"Petite correction du devoir surveillé."

1) Notons $u_1=(1,a,2,-1)$, $u_2=(-2,3,a,1)$, $u_3=(2,-1,a,1)$, $u_4=(-1,0,2,-1)$; $E=\mathbb{R}^4$ si $\{u_1,u_2,u_3,u_4\}$ famille libre. Soit $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ tq $\alpha_1u_1+\alpha_2u_2+\alpha_3u_3+\alpha_4u_4=0$;

soit $\begin{cases}\alpha_1-2\alpha_2+2\alpha_3-\alpha_4=0\\ a\alpha_1+3\alpha_2-\alpha_3=0\\ 2\alpha_1+a\alpha_2+a\alpha_3+2\alpha_4=0\\ -\alpha_1+\alpha_2+\alpha_3-\alpha_4=0\end{cases} \iff \begin{cases}\alpha_4=\alpha_1-2\alpha_2+2\alpha_3\\ a\alpha_1+3\alpha_2-\alpha_3=0\\ 4\alpha_1+(a-4)\alpha_2+(a+4)\alpha_3=0\\ -2\alpha_1+3\alpha_2-\alpha_3=0\end{cases} \iff \begin{cases}\alpha_4=\alpha_1-2\alpha_2+2\alpha_3\\ \alpha_3=3\alpha_2-2\alpha_1\\ (a+2)\alpha_1=0\\ (4a+8)\alpha_2-2(a+2)\alpha_1=0\end{cases}$

$\iff \begin{cases}\alpha_4=\alpha_1-2\alpha_2+2\alpha_3\\ \alpha_3=3\alpha_2-2\alpha_1\\ a+2=0 \text{ ou } \alpha_2=0\\ \cancel{2}(a+2)(2\alpha_2-\alpha_1)=0\end{cases}$; si $a\neq-2$, alors $\begin{cases}\alpha_1=0\\ \alpha_2=0\\ \alpha_3=0\\ \alpha_4=0\end{cases}$ et donc $E=\mathbb{R}^4$!!

Si $a=-2$, le système devient $\begin{cases}\alpha_4=\alpha_1-2\alpha_2+2\alpha_3\\ \alpha_3=3\alpha_2-2\alpha_1\end{cases}$, et donc $\{u_1,u_2,u_3,u_4\}$ est une famille liée, ce qui entraîne $E\subsetneq\mathbb{R}^4$.

2.1) Si $a=-2$, d'après la question précédente : $\alpha_1u_1+\alpha_2u_2+\alpha_3u_3+\alpha_4u_4=0$

$\iff \begin{cases}\alpha_4=\alpha_1-2\alpha_2+2\alpha_3\\ \alpha_3=3\alpha_2-2\alpha_1\end{cases} \iff \begin{cases}\alpha_3=3\alpha_2-2\alpha_1\\ \alpha_4=-3\alpha_1+4\alpha_2\end{cases}$; on obtient donc :

$\begin{cases}\text{si } \alpha_1=1 \text{ et } \alpha_2=0, \text{ alors } u_1=2u_3+3u_4, \text{ et donc } u_1\in\langle u_3,u_4\rangle\\ \text{si } \alpha_1=0 \text{ et } \alpha_2=1, \text{ alors } u_2=-3u_3-4u_4, \text{ et donc } u_2\in\langle u_3,u_4\rangle\end{cases}$;

ainsi $E=\langle u_3,u_4\rangle$. De plus, toujours d'après 1 : $\alpha_3u_3+\alpha_4u_4=0$ $\iff \alpha_3=\alpha_4=0$, ce qui entraîne $\{u_3,u_4\}$ base de E !!

2.2 $(x_1,x_2,x_3,x_4)\in E \iff \begin{cases}x_1=2\alpha_1-\alpha_2\\ x_2=-\alpha_1\\ x_3=-2\alpha_1+2\alpha_2=2(\alpha_2-\alpha_1)\\ x_4=\alpha_1-\alpha_2=(\alpha_1-\alpha_2).\end{cases} \iff \begin{cases}x_1=-2x_2-\alpha_2\\ x_2=-\alpha_1\\ x_4=\alpha_1-\alpha_2=-x_2-\alpha_2\\ x_4+x_3=0\end{cases}$

$\iff \begin{cases}x_1=-2x_2-\alpha_2\\ x_2=-\alpha_1\\ x_4=-x_2+x_1+2x_2=x_1+x_2\\ x_4+x_3=0\end{cases} \iff \begin{cases}x_3+x_4=0\\ x_1+x_2-x_4=0\end{cases}$;

d'où $(1,0,0,0)\notin E$, $(0,1,0,0)\notin E$, $(0,0,1,0)\notin E$, $(0,0,0,1)\notin E$!

