Some Convergence Results for the Generalized Padé-Type Approximants

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Abstract

The aim of this paper is to give some convergence results for some sequences of generalized Padé type approximants. We will consider two types of interpolatory functionals: one corresponding to Lagrange and Hermite interpolation and other corresponding to orthogonal expansions. For these two cases we will give sufficient conditions on the generating function $G(x,t)$ and on the linear functional c in order to obtain the convergence of the corresponding sequence of generalized Padé type approximants. Some examples are given.

1 Definition of the Generalized Padé-Type Approximants.

Let f be an analytic function defined on a set $A \subseteq \mathbb{C}$ by the series of functions

$$f(t) = \sum_{i=0}^{\infty} c_i g_i(t), \qquad t \in A. \tag{1}$$

Let $G(x,t)$ be a generating function of the family $\{g_i(t)\}_i$, that is, $G(x,t) = \sum_{i=0}^{\infty} e_i x^i g_i(t)$, with $e_i \neq 0$. In applications, $G(x,t)$ could be taken to be a standard generating function whose coefficients e_i are specified: see Section 3.1.3.

We define the linear functional c by its moments in the following way:

$$c(e_i x^i) = c_i \Leftrightarrow c(x^i) = \frac{c_i}{e_i} = d_i, \quad i \in \mathbb{N}\,. \tag{2}$$

Then formally we have $f(t) = c(G(x,t))$, $t \in A$, where the linear form c acts on the variable x (this will be the case along all the paper).

The generalized Padé-type approximant of order n of the series f, $(n)_f(t)$ – or in short GPTA – is defined in the following way [2]:

- we fix $t \in A$ and consider the polynomial $P_n(x;t)$ of degree less or equal n in x which satisfies the following interpolation conditions:

$$L_i(P_n(x;t)) = L_i(G(x,t)) \quad i = 0, \cdots, n, \tag{3}$$

where the L_i are linear functionals acting on the variable x;

- we replace G by its approximation P_n and we construct the approximant

$$(n)_f(t) = c(P_n(x;t)) \quad n \in \mathbb{N}\,.$$

It's easy to see that this approximant satisfies the following order condition: the expansion of $(n)_f(t)$ in terms of the family $\{g_i(t)\}$ coincides with the one of $f(t)$ up to the order n, that is

$$f(t) - (n)_f(t) = \sum_{i=n+1}^{\infty} b_i g_i(t).$$

A similar generalization has been proposed for the Padé approximants of a class of functions having some integral representation and corresponds to the Baker-Gammel approximants (see, for instance [1]).

The existence and unicity conditions for the GPTA have been studied in [2] and a program in Mathematica for the formal recursive computation of these approximants has been given in [4]. The good numerical results obtained there motivate us to study the convergence behavior of these approximants.

In this paper we will study the convergence of sequences of generalized Padé type approximants corresponding to two types of linear functionals L_i:

1. $L_i(f) = f(x_i)$ (if the point is repeated, we consider the derivatives);

   or

2. $L_i(f) = \int_C f(z)\overline{p_i(z)}w(z)\,|dz|$, where $\{p_i(z)\}$ is the family of orthonormal polynomials on C with respect to the weight function $w(z)$.

We will determine what conditions must the generating function G and the linear functional c satisfy in order to have

$$\lim_{n\to\infty} (n)_f(t) = f(t)$$

for all t in some set $B \in \mathbb{C}$

We illustrate our results with some examples for the case where $G(x,t)$ is a generating function of a family of classical orthogonal polynomials.

2 A general convergence result.

2.1 Continuity properties of the linear functional.

From the definition of the GPTA, we see that to study the convergence of a sequence of approximants, we have to obtain the conditions for the convergence of the sequence of interpolation polynomials $P_n(x;t)$ and the continuity of the linear form c. Let us begin by obtaining some properties of c, namely the domain of definition and continuity. We consider three normed spaces of functions that we will need in the sequel.

