Linear Difference Operators and Acceleration Methods

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Abstract

The aim of this paper is the study of the kernel and acceleration properties of sequence transformations of the form $T_n = L(S_n/D_n)/L(1/D_n)$, where (S_n) is the sequence for which we want to compute the limit, (D_n) is an error estimate and L is a linear difference operator. We will obtain those properties for sequence transformations corresponding to different classes of operators L by using some results of the theory of linear operators. We will then study the following problem: given a class of sequences with some asymptotic expansion of the error, how can we construct an operator L for which the corresponding transformation accelerates the convergence of that class. Some applications are given.

Keywords: extrapolation methods, convergence acceleration.

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1 Introduction.

In a recent paper [4], a new derivation of extrapolation methods has been proposed, based on a formalism first given by [10]. This approach provides a systematic construction of extrapolation algorithms, leading to a better understanding of the mechanism of extrapolation. We will begin by recalling that formalism.

Let us introduce some notations:

- $\mathcal{S}$ will be the set of all sequences of complex numbers;
- an element $s \in \mathcal{S}$ will be denoted by $s = (s_n)$ and s_n will be the n–th term of the sequence;
- $\begin{cases} T: & \mathcal{S} \longrightarrow \mathcal{S} \\ & s = (s_n) \longmapsto T(s) = (T(s_n)) \end{cases}$ is a sequence transformation and we will denote the n–th term of the transformed sequence $T(s)$ by $T(s_n)$;

- we will denote by $\mathcal{S}^*$ the set

$$\mathcal{S}^* = \{(D_n) \in \mathcal{S} : \quad \forall n \in \mathrm{N} \quad D_n \neq 0\}.$$

  The condition $D_n \neq 0 \;\; \forall n \in \mathrm{N}$ can be replaced by $\exists N \in \mathrm{N} \;\; \forall n \geq N \;\; D_n \neq 0$ without loss of generality. In this case the transformed sequence will be $(T_n, n \geq N)$.

Let us consider a sequence $(S_n) \in \mathcal{S}$ satisfying

$$S_n - S_\infty = a_n D_n \;\; \forall n \in \mathrm{N}, \tag{1}$$

where (a_n) is an unknown sequence, (D_n) a known one called *remainder (or error) estimate* and S_∞ an unknown complex number that we want to approach. If (S_n) converges to S_∞ then S_∞ is called its limit, if $\lim_{n\to\infty} a_n D_n = \pm\infty$ yet the accelerated series converges, then S_∞ is called its antilimit. In order to compute or to approach S_∞ we will use an annihilation difference operator.

Definition 1 *L is called an annihilation difference operator for a sequence $a = (a_n)$ if:*

1. *L is a linear mapping of the set of complex sequences $\mathcal{S}$ into itself*

$$L \quad : u = (u_n) \in \mathcal{S} \longmapsto L(u) = (L(u_n)) \in \mathcal{S}.$$

2. $\exists N \in \boldsymbol{N} \quad \forall n \geq N \quad L(a_n) = 0.$

The most general form is given by

$$L(u_n) = \sum_{i=-p_n}^{q_n} G_i(n) u_{n+i} \quad n \in \mathrm{N},$$

where p_n and q_n are nonnegative integers which can eventually depend on n, $u_i = 0$ for $i < 0$, and the G_i's are given functions of n which can also depend on auxiliary fixed sequences (if the auxiliary sequences, on which the G_i's could depend, also depend on some terms of the sequence (u_n) itself, then these terms are *fixed* in the G_i's and thus the operator L is still a linear one). To each linear mapping L from $\mathcal{S}$ to $\mathcal{S}$, and each sequence $(D_n) \in \mathcal{S}^*$ we can associate the following sequence transformation:

$$\begin{cases} T: \;\; \mathcal{S} & \longrightarrow \;\; \mathcal{S} \\ \quad\;\; S = (S_n) & \longmapsto \;\; T(S) = (T(S_n)) = (T_n): \quad T_n = \dfrac{L(S_n/D_n)}{L(1/D_n)}, \quad n \in \mathrm{N} \end{cases} \tag{2}$$

If L is an annihilation difference operator for (a_n) and (S_n) satisfies (1) then $T_n = S_\infty$. Let us give an example. We consider a sequence (S_n) satisfying

$$S_n = S_\infty + c_1 x_n + \cdots + c_k x_n^k$$

where (x_n) is a known sequence. If we set

$$D_n = x_n, \quad a_n = c_1 + \cdots + c_k x_n^{k-1} \quad n \in \mathrm{N}\,,$$

then the divided difference operator δ_k of order k at the points x_i is an annihilation operator for the sequence (a_n) (we recall that this operator is recursively defined by

$$\delta_{k+1}(u_n) = \frac{\delta_k(u_{n+1}) - \delta_k(u_n)}{x_{n+k+1} - x_n} \text{ with } \delta_0(u_n) = u_n.)$$

So, if we choose $L = \delta_k$, the sequence transformation (2) gives

$$T_n = S_\infty \quad \forall n \in \mathrm{N}\,.$$

This is the well–known Richardson extrapolation procedure [9].

In [4], different choices of the linear operator L (independent of (S_n), dependent on (S_n) or linear combination of several operators) and particular choices for the error estimates (D_n) have been considered. They lead to different well-known transformations and some generalizations. We remarked that a great majority of the most used extrapolation algorithms could be put into this framework. Composition of operators leading to iteration of sequence transformations has also been studied in a systematic way [4, 11]. In this paper, we will continue the work of those papers by studying the kernel and acceleration properties of sequence transformations of the form (2), using the properties of the linear difference operators L and of the solutions of the difference equations $L(a_n) = 0, n \in \mathrm{N}$.

In the general study of acceleration (or, for some authors, extrapolation) methods, we can proceed in two different ways:

1. Given a sequence transformation T (or, in our approach, given a difference operator L and an error estimate (D_n)), determine

   (a) its kernel, i.e., the set of sequences for which $T(S_n) = S_\infty \;\; \forall n \geq N$. In our approach, $\mathrm{Ker}(T)$ can be writen as

   $$\mathrm{Ker}(T) = \{(S_n) : \;\; \forall n \in \mathrm{N} \;\;\; S_n - S_\infty = a_n D_n, \text{ and } \exists N \in \mathrm{N} \;\;\; \forall n \geq N \;\;\; L(a_n) = 0\}\,.$$

   The structure of the kernel follows immediately from the choice of the linear operator L and it has been obtained in [4] for operators leading to well–known extrapolation algorithms and in [5] for the composition of operators.

   (b) The class of sequences that the transformation T accelerates, that is, the sequences (S_n) for which $\lim_{n\to\infty}(T(S_n) - S_\infty)/(S_n - S_\infty) = 0$. In our framework, as

   $$\frac{T_n - S_\infty}{S_n - S_\infty} = \frac{L((S_n - S_\infty)/D_n)}{L(1/D_n)} \frac{1}{a_n D_n} = \frac{L(a_n)/a_n}{L(1/D_n) D_n},$$

   the set of sequences accelerated by T and which we will denote by $\mathcal{A}(T)$ is given by

   $$\mathcal{A}(T) = \left\{(S_n) : \;\; S_n - S_\infty = a_n D_n \;\; n \in \mathrm{N} \;\; \text{ and } \lim_{n\to\infty} \frac{L(a_n)/a_n}{L(1/D_n) D_n} = 0\right\}.$$

   We will say that L is an *accelerating operator* for $(S_n) \in \mathcal{A}(T)$.

2. Given a class $\mathcal{C}$ of sequences (S_n) satisfying (1), with (D_n) a known sequence and (a_n) satisfying some given properties, construct a sequence transformation of the form (2) (or, in an equivalent way, find a difference operator L) for which

   (a) $\forall (S_n) \in \mathcal{C}$, $(S_n) \in \mathrm{Ker}(T)$;

   or, in the case where this is not possible,

   (b) $\forall (S_n) \in \mathcal{C}$, $(S_n) \in \mathcal{A}(T)$.

   Given a sequence (S_n) satisfying (1), from the properties of the sequence (a_n), it is sometimes possible to find an annihilation difference operator for (a_n) and so compute the exact value of S_∞. For instance, if we know that (a_n) is a polynomial of degree $k-1$ in n, then we can choose $L = \Delta^k$ (the k-th order forward difference operator); other examples can be seen in [4].

   But these are the simplest cases. The general case consists in supposing that (a_n) has an asymptotic expansion in some comparison scale $\{g_i(n)\}_{i=0}^{\infty}$, that is,

$$\forall k \ \ a_n = \sum_{i=0}^{k} \alpha_i g_i(n) + o(g_k(n)) \quad (n \to \infty),$$

   with $\{g_i(n)\}_{i=0}^{\infty}$ a family of sequences satisfying $g_{i+1}(n) = o(g_i(n)) \quad (n \to \infty) \quad \forall i$, and α_i unknown constants. It is impossible now to find an annihilation difference operator for (a_n) and so to find in this way a sequence transformation such that $T(S_n) = S_\infty \ \forall n \geq N$. So in this case we will be interested in the acceleration properties of the transformation, that is, in the characterization of the sequences (S_n) for which $\lim_{n\to\infty}(L(a_n)/a_n))/(L(1/D_n)D_n) = 0$.

These two points of view are complementary and will interest us all along the paper. As in our framework we have interpreted sequence transformations in terms of difference operators, we will use some properties of these operators to give a contribution to the solution of the previous problems.

We will be interested in both approaches. In the case of approach 1., we are going to consider several classes of difference operators L and, from the properties of the solutions of the associated difference equation $L(u_n) = 0$, we will study, for the corresponding sequence transformation T (defined by (2)):

1. the properties of the kernel, completing the study given in the above mentioned works;

2. the acceleration properties of T.

This will be the subject of sections 2. and 3.

There are many linear difference equations for which we don't have a compact form for the solution but only an asymptotic expansion of it (or its first terms). We will make use of this information for, using approach 2., constructing an extrapolation method for a given sequence (S_n) for which we know the first terms of the asymptotic expansion of (a_n), and we will obtain the speed of convergence of the transformed sequence.

Let us begin by considering the sequence transformations associated with a linear difference operator with constant coefficients.

