty} \dfrac{g_k}{f_k} = 0$, then, for $z \in \mathcal{D}_f \cap \mathcal{D}_g$, the sequence of partial sums of g converges faster than the one of f.

To study the speed of convergence of the columns of the Padé–Legendre table for a function given by its Legendre series, we are going to write the approximants as partial sums of a certain Legendre series and obtain the asymptotic behaviour of the corresponding coefficient sequence.

We will now fix q and consider the sequence:

$$[n/q]_f^L(z) = \left(\sum_{i=0}^{n} a_i^{(n)} L_i(z) \right) / \left(\sum_{j=0}^{q} b_j^{(n)} L_j(z) \right) = \sum_{i=0}^{n} c_i^{(n)}(z) L_i(z) \quad n \in \mathrm{N}\,,$$

where

$$c_i^{(n)}(z) = \frac{a_i^{(n)}}{\sum_{j=0}^{q} b_j^{(n)} L_j(z)}\ , \quad 0 \le i \le n. \tag{25}$$

So, to study the speed of convergence of $([n/q]_f^L(z))_n$, it is sufficient to know the asymptotic behaviour of the sequence $(c_i^{(n)}(z))_n$ when $i \to \infty$, $n \to \infty$ $(i \le n)$. For this, we begin by studying the behaviour of $(h_{k,j})_k$ when $k \to \infty$ (j fixed).

6.1 Asymptotic behaviour of sequences $(h_{k,j})_{k\ge 0}$

We are going to consider two different behaviours for $(f_n)_n$ and deduce from them the asymptotic behaviour of the sequences $(h_{n,j})_n$.

Proposition 1 *Let us consider a sequence $(f_n)_n$ satisfying:*

$$h_{n,0} = f_n = \rho^n \left[\alpha_0 + \frac{\beta_0}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right], \quad \rho \ne 1, \alpha_0 \ne 0. \tag{26}$$

Then

$$h_{n,j} = \rho^n \left[\alpha_j + \frac{\beta_j}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right] (n \to \infty) \text{ with} \tag{27}$$

$$\alpha_1 = \alpha_0(\rho + \frac{1}{\rho})/2, \text{ and } \alpha_j = \frac{2j-1}{j}\alpha_{j-1}(\rho + 1/\rho)/2 - \frac{j-1}{j}\alpha_{j-2}. \tag{28}$$

Moreover, for all j, $\alpha_j \ne 0$.

Proposition 2 *Let us consider now a sequence $(f_n)_n$ having the following asymptotic behaviour:*

$$h_{n,0} = f_n = \frac{\alpha}{n^p}\left(1 + \frac{\beta}{n} + \frac{1}{n^2}\left(\sum_{i=0}^{k} \frac{c_{0i}}{n^i} + \mathcal{O}\left(\frac{1}{n^{k+1}}\right)\right)\right) \quad (n \to \infty). \tag{29}$$

Then

$$\forall j \geq 1 \quad h_{nj} = \frac{\alpha}{n^p}\left(1 + \frac{\beta}{n} + \frac{1}{n^2}\left(\sum_{i=0}^{k} \frac{c_{ji}}{n^i} + \mathcal{O}\left(\frac{1}{n^{k+1}}\right)\right)\right) \quad (n \to \infty) \tag{30}$$

where $c_{10} = c_{00} + \frac{1}{2}(p+1)^2$ *and* $c_{j0} = \dfrac{2j-1}{j}(c_{j-1,0} + \dfrac{1}{2}(p+1)^2) - \dfrac{j-1}{j}c_{j-2,0}$ *for* $j \geq 2$.

*Proof:*The result of the two propositions is easily obtained by induction using the recurrence relation for the computation of the quantities $(h_{n,j})$.

△

6.2 Asymptotic behaviour of the sequence $(c_i^{(n)}(z))_i$ and convergence results

We will now derive, from the previous results, the asymptotic behaviour of the numerator coefficients $(a_i^{(n)})(n \to \infty)$, $(i \leq n)$ and denominator coefficients $(b_i^{(n)}), i = 0, \cdots q$ in order to estimate the order of convergence of the approximants. We will consider first the cases $q = 1, 2$ for which we can obtain their explicit form and then turn to the general case $q > 2$.

