Meta-stablity in stochastic partial differential equations induced by Lévy noise

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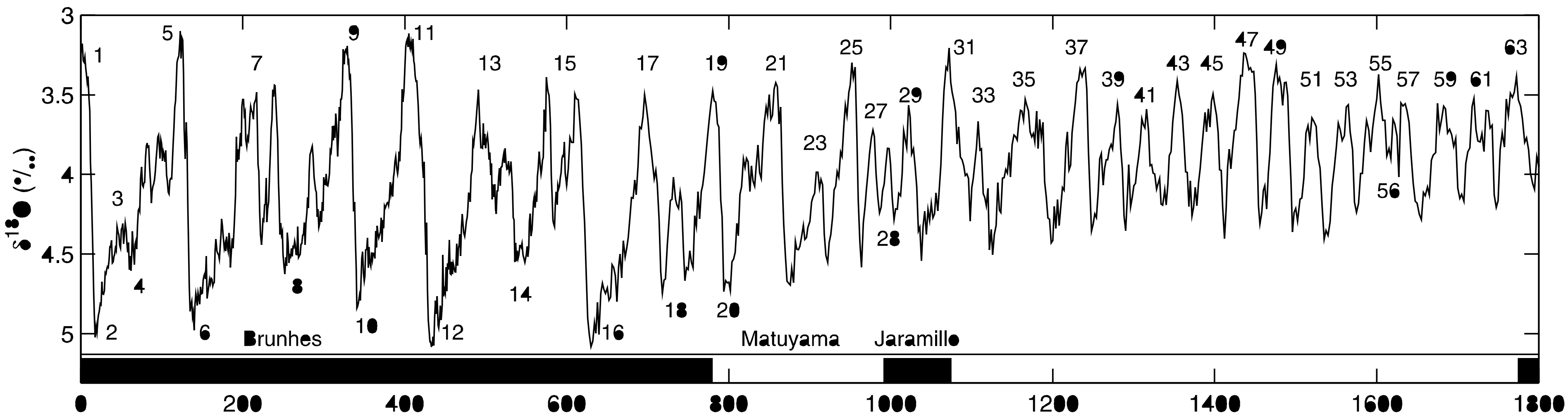
1. Paleoclimatic time series



Lisiecki, Raymo, Paleoceanography 2005 concentration variation of ^{18}O to ^{16}O taken from marine sediments at 57 globally distributed sites (e.g. Brunhes, Matuyama, Jaramillo):

global average temperature time series

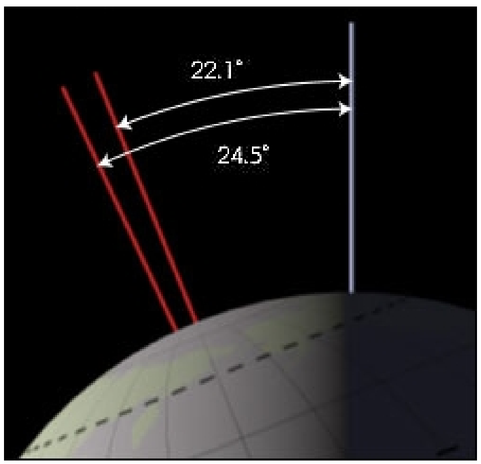
basic feature:

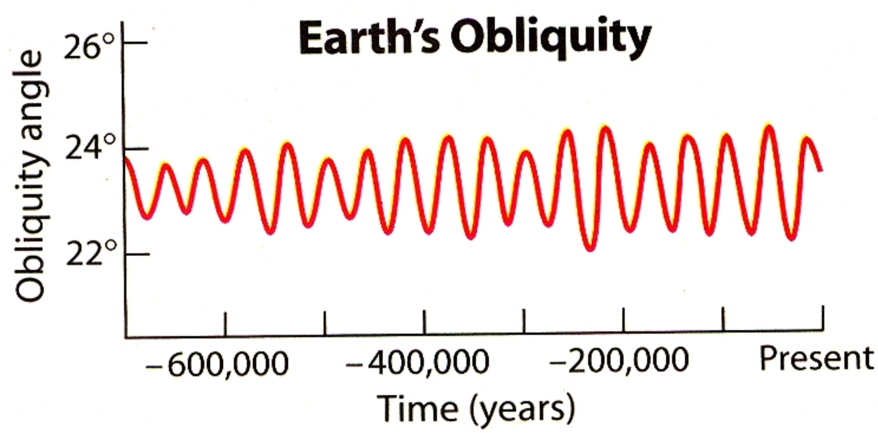
- from 0 to -1 Myr periodicity $\sim$ 100 000 y
- from -1 Myr to -1.8 Myr periodicity $\sim$ 44 000 y

2. Milankovich cycles

Milankovich (1920): astronomical perturbations of earth's trajectory give rise to basically three cycles: precession, obliquity, eccentricity

obliquity (axial tilt) (41 000 y)