(a) Let $B[a,b]$ be the set of all bounded functions defined on $[a,b]$. We define the norm function

$$\| f \| = sup_{a \leq x \leq b} |f(x)| .$$

We have a normed linear space. The linear functional c defined by (2) is defined on the set of all polynomials. We can extend its domain of definition to all the functions of $B[a,b]$ which are the uniform limit in $[a,b]$ of a sequence of polynomials ($\lim_{n\to\infty} p_n(x) = f(x)$ uniformly on $[a,b]$) and for which the limit $\lim_{n\to\infty} c(p_n(x))$ exists. We set $\mathcal{H}_1[a,b]$ such normed space of functions and we define then

$$c(f) = \lim_{n\to\infty} c(p_n(x)).$$

$c(f)$ is well defined if for any other sequence of polynomials (q_n) converging uniformly to f in $[a,b]$ we have: $\lim_{n\to\infty} c(q_n(x)) = \lim_{n\to\infty} c(p_n(x))$. This is clearly satisfied if

$$(\forall (p_n) \to 0 \text{ uniformly in } [a,b]) \quad \Rightarrow \quad (c(p_n) \to 0) \tag{4}$$

For instance, if the moments c_i satisfy

$$c_i = \int_a^b f(x) q_i(x) w(x) dx,$$

where $\{q_i(x)\}$ is the sequence of orthonormal polynomials in $[a,b]$ with respect to the weight function $w(x)$ then we can show that condition (4) is satisfied.

So we obtain from the definition of c:

Property 1 *If the linear functional c satisfies (4) then it is continuous in $\mathcal{H}_1[a,b]$.*

(b) The linear functional c can be also extended by continuity to a certain class of analytic functions. Let us suppose that the sequence $(d_i)_i$ defined by (2) satisfies

$$\overline{\lim_{n\to\infty}} |d_n|^{1/n} = r < \infty \tag{5}$$

Let h be a function analytic in a neighborhood of 0 for which we have the power series representation $h(z) = \sum_{i=0}^{\infty} a_i z^i$. If $\lim_{n\to\infty} \sum_{i=0}^{n} a_i d_i$ exists then we can define $c(h(x))$ by

$$c(h(x)) = \sum_{i=0}^{\infty} a_i d_i.$$

A sufficient condition for convergence of this series is $\overline{\lim}_{n\to\infty} |a_n|^{1/n} < 1/r$ and so we have the following result:

Proposition 1 *The linear functional c is well defined for all functions analytic in $D_r = \{z \ : |z| \leq r\}$*

Let B be a region in $\mathbb{C}$ and let

$$L^2(B) = \left\{ f \ : \ f \text{ is analytic in } B \text{ and} \int\int_B |f(z)|^2 \, dx dy < \infty \right\}.$$

With the scalar product $(f,g) = \int\int_B f\overline{g} dx dy$, $L^2(B)$ is a complete inner product space [3].

We choose $B = D_{R_*} = \{z \quad : \quad |z| \leq R_*\}$, with $R_* > r$, and then we have [3]

$$\forall g \in L^2(D_{R_*}) \quad \| g \| = \pi \sum_{n=0}^{\infty} |a_n|^2 \frac{R_*^{2n+2}}{n+1} < \infty \ ,$$

where $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Let c be the linear functional defined by (2) and $(d_n)_n$ satisfy (5). If we denote by $\mathcal{D}(c)$ the set of functions where c is well defined then $L^2(D_{R_*}) \subset \mathcal{D}(c)$, and for $g \in L^2(D_{R_*})$, we obtain

$$\begin{aligned}
|c(g)|^2 = \left| \sum_{i=0}^{\infty} a_i d_i \right|^2 &= \left| \sum_{i=o}^{\infty} \frac{a_i}{\sqrt{i+1}} \left(\frac{1}{R_*} \right)^{i+1} \sqrt{i+1} R_*^{i+1} d_i \right|^2 \leq \\
&\leq \left(\sum_{i=0}^{\infty} |a_i|^2 \frac{1}{i+1} R_*^{2i+2} \right) \left(\sum_{i=0}^{\infty} |d_i|^2 (i+1) \left(\frac{1}{R_*} \right)^{2i+2} \right) \\
&\leq \frac{1}{\pi} \left(\sum_{i=0}^{\infty} (i+1) |d_i|^2 \left(\frac{1}{R_*} \right)^{2i+2} \right) \| g \|^2 .
\end{aligned}$$

If we set $M = \frac{1}{\pi} \sum_{i=0}^{\infty} (i+1) |d_i|^2 \left(\frac{1}{R_*} \right)^{2i+2} < \infty$ then $M < \infty$ and

$$|c(g)| \leq M \| g \| \quad \forall g \in L^2(D_{R_*}).$$

As a bounded linear functional defined on a normed linear space is also continuous, we obtain the following result

Property 2

$$\forall (g_n)_n \quad \textit{such that} \quad g_n \in L^2(D_{R_*}) \ \textit{and} \ \exists g \in L^2(D_{R_*}) \lim_{n \to \infty} \int \int_{D_{R_*}} |g_n(z) - g(z)|^2 \, dx dy = 0$$

we have

$$c(g) = \lim_{n \to \infty} c(g_n).$$

We will set $\mathcal{H}_2(R_*) = L^2(D_{R_*})$.

