Laguerre Tessellations

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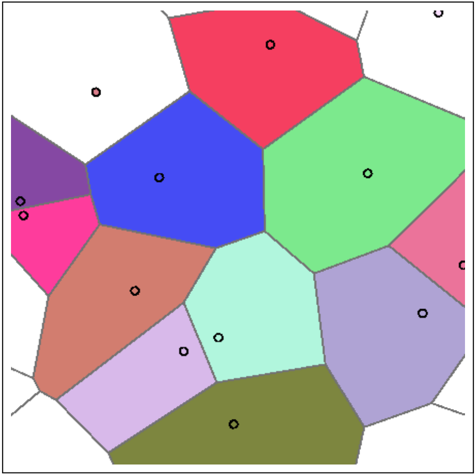
Voronoi Tessellation



Voronoi Cells

Given $\varphi = \{x_1, x_2, \dots\} \subset \mathbb{R}^d$ at most countable set of nuclei points, with each $x_i \in \varphi$ there is associated a

Voronoi Cell

$$C(x_i, \varphi) = \{y \in \mathbb{R}^d : \; \|y - x_i\| \leq \|y - x_j\| \text{ for all } x_j \in \varphi\}$$

Main Properties of Voronoi Tessellation

- Being intersection of half-spaces, each cell is a convex polytope
- Voronoi tessellation (VT) is normal, i. e. if nuclei φ are in a *general position*[1] then each k-dimensional face (k-face) *is* the intersection of exactly $d - k + 1$ cells (vertices are 0-dimensional faces)
- In particular, each k-face is the set of points equidistant from exactly $d - k + 1$ nuclei.

[1] No $k + 2$ nuclei lie in a k-dimensional plane and no $k + 3$ nuclei lie in a k-dimensional sphere

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Applications and Limitations of Voronoi Tessellation

- There are huge number of different applications of VT: modelling of cellular systems in Biology, material science, telecommunications, encoding algorithms, numeric integration, etc.
- Still, since the geometry of cells defined by inter-distances between nuclei only, all nuclei are ‘equal’. However, in many real situations this may be too restrictive. One may wish to assign some ‘weight’ to nuclei so that ‘mighty’ nuclei get larger cells.

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Power Function

Assume with each nuclei $x_i \in \varphi$ there is associated a non-negative weight w_i. A power of a point y with respect to the pair (x, w) is defined as

$$\mathrm{pow}(y,(x,w)) = \|y - x\|^2 - w\,.$$

In geometry, this is the power of a point with respect to a sphere of radius $r = \sqrt{w}$, hence the name.

Laguerre Cells

- Let $\varphi = \{(x_i, r_i)\}$ be (weighted) nuclei set. With every nucleus (x_i, r_i) there is associated

Laguerre cell

$$C((x_i, r_i), \varphi) = \{y \in \mathbb{R}^d : \ \mathrm{pow}(y, (x_i, r_i^2)) \leq \mathrm{pow}(y, (x_j, r_j^2)) \quad \text{for all } (x_j, r_j) \in \varphi\} .$$

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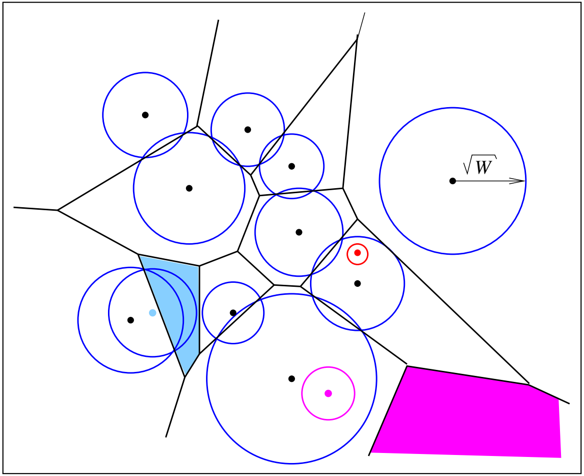
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Voronoi Cells vs. Laguerre Cells

Similarities

- Laguerre cells are also convex polytopes
- Laguerre tessellation (LT) formed by non-empty cells is normal

Differences

- Laguerre cell can be empty
- It may not contain its nucleus
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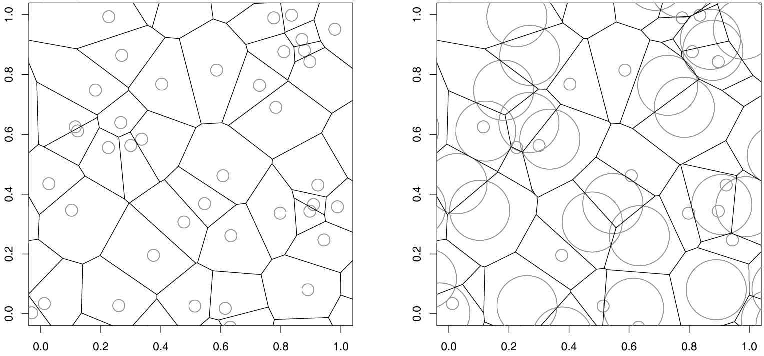
Voronoi and Laguerre Tessellations



Normal tessellations in $\mathbb{R}^d$ $(d \geq 3)$

Theorem

Every normal tessellation of $\mathbb{R}^d$ with convex cells for $d \geq 3$ is a Laguerre tessellation.