6.2.1 Case $q = 1$

We obtain the following result:

Proposition 3 *Let $f(z) = \sum_{i=0}^{\infty} f_i L_i(z)$ be a function given by a Legendre series in a domain $\mathcal{D}$ of convergence of the series. We suppose that $H_{2,1}^n \neq 0 \quad \forall n \in N$. We consider the first column of the Padé–Legendre table*

$$[n/1]_f^L(z) = \sum_{i=0}^{n} c_i^{(n)}(z) L_i(z).$$

Then the following holds:

1. *If the sequence $(f_i)_i$ satisfies (26), then*

$$c_i^{(n)}(z) \sim K(z)\frac{\rho^i}{i}(i \to \infty) \quad (i \leq n), \tag{31}$$

(where $K(z)$ independent of i). So, for $z \in \mathcal{D}$, the sequence $([n/1]_f^L(z))_n$ converges to $f(z)$ like the partial sums of a Legendre series whose coefficient sequence has order $\mathcal{O}\left(\frac{\rho^n}{n}\right)$.

2. *If the coefficient sequence $(f_i)_i$ satisfies (29), then the asymptotic behaviour of $c_i^{(n)}(z)$ is*

$$c_i^{(n)}(z) \sim K(z)\frac{1}{i^{p+2}} \ (i \to \infty), (i \le n), \tag{32}$$

where $K(z)$ is independent of i.

So, for $z \in \mathcal{D}$, the sequence $([n/1]_f^L(z))_n$ converges to $f(z)$ like the sequence of partial sums of a Legendre series whose coefficient sequence has order $\mathcal{O}\left(\frac{1}{n^{p+2}}\right)$.

Proof: From the results of the previous sections we know that the normality condition $H_{2,1}^n \neq 0 \ \forall n$ implies that $b_1^{(n)} \neq 0$ where $D^{[n/1]}(z) = b_0^{(n)} + b_1^{(n)} L_1(z)$. So we can choose, without loss of generality, $b_1^{(n)} = 1$. The other coefficients of the approximants are given by

$$\left\{ \begin{array}{l} b_0^{(n)} h_{n+1,0} + b_1^{(n)} h_{n+1,1} = 0 \\ a_i^{(n)} = b_0^{(n)} h_{i,0} + b_1^{(n)} h_{i,1} \quad i = 0, \cdots, n \end{array} \right. \Leftrightarrow \left\{ \begin{array}{lcl} b_0^{(n)} & = & -\frac{h_{n+1,1}}{h_{n+1,0}} \\ a_i^{(n)} & = & h_{i,0}\left(\frac{h_{i,1}}{h_{i,0}} - \frac{h_{n+1,1}}{h_{n+1,0}}\right) \quad i = 0, \cdots, n \end{array} \right. . \tag{33}$$

From the definition we have

$$a_i^{(n)} = h_{i,0} - \frac{h_{n+1,0}}{h_{n+1,1}} h_{i,1}.$$

1) From (27) and (28) we obtain

$$\frac{h_{i,1}}{h_{i,0}} = \frac{\alpha_1 + \frac{\beta_1}{i} + \mathcal{O}(1/i^2)}{\alpha_0 + \frac{\beta_0}{i} + \mathcal{O}(1/i^2)} = \frac{\alpha_1}{\alpha_0}\left(1 + (\frac{\beta_1}{\alpha_1} - \frac{\beta_0}{\alpha_0})\frac{1}{i}\right) + \mathcal{O}\left(\frac{1}{i^2}\right) (i \to \infty)$$

which leads to

$$a_i^{(n)} = h_{i,0}\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_0}{\alpha_0}\right)\frac{1}{i}\left(1 - \frac{i}{n+1}\right) + \mathcal{O}\left(\frac{1}{i^2}\right) \quad (i \to \infty), \quad (i \le n)$$

and

$$c_i^{(n)}(z) = \frac{a_i^{(n)}}{b_0^{(n)} + z} \sim K(z)\rho^i \frac{1}{i},$$

where $K(z) = (2\beta_1(\rho + \rho^{-1}) - \beta_0/(z - (\rho + \rho^{-1})/2)$, which enables us to obtain the asymptotic order (31) of $(c_i^{(n)}(z))_i$.

