Homogenization of a semi-linear PDE wit
discontinuous averaged coefficients via BSD

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We consider the sequence of semilinear PDEs index

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(s, x_1, x_2) = L^\varepsilon(x_1, x_2) v^\varepsilon(s, x_1, x_2) + f(\frac{x_1}{\varepsilon}, x_2, v^\varepsilon \\ \\ v^\varepsilon(0, x_1, x_2) = H(x_1, x_2) \end{cases}$$

where

$$L^\varepsilon(x_1, x_2) \;=\; a_{00}(\frac{x_1}{\varepsilon}, x_2)\frac{\partial^2}{\partial^2 x_1} + a_{ij}(\frac{x_1}{\varepsilon}, x_2)\frac{\partial^2}{\partial x_{2i}\partial x_{2j}}$$

with a_{ij} and $b_i^{(1)}$, $: \mathbb{R}^{d+1} \mapsto \mathbb{R}$

$f : \mathbb{R}^{d+1} \times \mathbb{R} \mapsto \mathbb{R}$, and $H : \mathbb{R}^{d+1} \mapsto \mathbb{R}$

The main goal is to show, by using BSDEs, that

if each $g \in \{a, b, f\}$ has a Ćesaro-average,

$$\overline{g}(x_1, x_2) := \frac{\lim_{x_1 \to +\infty} \frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) g(t, x_2)}{\lim_{x_1 \to +\infty} \frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) dt}$$

$$+ \frac{\lim_{x_1 \to -\infty} \frac{1}{-x_1} \int_0^{x_1} \rho(t, x_2) g(t, x_2}{\lim_{x_1 \to -\infty} \frac{1}{-x_1} \int_0^{x_1} \rho(t, x_2) dt}$$

With $\rho := \frac{1}{a_{00}}$

Then,

i) $v^{\varepsilon}(x_1, x_2)$ converges to $v(x_1, x_2)$,

ii) $v(x_1, x_2)$, is a unique L^p-viscosity solution to t
averaged PDE

$$\begin{cases} \frac{\partial v}{\partial s}(s,\, x_1,\, x_2) = \overline{L}(x_1,\, x_2)v(s, x_1,\, x_2) + \overline{f}(x_1,\, x_2,\, v(s \\ \\ v(0, x_1,\, x_2) = H(x_1,\, x_2) \end{cases}$$

where $\overline{L}(x_1,\, x_2) := \sum_{i,\, j} \overline{a}_{ij}(x_1,\, x_2) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \overline{b}_i(x_1,\, x$

In 2001, R.Z. Khashmiskii and N.V. Krylov, SPA 200
dered the following system of SDEs

$$\begin{cases} U_t^{1,\varepsilon} = U_1 + \frac{1}{\varepsilon}\int_0^t \varphi(U_s^{1,\varepsilon}, U_s^{2,\varepsilon})dW_s \\ U_t^{2,\varepsilon} = U_2 + \int_0^t b^{(1)}(U_s^{1,\varepsilon}, U_s^{2,\varepsilon})ds + \int_0^t \sigma^{(1)}(U_s^{1,\varepsilon}, U \end{cases}$$

They prove that if the averaged system

$$\begin{cases} X_t^1 = \int_0^t \overline{\varphi}(X_s^1, X_s^2)dW_s \\ X_t^2 = U_2 + \int_0^t \overline{b}^{(1)}(X_s^1, X_s^2)ds + \int_0^t \overline{\sigma}^{(1)}(X_s^1, X_s^2 \end{cases}$$

has a weakly unique solution. Then, $(\varepsilon U^{1,\varepsilon}, U^{2,\varepsilon}) \stackrel{la}{=}$

