MATH 202B 2ème session. juin 2008 SOLUTIONS.

Ex I. (cours)

Ex II.

(a). Déf : f est continue en $(0,0)$ ssi $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^2} - \overset{\nearrow 0}{y} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^2}$$

① passage en polaires
$$= \lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2(r^2\cos^2\theta+\sin^2\theta)} = \lim_{r\to 0} \underset{\downarrow 0}{r}\left(\frac{\cos\theta\sin^2\theta}{r^2\cos^2\theta+\sin^2\theta}\right) = 0$$
$\cos\theta$ bornée

ou

② majorations directes
$$\leq \lim_{(x,y)\to(0,0)} \frac{xy^2}{y^2} = \lim_{(x,y)\to(0,0)} x = 0$$
on a $x^4+y^2 \geq y^2$

(b).
$$\frac{\partial f}{\partial x}(0,0) := \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\left(\frac{0-0}{x}\right) = 0$$

$$\frac{\partial f}{\partial y}(0,0) := \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1$$

(c). Utilisons la déf de la différentiabilité en $(0,0)$.

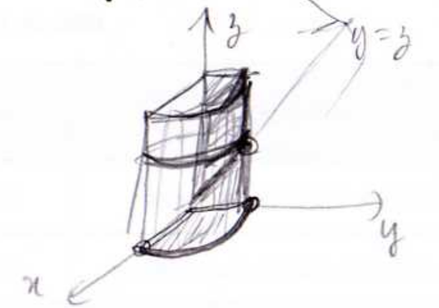
$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)\cdot x-\frac{\partial f}{\partial y}(0,0)\cdot y}{\sqrt{x^2+y^2}} = \lim_{\binom{x}{y}\to\binom{0}{0}} \frac{\frac{xy^2}{x^4+y^2}}{\sqrt{x^2+y^2}}$$

$$= \lim_{\binom{x}{y}\to\binom{0}{0}} \frac{xy^2}{(x^4+y^2)\sqrt{x^2+y^2}}$$; Calculons la limite suivant

la direction $x=y$ on obtient $\lim\limits_{x\to 0} \dfrac{x^3}{(x^4+x^2)\sqrt{2}\,|x|} = \lim\limits_{x\to 0} \dfrac{x^3}{x^2|x|(x^2+1)\sqrt{2}}$ $\to \sqrt{2}$

$= \lim \dfrac{x}{|x|}\cdot\dfrac{1}{\sqrt{2}}$ n'existe pas car $\dfrac{x}{|x|}$ n'a pas de limite en 0.

Ex III : $D = \{(x,y,z)\in\mathbb{R}^3 : x\geq 0,\ 0\leq z\leq y\ \ x^2+y^2\leq 1\}$



on a $0\leq z\leq 1$ $\quad 1\geq y\geq z$ $\quad 0\leq x\leq\sqrt{1-y^2}$

d'où
$$\iiint_D xyz\,dx\,dy\,dz = \int_0^1\left(\underbrace{\int_z^1\Big(\underbrace{\int_0^{\sqrt{1-y^2}} xyz\,dx}_{①}\Big)dy}_{②}\right)dz$$

Calculons ①.
$$① = \int_0^{\sqrt{1-y^2}} xyz\,dx = (yz)\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} = yz\cdot\frac{1-y^2}{2}$$

Calculons ②.
$$② = \int_z^1 yz\,\frac{1-y^2}{2}\,dy = \frac{z}{2}\cdot\int_z^1 y-y^3\,dy$$
$$= \frac{z}{2}\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_z^1 = \frac{z}{2}\left[\frac{1}{2}-\frac{1}{4}-\frac{z^2}{2}+\frac{z^4}{4}\right] = \frac{z-2z^3+z^5}{8}$$

Finalt l'intégrale cherchée vaut :
$$\frac{1}{8}\int_0^1 z-2z^3+z^5 = \frac{1}{8}\left[\frac{z^2}{2}-\frac{z^4}{2}+\frac{z^6}{6}\right]_0^1 = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)$$
$$= \frac{1}{8\times 6} = \boxed{\frac{1}{48}}$$

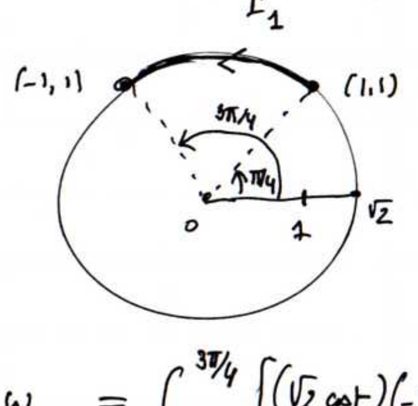
Ex IV.

1. $\underset{Pdx+Qdy}{\omega}$ est fermée sur $\mathbb{R}^2$ si $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

ici on a $\frac{\partial P}{\partial y}(x,y) = 0$ et $\frac{\partial Q}{\partial x}(x,y) = y$

Donc ω n'est pas fermée sur $\mathbb{R}^2$

2.



paramétrisons Γ_1 :

$\Gamma_1 : \begin{cases} \sqrt{2}\cos t \\ \sqrt{2}\sin t \end{cases}, \ t \in [\frac{\pi}{4}, \frac{3\pi}{4}]$