(c) Let us finally consider the case where

$$\overline{\lim_{n \to \infty}} |d_n|^{1/n} = \infty. \tag{6}$$

Let us denote by $\mathcal{H}_3(R)$ the space of entire functions $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for which $\sum_{i=0}^{\infty} a_i d_i$ converges with the norm

$$\| g \| = \sup_{|z| \leq R} |g(z)| \, .$$

The linear functional c can be extended to $\mathcal{H}_3(R)$ by setting

$$c(f) = \sum_{i=0}^{\infty} a_i d_i .$$

We easily obtain

Proposition 2 *If the linear functional c satisfies*

$$\lim_{n\to\infty} c(p_n) = 0 \quad \forall (p_n), \text{ such that } \quad (p_n) \text{ converges uniformly to the null function in } D_R \tag{7}$$

and p_n is a polynomial of degree $\leq n$, then for all sequence of polynomials (q_n) converging uniformly to $g(z)$ in $\mathcal{H}_3(R)$ we obtain

$$\lim_{n\to\infty} c(q_n) = c(g).$$

2.2 Convergence theorem.

From the definition of the GPTA and the continuity properties of the linear form c on $\mathcal{H}_1[a,b]$, $\mathcal{H}_2(R_*)$ and $\mathcal{H}_3(R)$ obtained in the previous section, we get:

Theorem 1 *Let f be an analytic function on $A \subset \mathbb{C}$ represented by the series of functions*

$$f(z) = \sum_{i=0}^{\infty} c_i g_i(z) \quad z \in A, \tag{8}$$

and $G(x,t)$ a generating function of the family $\{g_i(z)\}$. We define the linear functional c by its moments by (2) and $\mathcal{H}_1[a,b]$, $\mathcal{H}_2(R_*)$ and $\mathcal{H}_3(R)$ as in the previous section. Let us consider a sequence of polynomials of degree n in the variable x (t is a parameter), $P_n(x;t)$, satisfying the interpolating conditions (3). Let us suppose that one of the following situations holds:

1. *for t in some subset $A_* \subset A$ the generating function $G(\cdot,t)$ belongs to $\mathcal{H}_1[a,b]$, c satisfies (4) and the sequence $\{P_n(\cdot;t)\}$ converges uniformly in $[a,b]$ to $G(\cdot,t)$;*
2. *for t in some subset $A_* \subset A$ the generating function $G(\cdot,t)$ belongs to $\mathcal{H}_2(R_*)$, c satisfies (5) and the sequence $\{P_n(\cdot;t)\}$ converges to $G(\cdot,t)$ in $\mathcal{H}_2(R_*)$;*
3. *for t in some subset $A_* \subset A$ the generating function $G(\cdot,t)$ belongs to $\mathcal{H}_3(R)$, c satisfies (6) and (7) and the sequence $\{P_n(\cdot;t)\}$ converges uniformly in D_R to $G(\cdot,t)$.*

Then the corresponding sequence of GPTA converges to f in A_*:

$$\forall t \in A_* \quad \lim_{n\to\infty} (n)_f(t) = f(t).$$

3 Convergence results for different types of interpolatory conditions

Let us use the results of the interpolatory convergence theory and **Theorem 1** to obtain convergence results for particular sequences of GPTA. We will consider different types of interpolation functionals L_k.

3.1 Lagrange and Hermite interpolation.

3.1.1 Interpolation series.

Theorem 2 *Let $\zeta_1, \zeta_2, \cdots, \zeta_k$ be points in $\mathbb{C}$ and $\mathcal{L}_\rho$ the lemniscate interior defined by:*

$$|(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_k)| < \rho^k.$$

Let f be an analytic function of the form

$$f(z) = \sum_{i=0}^{\infty} c_i g_i(z) \quad z \in A,$$

and let $G(x,t)$ be the corresponding generating function. We suppose that the linear functional c defined by (2) satisfies (5) and that:

$$\forall t \in A_* \subset A \quad G(\cdot, t) \text{ is analytic in } \mathcal{L}_\rho \supseteq D_{R_*}, \quad R_* > r.$$