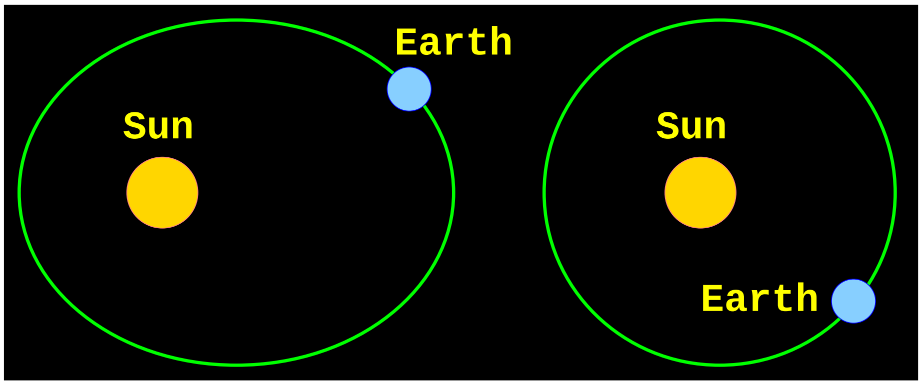


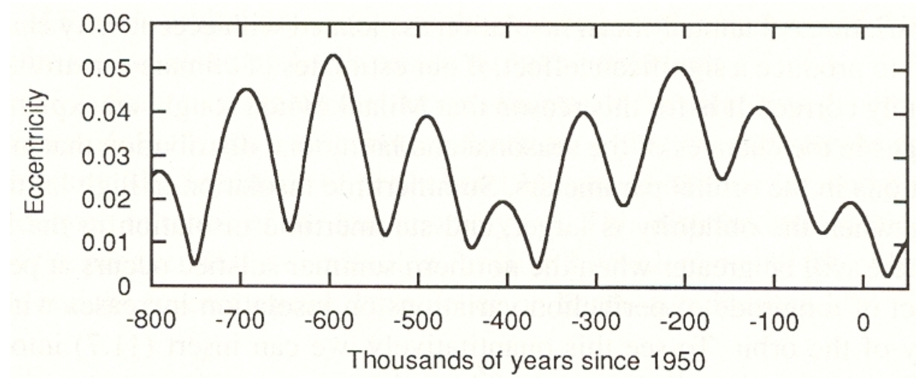
periodic wobbling of Earth's rotation axis: 2.4 deg change in tilt of axis with respect to plane of orbit; periodicity: 41 000 y; present tilt: 23.4 deg.

increase of obliquity: increase of amplitude of seasonal cycle in insolation, summers receiving more radiative flux, winters less

3. Milankovich cycles

eccentricity (100 000 y)





eccentricity: measure of deviation of ellipse from circularity; periodic variation between 0.005 and 0.058; reason: interactions with gravitational fields of Jupiter and Saturn; periodicity 100,000 y; present eccentricity: 0.017.

Orbital mechanics: eccentricity extreme means seasons on far side of orbit last substantially longer.

4. EBM: Gaussian SDE model; noise induced transitions

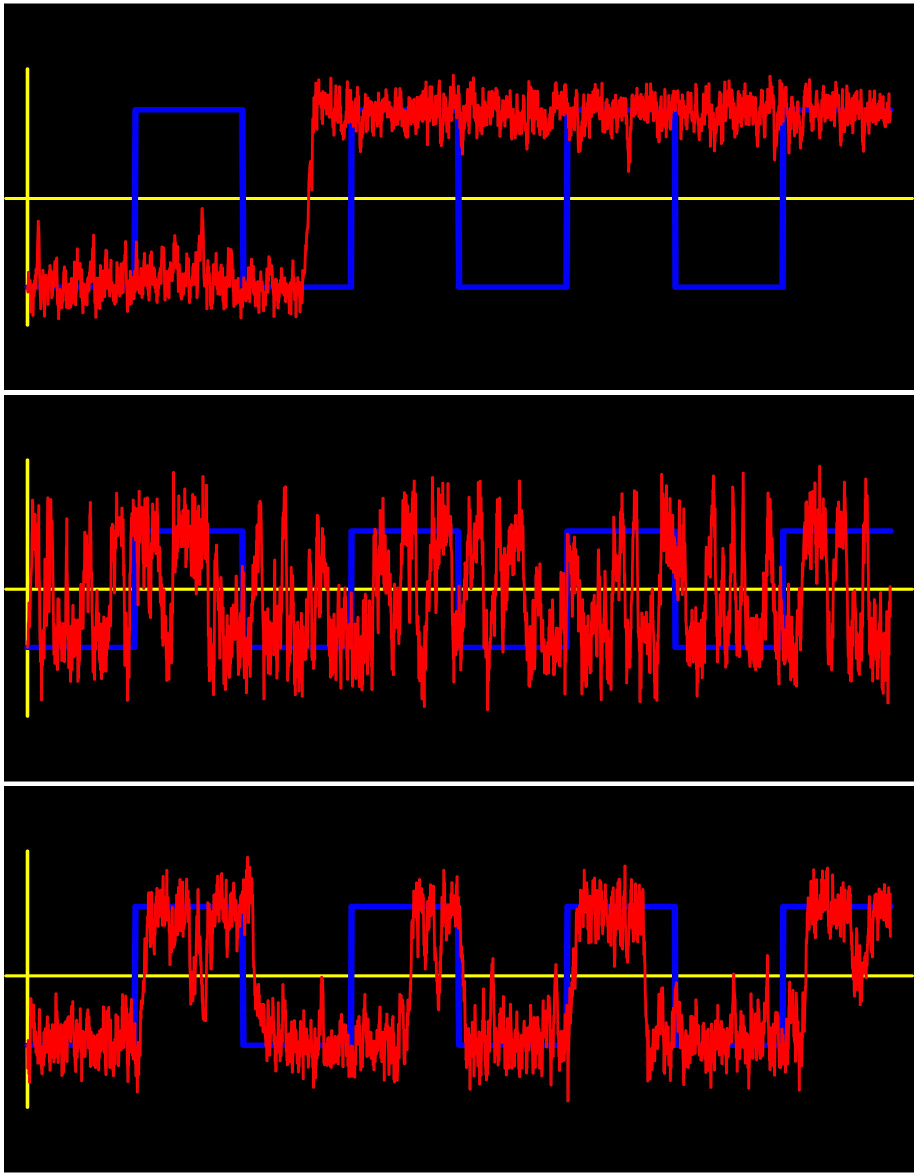
C. Nicolis; Benzi et al. 80’
Energy **B**alance **M**odel for global Earth temperature X: attempt of simple explanation of time series