2.3 d'après ce qui précède $(1,-1,0,0)\in E$!

Exercice 2

1)

Soit $u_1 = (1,0,1,-1,0)$, $u_2 = (2,1,2,1,1)$, $u_3 = (3,1,2,0,1)$; $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$

$$\Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_2 + \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = +\alpha_2 \\ 3\alpha_2 + 3\alpha_3 = 0 \\ 3\alpha_2 + 2\alpha_3 = 0. \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \alpha_2 \\ \alpha_3 = 0 \\ \alpha_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases};$$

$\{u_1, u_2, u_3\}$ est une famille libre $\Rightarrow$ $\{u_1, u_2, u_3\}$ base de F !

- Soit $w_1 = (1,1,3,-1,1)$, $w_2 = (2,-1,-4,4,-1)$, $w_3 = (0,1,2,0,1)$, $w_4 = (1,-2,-3,-1,-2)$;

$$\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 = 0 \Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_4 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4 = 0 \\ 3\alpha_1 - 4\alpha_2 + 2\alpha_3 - 3\alpha_4 = 0 \\ -\alpha_1 + 4\alpha_2 - \alpha_4 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -2\alpha_2 - \alpha_4 \\ -3\alpha_2 - 3\alpha_4 + \alpha_3 = 0 \\ -10\alpha_2 - 6\alpha_4 + 2\alpha_3 = 0 \\ 6\alpha_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_2 = 0 \\ \alpha_3 - 3\alpha_4 = 0 \\ 2\alpha_3 - 6\alpha_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_2 = 0 \\ \alpha_3 = 3\alpha_4 \end{cases};$$

si $\alpha_4 = 1$, alors $w_4 = w_1 - 3w_3 \Rightarrow w_4 \in \langle w_1, w_2, w_3 \rangle$. De plus,

si $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = 0$, alors - d'après ce qui précède - $\alpha_1 = \alpha_2 = \alpha_3 = 0$, et donc $\{w_1, w_2, w_3\}$ est une famille libre et $G = \langle w_1, w_2, w_3 \rangle$!

2) - $F + G = \langle u_1, u_2, u_3, w_1, w_2, w_3 \rangle$! De plus $(x_1, x_2, x_3, x_4, x_5) \in F$

$$\Leftrightarrow \begin{cases} x_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ x_2 = \alpha_2 + \alpha_3 \\ x_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ x_4 = -\alpha_1 + \alpha_2 \\ x_5 = \alpha_2 + \alpha_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = \alpha_1 + 2\alpha_2 + \alpha_3 \\ x_2 = \alpha_2 + \alpha_3 \\ x_3 = x_1 \\ x_4 = \alpha_2 - \alpha_1 \\ x_5 = x_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = x_1 - 2\alpha_2 - \alpha_3 \\ x_2 = \alpha_2 + \alpha_3 \\ x_4 = 3\alpha_2 - x_1 + \alpha_3 \\ x_1 = x_3 \\ x_5 = x_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha_1 = x_1 - 2\alpha_2 - \alpha_3. \\ \alpha_2 = x_2 - \alpha_3. \\ x_4 = 3x_2 - x_1 - 2\alpha_3 \\ x_1 = x_3 \\ x_5 = x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_3 \\ x_5 = x_2 \end{cases};$$

ainsi $w_1 \notin \langle u_1, u_2, u_3 \rangle$ et $\{u_1, u_2, u_3, w_1\}$ est une famille libre !!

$$(x_1, x_2, x_3, x_4, x_5) \in \langle u_1, u_2, u_3, w_1 \rangle \Leftrightarrow \begin{cases} x_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \\ x_2 = \alpha_2 + \alpha_3 + \alpha_4 \\ x_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 \\ x_4 = -\alpha_1 + \alpha_2 - \alpha_4 \\ x_5 = \alpha_2 + \alpha_3 + \alpha_4 \end{cases}$$

$$\iff \begin{cases} x_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \\ x_2 = \alpha_2 + \alpha_3 + \alpha_4 \\ x_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 \\ x_4 = -\alpha_1 + \alpha_2 - \alpha_4 \\ x_5 = x_2 \end{cases} \iff \begin{cases} \alpha_1 = x_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4 \\ \alpha_2 = x_2 - \alpha_3 - \alpha_4 \\ x_3 = x_1 - \alpha_3 + 2\alpha_4 \\ x_4 = -x_1 + 3\alpha_2 + 3\alpha_3 \\ x_5 = x_2 \end{cases} \iff \begin{cases} \alpha_1 = x_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4 \\ \alpha_2 = x_2 - \alpha_3 - \alpha_4 \\ x_3 = x_1 - \alpha_3 + 2\alpha_4 \\ x_4 = -x_1 + 3x_2 - 3\alpha_4 \\ x_5 = x_2 \end{cases}$$