If we consider a sequence of points $(z_i)_i$ satisfying

$$\lim_{n\to\infty} z_{nk+i} = \zeta_i, \quad 1 \le i \le k \quad ; \quad z_i \in \mathcal{L}_\rho \quad \forall i,$$

and if $P_n(x;t)$ is the polynomial of degree n that interpolates $G(x,t)$ at the points z_i, $i = 1, \cdots, n+1$, (t is considered as a parameter), then the corresponding sequence of GPTA satisfies

$$\lim_{n\to\infty} (n)_f(t) = f(t) \quad \forall t \in A_*.$$

Proof: It is known from the interpolation theory results (see for instance [3]) that in the conditions of this theorem, the sequence $\{P_n(\cdot;t)\}_n$ converges uniformly to $G(\cdot,t)$ in $\mathcal{L}_\rho$. As $\mathcal{L}_\rho \supseteq D_{R_*}$, part **2.** of **Theorem 1** holds and the result follows. △

What form have these generalized Padé type approximants?

(a) If the interpolation points are all distinct, by the Lagrange formula we obtain

$$c(P_n(\cdot,t)) = \sum_{i=0}^{n} c(l_i) G(\zeta_i, t)$$

where the $l_i(x)$ are the fundamental polynomials for pointwise interpolation and so the approximants are linear combinations of the functions $G(\zeta_i, t)$, $i = 1, \cdots, n+1$.

(b) If all the interpolation points coincide, the lemniscate reduces to a circle and $P_n(x;t)$ is the Taylor polynomial

$$P_n(x;t) = \sum_{i=0}^{n} \frac{\partial^i}{\partial x^i}(G(x,t))\,|_{x=\zeta}\ \frac{x^i}{i!}.$$

In this case the approximants are linear combinations of the functions $\left\{\frac{\partial^i}{\partial x^i}(G(\zeta,t)\right\}_{i=1}^{n+1}$.

(c) If we consider the sequence of interpolation points $z_{nk+i} = \zeta_i \quad \forall n \in \mathbb{N} \quad i = 1, \cdots, k$, then for all $m = nk + i$ it is easy to verify that $(m)_f(t)$ is a linear combination of the functions

$$\left\{ \frac{\partial^j}{\partial x^j} G(\zeta_l, t) \ \ l = 1, \cdots, k; \ \ j = 0, \cdots, n-1; \ \ \frac{\partial^n}{\partial x^n} G(\zeta_l, t) \ \ l = 1, \cdots i \right\}.$$

We remark that if $g_i(t) = t^i \quad \forall i$, the generating function is given by $G(x,t) = 1/(1 - xt)$ and the approximant will be a rational function - it is a Padé-type approximant.

3.1.2 A more general case: triangular set of interpolation points.

Let us consider now the more general case where the linear functionals L_k depend on n, that is, we consider the set of interpolation points

$$\left\{ \begin{array}{llll} \beta_1^{(0)} & & & \\ \beta_1^{(1)}, & \beta_2^{(1)} & & \\ \cdots & \cdots & \cdots & \\ \beta_1^{(n)}, & \beta_2^{(n)}, & \cdots & \beta_{n+1}^{(n)} \\ \cdots & \cdots & \cdots & \end{array} \right. \tag{9}$$

and the sequence of polynomials $p_n(z)$ of respective degrees n interpolating $f(z)$ at the points $\beta_1^{(n)}, \cdots, \beta_{n+1}^{(n)}$. We have the following general result for the convergence of the sequence $\{p_n(z)\}$ [6]:

Theorem 3 *Let C be a closed limited point set whose complement K is connected and regular. Let $w = \phi(z)$ map K onto the region $|w| > 1$ so that the points at infinity correspond to each other, and let Δ be the capacity of C. Let the points (9) have no limit point exterior to C and satisfy the relation*

$$\lim_{n\to\infty} \left|(z-\beta_1^{(n)})(z-\beta_2^{(n)})\cdots(z-\beta_{n+1}^{(n)})\right|^{1/(n+1)} = \Delta\,|\phi(z)|\,,$$

uniformly on any closed limited point set interior to K, and let f be an analytic and single valued function on C.

Then the sequence of polynomials $p_n(z)$ of respective degrees n found by interpolation of $f(z)$ in the points $\beta_1^{(n)}, \cdots, \beta_{n+1}^{(n)}$ satisfy

$$\lim_{n\to\infty} p_n(z) = f(z) \quad \textit{uniformly for } z \in C.$$

From this general result, considering particular sets (9) and applying **Theorem 1** we will obtain convergence results for the corresponding sequences of GPTA.

We begin by considering the case where the $\beta_i^{(n)}$ are the roots of classical orthogonal polynomials.

Jacobi abcissas.