2) From (29) and (30) we obtain

$$\frac{h_{i,1}}{h_{i,0}} = 1 + \frac{1}{2}(p+1)^2\frac{1}{i^2} + \mathcal{O}\left(\frac{1}{i^3}\right) \quad (i \to \infty) \text{ and so}$$

$$a_i^{(n)} = h_{i,0}\left[\frac{1}{2}(p+1)^2(\frac{1}{i^2} - \frac{1}{(n+1)^2}) + \mathcal{O}\left(\frac{1}{i^3}\right)\right] = \frac{\alpha_0}{2}(p+1)^2\frac{1}{i^{p+2}}\left(1 + \mathcal{O}\left(\frac{1}{i}\right)\right) \quad (i \to \infty)(i \le n).$$

So the result (32) follows with $K(z) = \frac{\alpha_0}{2}(p+1)^2/(z-1)$.

$\triangle$

As we couldn't show any change in the order of the sequences $(c_i^{(n)})$ for the subsequent columns of the Padé–Legendre table when the coefficients f_k satisfy (26), from now on we will only consider Legendre series for which (f_k) satisfy (29).

6.2.2 Case $q=2$

Let us study now the behaviour of the second column of the Padé–Legendre table. We suppose that $H^p_{3,2}\neq 0\ \ \forall p\geq 0$, which garantees the existence and uniqueness of the sequence of approximants $([n/2]^L_f)_n$ and enables us to choose, without loosing generality, $b_2(n)=1\ \ \forall n$. The coefficients of the denominator are the solution of the linear system:

$$\begin{cases} b_0^{(n)}h_{n+1,0}+b_1^{(n)}h_{n+1,1}=-h_{n+1,2}\\ b_0^{(n)}h_{n+2,0}+b_1^{(n)}h_{n+2,1}=-h_{n+2,2}\end{cases}$$

which gives

$$b_0^{(n)}=\frac{h_{n+1,2}}{h_{n+1,0}}\frac{\dfrac{h_{n+2,2}}{h_{n+1,2}}-\dfrac{h_{n+2,1}}{h_{n+1,1}}}{\dfrac{h_{n+2,1}}{h_{n+1,1}}-\dfrac{h_{n+2,0}}{h_{n+1,0}}},\quad b_1^{(n)}=\frac{h_{n+1,2}}{h_{n+1,1}}\frac{\dfrac{h_{n+2,0}}{h_{n+1,0}}-\dfrac{h_{n+2,2}}{h_{n+1,2}}}{\dfrac{h_{n+2,1}}{h_{n+1,1}}-\dfrac{h_{n+2,0}}{h_{n+1,0}}}. \tag{34}$$

From the expansion (29) and using the recurrence relation for the quantities $(h_{j,k})$, we can obtain explicitly the first coefficients of the $(h_{n,j})$'s expansion in terms of the powers of $\frac{1}{n}$, for $j=1,2$. With the help of some computer algebra system we easily obtain:

$$\begin{cases} c_{10}=c_{00}+r_1(p)\\ c_{11}=c_{01}+r_2(p)\\ c_{12}=c_{02}+r_3(p)c_{00}+r_4(p)\end{cases},\quad \begin{cases} c_{20}=c_{00}+3r_1(p)\\ c_{21}=c_{01}+3r_2(p)\\ c_{22}=c_{02}+3r_3(p)c_{00}+3r_4(p)+\frac{3}{2}r_3(p)r_4(p)\end{cases} \tag{35}$$

with

$$\begin{array}{ll} r_1(p)=\frac{1}{2}(p+1)^2, & r_2(p)=2b(1+\frac{p}{2})^2-\frac{1}{2}(1+\frac{p}{2})\\ r_3(p)=\frac{(p+3)^2}{2}, & r_4(p)=-\frac{b}{4}(p+3)+\frac{1}{24}(p^4+8p^3+23p^2+31p+21)\end{array}.$$