$$\iff \begin{cases} \alpha_1 = x_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4 \\ \alpha_2 = x_2 - \alpha_3 - \alpha_4 \\ x_3 = x_1 - \alpha_3 + 2\alpha_4 \\ x_4 = -x_1 + 3x_2 - 3\alpha_4 \\ x_5 = x_2 \end{cases} \iff x_5 = x_2 \text{ !!}$$

Par conséquent, $\begin{cases} w_2 \in \langle u_1, u_2, u_3, w_1 \rangle \\ w_3 \in \langle u_1, u_2, u_3, w_1 \rangle \end{cases}$

$\Rightarrow F + G = \langle u_1, u_2, u_3, w_1 \rangle$ avec $\{u_1, u_2, u_3, w_1\}$ base de $F+G$!

D'après le cours : $\dim(F+G) = 4 = \dim(F) + \dim(G) - \dim(F \cap G) = 6 - \dim(F \cap G)$

$\Rightarrow \dim(F \cap G) = 2$!!

Exercice III

$u = u_1, u_2$

$v = v_1, v_2$; $\begin{cases} f(u) = 2u \\ f(v) = u + 2v \end{cases} \iff \begin{cases} 4u_1 + 4u_2 = 2u_1 \\ -u_1 = 2u_2 \\ 4v_1 + 4v_2 = u_1 + 2v_1 \\ -v_1 = u_2 + 2v_2 \end{cases} \iff \begin{cases} u_1 + 2u_2 = 0 \\ 2v_1 + 4v_2 = u_1 \\ -v_1 = u_2 + 2v_2 \end{cases}$

$$\iff \begin{cases} u_1 = -2u_2 \\ 2v_1 + 4v_2 = -2u_2 \\ v_1 + 2v_2 = -u_2 \end{cases} \iff \begin{cases} u_1 = -2u_2 \\ v_1 + 2v_2 = -u_2 \end{cases}$$

Si $u = (-2, 1)$ et $v = (-1, 0)$ alors $\begin{cases} f(u) = 2u \\ f(v) = u + 2v \end{cases}$

- Si $\alpha u + \beta v = 0 \Rightarrow \alpha f(u) + \beta f(v) = 0 \Rightarrow 2\alpha u + \beta(u + 2v) = (2\alpha + \beta)u + 2\beta v = 0$,

et donc $\begin{cases} \alpha u + \beta v = 0 \\ (2\alpha + \beta)u + 2\beta v = 0 \end{cases} \Rightarrow \begin{cases} \alpha u + \beta v = 0 \\ \beta u = 0 \end{cases} \Rightarrow \begin{cases} \alpha u = 0 \Rightarrow \alpha = 0 \text{ car } u \neq 0 \\ \beta = 0 \text{ car } u \neq 0 \end{cases}$

(u, v) est donc libre !

$$M = \begin{array}{cc} & \begin{array}{cc} f(u) & f(v) \end{array} \\ & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{array}{c} u \\ v \end{array} \end{array} ;$$

Exercice IV

1) $\alpha u + \beta f(u) = 0 \Rightarrow \alpha f(u) + \beta f^2(u) = 0 \Rightarrow \alpha f(u) - \beta u = 0$, et donc : $\begin{cases} \alpha u + \beta f(u) = 0 \\ \alpha f(u) - \beta u = 0 \end{cases}$

TSVP →

- Si $\alpha \neq 0 \Rightarrow \begin{cases} f(u) = \frac{\beta}{\alpha} u \\ \alpha u + \frac{\beta^2}{\alpha} u = \frac{(\alpha^2+\beta^2)}{\alpha} u = 0 \Rightarrow \alpha = \beta = 0 \text{ car } u \neq 0 \text{ impossible !} \end{cases}$

d'après ce qui précède, $\alpha = 0$ et donc $-\beta u = 0 \Rightarrow \beta = 0$ car $u \neq 0$;
$(u, f(u))$ est donc libre !

2) $M = \begin{array}{cc} \begin{array}{cc} f(u) & f(f(u)) \end{array} & \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{array}{c} u \\ f(u) \end{array} \end{array}$