Proposition 3 *Let f be an analytic function satisfying (8), $G(x,t)$ the generating function and c the corresponding linear functional defined by (2). Let us suppose that:*

- *for $t \in A_* \subset A$ the generating function $G(\cdot,t)$ is continuous in $[-1,1]$ with modulus of continuity $w(\delta)$ satisfying $w(\delta) = o(|\log\delta|^{-1})$;*
- *the linear functional c satisfies (4) in $[-1,1]$;*
- *for $t \in A_*$ $G(\cdot,t)$ belongs to $\mathcal{H}_1[-1+\epsilon, 1-\epsilon]$ $(\epsilon < 1/2)$.*

Then the Lagrange polynomials interpolating $G(\cdot,t)$ at the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ converge uniformly to $G(\cdot,t)$ in $[-1+\epsilon, 1-\epsilon]$ and the corresponding sequence of GPTA satisfies

$$\lim_{n\to\infty} (n)_f(t) = f(t) \quad \forall t \in A_*.$$

Laguerre abcissas

Proposition 4 *Let f, $G(\cdot,t)$ and c be defined as in the previous proposition. Let us suppose that*

- *for $t \in A_* \subset A$ $G(\cdot,t)$ is continuous for $x \geq 0$ and $G(x,t) = O(x^m)$ $(x \to \infty)$ (m is a fixed positive number);*
- *$\exists a, b > 0$: c satisfies (4) in $[a,b]$ and $G(.,t)$ belongs to $\mathcal{H}_1[a,b]$ for $t \in A_*$.*

Then the sequence of GPTA corresponding to the sequence of interpolating polynomials of the function $G(\cdot,t)$ on the zeros of the Laguerre polynomials $L_n^{(\alpha)}$ $(\alpha > -1)$ converges to $f(t)$ for $t \in A_*$.

Lagrange polynomials for certain general classes of abcissas. Let $\beta_i^{(n-1)}$ $i = 1, \cdots, n$, denote the zeros of the n^{th} orthogonal polynomial $p_n(x)$ associated with the weight function $w(x)$ in the interval $-1 \leq x \leq 1$. We consider two classes of weight functions characterized by the following conditions:

A. $\exists \mu > 0 \ : \ W(x) \geq \mu, \quad -1 \leq x \leq +1.$

B. $\exists \mu > 0 \ : \ W(x) \geq \mu(1-x^2)^{-1/2}, \quad -1 < x < +1.$

Then we obtain:

Proposition 5 *Let f, $G(x,t)$, and c be in the conditions of the previous theorem and let us suppose that:*

- *for $t \in A_*$ $G(\cdot,t)$ is defined in $-1 \leq x \leq +1$;*
- *c belongs to $\mathcal{H}_1[-1,1]$ and satisfies (4).*

Let $P_n(x;t)$ be the sequence of Lagrange interpolation polynomials at the zeros of the orthogonal polynomials associated to a weight function $w(x)$, $-1 \leq x \leq 1$. Then, if

(a) w satisfies A and $G(\cdot,t)$ has a continuous derivative in $[-1,1]$,

or

(b) w satisfies B and $w(\delta) = o(\delta^{1/2})$,

the sequence of GPTA corresponding to the sequence $P_n(x;t)$ converges to $f(t)$ in A_*.

Regular distribution of the interpolation points $\beta_k^{(n)}$. Let us suppose that for $t \in A_* \subset A$ the function $G(\cdot,t)$ is analytic in a closed simply connected region R. Let C be a simple, closed, rectifiable curve that lies in R and contains the points $\beta_k^{(n)}$ in its interior. Then, if $p_n(x;t)$ is the interpolating polynomial in the points $\beta_k^{(n)}$, the interpolation error is given by

$$G(x,t) - p_n(x;t) = \frac{1}{2\pi i} \int_C \frac{(x-\beta_1^{(n)})(x-\beta_2^{(n)})\cdots(x-\beta_{n+1}^{(n)})}{(z-\beta_1^{(n)})(z-\beta_2^{(n)})\cdots(z-\beta_{n+1}^{(n)})} \frac{G(z,t)}{z-x} dz.$$