Replacing in (34), we obtain

$$b_0^{(n)}=2+\epsilon_1(n),\ \ b_1^{(n)}=-3+\epsilon_1(n),\ \text{with } \epsilon_0(n)=\mathcal{O}\left(\frac{1}{(n+1)^2}\right),\ \ \epsilon_1(n)=\mathcal{O}\left(\frac{1}{(n+1)^2}\right). \tag{36}$$

Replacing the quantities by their expansions, we obtain

$$\begin{aligned} &b_0^{(n)}h_{n+1,0}+b_1^{(n)}h_{n+1,1}+h_{n+1,2}=\\ &=\ \frac{\alpha}{n^p}\Bigg[(2+\epsilon_0(n)-3+\epsilon_1(n)+1)(1+\frac{\beta}{n+1})+(2c_{00}-3c_{10}+c_{20})\frac{1}{(n+1)^2}\\ &\quad+(c_{00}\epsilon_0(n)+c_{10}\epsilon_1(n)\frac{1}{(n+1)^2}+(2c_{01}-3c_{11}+c_{21})\frac{1}{(n+1)^3}+\\ &\quad+(2c_{02}-3c_{12}+c_{22})\frac{1}{(n+1)^4}+\mathcal{O}\left(\frac{1}{n^5}\right)\Bigg].\end{aligned}$$

Using the definitions (35) and the equation satisfied by $b_0^{(n)},b_1^{(n)}$, we find:

$$\frac{\alpha}{(n+1)^p}\left[\epsilon(n)(1+\frac{\beta}{n+1})+(2c_{02}-3c_{12}+c_{22})\frac{1}{(n+1)^4}+(c_{00}\epsilon_0(n)+c_{10}\epsilon_1(n))\frac{1}{(n+1)^2}+\mathcal{O}\left(\frac{1}{n^5}\right)\right]=0$$

which gives

$$\epsilon(n) = \mathcal{O}\left(\frac{1}{n^4}\right).$$

Finally, for the asymptotic behaviour of the $a_i^{(n)}$ we obtain

$$\begin{aligned} a_i^{(n)} &= b_0^{(n)} h_{i,0} + b_1^{(n)} h_{i,1} + h_{i,2} = \\ &= \frac{a}{i^{p+4}}\left[\frac{3}{2} r_3(p) r_4(p) + \left(r_1(p)\epsilon_1(n) + \epsilon(n) i^2\right) i^2 + \mathcal{O}\left(\frac{1}{i}\right)\right], \end{aligned}$$

which gives

$$c_i^{(n)}(z) = \frac{a_i^{(n)}}{b_0^{(n)} + b_1^{(n)} L_1(z) + L_2(z)} = \mathcal{O}\left(\frac{1}{i^{p+4}}\right) \quad (i \to \infty) \quad (i \leq n).$$

Summarising, we have shown the following result:

Proposition 4 *Let f be given by a Legendre series $f(z) = \sum_{i=0}^{\infty} f_i L_i(z)$ in a domain $\mathcal{D}$ of its convergence. We suppose that $(f_i)_i$ satisfies (29). Then the second column of the Padé –Legendre table, $[n/2]_f^L(z) = \sum_{i=0}^{n} c_i^{(n)}(z) L_i(z)$, converges to f in $\mathcal{D}$ like the partial sums of a Legendre series whose coefficient sequence has order $\mathcal{O}\left(\frac{1}{n^{p+4}}\right)$.*

6.2.3 Case $q > 2$

The determination of the first terms of the expansion of the denominator coefficients of $[n/q]_f^L$ $(i < q)$, $b_i^{(n)}$ from their explicit expression as a quotient of two determinants becomes very difficult when $q > 2$. So we are going to proceed in a different way.