This statement cannot be strengthened to include $d = 2$, a counter-example is given by F. Aurenhammer

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Poisson Laguerre Tessellations

- If the nuclei set φ is given by a realisation of a random point process, the corresponding LT becomes a random tessellation.
- From now on we concentrate on Poisson Laguerre tessellations (PLT) when φ is a realisation of a Poisson point process (PPP) in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda dx \rho(dr)$.
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The Condition of Existence

Theorem

Let Φ be a PPP with intensity measure $\lambda dx \rho(dr)$. Then the PLT exists, i.e. $\min_{(x,r)\in\Phi} \mathrm{pow}(y,(x,r)) > -\infty$ a.s. for all $y \in \mathbb{R}^d$,

$$\text{if and only if } \int_1^\infty r^d \rho(dr) < \infty .$$

Idea of the Proof

Generating functional for PPP: for a suitable function $v(x,r)$

$$G[v] = \mathbf{E}\Big[\prod_{(x,r)\in\Phi} v(x,r)\Big] = \exp\Big\{-\lambda \int (1-v(x,r))\,\lambda_d(dx)\,\rho(dr)\Big\}.$$

Let $y \in \mathbb{R}^d$ and $p(t)$ be the probability that the power from y to each point of Φ exceeds t:

$$\begin{aligned} p(t) &= \mathbf{E}\left[\prod_{(x,r)\in\Phi} \mathbb{1}\{\mathrm{pow}(y,(x,r)) > t\}\right] \\ &= G\big[\mathbb{1}\{\mathrm{pow}(y,(x,r)) > t\}\big] \\ &= \exp\left\{-\lambda\omega_d \int_0^\infty \left([t+r^2]^+\right)^{\frac{d}{2}} \rho(dr)\right\}, \end{aligned}$$

where $t^+ = \max(t,0)$ and ω_d is the volume of a unit ball in $\mathbb{R}^d$.

If $\int_1^\infty r^d \rho(dr) < \infty$, then

$$\mathbf{P}\Big(\inf_{(x,r)\in\Phi} \mathrm{pow}(y,(x,r)) = -\infty\Big)$$
$$= \lim_{t\to-\infty} \mathbf{P}\Big(\inf_{(x,r)\in\Phi} \mathrm{pow}(y,(x,r)) < t\Big) = \lim_{t\to-\infty}(1-p(t)) = 0.$$

Thus for each $y \in \mathbb{R}^d$ we have at least one point $(x,r) \in \Phi$ minimising $\mathrm{pow}(y,(x,r))$. Hence, $y \in C((x,r),\Phi)$.
On the other hand, $\int_1^\infty r^d \rho(dr) = \infty$ implies $p(t) = 0$ for each t and therefore $\inf_{(x,r)\in\Phi} \mathrm{pow}(y,(x,r)) = -\infty$ with probability 1.

Intensities of faces

There are two types of intensities to be defined: counting and area-weighted.

- Let γ_k be intensity of k-dimensional faces of the PVT ($k = 0$ corresponds to vertices, $k = d$ – to the cells). So that in a large volume A there is on average $\gamma_k A$ k-dimensional faces.
- Let μ_k be the mean k-content of k-faces per unit volume. So that in a large volume A the k-content of k-dimensional faces is on average $\mu_k A$.

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Expressions for intensities

There are expressions for γ_k and μ_k for $k = 0, \dots, d$ and all $d \geq 2$ which involve integration w.r.t. $\rho(dr)$ and the integrals

$$V_{m,k}(w_0, \dots, w_m) = (m!)^{k+1} \int_{(\mathbb{S}^{m-1})^{m+1}} \Delta_m^{k+1}(w_0 u_0 \dots w_m u_m)\, \sigma^{m+1}(du_0 \dots du_m)\,,$$

where $\Delta_m(x_0, \dots, x_m)$ is the m-dimensional volume of the convex hull of $x_0, \dots, x_m$ in $\mathbb{R}^m$ and σ is the surface measure.

In some cases, there are explicit expressions for $V_{m,k}$. E.g. if the weights are equal (Voronoi tessellation case)

$$V_{m,k}(1,\ldots,1) = 2^{m+1}\pi^{\frac{m(m+1)}{2}} \times \frac{\Gamma\left(\frac{1}{2}(m+1)(d+1)-m\right)}{\Gamma(\frac{md}{2})\Gamma(\frac{d+1}{2})^{m+1}} \prod_{i=1}^{m} \frac{\Gamma\left(\frac{1}{2}(k+1+i)\right)}{\Gamma(\frac{i}{2})}.$$

Planar Laguerre Tessellations

For planar normal tessellations one has

$$\gamma_1 = 3/2\gamma_2 \qquad \gamma_0 = 1/2\gamma_2$$
$$\gamma_2 = p_0\lambda \qquad \mu_0 = \gamma_0 = 2\gamma_2\,,$$

where p_0 is the (Palm) probability that a typical Laguerre cell is non-empty.
So it is sufficient to evaluate explicitly at least one of the above quantities (and μ_1) for the others to come explicit too.

Idea for computation of vertex intensity

At a vertex, the powers from nuclei $z_i = (x_i, r_i)$ of 3 neighbouring Laguerre cells have the same value, say t. Powers from all other nuclei should be larger than t. The prob. of the last is $p(t)$. Now integrate w.r.t t and possible position of z_i's.

$$\mu_0 = \frac{\lambda^3}{12} \iiint_{\mathbb{R}^3_+} \int_{-\min_i r_i^2}^{\infty} e^{-\lambda\pi \int_0^\infty [t+r^2]^+ \rho(dr)} \times V_{2,0}\left((t+r_0^2)^{\frac{1}{2}}, (t+r_1^2)^{\frac{1}{2}}, (t+r_2^2)^{\frac{1}{2}}\right) dt \times \rho(dr_0)\rho(dr_1)\rho(dr_2)$$

Other results

- Mean k-content of typical k-faces
- Spherical and linear contact distribution functions
- Convergence to VT under different schemes of power distributions
- Characteristics of sectional tessellations
- Statistical applications: modelling and parameter fitting

Happy reading and applying!

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References

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