2 Linear difference operators with constant coefficients.

Let us consider first a difference operator of order k of the form

$$L^{(k)}(u_n) = \sum_{i=0}^{k} p_i^{(k)} u_{n+i}, \tag{3}$$

where the quantities $p_i^{(k)}$ are independent of n. If its characteristic polynomial is given by

$$P_k(x) = \sum_{i=0}^{k} p_i^{(k)} x^i = (x-\lambda_1)^{s_1} \cdots (x-\lambda_m)^{s_m}, \tag{4}$$

then it is well–known (see, for instance, [7]) that a set of k linearly independent solutions of the associated difference equation $L^{(k)}(u_n) = 0$ is given by

$$\forall n \in \mathrm{N} \quad u_n^{(i,j)} = n^i \lambda_j^n, \text{ for } 0 \le i \le s_j - 1, \quad 1 \le j \le m.$$

Let $(S_n) \in \mathcal{S}$ satisfying (1) with $(D_n) \in \mathcal{S}^*$ and $T^{(k)}$ be the corresponding sequence transformation as defined in the previous section:

$$\begin{cases} T^{(k)}: \; \mathcal{S} & \longrightarrow \; \mathcal{S} \\ \quad S = (S_n) & \longmapsto \; T^{(k)}(S) = (T^{(k)}(S_n)) = (T_n^{(k)}): \; T_n^{(k)} = \dfrac{L^{(k)}(S_n/D_n)}{L^{(k)}(1/D_n)}, \quad n \in \mathrm{N} \end{cases} . \tag{5}$$

We will obtain, from the the form of the solutions of the difference equation, the description of the kernel and the acceleration properties of these transformations.

2.1 Kernel.

We have trivially the following result:

Proposition 1 *Let $L^{(k)}$ be an operator of the form (3) with characteristic polynomial (4), $(D_n) \in \mathcal{S}^*$ and $T^{(k)}$ the corresponding transformation (5). Then*

$$(S_n) \in Ker(T^{(k)}) \Leftrightarrow \textit{ for } n \in N \quad S_n - S_\infty = a_n D_n \quad \textit{with } a_n = \sum_{i=1}^{m} q_i(n)\lambda_i^n \textit{ and}$$

$q_i(n) \in \Pi_{s_{i-1}}$ (the set of polynomials of degree less or equal s_{i-1}).

Corollary 1 *Let us suppose that $s_1 \le s_2 \le \cdots \le s_m$ and consider a sequence (S_n) satisfying:*

$$S_n = S_\infty + r_n \times \left[\lambda_1^n \left(c_0^{(1)} + c_1^{(1)}\frac{1}{n} + \cdots c_{s_1-1}^{(1)}\frac{1}{n^{s_1-1}}\right) + \cdots + \lambda_m^n \left(c_0^{(m)} + \cdots + c_{s_1-1}^{(m)}\frac{1}{n^{s_1-1}}\right)\right], \tag{6}$$

with (r_n) a known sequence ($(r_n) \in \mathcal{S}^*$) and $c_i^{(j)}$ unknown constants. Let us choose $D_n = n^{-s_1+1} r_n$ for $n \in N$. If $L^{(k)}(1/D_n) \neq 0$ for $n \in N$ then, applying to (S_n) the transformation $T^{(k)}$ given by (5), we obtain

$$T_n^{(k)} = S_\infty \quad \forall n \in N.$$

Proof: It is sufficient to remark that (6) can be writen in the form $S_n - S_\infty = r_n n^{-s_1+1} a_n$, where (a_n) is a linear combination of solutions of $L^{(k)}(u_n) = 0$.

$\triangle$

2.2 Acceleration properties.

The error of the transformed sequences (5), that is, the difference between S_∞ (the value we want to compute) and its approximation $T_n^{(k)}$ can be writen

$$T_n^{(k)} - S_\infty = \frac{L^{(k)}((S_n - S_\infty)/D_n)}{L^{(k)}(1/D_n)} = \frac{L^{(k)}(a_n)}{L^{(k)}(1/D_n)} \quad n \in \mathrm{N}\,, \tag{7}$$

and to get the speed of convergence of this error sequence we have to consider two cases.

2.2.1 $\lim_{n\to\infty} D_n/D_{n+1} = \sigma$ and $P_k(\sigma) \neq 0$

In this case we obtain

$$\lim_{n\to\infty} D_n L^{(k)}(1/D_n) = P_k(\sigma) \neq 0 \tag{8}$$

and so

$$\lim_{n\to\infty} (T_n^{(k)} - S_\infty) = \frac{1}{P_k(\sigma)} \lim_{n\to\infty} D_n L^{(k)}(a_n).$$

Therefore we will have convergence acceleration if

$$\lim_{n\to\infty} \frac{L^{(k)}(a_n)}{a_n} = 0. \tag{9}$$

We obtain the following acceleration result:

Theorem 1 *Let (S_n) be a sequence of complex numbers satisfying:*

$$S_n - S_\infty = r_n \times \left[\lambda_1^n \sum_{i=0}^{\infty} c_i^{(1)} \frac{1}{n^i} + \cdots + \lambda_m^n \sum_{i=0}^{\infty} c_i^{(m)} \frac{1}{n^i}\right], \tag{10}$$

where

- $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_m| = 1$;
- $\lim_{n\to\infty} r_n/r_{n+1} = \sigma$ *with* $P_k(\sigma) \neq 0$.

Then, if in (5) the operator $L^{(k)}$ satisfies (3) and (4) with $s_1 \leq s_2 \leq \cdots \leq s_m$, and we choose $D_n = n^{-s_1+1} r_n$, we obtain

$$\left|T_n^{(k)} - S_\infty\right| = O(n^{-2s_1} |r_n|) \quad (n \to \infty).$$

In order to prove this theorem we will need a preliminary lemma

Lemma 1 *Let $P_k(x)$ be the polynomial defined by (4) and (A_n) the sequence defined by*

$$A_n = \sum_{j=0}^{k} p_j^{(k)} \frac{\lambda_i^{n+j}}{n+j} \quad n \in \boldsymbol{N}\,, \tag{11}$$

with λ_i such that $P_k(\lambda_i)=0$.

Then the order of convergence of (A_n) is given by

$$A_n = C^* \lambda_1^{n+s_1} \frac{1}{n^{s_1+1}} \left(1 + O\left(\frac{1}{n}\right)\right) \quad (n \to \infty) \quad (C^* \textit{ independent of } n).$$

Proof of the lemma:

As $(n+j)^{-1} = \int_0^\infty e^{-t(n+j)} dt$ we can write

$$\begin{aligned} A_n &= \sum_{j=0}^{k} p_j^{(k)} \lambda^{n+j} \int_0^\infty e^{-t(n+j)} dt = \lambda^n \int_0^\infty e^{-tn} \left(\sum_{j=0}^{k} p_j^{(k)} \lambda^j e^{-tj} \right) dt = \\ &= \lambda^n \int_0^\infty e^{-tn} P_k(\lambda e^{-t}) dt. \end{aligned}$$

If P_k is given by (4) and, for instance, $\lambda = \lambda_1$ then

$$\begin{aligned} A_n &= \lambda_1^n \int_0^\infty e^{-tn} (\lambda_1 e^{-t} - \lambda_1)^{s_1} \cdots (\lambda_1 e^{-t} - \lambda_m)^{s_m} dt = \\ &= \lambda_1^n \int_0^\infty e^{-t} \lambda_1^{s_1} (e^{-t} - 1)^{s_1} \cdots (\lambda_1 e^{-t} - \lambda_m)^{s_m} dt. \end{aligned}$$

But we can write $\lambda_j = \rho e^{i\theta_j} \quad 0 \le \theta_j < 2\pi \quad j = 1, \cdots, m$ and so

$$\begin{aligned} A_n &= \lambda_1^{n+s_1} \rho^{s_2+\cdots+s_m} \int_0^\infty e^{-tn} (e^{-t} - 1)^{s_1} (e^{-t+i\theta_1} - e^{i\theta_2})^{s_2} \cdots (e^{-t+i\theta_1} - e^{i\theta_m})^{s_m} dt = \\ &= C \lambda_1^{n+s_1} \int_0^\infty e^{-tn} t^{s_1} \left(\sum_{j=0}^{\infty} \alpha_j t^j \right) dt \text{ (where } C, \alpha_j \text{ are coefficients independent of } n). \end{aligned}$$

As $\int_0^\infty e^{-tn} t^s dt = s!/n^{s+1}$ we finally get

$$A_n = C^* \lambda_1^{n+s_1} \frac{1}{n^{s_1+1}} \left(1 + O\left(\frac{1}{n}\right)\right) \quad (n \to \infty) \quad (C^* \text{ independent of } n).$$

$\triangle$

Proof of the theorem:

We set

$$a_n = n^{s_1-1} \left[\lambda_1^n \sum_{i=0}^{\infty} c_i^{(1)} \frac{1}{n^i} + \cdots + \lambda_m^n \sum_{i=0}^{\infty} c_i^{(m)} \frac{1}{n^i} \right].$$

Then, because the operator annihilates the first s_1 terms in each summation, we have

$$D_n L^{(k)}(a_n) = D_n \sum_{j=0}^{k} p_j^{(k)} \left(\lambda_1^{n+j} \sum_{i=0}^{\infty} c_{s_1+i}^{(1)} \frac{1}{(n+j)^i} + \cdots + \lambda_m^{n+j} \sum_{i=0}^{\infty} c_{s_1+i}^{(m)} \frac{1}{(n+j)^i} \right).$$

This can also be writen in the following form:

$$\begin{aligned} D_n L^{(k)}(a_n) &= D_n \left[\sum_{j=0}^{k} p_j^{(k)} \left(c_{s_1}^{(1)} \frac{\lambda_1^{n+j}}{n+j} + \cdots + c_{s_1}^{(m)} \frac{\lambda_m^{n+j}}{n+j} \right) \right] \left(1 + O\left(\frac{1}{n}\right) \right) \\ &= D_n \left[c_{s_1}^{(1)} \left(\sum_{j=0}^{k} p_j^{(k)} \frac{\lambda_1^{n+j}}{n+j} \right) + \cdots + c_{s_1}^{(m)} \left(\sum_{j=0}^{k} p_j^{(k)} \frac{\lambda_m^{n+j}}{n+j} \right) \right] \left(1 + O\left(\frac{1}{n}\right) \right). \quad (12) \end{aligned}$$

In order to obtain the asymptotic behavior of this sequence we use the result of the previous lemma and we obtain

$$D_n L^{(k)}(a_n) = D_n \left(C_1 \frac{\lambda_1^{n+s_1}}{n^{s_1+1}} + C_2 \frac{\lambda_2^{n+s_2}}{n^{s_2+1}} + \cdots + C_m \frac{\lambda_m^{n+s_m}}{n^{s_m+1}} \right) \left(1 + O\left(\frac{1}{n}\right) \right). \quad (13)$$

So

$$\left| D_n L^{(k)}(a_n) \right| = |D_n| \frac{1}{n^{s_1+1}} \left(1 + O\left(\frac{1}{n}\right) \right).$$

Finally, as $\lim_{n\to\infty} D_n/D_{n+1} = \lim_{n\to\infty} r_n/r_{n+1} = \sigma$, (8) is satisfied and from (7) we obtain

$$\left| T_n^{(k)} - S_\infty \right| = O\left(n^{-s_1+1} r_n n^{-s_1-1} \right) = O(n^{-2s_1} r_n).$$

△

2.2.2 $\lim_{n\to\infty} D_n/D_{n+1} = \sigma$ and $P_k(\sigma) = 0$

This case is less simple to study because now, as $\lim_{n\to\infty} D_n L^{(k)}(1/D_n) = 0$, (9) is not a sufficient condition of acceleration. So we have to impose some supplementary conditions on (D_n) and on the characteristic polynomial P_k in order to obtain the order of convergence of $(D_n L^{(k)}(1/D_n))$.