From this expression we see that the convergence of the sequence $p_n(x;t)$ is determined by the behavior of the polynomials $(x-\beta_1^{(n)})\cdots(x-\beta_{n+1}^{(n)})$. Let us suppose that the points $\beta_k^{(n)}$ satisfy:

$$\lim_{n\to\infty} \left|(z-\beta_1^{(n)})\cdots(z-\beta_{n+1}^{(n)})\right|^{1/(n+1)} = \sigma(z)$$

for z on certain sets of the complex plane. If we can choose C as the contour $|\sigma(z)| = K$ (constant) then we obtain

$$\lim_{n\to\infty} p_n(x;t) = G(x,t)$$

uniformly for x on the compact subsets of $\{x \in R : |\sigma(x)| < K\}$. Let us consider the following example. Let $\beta_1^{(n)}, \cdots, \beta_{n+1}^{(n)}$ be the $(n+1)$ roots of the $(n+1)$st Tchebyshev polynomial $T_{n+1}(z)$. Then

$$T_{n+1}^*(z) = (z - \beta_1^{(n)}) \cdots (z - \beta_{n+1}^{(n)}) = \frac{1}{2^n} T_{n+1}(z)$$

and we can show that [3]

$$\sigma(z) = \lim_{n\to\infty} |T_n^*(z)|^{1/n} = \rho/2 \quad \text{for } z \in \mathcal{E}_\rho \text{ uniformly }, \tag{10}$$

where $\mathcal{E}_\rho$ is the ellipse with foci ± 1 and semi-axes $a = (\rho + \rho^{-1})/2$ and $b = (\rho - \rho^{-1})/2$. Applying **Theorem 1** we obtain the following result

Proposition 6 *Let f be an analytic function defined by (8), c the linear form defined by (2) and $G(x,t)$ the generating function. We suppose that c satisfies (5) and that for $t \in A_* \subset A$, $G(\cdot, t)$ is analytic in a region S and let $\mathcal{E}_\rho$ ($\rho > 1$) be the ellipse such that:*

$$\mathcal{E}_\rho \subset S \text{ and } \forall \rho' > \rho \;\; \mathcal{E}_{\rho'} \subset S.$$

We denote by E_ρ the interior of the ellipse $\mathcal{E}_\rho$ .If $\exists R_* > r$ (r given by (5)): $D_{R_*} \subset E_\rho$ and we choose the interpolation points $\beta_k^{(n)}$ as the roots of the Tchebyshev polynomials, the sequence of corresponding GPTA, $(n)_f(t)$, converges to $f(t)$ in A_*.

Remark: We obtain a similar result if we consider the roots of the Legendre polynomials, because $\lim_{n\to\infty} |P_n(z)|^{1/n} = \rho \;\; \forall z \in \mathcal{E}_\rho$ ($P_n(z)$ is the n-th Legendre polynomial).
*Proof:*From the conditions of the proposition we get that $G(\cdot, t) \in \mathcal{H}_2(R_*)$ for $t \in A_*$ and from (10) we obtain the uniform convergence of the interpolating polynomials in D_{R_*}. Result follows by applying **Theorem 1**. $\triangle$

Interpolation in the roots of unity.

Proposition 7 *Let $f(t)$, $G(x,t)$ and c be defined as in the previous theorems, and let us suppose that*

$$\forall t \in A_* \subset A \text{ fixed } G(\cdot, t) \text{ is analytic for } |x| < \rho > 1.$$

Let $P_n(x;t)$ be the polynomial of degree n which interpolates $G(\cdot, t)$ in the $(n+1)$st roots of unity. If $\rho > R_*$ then the corresponding sequence of GPTA converges to $f(t)$ for all $t \in A_*$.

This result follows immediately from the convergence result for interpolating polynomials in the roots of unity [6] and from **Theorem 1**.

3.1.3 Examples.

a) Function given by a Legendre series.

Let $f(z)$ be given by the series of functions

$$f(z) = \sum_{n=0}^{\infty} c_n P_n(z), \tag{11}$$

where $P_n(z)$ are the Legendre polynomials and $c_n = \int_{-1}^{1} f(x)P_n(x)dx$, $n = 0, 1, \cdots$. Let us suppose that $\lim_{n\to\infty} \sup |c_n|^{1/n} = 1/\rho$. Then the expansion (11) converges for t in the ellipse $\mathcal{E}_\rho$. A generating function for the Legendre polynomials is

$$G(x,t) = \sum_{n=0}^{\infty} x^n P_n(t) = (1 - 2tx + x^2)^{-1/2}.$$

For t fixed in $\mathcal{E}_\rho$, $G(x,t)$ is analytic in D_R, $\forall R \leq 1$ and so belongs to $\mathcal{H}_2(R)$. Let us choose R_* such that $1 > R_* > 1/\rho$, and a sequence of interpolation points $(x_i)_i$ in D_{R_*} converging to 0.