$$c\frac{dX(t)}{dt} = R_{\text{in}} - R_{\text{out}}$$

Gaussian SDE

$$dX(t) = -U'\left(\frac{t}{T}, X(t)\right) dt + \varepsilon\, dW(t)$$

Double-well potential with periodically varying wells’ depths
$U'(t,\pm 1) = U'(t,0) = 0$
Period T – large; noise ε – small.
Different noise intensities lead to different regimes.



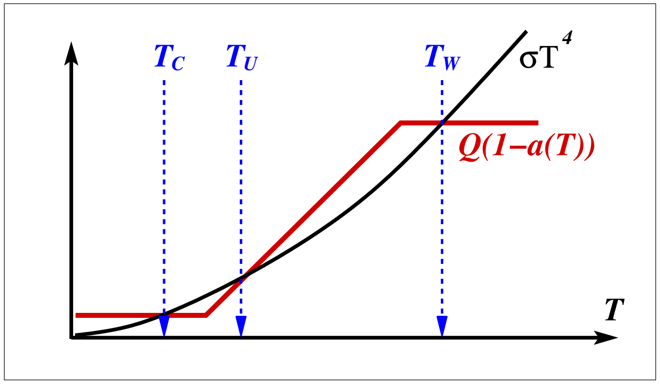
5. Gaussian transition times

simplest model

slowly time varying potential gradient

$$U'(t,x) = Q_t(1 - a(x)) - \sigma x^4,$$

first omit time variation; double well potential U;



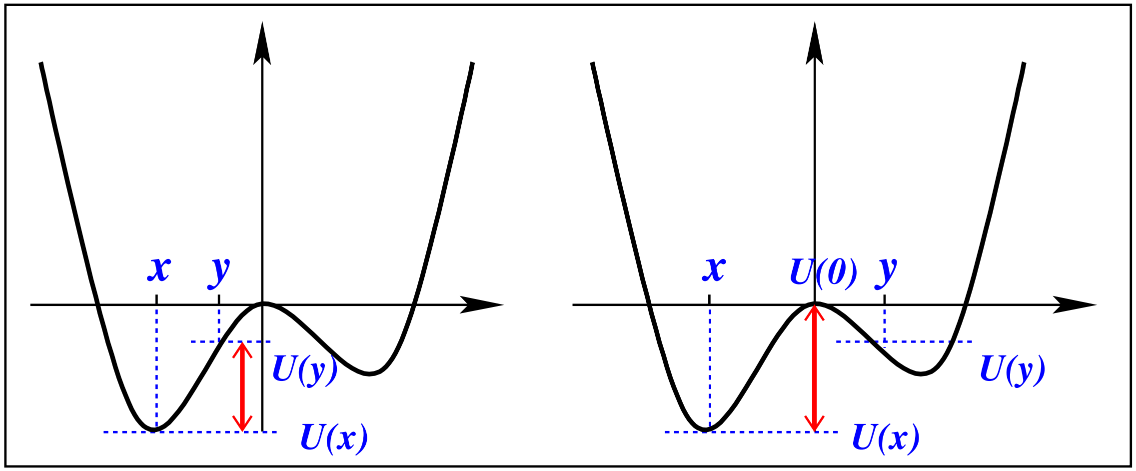
consider sde

$$dX^\varepsilon(t) = -U'(X^\varepsilon(t))dt + \varepsilon dW(t)$$

Under which conditions transitions can be expected?
How long do they take in the small noise limit?

6. Gaussian transition times

action functional $\Rightarrow$ pseudopotential $V(x,y) = 2[\text{work } x \to y]$,