Proposition 2 *Let us suppose that the sequence (D_n) satisfies:*

$$\frac{D_n}{D_{n+1}} = \sigma + \sigma_n \quad \textit{with } P_k(\sigma) = p_0^{(k)} + p_1^{(k)}\sigma + \cdots + p_k^{(k)}\sigma^k = 0, \lim_{n\to\infty} \sigma_n = 0.$$

Moreover we suppose that one of the following conditions is satisfied:

a) $\lim_{n\to\infty} \sigma_{n+1}/\sigma_n = 1$ *and σ is a simple root of P_k;*

or

b) $\lim_{n\to\infty} \sigma_{n+1}/\sigma_n = \sigma_* \neq 1$ *and* $q_{k-1}(\sigma) - \sigma_* q_{k-1}(\sigma\sigma_*) \neq 0$

(where $q_{k-1}(x) = p_1^{(k)} + p_2^{(k)} x + \cdots + p_k^{(k)} x^{k-1}$).
Then the difference operator $L^{(k)}$ with characteristic polynomial P_k satisfies

$$L^{(k)}(\frac{1}{D_n}) \sim C\frac{\sigma_n}{D_n} \quad (n \to \infty) \quad (C \neq 0). \tag{14}$$

Proof: From the conditions on the sequence (D_n) we get

$$\frac{D_n}{D_{n+i}} = \prod_{j=0}^{i-1}(\sigma + \sigma_{n+j}) \sim \sigma^i + \sigma^{i-1}\left(\sum_{j=0}^{i-1}\sigma_{n+j}\right)$$

Then, for the sequence $L^{(k)}(1/D_n)$ we have:

$$\begin{aligned} L^{(k)}(1/D_n) &= \frac{1}{D_n}\left[p_0^{(k)} + p_1^{(k)}\frac{D_n}{D_{n+1}} + \cdots + p_k^{(k)}\frac{D_n}{D_{n+k}}\right] \sim \\ &\sim \frac{1}{D_n}\left[p_0^{(k)} + p_1^{(k)}(\sigma + \sigma_n) + \cdots + p_k^{(k)}(\sigma^k + \sigma^{k-1}\sum_{i=0}^{k-1}\sigma_{n+i})\right] \end{aligned}$$

As $p_k(\sigma) = 0$ we obtain

$$L^{(k)}(1/D_n) \sim \frac{\sigma_n}{D_n}\left[p_1^{(k)} + p_2^{(k)}\frac{\sigma}{\sigma_n}\sum_{i=0}^{1}\sigma_{n+i} + \cdots + p_{k-1}^{(k)}\frac{\sigma^{k-2}}{\sigma_n}\sum_{i=0}^{k-2}\sigma_{n+i} + p_k^{(k)}\frac{\sigma^{k-1}}{\sigma_n}\sum_{i=0}^{k-1}\sigma_{n+i}\right] \tag{15}$$

- If conditions in *a)* are satisfied then (15) becomes

$$L^{(k)}(1/D_n) \sim \frac{\sigma_n}{D_n}\left[p_1^{(k)} + 2p_2^{(k)}\sigma + \cdots + (k-1)p_{k-1}^{(k)}\sigma^{k-2} + kp_k^{(k)}\sigma^{k-1}\right] = \frac{\sigma_n}{D_n}P_k'(\sigma)$$

with $P_k'(\sigma) \neq 0$ and so (14) follows.
- If conditions in *b)* are satisfied, then from (15) we obtain

$$\begin{aligned} L^{(k)}(1/D_n) &\sim \frac{\sigma_n}{D_n}\left[p_1^{(k)} + p_2^{(k)}\sigma(1+\sigma_*) + p_{k-1}^{(k)}\sigma^{k-2}\sum_{i=0}^{k-2}\sigma_*^i + p_k^{(k)}\sigma^{k-1}\sum_{i=0}^{k-1}\sigma_*^i\right] = \\ &= \frac{\sigma_n}{D_n}\left[p_1^{(k)} + p_2^{(k)}\sigma\frac{1-\sigma_*^2}{1-\sigma_*} + \cdots + p_k^{(k)}\sigma^{k-1}\frac{1-\sigma_*^k}{1-\sigma_*}\right] = \\ &= \frac{q_{k-1}(\sigma) - \sigma_* q_{k-1}(\sigma\sigma_*)}{1-\sigma_*}\frac{\sigma_n}{D_n} \end{aligned}$$

which gives (14).

△

From this result we easily obtain:

Theorem 2 *Let (S_n) be a sequence that can be writen in the form $S_n - S_\infty = a_n D_n, \quad n \in N$ with*

- (D_n) *satisfying the conditions of the previous proposition;*
- $a_n = n^{s_1-1}\left(\sum_{i=0}^{\infty} c_i^{(1)} \frac{\lambda_1^n}{n^i} + \cdots + \sum_{i=0}^{\infty} c_i^{(m)} \frac{\lambda_m^n}{n^i}\right)$ *for* $n \in \mathbf{N}$ *;*
- $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_m|$.

Then the transformed sequence $(T_n^{(k)})$ *given by (5), where the operator* $L^{(k)}$ *is given by (3) and (4), satisfies:*

$$\frac{T_n^{(k)} - S_\infty}{S_n - S_\infty} \sim \frac{C}{n^{2s_1}\sigma_n} \quad (n \to \infty),$$

and so

$$(S_n) \in \mathcal{A}(T) \text{ if and only if } \lim_{n\to\infty} n^{2s_1}\sigma_n = \infty.$$

Proof: We have already seen, in (13) that

$$L^{(k)}(a_n) = \left(C_1 \frac{\lambda_1^{n+s_1}}{n^{s_1+1}} + \cdots + C_m \frac{\lambda_m^{n+s_m}}{n^{s_m+1}}\right)\left(1 + O\left(\frac{1}{n}\right)\right).$$

So we obtain $L^{(k)}(a_n)/a_n \sim C^* n^{-2s_1} \quad (n \to \infty)$. By the previous proposition we know that $L^{(k)}(1/D_n)D_n \sim C'\sigma_n \ (n \to \infty)$. As the ratio of the error sequences $(T_n^{(k)} - S_\infty)$ and $(S_n - S_\infty)$ is given by (7), the result follows immediately.

△

3 Some classes of general difference operators.

We will now study the structure of the kernel and acceleration properties of extrapolation methods corresponding to difference operators with coefficients depending on n.

3.1 Linear operator of first order.

Let $p(n)$ and $q(n)$ two polynomials of degree r and s respectively. We can write them in the form

$$\begin{cases} p(n) &= a\prod_{i=1}^{r}(n-\alpha_i), \quad \alpha_i \in \mathbf{R}\ , \\ q(n) &= b\prod_{j=1}^{s}(n-\beta_j), \quad \beta_j \in \mathbf{R}\ . \end{cases}$$

We set

$$\alpha = \max_{1\le i\le r} \alpha_i \ , \beta = \max_{1\le j\le s} \beta_j, \text{ and } N \in \mathrm{N} \text{ such that } N-1 \le \max(\alpha,\beta) < N. \tag{16}$$

Let us consider a linear operator of the form

$$L(u_n) = u_{n+1} - \frac{p(n)}{q(n)}u_n, \text{ for } n \ge N. \tag{17}$$

The solution of the difference equation $L(u_n) = 0, n \geq N$ is given by [8]

$$u_n = \left(\frac{a}{b}\right)^n \prod_{i=1}^{r} \Gamma(n-\alpha_i) / \prod_{j=1}^{s} \Gamma(n-\beta_j), n \geq N$$

and so we immediately obtain the kernel of the corresponding sequence transformation:

Theorem 3 *Let $(D_n) \in \mathcal{S}^*$ and let L be a linear operator of the form (17). If T is the corresponding transformation defined by (5) then*
Ker $(T) = \{(S_n) \in \mathcal{S} \;\;:\;\; S_n - S_\infty = a_n D_n, n \geq N$ with a_n satisfying

$$a_n = \left(\frac{a}{b}\right)^n \prod_{i=1}^{r} \Gamma(n-\alpha_i) / \prod_{j=1}^{s} \Gamma(n-\beta_j)$$

for some $r, s, a, b, \alpha_i, \beta_j \in \boldsymbol{R}$ and N defined as in (16)$\}$

As we can write

$$\frac{T_n - S_\infty}{S_n - S_\infty} = \frac{L(a_n)}{a_n} \frac{1/D_n}{L(1/D_n)} = \frac{\dfrac{a_{n+1}}{a_n} - \dfrac{p(n)}{q(n)}}{\dfrac{D_n}{D_{n+1}} - \dfrac{p(n)}{q(n)}} \tag{18}$$

we easily get:

Theorem 4 *Let (S_n) be a sequence of the form $S_n - S_\infty = a_n D_n$, $n \geq N$, with $(D_n) \in \mathcal{S}^*$ and* $\lim_{n\to\infty} a_{n+1}/a_n = 1 \neq \lim_{n\to\infty} D_n/D_{n+1}$. *We consider the sequence transformation*

$$T_n = \frac{L(S_n/D_n)}{L(1/D_n)} \quad n \geq N, \tag{19}$$

where L has the form (17). Then, if $\lim_{n\to\infty} p(n)/q(n) = 1$ *we obtain*

$$\lim_{n\to\infty} (T_n - S_\infty)/(S_n - S_\infty) = 0,$$

that is, if S_∞ is finite then $(S_n) \in \mathcal{A}(T)$.