As a consequence, Khashmiskii & Krylov prove tha
$\psi(x_1, x_2) \in \mathcal{C}_b^\infty$, the problem

$$\begin{cases} \frac{\partial \overline{v}}{\partial s}(s, x_1, x_2) = \overline{L}(x_1,\, x_2) \\ \\ \overline{v}(0,\, x_1,\, x_2) = \psi(x_1, x_2) \end{cases}$$

has a unique bounded solution $\overline{v}(t, x_1, x_2) \in W^{1,2}_{d+1,l}$
any bounded solution $v_\varepsilon(t, x_1, x_2) \in W^{1,2}_{d+1,loc}$ of the

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(s, x_1, x_2) = \mathcal{L}^\varepsilon(x_1,\, x_2) v^\varepsilon(s, x_1, x_2) \\ \\ v^\varepsilon(0,\, x_1,\, x_2) = \psi(x_1, x_2) \end{cases}$$

we have, $\lim\limits_{\varepsilon \to 0} v_\varepsilon(t, x_1, x_2) = v(t, x_1, x_2)$

We use the idea of Khashmiskii & Krylov, to solve

We put $B := (W, \widetilde{W}) := \mathbb{R} \times \mathbb{R}^{\mathsf{d}}-$Brownian motion.

We denote, $b := (0, b^{(1)})^*$, $\quad a_{00} := \frac{1}{2}\varphi^2$,

$a_{ij} := \frac{1}{2}(\sigma^{(1)}\sigma^{(1)\,*})_{ij}$, $i, j = 1, ..., d$, and $\sigma := \Big($

One has $\sigma \in \mathbb{R}^{(\mathsf{d}+1)\times\mathsf{k}}$ with

$$\begin{cases} \sigma_{00} = \varphi, \\ \sigma_{0j} = 0,\ j = 1, ..., k-1 \\ \sigma_{i0} = 0,\ i = 1, ..., d \\ \sigma_{ij} = \sigma^{(1)}_{ij},\ i = 1, ..., d,\ j = 1, ..., k-1 \end{cases}$$

The PDEs (1) is connected to the sequence of decou

$$\begin{cases} X_s^\varepsilon = X_0^\varepsilon + \int_0^s b(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})du + \int_0^s \sigma(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}) \\ \\ Y_s^\varepsilon = H(X_t^\varepsilon) + \int_s^t f(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du - \int_s^t Z_u^\varepsilon dM \end{cases}$$

where M^{X^ε} is a martingale part of the process $X^\varepsilon :=$ (

We show that :

1) the sequence of proces $(X_t^\varepsilon, Y_t^\varepsilon, \int_s^t Z_u^\varepsilon dM_u^{X^\varepsilon}) \overset{law}{\Longrightarrow} (X_t$

which is the unique solution to the FBSDE,

$$\begin{cases} X_s = x + \int_0^s \overline{b}(X_u)du + \int_0^s \overline{\sigma}(X_u)dB_u,\ 0 \leq s \leq t. \\ \\ Y_s = H(X_t) + \int_s^t \overline{f}(X_u, Y_u)du - \int_s^t Z_u dM_u^X, 0 \leq s \end{cases}$$

where $\overline{\sigma}$, $\overline{b}$ and $\overline{f}$ (defined below) are the averages o

2) As a consequence, we establish that $v^{\varepsilon}(x_1, x_2)$ –
which solves the following PDE in the L^p-viscosity s

$$\begin{cases} \frac{\partial v}{\partial s}(s,\, x_1,\, x_2) = \overline{L}(x_1,\, x_2)v(s, x_1,\, x_2) + \overline{f}(x_1,\, x_2,\, v(s \\ \\ v(0, x_1,\, x_2) = H(x_1,\, x_2) \end{cases}$$

where $\overline{L}(x_1,\, x_2) = \sum_{i,\,j} \overline{a}_{ij}(x_1,\, x_2) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \overline{b}_i(x_1,\, x$
averaged operator.

L^p**-viscosity solution** (L. Caffarelli et al. (CPAM 19

Let p be an integer such that $p > N = d + 2$.