Let us suppose that we can compute quantities $(b_{i,0}, i = 0, \cdots, q-1)$ such that

$$h_{n,0} b_{0,0} + h_{n,1} b_{1,0} + h_{n,q-1} b_{q-1,0} + h_{n,q} = \epsilon(n) \text{ with } \epsilon(n) = \mathcal{O}\left(\frac{1}{n^{\alpha+q+1}}\right) \quad (n \to \infty).$$

Using the expansion (30) of $(h_{ni})_n$, $i \leq q$, and collecting the coefficients of the terms with the same power of $(1/n)$, this is equivalent to suppose that the system

$$\left\{ \begin{array}{l} b_{10} + b_{20} + \cdots + b_{q0} + 1 = 0 \\ b_{10} c_{1i} + b_{20} c_{2i} + \cdots + b_{q0} c_{qi} + c_{0i} = 0, \qquad i = 1, \cdots, q-1 \end{array} \right. \tag{37}$$

has a solution. If we define the quantities $(\epsilon_i(n), i = 0, \cdots, q-1)$ by

$$b_i^{(n)} = b_{i0} + \epsilon_i(n) \quad \text{for } i = 0, \cdots, q-1,$$

then, from the system of equations satisfied by the $b_i^{(n)}$'s, we immediately get that the $\epsilon_i(n)$'s satisfy

$$h_{n+j,0}\epsilon_0(n) + h_{n+j,1}\epsilon_1(n) + \cdots + h_{n+j,q-1}\epsilon_{q-1}(n) = \epsilon(n+j) \quad \text{for } j = 1, \cdots, q.$$

As

$$\epsilon(n) \sim \frac{C}{n^{\alpha+q+1}} \sim \frac{C_i}{n^{q+1}} h_{n,i} \quad (n \to \infty),$$

solving the previous system we get, for $i = 0, \cdots, q-1$,

$$\epsilon_i(n) = \frac{|h_{n+1,0} \;\; h_{n+1,1} \;\; \cdots \;\; h_{n+1,i-1} \;\; \epsilon(n+1) \;\; h_{n+1,i+1} \cdots \;\; h_{n+1,q-1}|}{|h_{n+1,0} \;\; h_{n+1,1} \;\; \cdots \;\; h_{n+1,i-1} \;\; h_{n+1,i} \;\; h_{n+1,i+1} \cdots \;\; h_{n+1,q-1}|} \sim \frac{K_i}{n^{q+1}} \quad (n \to \infty).$$

Then, for the coefficients $(a_i^{(n)})$, we find

$$\begin{aligned} a_i^{(n)} &= h_{i,0} b_0^{(n)} + h_{i,1} b_1^{(n)} + \cdots + h_{i,q-1} b_{q-1}^{(n)} + h_{i,q} = \\ &= h_{i,0} b_{0,0} + h_{i,1} b_{1,0} + \cdots + h_{i,q-1} b_{q-1,0} + h_{i,q} + h_{i,0}\epsilon_0(n) + \cdots + h_{i,q-1}\epsilon_{q-1}(n) \\ &= h_{i,0}(\epsilon_0(n) - \epsilon_0(i)) + \cdots + h_{i,q-1}(\epsilon_{q-1}(n) - \epsilon_{q-1}(i)). \end{aligned}$$

But as $\epsilon_k(n) - \epsilon_k(i) = \mathcal{O}\left(\frac{1}{i^{q+2}}\right)$ for $n \to \infty, i \to \infty, i < n$, we finally obtain

$$a_i^{(n)} = \mathcal{O}\left(\frac{1}{i^{p+q+2}}\right) (i \to \infty),$$

and the same asymptotic behaviour holds for the $(c_i^{(n)}(z))$.

We can gather the previous results in the following theorem:

Theorem 4 *Let f be given by a Legendre series $f(z) = \sum_{k=0}^{\infty} f_k L_k(z)$ for $z \in \mathcal{D}$, a domain in the complex plane. Let us suppose that $(f_k)_k$ satisfies (29). We construct the Padé–Legendre table $([n/q]_f^L)_{n,q\geq 0}$ and we set*

$$[n/q]_f^L(z) = \sum_{i=0}^{n} c_i^{(n)}(z) L_i(z).$$

For $q > 2$ fixed, if the system (37) has a solution, then the corresponding $(c_i^{(n)}(z))_i$ satisfy

$$c_i^{(n)}(z) \sim \frac{K}{i^{p+q+2}} \quad (i \to \infty) \quad (i \leq n).$$

For $z \in \mathcal{D}$ fixed, each column of the Padé–Legendre table converges to f as the partial sums of a Legendre series whose coefficient sequence has order $\mathcal{O}\left(\frac{1}{n^{p+q+2}}\right)$ $(n \to \infty)$.