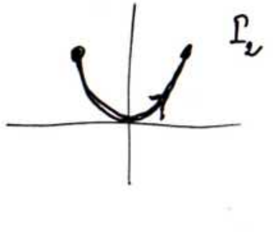
$$\int_{\Gamma_1} \omega = \int_{\pi/4}^{3\pi/4} \left[(\sqrt{2}\cos t)(-\sqrt{2}\sin t) + (\sqrt{2}\cos t)(\sqrt{2}\sin t)\sqrt{2}\cos t \, dt\right.$$

$$= \int_{\pi/4}^{3\pi/4} \left(\underset{2uu'}{-2\cos t \sin t} + \underset{-2\sqrt{2}\, v^2 u'}{2\sqrt{2}\cos^2 t \sin t}\right) dt$$

$$= \left[\cos^2 t - \frac{2\sqrt{2}}{3}\cos^3 t\right]_{\pi/4}^{3\pi/4} = \cancel{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{2\sqrt{2}}{3}\left(-\frac{\sqrt{2}}{2}\right)^3$$

$$- \cancel{\left(\frac{\sqrt{2}}{2}\right)^2} + \frac{2\sqrt{2}}{3}\left(\frac{\sqrt{2}}{2}\right)^3 = \cancel{\frac{4\sqrt{2}}{3}} \cdot \cancel{\frac{2\sqrt{2}}{8}} = \frac{2}{3}$$

3.

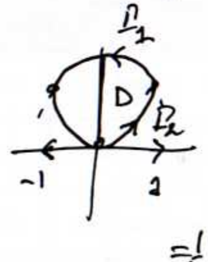


paramétrisons Γ_2 : $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \quad t \in [-1,1]$

$$\int_{\Gamma_2} \omega = \int_{-1}^{1} t \, dt + \frac{t^3}{2t^4} 2t \, dt = \left[\frac{t^2}{2} + \frac{2t^5}{5}\right]_{-1}^{1} = \frac{4}{5}$$

4. $D = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \ ; \ x^2 + y^2 \leq 2\}$

$$\iint_D y \, dx\, dy = \int_{-1}^{1} \left(\int_{x^2}^{\sqrt{2-x^2}} y \, dy\right) dx$$



$$= \int_{-1}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{2-x^2}} dx = \int_{-1}^{1} \frac{1}{2}(2 - x^2 - x^4) \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (2 - x^2 - x^4) \, dx = \frac{1}{2}\left[2x - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \cancel{\frac{1}{2}}\cancel{\times 2}\left(2 - \frac{1}{3} - \frac{1}{5}\right) = \frac{30-5-3}{15}$$

$$\boxed{= \frac{22}{15}}$$

5. La formule de Green - Riemann appliquée à ω sur le contour fermé $\Gamma_1 \cup \Gamma_2$ s'écrit :

$$\int_{\Gamma_1 \cup \Gamma_2} \omega = \iint_D d\omega \qquad d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = y \, dx\,dy$$

càd $\underbrace{\int_{\Gamma_1} \omega}_{} + \int_{\Gamma_2} \omega = \iint_D y \, dx\,dy$

$$\frac{2}{3} + \frac{4}{5} = \frac{22}{15} \quad \text{ok.}$$

Ex V.

1. Montrons que φ est une bijection de U dans V

$$\begin{cases} u = xy \\ v = x+y \end{cases} \iff \begin{cases} u = x(v-x) \iff x^2 - vx + u = 0 \quad (1) \\ y = v - x \end{cases}$$

<u>résolution de (1)</u> : $x^2 - vx + u = 0 \quad \Delta = v^2 - 4u > 0$ car $u < 0$

donc $x_1 = \frac{v + \sqrt{v^2 - 4u}}{2}$ ou $x_2 = \frac{v - \sqrt{v^2-4u}}{2}$

comme $u < 0$ $v^2 - 4u > v^2$ donc $\sqrt{v^2 - 4u} > |v|$ donc $\begin{cases} x_2 < 0 \to y = v - x_2 = x_1 \\ x_1 > 0 \to \end{cases}$

<u>Par conséquent</u> étant donné $(u,v) \in V$, $\exists !\,(x,y)$ avec $\begin{cases} x < 0 \\ y > 0 \end{cases}$ tq $\varphi(u,v) = (x,y)$

Ceci montre que $V \subseteq \varphi(U)$

Vérifions que $\varphi(U) \subseteq V$ en effet $u = xy < 0$.

φ est C^1 : φ est une f^n polynomiale

$$\left|(\mathrm{Jac}\,\varphi)_{(x,y)}\right| = \left|\begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}\right| = y - x$$ ne s'annule jamais sur U

donc φ est bien un $\mathcal{C}^1$ difféo et $\varphi^{-1}(u,v) =$

$$\left(\frac{v - \sqrt{v^2 - 4u}}{2}, \frac{v + \sqrt{v^2 - 4u}}{2}\right).$$

2. (i). ∘ $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial F(u(x,y), v(x,y))}{\partial x}$

$F(u,v) = f(x,y)$

$$= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial u} \cdot (y) + \frac{\partial F}{\partial v}$$

∘ $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{\partial F}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial F}{\partial v} \cdot \dfrac{\partial v}{\partial y}$

$$= \frac{\partial F}{\partial u}(x) + \frac{\partial F}{\partial v}.$$

L'eq s'écrit : $y \dfrac{\partial F}{\partial u} + \dfrac{\partial F}{\partial v} - \left(x \dfrac{\partial F}{\partial u} + \dfrac{\partial F}{\partial v}\right) = 3(y-x)$

$$(y-x)\frac{\partial F}{\partial u} = 3(y-x)$$

càd $\dfrac{\partial F}{\partial u} = 3$.

(ii). $F(u,v) = 3u + g(v)$. g f^n C^1 qcq

et par conséquent $f(x,y) = 3xy + g(x+y)$.