From (18) we easily see that we will get good acceleration properties for (T_n) if we can obtain a good approximation of (a_{n+1}/a_n) by the rational function of n, $(p(n)/q(n))$. In fact, we obtain:

Theorem 5 *Let (S_n) be a sequence of the form $S_n - S_\infty = a_n D_n$ for $n \geq N$, where*
a) (a_n) satisfies:

$$a_{n+1}/a_n \sim \sum_{i=0}^{\infty} d_i n^{-i} \text{ where } d_i, \quad i = 0, \cdots, 2k \text{ are known;}$$

b) (D_n) satisfies:

$$D_n/D_{n+1} \sim \sum_{i=0}^{\infty} e_i n^{-i} \text{ with } e_i = d_i, \quad i = 0, \cdots l-1; \quad e_l \neq d_l \quad (l < 2k).$$

Let $[k/k]$ be the Padé-approximant [3] of the function $f(x) = \sum_{i=0}^{\infty} d_i x^i$ and set

$$[k/k]_f(x) = \frac{P(x)}{Q(x)}.$$

We consider the operator L of the form (17) with $r = s = k$ and

$$\begin{aligned} p(n) &= P(1/n)n^k, \\ q(n) &= Q(1/n)n^k. \end{aligned} \tag{20}$$

Then the corresponding sequence transformation defined by (19) accelerates the convergence of (S_n). Moreover the speed of convergence can be measured by

$$\frac{T_n - S_\infty}{S_n - S_\infty} = O(n^{l-2k-1}) \quad (n \to \infty).$$

Proof: By definition of the Padé approximants we know that $f(x) - [k/k](x) = O(x^{2k+1}) \quad (x \to 0)$ which implies

$$\frac{a_{n+1}}{a_n} - \frac{P(1/n)}{Q(1/n)} = O\left(\frac{1}{n^{2k+1}}\right) \quad (n \to \infty).$$

Defining the polynomials $p(n)$ and $q(n)$ of degree k by (20), from condition b) we get

$$\frac{D_n}{D_{n+1}} - \frac{p(n)}{q(n)} \sim (e_l - d_l)\frac{1}{n^l} \quad (n \to \infty),$$

and replacing in (18) the result follows.

△

If (D_n/D_{n+1}) doesn't have an asymptotic expansion in the powers of $(1/n)$ but satisfies

$$D_n/D_{n+1} = e_0 + \sum_{i=0}^{l} e_i g_i(n) + o(g_l(n)) \quad (g_{i+1}(n) = o(g_i(n)) \quad (n \to \infty))$$

(for instance, if $D_n = n^\alpha (\log n)^\beta$, then $D_n/D_{n+1} = 1 + c_1/n + c_2/(n \log n) + o(n^{-1}(\log n)^{-1})$),
we can still determine the asymptotic behavior of $(D_n/D_{n+1} - p(n)/q(n))$ and obtain the corresponding acceleration results.

3.2 Difference operators with polynomial coefficients.

Let us consider now some classes of difference operators of the form

$$L(u_n) = \sum_{i=0}^{l} \lambda_i(n) u_{n+i} \ , \quad \text{with} \quad \lambda_i(n) \text{ polynomials in } n,$$

for which we can obtain l linearly independent solutions for the homogeneous difference equation $L(u_n) = 0$. We will give, for the corresponding sequence transformations

$$T_n = \frac{L(S_n/D_n)}{L(1/D_n)}, \tag{21}$$

the kernel and acceleration properties. We will see that some of these extrapolation methods are very well suited for accelerating the convergence of logarithmic sequences (that is, sequences that satisfy $\lim_{n\to\infty}(S_{n+1} - S_\infty)/(S_n - S_\infty) = 1$ and so converging very slowly) for which we have an asymptotic expansion of the error in terms of the sequences $\left\{\frac{1}{n(n+1)\cdots(n+i)}\right\}_{i=0}^{\infty}$.

In order to simplify notations, we introduce the following elementary operators:

$$E(u_n) = u_{n+1} \ , \ \Omega(u_n) = n u_{n+1} \ , \ \pi(u_n) = n\Delta u_n.$$

The composition of these operators gives

$$E^r(u_n) = u_{n+r} \ , \ \Omega^r(u_n) = n(n+1)\cdots(n+r-1)u_{n+r} \ , \ \pi^r(u_n) = \pi(\pi^{r-1}(u_n)).$$

First, let us consider the following class of operators L

a) $L(u_n) = (\Omega^l + \lambda_1\Omega^{l-1} + \cdots + \lambda_{l-1}\Omega + \lambda_l)(u_n)$, λ_i's given constants.

If we compute the roots of the polynomial $p_l(x) = x^l + \lambda_1 x^{l-1} + \cdots + \lambda_l$ we can write L as a composition of simple operators. Let us consider the two following cases:

$$L_1(u_n) = (\Omega - \alpha_1)(\Omega - \alpha_2)\cdots(\Omega - \alpha_l)(u_n) \text{ with } \alpha_i \neq \alpha_j \ \forall i \neq j; \tag{22}$$

$$L_2(u_n) = (\Omega - \alpha)^l(u_n). \tag{23}$$

For the general case $L(u_n) = (\Omega - \alpha_1)^{s_1}(\Omega - \alpha_2)^{s_2}\cdots(\Omega - \alpha_l)^{s_l}(u_n)$ the results are trivially obtained from the ones for these two types of operators.

From the theory of linear difference operators (see for instance [2]) we obtain:

$$L_1(u_n) = 0 \Leftrightarrow u_n = \frac{1}{(n-1)!}\sum_{i=1}^{l} A_i\alpha_i^n, \quad A_i \ \text{ constants};$$

$$L_2(u_n) = 0 \Leftrightarrow u_n = \frac{\alpha^n}{(n-1)!}\sum_{i=0}^{l-1} A_i n^i, \quad A_i \ \text{ constants}.$$

For the corresponding transformations given by (21), the structure of the kernels follows immediately:

Theorem 6 *Let us consider the sequence transformations*

$$T_n^{(i)} = \frac{L_i(S_n/D_n)}{L_i(1/D_n)}, \quad n \in \boldsymbol{N}\,, \quad i = 1, 2, \tag{24}$$

with L_i, $i = 1, 2$ given by (22) and (23). Then

$$a)\ (S_n) \in Ker(T^{(1)}) \Leftrightarrow S_n - S_\infty = \frac{D_n}{(n-1)!} \sum_{i=1}^{l} A_i \alpha_i^n \text{ for } n \in \boldsymbol{N}\,;$$

$$b)\ (S_n) \in Ker(T^{(2)}) \Leftrightarrow S_n - S_\infty = \frac{D_n \alpha^n}{(n-1)!} \sum_{i=0}^{l-1} A_i n^i \text{ for } n \in \boldsymbol{N}\,.$$

In order to obtain the acceleration properties of these transformations we have to determine the asymptotic behavior of the sequences $(L_i(u_n))$ $i = 1, 2$ from some properties of the sequence (u_n).

Lemma 2 *Let (u_n) be a sequence satisfying*

$$\lim_{n\to\infty} n \frac{u_{n+1}}{u_n} = \beta.$$

If we choose $\alpha_i \neq \beta$, $i = 1, \cdots l$ then the asymptotic behavior of the sequences $(L_1(u_n))$, where L_1 is given by (22) is

$$L_1(u_n) \sim (\beta - \alpha_1)(\beta - \alpha_2) \cdots (\beta - \alpha_l) u_n \quad (n \to \infty).$$

Proof: We will prove this result by induction on l. For $l = 1$ we obtain

$$(\Omega - \alpha_1) u_n = n u_{n+1} - \alpha u_n = u_n \left(n \frac{u_{n+1}}{u_n} - \alpha_1 \right) \sim (\beta - \alpha_1) u_n.$$

Let us suppose it is true up to $l - 1$, that is

$$(\Omega - \alpha_1) \cdots (\Omega - \alpha_{l-1}) u_n = \prod_{i=1}^{l-1} (\beta - \alpha_i) u_n (1 + \epsilon_n) \qquad \lim_{n\to\infty} \epsilon_n = 0$$

Then

$$(\Omega - \alpha_1) \cdots (\Omega - \alpha_{l-1})(\Omega - \alpha_l) u_n = (\Omega - \alpha_l)(\prod_{i=1}^{l-1} (\beta - \alpha_i) u_n (1 + \epsilon_n)) =$$

$$\prod_{i=1}^{l-1} (\beta - \alpha_i) u_n (1 + \epsilon_n) \left(n \frac{\prod_{i=1}^{l-1} (\beta - \alpha_i) u_{n+1} (1 + \epsilon_{n+1})}{\prod_{i=1}^{l-1} (\beta - \alpha_i) u_n (1 + \epsilon_n)} - \alpha_l \right)$$

$$\Longrightarrow L_1(u_n) \sim \prod_{i=0}^{l-1} (\beta - \alpha_i) u_n \times (\beta - \alpha_l) \quad (n \to \infty),$$

and the result follows.

△

In the same way, we can easily show:

Lemma 3 *Let* (u_n) *be a sequence such that* $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = C \neq 0$. *Then, for*

$$L_1(u_n) = \prod_{i=1}^{l} (\Omega - \alpha_i)(u_n)$$

we obtain

$$L_1(u_n) \sim C^l n^l u_n \quad (n \to \infty).$$

Lemma 4 *If the sequence* (u_n) *is given by* $u_n = \dfrac{\alpha^n}{(n-1)!}\dfrac{1}{n^i}$ *then*

$$(\Omega - \alpha)(u_n) \sim -\alpha i \frac{u_n}{n} \quad (n \to \infty).$$

Proof: In fact,

$$\begin{aligned}(\Omega - \alpha)(u_n) &= n\frac{\alpha^{n+1}}{n!}\frac{1}{(n+1)^i} - \frac{\alpha^{n+1}}{(n-1)!}\frac{1}{n^i} \\ &= \frac{\alpha^{n+1}}{(n-1)!}\frac{1}{n^i}\frac{n^i - (n+1)^i}{(n+1)^i} \sim -\alpha i \frac{u_n}{n} \quad (n \to \infty).\end{aligned}$$

△

Lemma 5 *Let* (u_n) *be a sequence of the form*

$$u_n = \frac{1}{(n-1)!}\sum_{i=1}^{m} A_i \alpha_i^n \quad \textit{with } |\alpha_1| > |\alpha_2| > \cdots > |\alpha_m|\,.$$

Let us consider the operator $L_1(u_n) = (\Omega - \alpha_1)\cdots(\Omega - \alpha_l)(u_n)$ *with* $m > l$. *Then the asymptotic behavior of* $(L_1(u_n))$ *is given by:*

$$L_1(u_n) \sim C\frac{\alpha_{l+1}^n}{(n-1)!} \quad (n \to \infty).$$

Proof: The result follows immediately from the fact that the operators $(\Omega - \alpha_i)$ commute and

$$\begin{aligned}(\Omega-\alpha)\left(\frac{\alpha^n}{(n-1)!}\right) &= 0 \quad n \in \mathrm{N} \quad \text{and} \\ (\Omega-\alpha)\left(\frac{\alpha_*^n}{(n-1)!}\right) &= (\alpha_*-\alpha)\frac{\alpha_*^n}{(n-1)!} \quad \alpha \neq \alpha_*.\end{aligned}$$