7 Numerical examples

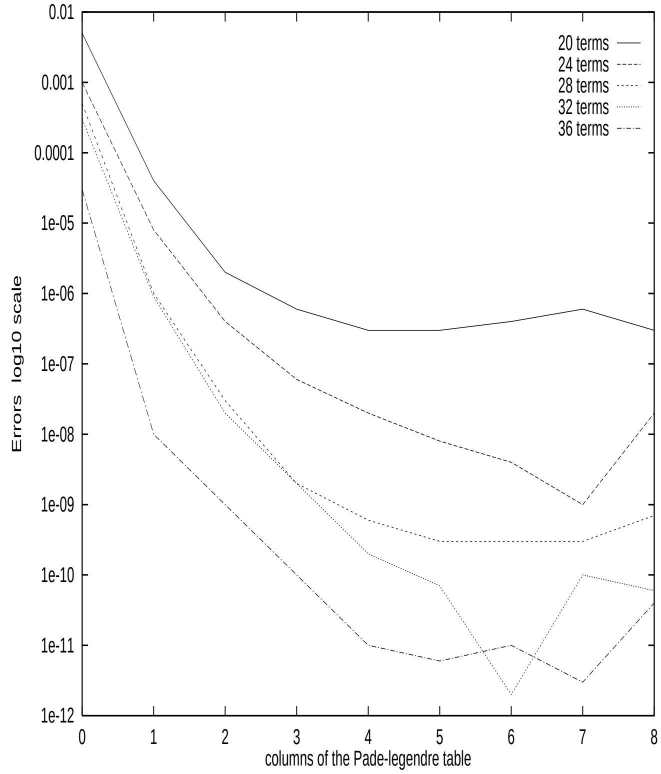
We have programmed the algorithms given in the previous sections and obtained some numerical results which illustrate the acceleration properties given below.

Example 1: Let us consider the generating function for the Legendre polynomials

$$f(z) = \frac{1}{\sqrt{1 - 2az + a^2}} = \sum_{l=0}^{\infty} a^l P_l(z)$$

For values $a < 1$ the coefficient sequence of this Legendre series converges linearly and conditions of **Proposition 3** are satisfied for $-1 < z < 1$. The figures 1 and 2 show that if we consider approximants computed with the the same number n of coefficients of the Legendre series, the error $|f(z) - [n-2p/p](z)|$ decreases with p. So each column of the Padé–Legendre converges faster than the previous one.

Figure 1: Generating function, $a = 0.8$, $z = 0.9$

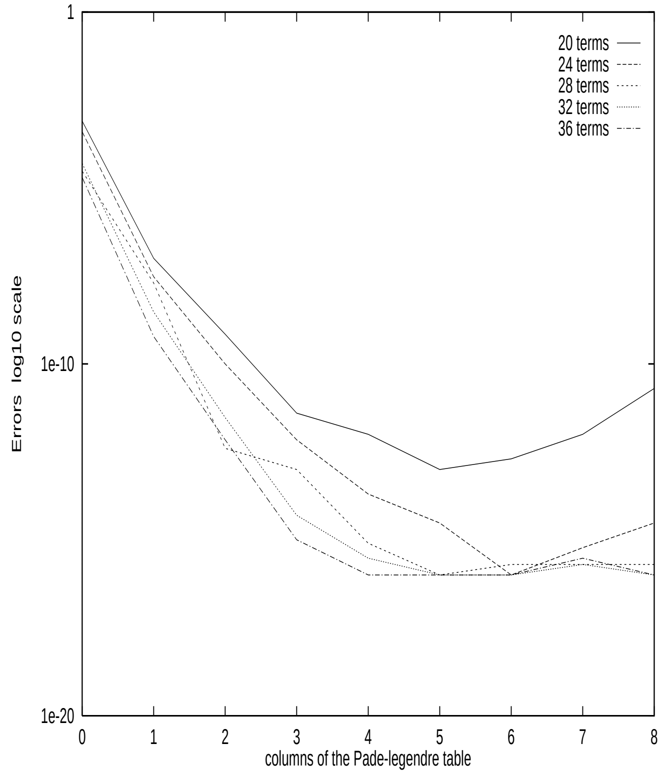


Example 2: Let us consider now a function for which the coefficient sequence of the corresponding Legendre series is logarithmic. It can be shown that [7]

$$f(x) = \sqrt{\frac{1-x}{2}} = \frac{2}{3}P_0(x) - 2\sum_{n=1}^{\infty} \frac{P_n(x)}{(2n-1)(2n+3)}, \quad -1 < x < 1$$