$V(x,y) = 2[U(y) - U(x)]^+ \quad V(x,y) = 2[U(0) - U(x)]^+$

$\tau_y^\varepsilon = \inf\{t \geq 0 : X^\varepsilon(t) = y\}$ transition time

Thm 1(Freidlin, transition law)
for all δ

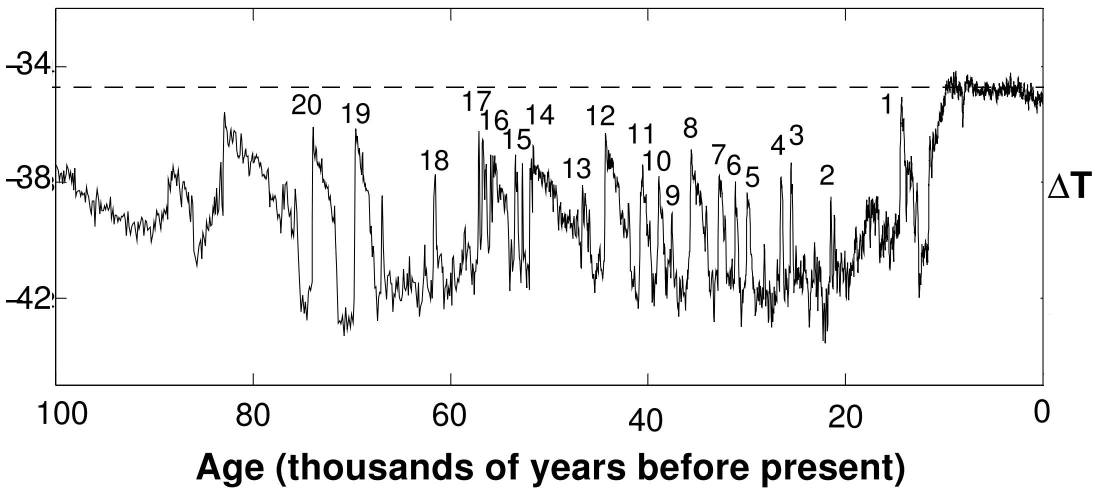
$$\mathbf{P}_x\left[\exp\left(\frac{V(x,y)-\delta}{\varepsilon^2}\right) \leq \tau_y^\varepsilon \leq \exp\left(\frac{V(x,y)+\delta}{\varepsilon^2}\right)\right] \to_{\varepsilon\to 0} 1$$

interprets Kramers-Eyring law: $E_x(\tau_y^\varepsilon) \sim \exp(\frac{V(x,y)}{\varepsilon^2})$

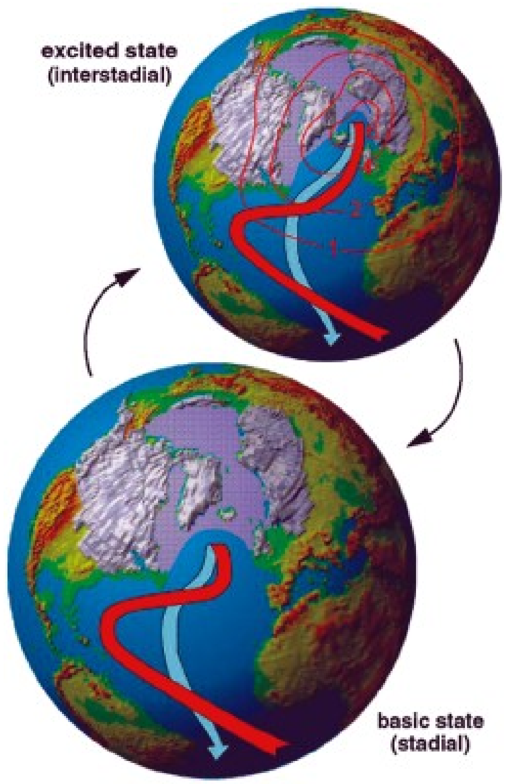
7. Dansgaard-Oeschger events

temperature indicators: ^{18}O, ^{16}O, methane, calcium etc.

GRIP ice core data: 20 abrupt changes in climate of Greenland during last ice age (-91 000 to -11 000 y) (D/O events).



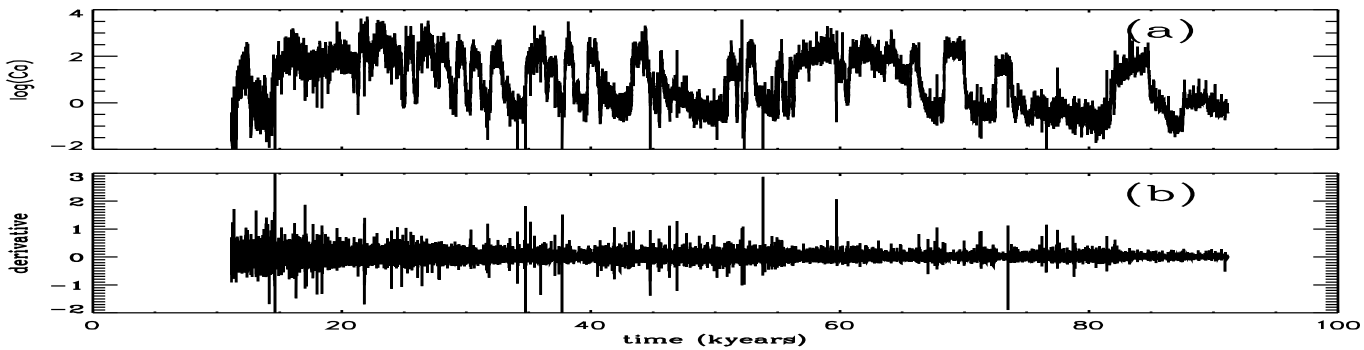
- rapid warming by 5-10°C within one decade
- subsequent slower cooling within a few centuries
- fast return to stable cold ground state



simulations: Ganopolsky/Rahmstorf, Potsdam Institute for Climate Impact Research

8. Dansgaard-Oeschger events. Statistical analysis

Calcium signal from GRIP: about 80 000 samples for 80 000 y



typical waiting time between D/O events: 1000 – 2000 y,
waiting times between D/O events: multiples of $\sim$1470 years.