△

From these results we can obtain the following acceleration properties:

Theorem 7 *Let (S_n) be a sequence satisfying*

$$S_n - S_\infty = r_n \sum_{i=0}^{\infty} A_i \frac{\alpha_i^n}{(n-1)!} \quad n \in \boldsymbol{N} \;, \; with \; |\alpha_1| > |\alpha_2| > \cdots$$

We consider the sequence transformation $T^{(1)}$ given by (24) with L_1 defined by (22) and $D_n = r_n$ for $n \in \boldsymbol{N}$. Then

$$\frac{T_n^{(1)} - S_\infty}{S_n - S_\infty} = \begin{cases} O\left(\dfrac{\alpha_{l+1}^n}{n^l (n-1)!}\right) & \textit{if} \quad \lim_{n\to\infty} \dfrac{r_n}{r_{n+1}} = c \neq 0 \\ O(\dfrac{\alpha_{l+1}^n}{(n-1)!}) & \textit{if} \quad \lim_{n\to\infty} \dfrac{n r_n}{r_{n+1}} = \beta \neq \alpha_i \quad i = 1, \cdots l. \end{cases}$$

Proof: If we set $a_n = \sum_{i=0}^{\infty} A_i \alpha_i^n/(n-1)!$ then, from lemma 5, we know that

$$L^{(1)}(a_n) \sim C\alpha_{l+1}^n/(n-1)! \quad (n \to \infty)$$

and using lemmas 2 and 3 to estimate the asymptotic behavior of $(L^{(1)}(1/D_n))$ the result follows.
△

Theorem 8 *Let us consider a sequence (S_n) satisfying:*

$$S_n - S_\infty = \frac{r_n \alpha^n}{(n-1)!} \sum_{i=0}^{\infty} A_i \frac{1}{n^i} \quad with \quad \lim_{n\to\infty} \frac{r_n}{r_{n+1}} = r \neq 0$$

and apply to (S_n) the sequence transformation $T^{(2)}$ given by (24) where L_2 has the form (23) and (D_n) is chosen by $D_n = r_n/n^l$, $n \in \boldsymbol{N}$. Then $(T_n^{(2)})$ converges to S_∞ faster than (S_n) and moreover the speed of convergence can be measured by:

$$\frac{T_n^{(2)} - S_\infty}{S_n - S_\infty} = O\left(\frac{1}{n^{2l+2}}\right) \quad (n \to \infty).$$

Proof: For this choice of (D_n), $(S_n - S_\infty)$ can be written in the form

$$S_n - S_\infty = a_n D_n \quad \text{with } a_n = \frac{\alpha^n}{(n-1)!} \sum_{i=0}^{l} A_i n^i + \frac{\alpha^n}{(n-1)!} \sum_{i=l+1}^{\infty} \frac{A_i}{n^{i-l}}.$$

But

a) $L_2(\frac{\alpha^n}{(n-1)!} \sum_{i=0}^{l} A_i n^i) = 0$ because we have a linear combination of the solutions;

b) $L_2(\frac{\alpha^n}{(n-1)!} \sum_{i=l+1}^{\infty} \frac{A_i}{n^{i-l}}) \sim \frac{\alpha^n}{(n-1)!} \frac{C}{n^2}$ $(n \to \infty)$ by Lemma 5.

Moreover, $\lim_{n\to\infty} D_n/D_{n+1} = \lim_{n\to\infty} \frac{(n+1)^l}{n^l} \frac{r_n}{r_{n+1}} = r \neq 0$ and by lemma 4

$$L_2(1/D_n) \sim r^l \frac{n^l}{D_n} \quad (n \to \infty).$$

So

$$T_n^{(2)} - S_\infty = O\left(\frac{\alpha^n}{(n-1)!} \frac{1}{n^2} \frac{r_n}{n^{2l}}\right) \quad (n \to \infty),$$

and the result follows.

△

Let us consider now another class of operators L:

b) $L(u_n) = (\pi - \alpha_1)(\pi - \alpha_2) \cdots (\pi - \alpha_l)(u_n)$ α_i constants

It can be shown that the sequences (u_n) for which $L(u_n) = 0$ have the form [8]:

$$u_n = \frac{1}{(n-1)!} \sum_{i=1}^{l} A_i \Gamma(n + \alpha_i),$$

which gives the following result:

Theorem 9 *Let T be the sequence transformation given by (21) corresponding to an operator L of the form*

$$L(u_n) = (\pi - \alpha_1)(\pi - \alpha_2) \cdots (\pi - \alpha_l)(u_n) \ , \ \alpha_i \ \ \text{constants} \ . \tag{25}$$

Then the kernel of T is given by

$$Ker(T) = \left\{ (S_n) : \quad S_n - S_\infty = \frac{D_n}{(n-1)!} \sum_{i=1}^{l} A_i \Gamma(n + \alpha_i) \right\}.$$

These operators L, also called Euler difference operators, can be writen in the more explicit form

$$L(u_n) = \frac{\Gamma(n+l)}{\Gamma(n)} \Delta^l u_n + p_{l-1} \frac{\Gamma(n+l-1)}{\Gamma(n)} \Delta^{l-1} u_n + \cdots + p_1 n \Delta u_n + p_0 u_n,$$

and the relation between the p_i's and the α_i's is the following: $\alpha_1, \alpha_2, \cdots, \alpha_r$ are the roots of the polynomial

$$p_*(x) = x(x-1)\cdots(x-r+1) + p_{r-1}x(x-1)\cdots(x-r+2) + \cdots + p_1 x + p_0.$$

Let us now study the asymptotic behaviour of the sequences $(L(u_n))$ with L given by (25) for sequences (u_n) satisfying some conditions.

Lemma 6 *Let (u_n) be a logarithmic sequence such that*

$$\frac{u_{n+1}}{u_n} = 1 + \frac{C}{n} + \alpha_n \ , \quad \alpha_n \sim \frac{C_1}{n^2} \quad (n \to \infty). \tag{26}$$

If $L(u_n) = (\pi - \alpha_1)\cdots(\pi - \alpha_l)(u_n)$ with $\alpha_i \neq C$ $\forall i$ then

$$L(u_n) \sim (C - \alpha_1)\cdots(C - \alpha_r)u_n \quad (n \to \infty).$$

Proof: We will prove this result by induction on l. For $l = 1$ we have

$$L(u_n) = (\pi - \alpha_1)u_n = n\Delta u_n - \alpha_1 u_n = u_n\left[n\left(\frac{u_{n+1}}{u_n} - 1\right) - \alpha_1\right],$$

and as (u_n) satisfies (26) we obtain $L(u_n) = u_n[(C - \alpha_1) + \gamma_n^{(1)}]$, $\gamma_n^{(1)} \sim \dfrac{K_1}{n}$ $(n \to \infty)$.

Let us suppose that:

$$(\pi - \alpha_1)\cdots(\pi - \alpha_{l-1})(u_n) = u_n(A_{l-1} + \gamma_n^{(l-1)}), \quad \gamma_n^{(l-1)} \sim \frac{K_{l-1}}{n} \quad (n \to \infty),$$

with $A_{l-1} = (C - \alpha_1)\cdots(C - \alpha_{l-1})$. Then

$$\begin{aligned}
&(\pi - \alpha_1)\cdots(\pi - \alpha_{l-1})(\pi - \alpha_l)(u_n) = (\pi - \alpha_l)\left[u_n(A_{l-1} + \gamma_n^{(l-1)})\right] = \\
&= u_n(A_{l-1} + \gamma_n^{(l-1)})\left[n\left(\frac{u_{n+1}(A_{l-1} + \gamma_n^{(l-1)})}{u_n(A_{l-1} + \gamma_n^{(l-1)})} - 1\right) - \alpha_l\right] = \\
&= u_n(A_{l-1}+\gamma_n^{(l-1)})\left[n\left(1 + \frac{C}{n} + O\left(\frac{1}{n^2}\right) - 1\right) - \alpha_l\right] = u_n(A_{l-1}+\gamma_n^{(l-1)})\left(C - \alpha_l + O\left(\frac{1}{n}\right)\right) = \\
&= u_n \textstyle\prod_{i=1}^{l}(C - \alpha_i)(1 + \gamma_n^{(l)}) \ , \ \gamma_n^{(l)} \sim \dfrac{K_l}{n} \quad (n \to \infty).
\end{aligned}$$

△

Lemma 7 *If (u_n) is a sequence satisfying*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \rho \neq 1, \tag{27}$$

then

$$L(u_n) = (\pi - \alpha_1)\cdots(\pi - \alpha_l)(u_n) \sim (\rho - 1)^l n^l u_n \quad (n \to \infty).$$

Proof: Again we proceed by induction on l. If (u_n) satisfies (27) then : $u_{n+1}/u_n = \rho + \beta_n, \ \lim_{n\to\infty} \beta_n = 0$. For $l = 1$ we get

$$(\pi - \alpha_1)u_n = u_n[n(\rho - 1 + \beta_n) - \alpha_1] \sim (\rho - 1)nu_n \quad (n \to \infty).$$

We suppose that

$$(\pi - \alpha_1)\cdots(\pi - \alpha_{l-1})(u_n) = (\rho - 1)^{l-1}n^{l-1}u_n(1 + \gamma_n^{(l-1)}), \quad \lim_{n\to\infty} \gamma_n^{(l-1)} = 0.$$

Then

$$\begin{aligned}
&(\pi - \alpha_1)\cdots(\pi - \alpha_{l-1})(\pi - \alpha_l) = (\pi - \alpha_l)((\rho - 1)^{l-1}n^{l-1}u_n(1 + \gamma_n^{(l-1)}) = \\
&= (\rho - 1)^{l-1}n^{l-1}u_n(1 + \gamma_n^{(l-1)}) \left[n\left(\frac{(n+1)^{l-1}u_{n+1}}{n^{l-1}u_n}\frac{1 + \gamma_{n+1}^{(l-1)}}{1 + \gamma_n^(l - 1)} - 1\right) - \alpha_l\right] \\
&= (\rho - 1)^{l-1}n^{l-1}u_n(1 + \gamma_n^{(l-1)})\left[n(\rho - 1 + o(1)) - \alpha_l\right] \\
&\sim (\rho - 1)^l n^l u_n.
\end{aligned}$$

△

These lemmas enable us to obtain the following acceleration results:

Theorem 10 *Let (S_n) be a sequence of the form:*

$$S_n - S_\infty = D_n\left(A_1\frac{\Gamma(n + \alpha_1)}{(n-1)!} + \cdots + A_r\frac{\Gamma(n + \alpha_r)}{(n-1)!} + \cdots\right) \quad n \in \boldsymbol{N},$$

where

a) $\lim_{n\to\infty} \dfrac{\Gamma(n + \alpha_i)}{\Gamma(n + \alpha_{i-1})} = 0 \quad \forall i;$

b) (D_n) *satisfies:* $\lim_{n\to\infty} \left(\dfrac{D_n}{D_{n+1}} - 1\right) n \neq \alpha_i \quad i = 1, \cdots r.$

If we apply to (S_n) the sequence transformation $T^{(r)}$ given by (21) with $L^{(r)}(u_n) = (\pi - \alpha_1)\cdots(\pi - \alpha_r)(u_n)$, we obtain

$$\frac{T_n^{(r)} - S_\infty}{S_n - S_\infty} \sim C\frac{\Gamma(n + \alpha_{r+1})}{\Gamma(n + \alpha_1)} \quad (n \to \infty).$$

Proof: We set $a_n = A_1\dfrac{\Gamma(n + \alpha_1)}{(n-1)!} + \cdots + A_r\dfrac{\Gamma(n + \alpha_r)}{(n-1)!} + \cdots$. By the definition of $L^{(r)}$ and theorem 9 we obtain $L^{(r)}\left(\dfrac{\Gamma(n + \alpha_i)}{(n-1)!}\right) = 0 \quad i = 0, \cdots, r$. Moreover we can easily see that

$$L^{(r)}\left(\frac{\Gamma(n + \alpha_i)}{(n-1)!}\right) \sim C_i\frac{\Gamma(n + \alpha_i)}{(n-1)!} \ (n \to \infty) \ \text{ for } i > r.$$

From the conditions on (D_n) and lemma 6 we know that $L^{(r)}(1/D_n) \sim D_n \quad (n \to \infty)$. So, from $L^{(r)}(a_n)/a_n \sim C\Gamma(n + \alpha_{r+1})/\Gamma(n + \alpha_1) \quad (n \to \infty)$, the result follows.

△

Theorem 11 *Let (S_n) be a sequence having the following asymptotic expansion of the error:*

$$S_n - S_\infty = \frac{C_1}{n} + \frac{C_2}{n(n+1)} + \cdots + \frac{C_r}{n(n+1)\cdots(n+r-1)} + \cdots \qquad n \in \boldsymbol{N}\,.$$

We consider the sequence transformation $T^{(r)}$ given by (21) with the choices

a) $D_n = \dfrac{1}{n(n+1)\cdots(n+r-1)}$ for $n \in \boldsymbol{N}$;

b) $L^{(r)}(u_n) = \pi(\pi-1)\cdots(\pi-(r-1))(u_n)$ (that is, $\alpha_i = i,\ i = 0,\cdots r-1$).

Then the transformed sequence $(T_n^{(r)} = T^{(r)}(S_n))$ converges to S_∞ faster than (S_n). Moreover we obtain:

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim O\left(\frac{1}{n^r}\right) \ (n \to \infty).$$

Proof: From the form of (S_n) and the choice of (D_n) , $(S_n - S_\infty)$ can be writen as

$$S_n - S_\infty = D_n[C_r + C_{r-1}(n+r-1) + \cdots + C_2(n+2)\cdots(n+r-1)+$$
$$+C_1(n+1)\cdots(n+r-1) + C_{r+1}\frac{1}{n+r} + \cdots].$$

The first r terms in this sum constitute a polynomial of degree $r-1$ in n and so we can write it in another basis:

$$S_n - S_\infty = D_n[C_0' + C_1'\frac{\Gamma(n+1)}{(n-1)!} + C_2'\frac{\Gamma(n+2)}{(n-1)!} + \cdots + C_{r-1}'\frac{\Gamma(n+r-1)}{(n-1)!}$$
$$+C_{r+1}\frac{\Gamma(n+r)}{(n+r)!} + C_{r+2}\frac{\Gamma(n+r)}{(n+r+1)!} + \cdots] =$$
$$= D_n A_n.$$

If $L^{(r)}$ is chosen like in b), then

$L^{(r)}\left(C_0' + C_1'\dfrac{\Gamma(n+1)}{(n-1)!} + \cdots + C_{r-1}'\dfrac{\Gamma(n+r-1)}{(n-1)!}\right) = 0$ because it is a linear combination of the solutions of $L^{(r)}(u_n) = 0$.

Let us now obtain the asymptotic behavior of $\left(L^{(r)}\left(\dfrac{\Gamma(n+r)}{(n+r+i)!}\right)\right)$. We obtain

$$(\pi - \alpha)\left(\frac{\Gamma(n+r)}{(n+r+i)!}\right) = n\frac{\Gamma(n+r+1)}{(n+r+i+1)!} - n\frac{\Gamma(n+r)}{(n+r+i)!} - \alpha\frac{\Gamma(n+r)}{(n+r+i)!} =$$

$$\left(\frac{n(n+r)}{n+r+i+1} - n - \alpha\right)\frac{\Gamma(n+r)}{(n+r+i)!} = -\left(\alpha + i + 1 - \frac{(i+1)(r+i+1)}{n+r+i+1}\right)\frac{\Gamma(n+r)}{(n+r+i)!}$$

So if $\alpha \neq -i-1$ then $(\pi-\alpha)\left(\dfrac{\Gamma(n+r)}{(n+r+i)!}\right) \sim C\dfrac{\Gamma(n+r)}{(n+r+i)!}$ $(C \neq 0)$ $(n \to \infty)$.

Therefore, for our choice of $L^{(r)}$, we obtain

$$L^{(r)}\left(\frac{\Gamma(n+r)}{(n+r+i)!}\right) \sim C\frac{\Gamma(n+r)}{(n+r+i)!}(n\to\infty), \tag{28}$$

and so $L^{(r)}(a_n) \sim K/n \quad (n\to\infty)$. As we have the following asymptotic behaviors:

$$a_n \sim C_1 n^{r-1} \quad (n\to\infty), \quad D_n \sim n^{-r} \quad (n\to\infty)\ ,$$

$$L^{(r)}(1/D_n) = L^{(r)}\left(\frac{\Gamma(n+r)}{(n-1)!}\right) \sim \frac{1}{n^r} \quad (n\to\infty)\ (\text{ by (28) with } i=-r-1),$$

we obtain

$$\frac{T_n - S_\infty}{S_n - S_\infty} = \frac{L(a_n)/a_n}{D_n L(1/D_n)} = O(n^{-r})\ (n\to\infty).$$

△

From the two theorems above we see that the extrapolation methods corresponding to the operators of the form (25) have good acceleration properties when applied to some classes of logarithmic convergent sequences.

4 Acceleration properties for the transformation (T_n) based on the asymptotic behavior of solutions of linear difference equations.

For the two classes of difference operators L considered in the previous section, a basis of solutions in explicit form for the associated homogeneous linear difference equation $L(u_n)=0$ was known, which enabled us to study the acceleration properties of the corresponding extrapolation method T. But for the general operator L of degree k and nonconstant coefficients there is not a general solution in compact form. For a given operator L not belonging to those classes we may be able to find an independent set of solutions for $L(u_n)=0$ using a method of the difference calculus (see, for instance [1, 7]): reduction of order, generating functions, etc. And from a basis of solutions we can proceed in the same way that above to get the kernel and acceleration properties of the extrapolation method T.

But, if we can't find a basis of solutions, we can obtain, using different techniques than above, the asymptotic behavior of the solution when $(n\to\infty)$ and, for some classes of operators, also the asymptotic expansion for a linearly independent set of solutions [2]. From the knowledge of this asymptotic behavior, we are going to:

a) give the acceleration properties for the sequence transformation T corresponding to a given operator L;

b) propose a method of constructing an operator L and the corresponding sequence transformation T to accelerate a class of sequences (S_n) for which the error sequence $(S_n - S_\infty)$ has a given asymptotic expansion.

From now, we will consider a difference operator L of the form:

$$L(u_n) = \Delta^k u_n + p_{k-1}(n)\Delta^{k-1} u_n + \cdots + p_1(n)\Delta u_n + p_0(n) u_n, \tag{29}$$

where $p_i(n),\ \ i = 0, \cdots, k-1$ satisfy the following condition: the functions f_i defined by $f_i(t) = p_i(1/t)t^{-k+i}$, $i = 0, \cdots, k-1$ are analytic in the neighborhood of 0. So the $p_i(n)$'s have the following asymptotic expansion

$$p_i(n) = \frac{1}{n^{k-i}}\left(C_0^{(i)} + \frac{C_1^{(i)}}{n} + \cdots\right) \quad i = 0, \cdots, k-1. \tag{30}$$

Before discussing the asymptotic behavior of the solutions of $L(u_n) = 0$, we will obtain the asymptotic behavior of $(L(u_n))$ when the sequence (u_n) satisfies some conditions.

Proposition 3 *Let (u_n) be a sequence satisfying* $\lim_{n\to\infty} u_{n+1}/u_n = \lambda \neq 1$. *Then*

$$L(u_n) \sim (\lambda - 1)^k u_n \quad (n \to \infty).$$

Proof: As we can write $\Delta^i u_n = \sum_{j=0}^{i} (-1)^{i-j} \dfrac{i!}{j!(i-j)!} u_{n+j}$, we obtain $\lim_{n\to\infty} \Delta^i u_n / u_n = (\lambda - 1)^i$. As $\lim_{n\to\infty} p_i(n) = 0$, replacing in (29) the result follows.