We replace $G(x,t)$ by the Lagrange interpolation polynomials $P_n(x;t) = \sum_{i=0}^{n} l_i(x)(1 - 2tx_i + x_i^2)^{-1/2}$ and so $f(t)$ is approached by the sequence

$$(n)_f(t) = c(P_n(\cdot;t)) = \sum_{i=0}^{n} c(l_i(\cdot))(1 - 2tx_i + x_i^2)^{-1/2}.$$

By applying **Theorem 2** we know that it converges to $f(t)$ for $t \in \mathcal{E}_\rho$.

b) Function given by an Hermite series.

Let $f(t)$ be given by the following Hermite series

$$f(t) = \sum_{n=0}^{\infty} c_n H_n(t) \quad t \in A, \text{ with } c_n = \int_{-\infty}^{+\infty} f(x)e^{-x^2} H_n(x)dx.$$

A generating function for the Hermite polynomials is

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n H_n(t) = e^{2xt - x^2}$$

For all $t \in \mathbb{C}$, $G(\cdot,t)$ is an entire function. In this case we have $d_n = n! c_n$ and so, if $\lim_{n\to\infty} |c_n|^{1/n} = 1/r$, then $\lim_{n\to\infty} |d_n|^{1/n} = \infty$ and so $G(\cdot,t)$ belongs to $\mathcal{H}_3(R)$ ($R > 0$).

If (x_i) is a sequence of interpolation points satisfying $\lim_{n\to\infty} x_n = 0$, $x_n \in D_R$ $\forall n$ then the corresponding sequence of approximants is a linear combination of exponentials

$$(n)_f(t) = c(P_n(\cdot,t)) = \sum_{i=0}^{n} e^{-x_i^2} c(l_i(\cdot)) e^{2x_i t}.$$

The sequence of GPTA $(n)_f(t)$ converges to $f(t)$ in A if c satisfies (7).

3.2 Orthogonal expansions

We will consider now a second type of interpolation conditions: the functionals L_k are defined by

$$L_k(g) = \int_C g(z)\overline{p_k(z)} w(z) \, |dz| \, ,$$

where C is a rectifiable Jordan arc or curve, $w(z)$ is a real, non-negative and uniformly limited function on C (not a null function), and $\{p_k(z)\}$ is the sequence of orthonormal polynomials on C with respect to the weight function $w(z)$. Then, the polynomial of degree n, $P_n(x;t)$, that satisfies the conditions

$$L_k(P_n(x;t)) = L_k(G(x,t)) \quad k = 0, \cdots, n, \tag{12}$$

(as before, L_k acts on the variable x) is given by

$$P_n(x;t) = \sum_{k=0}^{n} a_k p_k(x), \quad a_k = \int_C G(z,t)\overline{p_k(z)}w(z)\,|dz|\,,$$

which is also the polynomial of degree n of best approximation of $G(x,t)$ (function of x) in the sense of the least squares norm, that is, the polynomial of degree n that minimizes

$$\int_C |G(x,t) - p_n(x;t)|^2\, w(z)\,|dz|\,.$$

So the convergence of the sequence $(P_n(\cdot;t))_n$ to $G(\cdot,t)$ depends on the convergence of orthogonal expansions. We have the following result which determines the region of convergence of orthogonal expansions [6]:

Theorem 4 *Let C, $\{p_k(z)\}$ and $w(z)$ be defined as above.*

(i) *Let the numbers a_n satisfy the relation: $\lim_{n\to\infty} sup\,|a_n|^{1/n} = 1/\rho < 1$. Then the function $f(z)$ defined on C by the equation*

$$f(z) \equiv \sum_{n=0}^{\infty} a_n p_n(z)$$

is analytic interior to C_ρ, but if not identically constant with $\rho = \infty$ has a singularity on C_ρ.

(ii) *Conversely, if $f(z)$ is analytic interior to C_ρ but has a singularity at C_ρ then*

$$\lim_{n\to\infty} sup\,|a_n|^{1/n} = 1/\rho \quad , \quad a_n = \int_C f(z)\overline{p_k(z)}w(z)\,|dz|\,.$$

Let us recall the definition of C_R: C_R is the equipotential locus $|\phi(z)| = R$, where $\phi(z)$ is a function which maps the complement of the region bounded by C conformally onto the exterior of the unit circle, so that the points at infinity correspond to each other.

From this theorem, and proceeding like in the previous cases, we will obtain the convergence of a sequence of GPTA corresponding to interpolation conditions of the form (12) if the generating function satisfies the conditions of this theorem and if we can apply **Theorem 1**.