What triggers the transitions?

modeling by Langevin equation:

$$dX(t) = -U'(t, X(t))dt + \text{NOISE}$$

U — multi well potential, wells correspond to climate states

P. Ditlevsen (*Geophys. Res. Lett. 1999*): power spectrum analysis of time series:
NOISE contains strong α-stable component with $\alpha \approx 1.75$.

9. Simple system with Levy noise

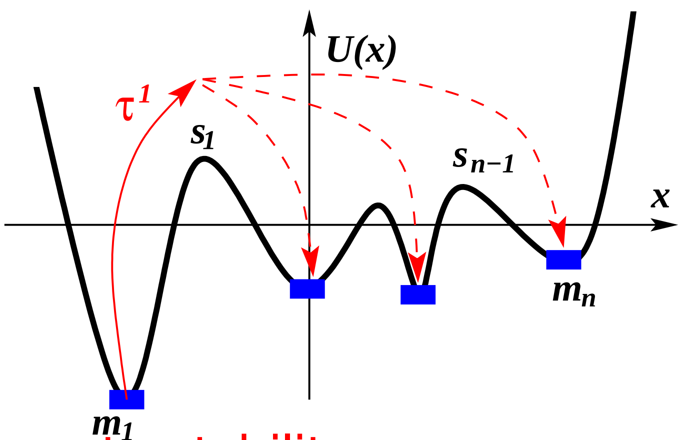
consider SDE driven by α-stable Lévy noise of small intensity

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\, ds + \varepsilon L(t), \quad \varepsilon \downarrow 0.$$

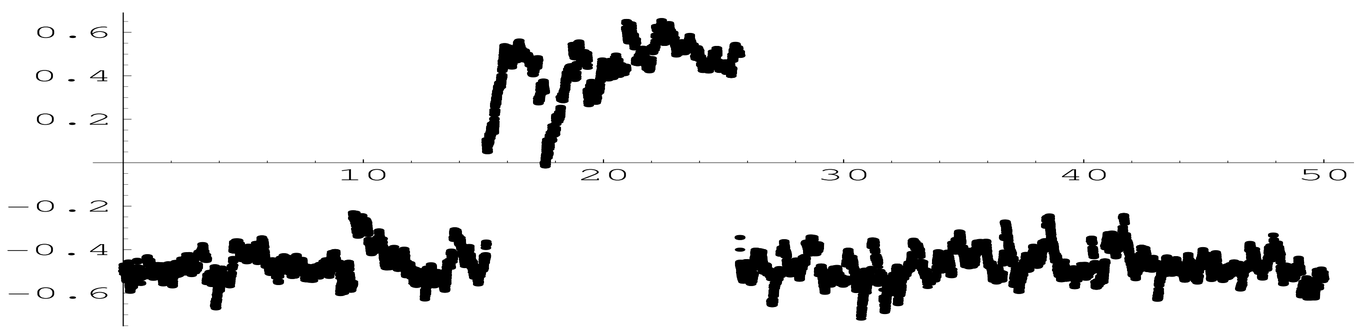
- L is α-stable symmetric Lévy process, $\alpha \in (0,2)$

multi well potential U

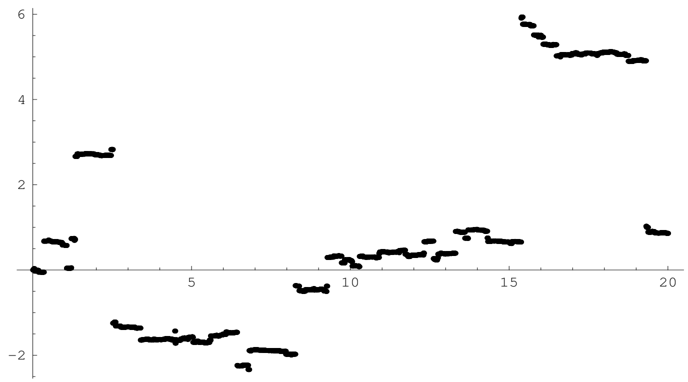
- n local minima m_i
- $n-1$ local maxima s_i
- $U''(m_i) > 0$, $U''(s_i) < 0$



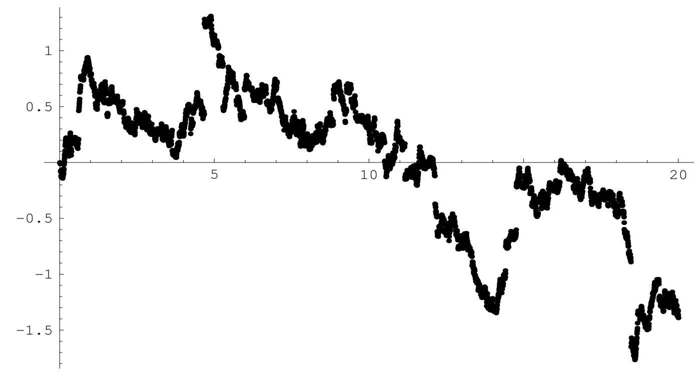
aim: investigate exit and transition rates, meta-stability.