Proposition 4 *Let (u_n) satisfy*

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\alpha}{n} + r_n \quad \text{with } r_n = o\left(\frac{1}{n}\right) \ (n \to \infty)\ ,\ \Delta^i r_n = o\left(\frac{1}{n^{i+1}}\right) \ (n \to \infty). \tag{31}$$

Then, if L is given by (29) we obtain:

$$L(u_n) \sim \frac{A_k}{n^k} u_n \ \ (n \to \infty) \quad \text{with } A_k = \sum_{i=0}^{k} \alpha^{(i)} C_0^{(i)},$$

where $\alpha^{(0)} = 1,\ \alpha^{(i)} = \prod_{j=0}^{i-1} (\alpha - j);\ \ C_0^{(k)} = 1;\ \ C_0^{(i)}$ *given by (30).*

Proof: Let us begin by showing by induction that

$$\frac{\Delta^i u_n}{u_n} = \frac{\alpha^{(i)}}{n^i} + r_n^{(i)},\ \ r_n^{(i)} = o\left(\frac{1}{n^i}\right),\ \ \Delta^j r_n^{(i)} = o\left(\frac{1}{n^{i+j}}\right). \tag{32}$$

It is trivially true for $i = 1$ because we have $\Delta u_n / u_n = \alpha/n + r_n$. Let us suppose that it is true for i and compute $\Delta^{i+1} u_n / u_n$.

$$\frac{\Delta^{i+1} u_n}{u_n} = \frac{\Delta^i u_{n+1}}{u_{n+1}} \frac{u_{n+1}}{u_n} - \frac{\Delta^i u_n}{u_n} = \left(\frac{\alpha^{(i)}}{(n+1)^i} + r_{n+1}^{(i)}\right) \times \left(1 + \frac{1}{\alpha} + r_n\right) - \frac{\alpha^{(i)}}{n^i} - r_n^{(i)} =$$

$$\alpha^{(i)}\left(\frac{1}{(n+1)^i}-\frac{1}{n^i}\right)+\Delta r_n^{(i)}+\frac{\alpha\alpha^{(i)}}{(n+1)^i n}+r_n^* n, \quad r_n^*=o\left(\frac{1}{n^{i+1}}\right).$$

So

$$\frac{\Delta^{i+1}u_n}{u_n}=\frac{\alpha^{(i)}(\alpha-i)}{n^{i+1}}+r_n^{(i+1)}, \quad r_n^{(i+1)}=o\left(\frac{1}{n^{i+1}}\right), \quad \Delta^j r_n^{(i+1)}=o\left(\frac{1}{n^{i+j+1}}\right)$$

Replacing (32) in the operator L and, as the $p_i(n)$ satisfy (30), we obtain

$$\begin{aligned}\frac{L(u_n)}{u_n} &= \frac{\Delta^k u_n}{u_n}+p_{k-1}(n)\frac{\Delta^{k-1}u_n}{u_n}+\cdots+p_1(n)\frac{\Delta u_n}{u_n}+p_0(n)\\ &= \frac{\alpha^{(k)}}{n^k}+\frac{\alpha^{(k-1)}}{n^{k-1}}\frac{C_{k-1}^{(0)}}{n}+\cdots+\frac{\alpha^{(1)}}{n}\frac{C_1^{(0)}}{n^{k-1}}+\frac{C_0^{(0)}}{n^k}+o\left(\frac{1}{n^k}\right)=\frac{A_k}{n^k}+o\left(\frac{1}{n^k}\right)\end{aligned}$$

and the result follows.

Remark: if $\frac{u_{n+1}}{u_n}=1+\frac{\alpha_1}{n}+\frac{\alpha_2}{n^2}+\cdots$ $(n\to\infty)$ the condition of the previous proposition is satisfied.

Let us suppose that (u_n) is a solution of $L(u_n)=0$ for which we know the first terms of its asymptotic expansion

$$u_n=c_1g_1(n)+c_2g_2(n)+r_n \quad n\in\mathrm{N}, \quad g_2(n)=o(g_1(n)), r_n=o(g_2(n)) \quad (n\to\infty), \tag{33}$$

with $(g_i(n))$, $i=1,2$, (r_n) satisfying (31) and $c_1\neq 0$. Then, as $L(u_n)=0$ we get

$$L(u_n)=0\Leftrightarrow L(g_1(n))=-\frac{c_2}{c_1}L(g_2(n))-\frac{1}{c_1}L(r_n),$$

and so

$$L(g_1(n))\sim Cn^{-k}g_2(n) \quad (n\to\infty), \tag{34}$$

which gives the following acceleration result:

Theorem 12 *Let (S_n) be a sequence satisfying $S_n-S_\infty=a_nD_n$ with*

$$a_n=b_1g_1(n)+\rho_n, \ \rho_n=o(g_1(n)) \ (n\to\infty), \quad (g_1(n)),(\rho_n) \ \textit{satisfying (31)}.$$

Let L be an operator of the form (29) for which we know an asymptotic expansion of a solution $(u_n)_n$ of $L(u_n)=0$,

$$u_n\sim\sum_{i=0}^{\infty}\alpha_i g_i(n), \quad \textit{with } (g_i(n)) \quad i=1,\cdots \quad \textit{satisfying (31).}$$

We suppose that $\rho_n\sim Kg_2(n)$ $(n\to\infty)$ and we consider the sequence transformation corresponding to this operator,

$$T_n=L(S_n/D_n)/L(1/D_n). \tag{35}$$

a) If $\lim_{n\to\infty} D_n/D_{n+1} = \lambda \neq 1$ *then*

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim C n^{-k} \frac{g_2(n)}{g_1(n)} \quad (n \to \infty) \quad (C \textit{ is a constant });$$

b) If $\lim_{n\to\infty} D_n/D_{n+1} = 1$ *and if* $(1/D_n)$ *satisfies (31) with* $A_k \neq 0$ *then*

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim C \frac{g_2(n)}{g_1(n)} \quad (n \to \infty) \quad (C \textit{ is a constant }).$$

Proof: From Proposition 4, the condition on (ρ_n) and (34) we obtain

$$L(a_n) = b_1 L(g_1(n)) + L(r_n) \sim b_1 C n^{-k} g_2(n) + C' n^{-k} r_n \sim C^* n^{-k} g_2(n) \quad (n \to \infty).$$

So, as $L(a_n)/a_n \sim C n^{-k} g_2(n)/g_1(n) \quad (n \to \infty)$ and

$$\frac{T_n - S_\infty}{S_n - S_\infty} = \frac{L(a_n)}{a_n} \frac{1/D_n}{L(1/D_n)}, \tag{36}$$

- if a) is satisfied, from Proposition 3 we get $\lim_{n\to\infty} D_n L(1/D_n) \neq 0$ and the result follows.
- if b) is satisfied, then $\lim_{n\to\infty} D_n L(1/D_n) n^k \neq 0$ which replaced in (36) gives the result.

△

This result can be generalized to the case when we know k linearly independent solutions of the difference equation $L(u_n) = 0$.

Theorem 13 *Let* (S_n) *be a sequence such that*

$$S_n - S_\infty = D_n \left(a_1 g_1^{(1)}(n) + a_2 g_1^{(2)}(n) + \cdots + a_k g_1^{(k)}(n) + \rho_n \right),$$

with $g_1^{(i+1)}(n) = o(g_1^{(i)}(n)),\ \rho_n = o(g_1^{(k)}(n)) \quad (n \to \infty)$.

Let us consider an operator L *of the form (29) for which we know a basis of solutions* $(u_n^{(i)})$ $i = 1, \cdots k$, *and each one can be writen*

$$u_n^{(i)} \sim \sum_{j=1}^{\infty} \alpha_j^{(i)} g_j^{(i)}(n), \quad g_{j+1}^{(i)}(n) = o(g_j^{(i)}(n)) \ (n \to \infty) \ \ \forall j \in \boldsymbol{N} \quad , j = 1, \cdots k.$$

We suppose that

a) $g_2^{(i+1)}(n) = o(g_2^{(i)}(n)) \quad (n \to \infty), \quad i = 1, \cdots k-1$

b) $g_2^{(1)}(n) = o\left(g_1^{(k)}(n)\right) \quad (n \to \infty); \quad \rho_n \sim K g_2^{(1)}(n) \quad (n \to \infty)$

c) $(g_j^{(i)}(n)) \quad i = 1, \cdots k, \quad j = 1, 2, \cdots$ *satisfy (31).*

Then

1. *If* (D_n) *satisfies* $\lim_{n\to\infty} D_n/D_{n+1} = \lambda \neq 1$, *the sequence transformation (35) accelerates the convergence of* (S_n). *Moreover the acceleration can be measured by:*

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim C n^{-k} \frac{g_2^{(1)}(n)}{g_1^{(1)}(n)} \quad (n \to \infty).$$

2. If $(1/D_n)$ satisfies (31), then the speed of convergence of (T_n) can be measured by

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim C \frac{g_2^{(1)}(n)}{g_1^{(1)}(n)} \quad (n \to \infty).$$

Proof: We set $a_n = a_1 g_1^{(1)}(n) + \cdots a_k g_1^{(k)}(n) + \rho_n$. Then

$$L(a_n) = a_1 L(g_1^{(1)}(n)) + \cdots a_k L(g_1^{(k)}(n)) + L(r_n)$$

From (34) and condition c) we obtain $L(g_1^{(i)}(n)) \sim C_i n^{-k} g_2^{(i)}(n) \quad (n \to \infty) \quad i = 1, \cdots, k$ and from properties a) and b) we obtain $L(a_n) \sim n^{-k} g_2^{(1)}(n) \quad (n \to \infty)$. The result follows as in the previous theorem.

△

In order to be able to give the form of the sequences (S_n) which can be accelerated by the extrapolation method (35) corresponding to a given difference operator L of the form (29) or to construct the operator L for which (T_n) accelerates a given class of sequences (S_n), we must obtain the asymptotic behavior of the solutions of $L(u_n) = 0$. We will follow the ideas given in [2].

To the difference equation

$$\Delta^k u_n + p_{k-1}(n)\Delta^{k-1} u_n + \cdots + p_1(n)\Delta u_n + p_0(n) u_n = 0, \tag{37}$$

we can associate, by the correspondence $\Delta^i u_n \longleftrightarrow \dfrac{d^i y}{dx^i}$, the differential equation

$$\frac{d^k y}{dx^k} + p_{k-1}(x)\frac{d^{k-1}y}{dx^{k-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y(x) = 0. \tag{38}$$

Let us make the substitution $x = 1/t$. The derivatives with respect to x can be expressed in terms of d^k/dt^k in the following way:

$$\frac{d^i y}{dx^i} = (-1)^i t^{2i} \frac{d^i y}{dt^i} + \sum_{j=1}^{i-1} d_{ji} t^{i+j} \frac{d^j y}{dt^j}. \tag{39}$$

Replacing in the previous equation we obtain

$$\sum_{i=1}^{k} l_i(t) y^{(i)}(t) + p_0(1/t) y(t) = 0$$

with $l_i(t) = t^{k+i} \sum_{j=i}^{k} d_{ij} p_j(1/t) t^{-k+j} \quad i = 1 \cdots k \ , \ p_k(1/t) = 1; \quad d_{ii} = (-1)^i$.