The conditions of **Theorem 4** are then satisfied and the good acceleration properties of the column sequences of the Padé–Legendre table can be seen in Figures 3 and 4, where we compare the precision of the approximants computed with the same number of coefficients of the initial series, that is, $\log_{10}\left|f(z) - [n-2p/p]_f^L(z)\right|$ for the values $p = 0, 1, \cdots, 8$ (for $p > 8$ the rounding errors become important and we do not gain any precision) and for two values of z, one positif and one negatif.

Figure 2: Generating function, $a = 0.8$, $z = -0.5$



Example 3: We will now consider a function which has a singularity in the interval $[-1, 1]$:

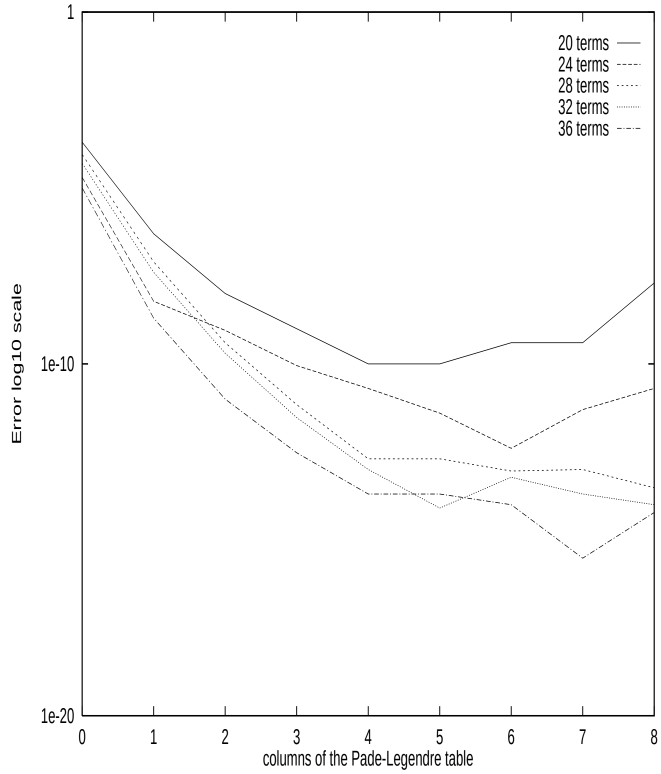
$$f(z) = \begin{cases} 0 & \text{if} \quad -1 < z < a \\ 1 & \text{if} \quad a < z < 1 \end{cases}$$

It is known [7] that its expansion in a Legendre series is given by

$$f(x) = \frac{1}{2}(1-a) - \frac{1}{2}\sum_{n=1}^{\infty} \left(P_{n+1}(a) - P_{n-1}(a)\right) P_n(x), \quad -1 < x < 1$$

In Figure 5 we compare the precision of the approximants $[n-2i/i]_f^L$, $i = 0, \cdots, 6$, in the interval $[-1, 1]$. These approximants are computed with the same number of coefficients of the Legendre series (in this example we consider $n = 140$ coefficients). We remark the gain of precision as we progress through the columns of the Padé–Legendre table, even near the singularity of the function. However, for $i > 6$, the numerical instabilities prevent better results.

Figure 3: $f(x) = \sqrt{\frac{1-x}{2}}, x = 0.9$



8 Conclusions and further work

In this paper we have shown that Padé–Legendre approximation can be an interesting tool to improve the convergence properties of the partial sums of the Legendre series. The previous numerical results confirm the good acceleration properties of the Padé–Legendre approximants and the algorithms proposed seem to have good stability properties.