10. The symmetric α-stable Lévy process L



$\alpha = 0.75$



$\alpha = 1.75$

Lévy measure $\nu(dy) = \frac{dy}{|y|^{1+\alpha}}$, trajectories have countably many (small) jumps on finite time interval, jump times dense.

$$\mathbf{E}e^{i\lambda L(t)} = \exp\{-c(\alpha)|\lambda|^\alpha t\}, \quad c(\alpha) = 2\int_0^\infty \frac{1-\cos y}{y^{1+\alpha}}\,dy.$$

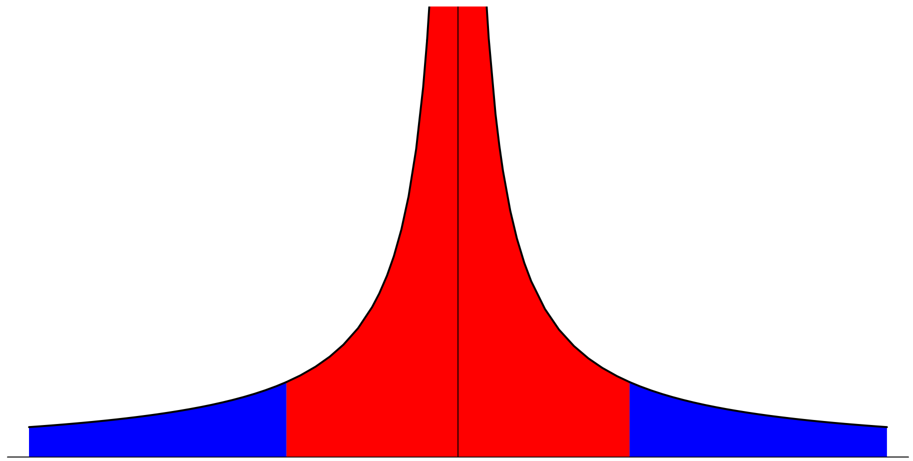
$\alpha = 1$ Cauchy process $\frac{1}{\pi}\frac{1}{1+x^2}$

$\alpha = 2$ Brownian motion $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

In physics: *anomalous diffusion* or *Lévy flights*

11. Probabilistic approach of exit times

$$L(t) = \xi^\varepsilon(t) + \eta^\varepsilon(t)$$



$$\nu_\xi^\varepsilon = \nu\big|_{[-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]}, \qquad \nu_\eta^\varepsilon = \nu\big|_{[-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]^c}$$

$$\nu_\xi^\varepsilon(\mathbb{R}) = \infty \qquad \nu_\eta^\varepsilon(\mathbb{R}) = \frac{2}{\alpha}\varepsilon^{\alpha/2} = \beta_\varepsilon$$

$\varepsilon\xi^\varepsilon$ sum of ε·BM and small jump ($\leq \sqrt{\varepsilon}$) process

$\varepsilon\eta^\varepsilon$ big jump ($\geq \sqrt{\varepsilon}$) compound Poisson process

big jumps at τ_k, inter-jump time T_k with exponential law

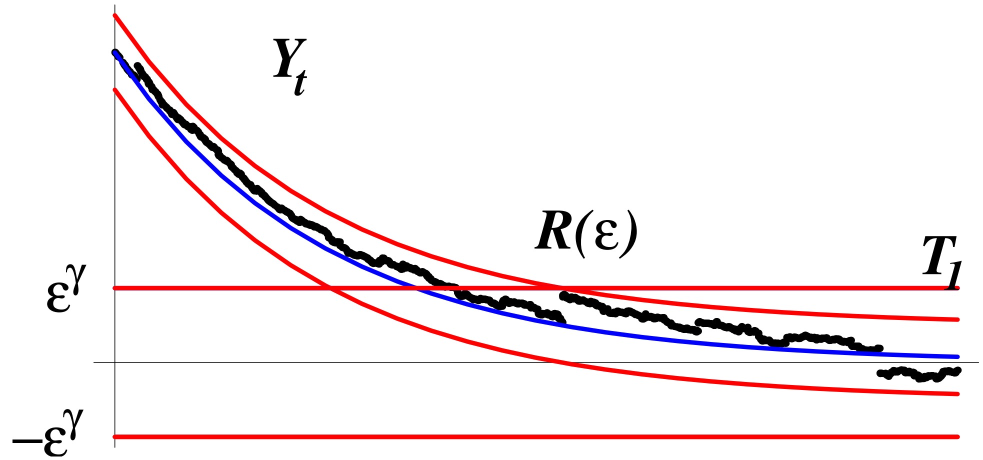
$$E(T_k) = (\beta^\varepsilon)^{-1} = \tfrac{\alpha}{2}\varepsilon^{-\alpha/2}$$

12. The small and large jump parts

U with stable state 0, exit from $[-b, a]$ for $a, b > 0$

between big jumps X^ε is Y perturbed by $\varepsilon\xi^\varepsilon$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\, ds + \varepsilon\xi^\varepsilon(t), \quad t \in [0, T_1), \quad Y(t) = x - \int_0^t U'(Y(s))\, ds$$



deviation $$\mathbf{P}\left(\sup_{[0,T_1)} |X^\varepsilon(t) - Y(t)| \geq \frac{\varepsilon^\gamma}{2}\right) \leq \mathbf{P}\left(\sup_{[0,T_1)} |\varepsilon\xi^\varepsilon(t)| \geq \frac{\varepsilon^\gamma}{C}\right) \leq e^{-1/\varepsilon^\delta}$$