Dividing by $l_k(t) = (-1)^k t^{2k}$ the equation can be writen

$$\frac{d^k y}{dt^k} + \frac{q_{k-1}(t)}{t}\frac{d^{k-1}y}{dt^{k-1}} + \frac{q_{k-2}(t)}{t^2}\frac{d^{k-2}y}{dt^{k-2}} + \cdots + \frac{q_o(t)}{t^k} y(t) = 0 \tag{40}$$

with

$$\begin{aligned} q_i(t) &= t^{k-i}\frac{l_i(t)}{l_k(t)} = (-1)^k \sum_{j=i}^{k} d_{ij}\frac{p_j(1/t)}{t^{k-j}} \quad i = 1,\cdots,k-1 \\ q_0(t) &= (-1)^k p_0(1/t)/t^k \end{aligned}$$

and so $q_i(t)$, $i = 0, \cdots k-1$ are analytic functions in the neighborhood of 0. The theory of linear differential equations enables us to predict the local behavior of the solutions near a point t_0 without knowing how to solve the equation. In the present case, and as $q_i(t)$ are analytic at 0, the point $t_0 = 0$ (or, equivalently, $x = \infty$) is an *ordinary* or *regular singular point* for the differential equation (40) (or (38)) [2]. In order to obtain the local behavior of the solutions we construct the *indicial polynomial* $P(\alpha)$:

$$P(\alpha) = \alpha(\alpha-1)\cdots(\alpha-k+1) + q_{k-1}(0)\alpha(\alpha-1)\cdots(\alpha-k+2) + \cdots + q_1(0)\alpha + q_0(0) \quad (41)$$

and we compute its k roots $\alpha_1, \cdots \alpha_k$. The asymptotic expansion of the solutions of (40) is given by:

a) If $(\alpha_i - \alpha_j)$ is not an integer for all $i \neq j$, then we obtain k linearly independent solutions of (40) which satisfy:

$$y_i(t) = t^{\alpha_i} A_i(t), \quad \text{with } A_i(t) \text{ analytic at 0}, \quad i = 1, \cdots, k.$$

b) b.1. If $\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i+j}$ then we obtain a general solution of the form:

$$y(t) = t^{\alpha_i} \sum_{k=0}^{j-1} (\log(t))^k A_k(t), \quad \text{with } A_k(t) \text{ analytic in 0} \quad k = 0, \cdots, j-1.$$

b.2. If $\alpha_{i+l} - \alpha_{i+l-1} = n_i \in Z$, $l = 1, \cdots, j$, then we obtain $j+1$ linearly independent solutions of the form.

$$\begin{aligned} y_i(t) &= t^{\alpha_i} A_i(t) \\ y_{i+1}(t) &= y_i(t)\log(t) + t^{\alpha_{i+1}} A_{i+1}(t) \\ \cdots & \quad \cdots \\ y_{i+j}(t) &= y_{i+j-1}(t)\log(t) + t^{\alpha_{i+j}} A_{i+j}(t) \end{aligned}$$

where $A_k(t)$ are analytic at 0.

We return now to the difference equation (37). The techniques of local asymptotic analysis are similar to those used above with differential equations. The point $n = \infty$ can be classified as an *ordinary* or *regular singular point* if the point $x = \infty$ in the corresponding differential equation is so classified. The procedure for constructing the Taylor, Frobenius or asymptotic series valid near the point $n = \infty$ is in close analogy with the corresponding procedure for differential equations [2]. We obtain:

a) If $(\alpha_i - \alpha_j)$ are not integers for all $i \neq j$, we will have k linearly independent solutions that have an asymptotic expansion in a Frobenius series:

$$u_n^{(i)} \sim \sum_{k=0}^{\infty} A_k^{(i)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)} \quad (n \to \infty).$$

As $\dfrac{\Gamma(n)}{\Gamma(n+k-\alpha)} \sim n^{\alpha-k} \quad (n \to \infty)$ we get $u_n^{(i)} \sim C_i n^{\alpha_i} \quad (n \to \infty) \quad i = 1, \cdots, k.$

b) We will have to replace the logarithm by its discrete analogue: the Digamma function $\Psi(n) = \Gamma'(n)/\Gamma(n)$ (which admits the asymptotic expansion $\Psi(n) \sim \log(n) + c + \sum_{i=1}^{\infty} b_i n^{-i} \quad (n \to \infty)$). We will get the following generalizations of the Frobenius series.

b.1. If $\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i+j}$ then equation (37) will have $j+1$ linearly independent solutions that possess the following asymptotic expansions:

$$
\begin{aligned}
u_n^{(i)} &\sim \sum_{k=0}^{\infty} A_k^{(i)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)} \quad (n \to \infty) \\
u_n^{(i+1)} &\sim \sum_{k=0}^{\infty} A_k^{(i+1)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)} + \sum_{k=0}^{\infty} B_k^{(i)} \Psi(n) \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)} \\
\cdots &\quad \cdots \\
u_n^{(i+j)} &\sim \sum_{k=0}^{\infty} A_k^{(i+j)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)} + \cdots + \sum_{k=0}^{\infty} B_k^{(i,j)} (\Psi(n))^j \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)}
\end{aligned}
$$

b.2. Finally, if $\alpha_{l+1} - \alpha_l = n_l$ integer, $l = i, \cdots, i+j$, equation (37) will have the following $j+1$ linearly independent solutions:

$$
\begin{aligned}
u_n^{(i)} &\sim \sum_{k=0}^{\infty} A_k^{(i)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)}; \\
u_n^{(i+1)} &\sim \sum_{k=0}^{\infty} A_k^{(i+1)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_{i+1})} + \sum_{k=0}^{\infty} B_k^{(i,1)} \Psi(n) \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)}; \\
\cdots &\quad \cdots \\
u_n^{(i+j)} &\sim \sum_{k=0}^{\infty} A_k^{(i+j)} \frac{\Gamma(n)}{\Gamma(n+k-\alpha_{i+j})} + \sum_{k=0}^{\infty} B_k^{(i,1)} \Psi(n) \frac{\Gamma(n)}{\Gamma(n+k-\alpha_{i+j-1})} + \cdots + \\
&\quad + \sum_{k=0}^{\infty} B_k^{(i,j)} (\Psi(n))^j \frac{\Gamma(n)}{\Gamma(n+k-\alpha_i)}.
\end{aligned}
$$

In the two last cases we obtain the following asymptotic behavior

$$u_n^{(i+l)} \sim C_{i+l} (\log(n))^l n^{\alpha_i} \quad (n \to \infty) \quad l = 0, \cdots, j.$$

Applications:

Let us now give some examples of how to construct a sequence transformation based on the techniques explained above and suitable for accelerating the convergence of a given family of sequences.

a) Let us consider a family of sequences of the form

$$S_n - S_\infty = D_n \left(\frac{b_1}{n^{\alpha_1}} + \frac{b_2}{n^{\alpha_2}} + \cdots + \frac{b_k}{n^{\alpha_k}} + \rho_n \right) \quad n \in \mathrm{N}\,,$$

with

$$0 < \alpha_1 < \alpha_2 < \cdots \alpha_k, \quad \alpha_k - \alpha_1 < 1, \quad \rho_n = o\left(\frac{1}{n^{\alpha_k}}\right)(n \to \infty).$$

We can easily construct an operator L of the form (29) that has k linearly independent solutions satisfying

$$u_n^{(i)} \sim \sum_{k=0}^{\infty} A_k^{(i)} \frac{\Gamma(n)}{\Gamma(n+k+\alpha_i)} \quad (n \to \infty), \quad i = 1, \cdots, k.$$

In fact, let us consider the polynomial

$$P(\alpha) = (\alpha+\alpha_1)\cdots(\alpha+\alpha_k) = \alpha(\alpha-1)\cdots(\alpha-k+1) + a_{k-1}\alpha\cdots(\alpha-k+2) + \cdots a_1\alpha + a_0.$$

We construct recursively the set of functions analytic in 0

$$p_i^*(t) = (-1)^k q_i(t) + (-1)^k \sum_{j=i+1}^{k} d_{ij}(-1)^j p_j^*(t) \ , \quad i = k-1, \cdots, 0$$

with

$p_k^*(t) = (-1)^k$;
$q_i(t)$ are arbitrary analytic functions in 0 satisfying $q_i(0) = a_i$;
d_{ij} are the quantities defined by (39) .

Now we set

$$p_i(t) = (-1)^i t^{i-k} p_i^*\left(\frac{1}{t}\right), \quad i = 0, \cdots, k,$$

and consider the corresponding operator L given by (29). If $\lim_{n\to\infty} D_n/D_{n+1} \neq 1$ then conditions of Theorem 13 are satisfied and we can conclude that the sequence transformation (T_n) given there accelerates the convergence of (S_n); moreover

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim Cn^{-k-1} \quad (n \to \infty) \text{ where C is a constant .}$$

b) We consider now the family of sequences of the form

$$S_n - S_\infty = D_n\left(b_1\frac{(\log n)^l}{n^a} + b_2\frac{(\log n)^{l-1}}{n^a} + \cdots + b_{l+1}\frac{1}{n^a} + \rho_n\right) \quad n \in \mathbf{N},$$

$$\text{where } \rho_n = o\left(\frac{1}{n^a}\right) \quad (n \to \infty).$$

We consider the polynomial $P(\alpha)$ of degree l whose roots are $a, a+1, \cdots, a+l-1$ and construct the operator L as in the previous example. In this case, a basis of solutions of $L(u_n) = 0$ is given by **b.2.** and applying again Theorem 13 we obtain

$$\frac{T_n - S_\infty}{S_n - S_\infty} \sim Cn^{-k-1} \ (n \to \infty) \text{ where C is a constant .}$$

5 Conclusions

As shown in [4], [5], [6] and [10], this new approach to sequence transformations provides a formalism where we can include a great variety of very well–known acceleration methods and give a better understanding of extrapolation methods.

In this paper we continued to explore this formalism and we gave, for a large class of difference operators L, the structure of the kernel and the acceleration properties of the corresponding sequence transformation (2). This study was based on the properties of the solution of linear difference equations $L(a_n) = 0$ for operators L for which

- we know a basis of solutions in explicit form

  or

- we can obtain the asymptotic behavior of the solutions.

We also proposed a method for, given a sequence (S_n) for which we know an asymptotic expansion of the error, constructing a linear difference operator L such that the corresponding sequence transformation (2) accelerates the convergence of (S_n)

In conclusion, this formalism seems to be very interesting for obtaining general theoretical acceleration results and for the construction of an extrapolation method suitable for a given class of sequences.

The acceleration properties of the iteration of the procedure

$$\begin{cases} T: \ \mathcal{S} & \longrightarrow \ \mathcal{S} \\ \quad S = (S_n) & \longmapsto \ T(S) = (T(S_n)) = (T_n): \quad T_n = \dfrac{L(S_n/D_n)}{L(1/D_n)}, \quad n \in \mathbb{N} \end{cases}$$

are under study and will be the subject of a forthcoming paper. Numerical comparisons of the new methods proposed here and their stability properties are also being studied.

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