Let us consider in more detail the particular case of $C = [-1,+1]$. Then C_R is the ellipse with foci ± 1 and semi-axes $a = (R + 1/R)/2$ and $b = (R - 1/R)/2$ [6]. We obtain

Corollary 1 *Let f be an analytic function defined by $f(z) = \sum_{i=0}^{\infty} c_i g_i(z)$, $z \in A$, and let $G(x,t)$ be a generating function of the family $\{g_i(z)\}$. Let c be the functional defined by (2). We suppose that*

- *for $t \in A_* \subset A$ fixed, the function $G(x,t)$ of the x variable is analytic in the closed segment $[-1,1]$ and we set $\mathcal{E}_\rho$ the greatest ellipse with foci ± 1 in which $G(\cdot,t)$ is analytic for all $t \in A_*$;*
- *c satisfies (5);*
- *we can choose $R_* > r$ such that $D_{R_*} \subset \mathcal{E}_\rho$.*

Let us consider the polynomial of degree n in x, $P_n(x;t)$, satisfying relations (12) with

$$L_k(g) = \int_{-1}^{1} g(x)\frac{P_n^{(\alpha,\beta)}(x)}{h_n^{(\alpha,\beta)}}(1-x)^\alpha(1+x)^\beta dx,$$

that is, the partial sum of order n of the expansion of $G(\cdot,t)$ in a Jacobi series ($P_n^{(\alpha,\beta)}$ are the Jacobi polynomials, $h_n^{(\alpha,\beta)}$ their norms).

Then we obtain

$$\forall t \in A_* \quad \lim_{n\to\infty} (n)_f(t) = f(t).$$

Let us consider an example of application of this corollary.

Let f be an analytic function represented by a power series

$$f(t) = \sum_{i=0}^{\infty} c_i t^i \text{ with } \lim_{n\to\infty} |c_n|^{1/n} = r < 1.$$

Then the generating function is given by

$$G(x,t) = \sum_{i=0}^{\infty} x^i t^i = \frac{1}{1-xt}, \quad x \neq \frac{1}{t}.$$

For $|t| < \alpha < 1$, we have $1/|t| > 1/\alpha$ and so $G(x,t)$ as function of x is analytic for $|x| < 1/\alpha$. Let us replace $G(x,t)$ by the sequence of polynomials

$$P_n(x;t) = a_0 p_0(x) + a_1 p_1(x) + \cdots + a_n p_n(x)$$

where $\{p_i(x)\}$ is a system of orthonormal polynomials on C with respect to the weight function w, and the a_i are given by

$$a_i = \int_C G(z,t)\overline{p_i(z)} w(z)\,|dz| \quad i = 0,\cdots,n.$$

These polynomials satisfy the interpolation conditions (12) with

$$L_k(f) = \int_C f(z)\overline{p_k(z)} w(z)\,|dz| \quad k = 0,\cdots,n.$$

The corresponding sequence of GPTA is

$$c(P_n(\cdot,t)) = \sum_{i=0}^{n} a_i c(p_i(\cdot)), \text{ where } a_i = \int_C \frac{\overline{p_i(z)}}{1-zt} w(z)\,|dz|\,.$$

Let us consider the particular case where $C = [-1,1]$ and the $\{p_i(x)\}$ are the Legendre polynomials and let us obtain the form of the approximants.

By simple computations we obtain

$$\int_{-1}^{1} \frac{z^i}{1-zt}dz = R^i_{\log}(t) = \frac{\log\left(\frac{1+t}{1-t}\right) - S_i(t)}{t^{i+1}},$$

where $S_i(t)$ is the ith partial sum of the Taylor series of $\log\left(\frac{1+t}{1-t}\right)$, and so the approximants have the form

$$c(P_n(\cdot;t)) = \sum_{k=0}^{n} \alpha_k R^k_{\log}(t). \tag{13}$$

We can easily verify that for $|t| < \alpha < 1$, the conditions of the corollary are satisfied and the sequence (13) of GPTA converges to $f(t)$.

4 Conclusion.

The convergence results presented in this paper show that some classes of generalized Padé type approximants can have interesting approximation properties. They are very general and this opens the way to a more detailed study for particular cases. For functions given by an expansion in a series of a family of classical orthogonal polynomials, the problem of how to choose the interpolation points in order to have better convergence results and even acceleration of convergence is under study. Also the effect of choosing different generating functions for the same family has to be considered. This will be the subject of a forthcoming paper.

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