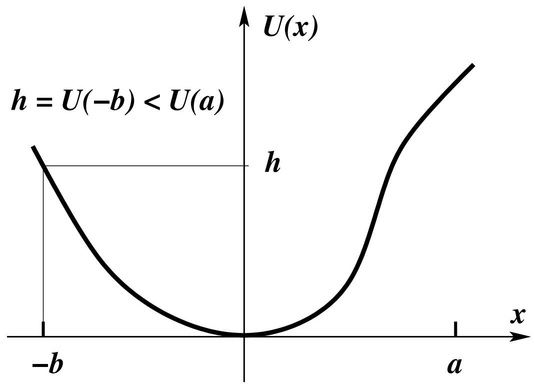
relaxation $$T(x,\varepsilon) = \int_{\varepsilon^\gamma/2}^x \frac{dy}{|U'(y)|} \approx \int_\delta^x \frac{dy}{|U'(y)|} + \int_{\varepsilon^\gamma/2}^\delta \frac{dy}{My}$$

$$\approx \text{Const} + \frac{\gamma}{M}|\ln\varepsilon| \leq R(\varepsilon) = \mathcal{O}(|\ln\varepsilon|)$$

asymptotically, big jumps coincide with exits

13. Exit times: comparison of Gaussian and Lévy dynamics

$$\hat{\sigma} = \inf\{t \geq 0 \,:\, \hat{X}^\varepsilon(t) \notin [-b, a]\}$$



$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(t))\,ds + \varepsilon W(t)$$

Thm 1 (Freidlin-Wentzell):

$$\mathbf{P}_x(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma} < e^{(2h+\delta)/\varepsilon^2}) \to 1$$

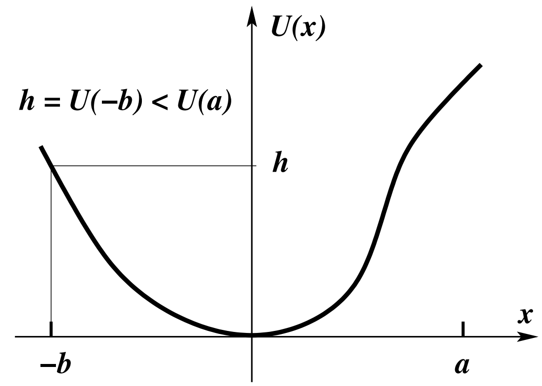
Kramers' law ('40, Williams, Bovier et al.):

$$\mathbf{E}_x\hat{\sigma} \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\varepsilon^2}$$

Exponential law (Day, Bovier et al.)

$$\mathbf{P}_x(\tfrac{\hat{\sigma}}{\mathbf{E}_x\hat{\sigma}} > u) \sim \exp(-u)$$

$$\sigma = \inf\{t \geq 0 \,:\, X^\varepsilon(t) \notin [-b, a]\}$$



$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\,ds + \varepsilon L(t)$$

Thm 2

$$\mathbf{P}_x(\tfrac{1}{\varepsilon^{\alpha-\delta}} < \sigma < \tfrac{1}{\varepsilon^{\alpha+\delta}}) \to 1$$

$$\mathbf{E}_x\sigma \approx \frac{1}{\varepsilon^\alpha}\Big(\int_{\mathbb{R}\setminus[-b,a]} \frac{dy}{|y|^{1+\alpha}}\Big)^{-1}$$

$$\mathbf{P}_x(\tfrac{\sigma}{\mathbf{E}_x\sigma} > u) \sim \exp(-u)$$

14. p-Variation as test statistic

Ditlevsen's analysis: power spectrum of residua of time series

Problem: Stationarity?

Aim: better test statistics than peaks of power spectrum.

Model assumption: with some U interpret data as

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))ds + \varepsilon L(t) = Y^\varepsilon(t) + L^\varepsilon(t)$$

L Lévy process containing α-stable component with unknown α, Y^ε of bounded variation; test α

Idea: p-variation characteristic for fluctuation behavior of noise processes.

$$V_t^{p,n}(X) = \sum_{i=1}^{[nt]} |X(\frac{i}{n}) - X(\frac{i-1}{n})|^p, \quad V_t^p = \lim_{n\to\infty} V_t^{p,n}$$

15. Stable Processes and their p-Variation

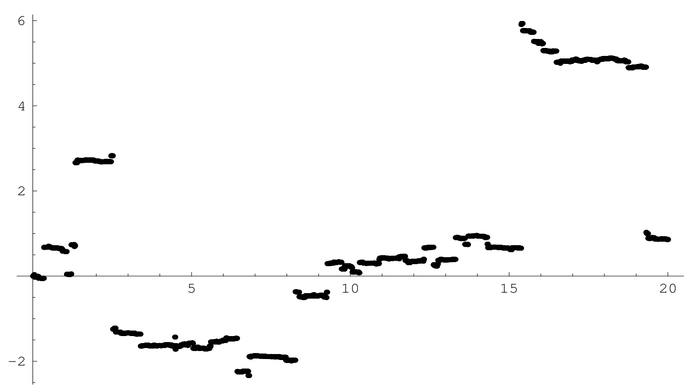
L Lévy process with characteristics (d, γ, ν) iff

$$E(\exp(iuL(t))) = \exp(t(-\frac{1}{2}du^2 + i\gamma u + \int_{\mathbf{R}} [e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}]\nu(dy))), \;\; u \in \mathbf{R}, t \geq 0,$$