The extension of the theoretical results to other classes of functions is under study. Another interesting problem is to obtain some convergence results for sequences of Padé–Legendre approximants for values of z such that the sequence of partial sums of the Legendre series doesn't converge (like in the classical Padé case).

A comparison between the different algorithms proposed below and a more detailed study of their stability is under progress. The following problems are also being treated:

- construction of new algorithms in order to progress in the Padé–Legendre table along any path;
- in the case of non–normality of the table, that is, when the condition $H^{p+1}_{q,q-1} \neq 0 \quad \forall p \geq 0, q \geq 1$ is not satisfied, how to detect the breakdown and modify the algorithms in order to be able to compute the existing approximants.

Figure 4: $f(x) = \sqrt{\frac{1-x}{2}}, x = -0.5$

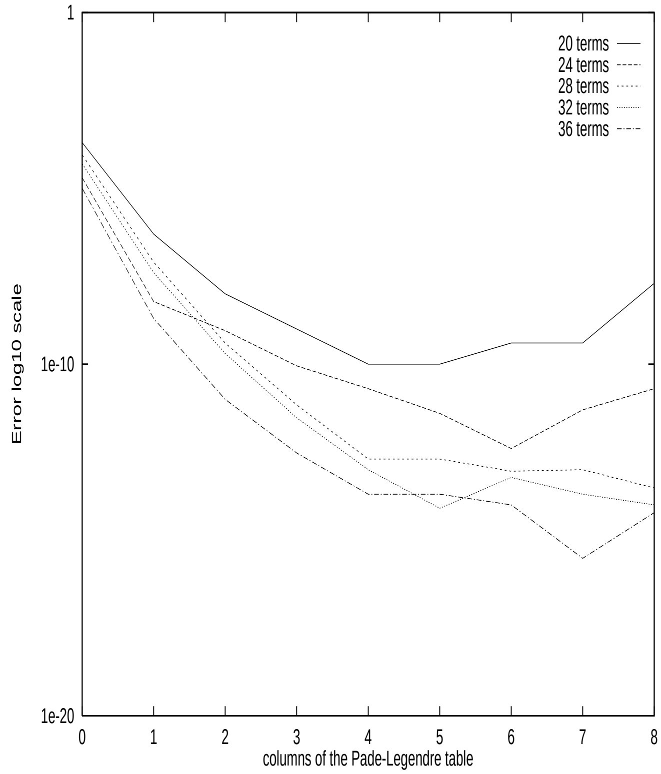


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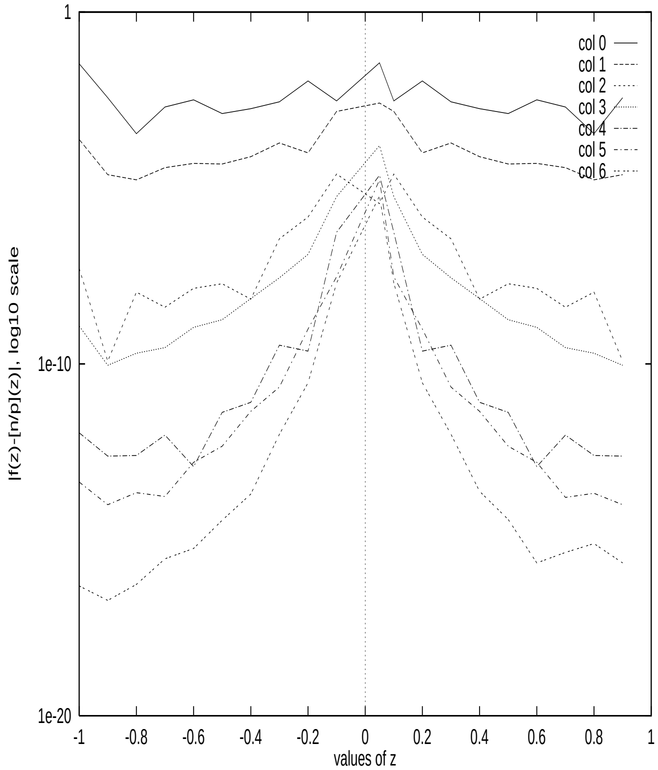
Figure 5: the step function with singularity in $a = 0.1$



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