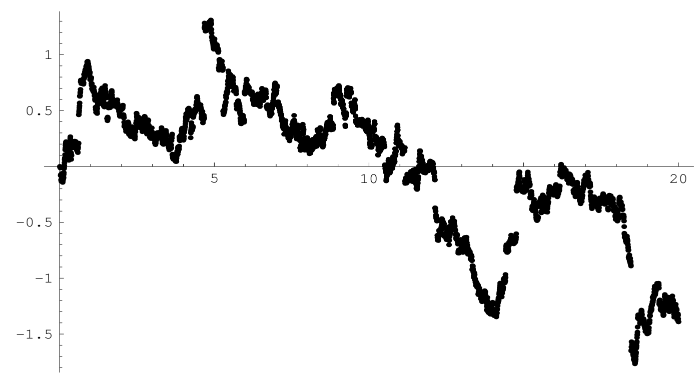
ν measure on Borel sets in $\mathbf{R}$ with $\nu(\{0\}) = 0$, $\int_{\mathbf{R}} [|y|^2 \wedge 1]\nu(dy) < \infty$.

L α-stable symmetric Lévy process if

$$E(\exp(iuL(t))) = \exp(-c(\alpha)t|u|^\alpha), \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}}dy, \quad u, y \in \mathbf{R}.$$



$\alpha = 0.75$



$\alpha = 1.75$

16. p-Variation and the Blumenthal-Getoor Index

L α-stable process with jump measure ν; then p-variation identified by

Blumenthal-Getoor index

$$\beta_L = \inf\{s \geq 0 : \int_{\{|y|\leq 1\}} |y|^s \nu(dy) < \infty\}$$

$$\gamma_L = \inf\{p > 0 : V_1^p(L) < \infty\}$$

Thm 1

L symmetric α-stable. Then

$$\gamma_L = \beta_L = \alpha.$$

Problem: How to read $\gamma_L = \alpha$ off the sequence $(V_t^{p,n}(L))_{n\in\mathbf{N}}$?

Calls for results about the asymptotic behavior of the sequence.

17. The case $\alpha = 2$: Brownian motion

For $n \in \mathbf{N}$ $V_1^{p,n}(W)$ consists of n independent increments and

$$E(V_1^{p,n}(W)) = n^{1-\frac{p}{2}} E(|W(1)|^p)$$

Thm 2 (LLN type)

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W) \to tE(|W(1)|^p) \quad \text{in probability,}$$

Y of bounded variation. Then also

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W+Y) \to tE(|W(1)|^p) \quad \text{in probability.}$$

Thm 3 (CLT type)

$$\begin{aligned}(n^{\frac{1}{2}}[n^{-1+\frac{p}{2}} V_t^{p,n}(W) - tE(|W(1)|^p)])_{t\geq 0} &= (n^{-\frac{1}{2}+\frac{p}{2}} V_t^{p,n}(W) - n^{\frac{1}{2}} tE(|W(1)|^p))_{t\geq 0} \\ &\to ((var(|W(1)|^p))^{\frac{1}{2}} \tilde{W}(t))_{t\geq 0}\end{aligned}$$

weakly with respect to the Skorokhod metric, and an independent Brownian motion $\tilde{W}$.

18. The case $\alpha < 2$

(Lit: Corcuera, Nualart, Wörner '07; case $p < \alpha$ for LLN type, $p < \frac{\alpha}{2}$ for CLT type)

Problem: $p < \frac{\alpha}{2} < 1$ not satisfactory for paleo-climatic data! Beyond $\frac{\alpha}{2}$ no CLT type result available, no asymptotic normality, but asymptotically of different type.

Thm 4 (LT type)
L α-stable with $\alpha \in]0,2[$. Then

$$(V_t^{p,n}(L) - B_t^n(\alpha,p))_{t \geq 0} \to \tilde{L}$$

weakly with respect to the Skorokhod metric, and an independent $\frac{\alpha}{p}$-stable process $\tilde{L}$. Here

$$B_t^n(\alpha,p) = \begin{cases} n^{1-\frac{p}{\alpha}} t E(|L(1)|^p), & \frac{\alpha}{2} < p < \alpha, \\ n t^2 E(\sin((nt)^{-1}|L(1)|^p)), & p = \alpha, \\ 0, & \alpha < p. \end{cases}$$

Same result with $L + Y$ instead of L if Y is of finite p-variation and $\frac{\alpha}{2} < p < 1$ or $p > \alpha$.

19. Methods of Proof

- show that $(|L(n) - L(n-1)|^p)_{n\in\mathbf{N}}$ is in domain of attraction of $\frac{\alpha}{p}$-stable law; use tails and characteristic functions

- use Aldous' criterion for tightness of sequence $(X^n)_{n\in\mathbf{N}}$ in Skorokhod metric:

  (i) $\lim_{K\to\infty} \sup_{n\in\mathbf{N}} P(\sup_{t\geq N} |X^n(t)| > K) = 0$ for all $N \geq 0,$

  $(ii) \lim_{\theta\downarrow 0} \limsup_{n\to\infty} \sup_{S\leq T\leq S+\theta} P(|X^n(T) - X^n(S)| \geq \varepsilon) = 0,$ for all $N \geq 0, \varepsilon > 0$

  S, T stopping times

- adding processes of smaller variation: new notion of Lipschitz continuity on large sets; comparison of small and large jumps