-(a)- A continuous function v is a L^p-viscosity sub

PDE (9), if

$v(T, x) \leq H(x)$, $x \in \mathbb{R}^{d+1}$

and

for every $\varphi \in W^{1,2}_{p,loc}\Big([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}\Big)$ and $(t_0, x_0$

$\mathbb{R}^{d+1}$ at which $v - \varphi$ has a local maximum, one has

$$ess \lim_{(s,x)\to(t_0,x_0)} \inf \left\{-\frac{\partial\varphi}{\partial s}(s, x_1, x_2) - G(s, x, \varphi(s, x)\right.$$

Here

$$G(s, x, \varphi(s, x)) = \overline{L}(x_1, x_2)\varphi(s, x_1, x_2) + \overline{f}(s, x_1, x_2,$$

is assumed to be merely measurable on the variable

-(b)- A function $v \in \mathcal{C}\left([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ is a L^p-visc
solution of PDE (9), if $v(T, x) \geq H(x)$, $x \in \mathbb{R}^{d+1}$ a
for every $\varphi \in W^{1,2}_{p, loc}\left([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}\right)$ and $(t_0, x_0$
$\mathbb{R}^{d+1}$ at which $v_l - \varphi$ has a local minimum, one has

$$ess \lim_{(s,x)\to(t_0,x_0)} \sup \left\{ -\frac{\partial \varphi}{\partial s}(s, x_1, x_2) - G(s, x, \varphi(s, x\right.$$

-(c)- A function $v \in \mathcal{C}\left([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}^{L}\right)$ is a L^p-
lution to PDE (9) if it is both a L^p-viscosity sub-
super-solution.

Proofs. Step 1

Assume that **(A), (B)** hold. Then,

• By Khasminskii & Krylov (SPA 2001) : *the pro*
$(X^{1,\varepsilon}, X^{2,\varepsilon})$ *converges in law to the process* $X :=$ (

and

• By Krylov (SPA 2004) : The limit $X = (X^1, X^2)$
weak solution to the forward component of FBSDE

Step 2 Let $M^\varepsilon :=$ the mart. part of Y^ε.
Arguing as in Pardoux (1999), we show that :

There exists $(Y,\, M)$ and a countable subset D of [0,
along a subsequence of ε,

(i) $(X^\varepsilon, Y^\varepsilon, M^\varepsilon) \overset{law}{\Longrightarrow} (X, Y, M)$ on $\mathcal{C} \times \mathcal{D}([0,\, t], \mathbb{R}) \times$
The space $\mathcal{D}$ is endowed with the Jakubowski $\mathbf{S}$-top

(ii) $(Y^\varepsilon,\, M^\varepsilon) \longrightarrow (Y,\, M)$ in finite-distribution on D^c.

(iii) $Y_s = H(X_t) + \int_s^t \bar{f}(X_u^1,\ X_u^2, Y_u)du - (M_t - M_s)$

The strong uniqueness of the BSDE $(\bar{f}, H(X_t))$ allo
that, $M_r = \int_0^r Z_u\, dM_u^X$.

Step 3 The function $v(t,x) := Y_0^{(t,x)}$ is continuo
L^p-viscosity solution to PDE (9).

Remark : The main difficulty, in the proof, stays in t
1) The identification of the the limit as $\int \bar{f}(...)$
2) The continuity of $Y_0^{(t,x)}$ in (t,x).
3) the fact that $Y_0^{(t,x)}$ is a L^p viscosity solution

The point 1) can be proved by using the almost
of Skorokhod's representation theorem (proved by
and the following lemma which is an extension, of K
Krylov result, to FBSDEs.

Lemma 1 *Assume* **(A), (B)**, *(C2) and (C3). For*
$V^\varepsilon(x_1, x_2, y)$ *denote the solution of the following eq*

$$\begin{cases} a_{00}(\frac{x_1}{\varepsilon}, x_2)D^2_{x_1}u(x_1, x_2, y) = f(\frac{x_1}{\varepsilon}, x_2, y) - \bar{f}(x_1, \\ \\ u(0, x_2) = D_{x_1}u(0, x_2) = 0. \end{cases}$$

Then,

(i) $\quad D_{x_1}V^\varepsilon(x_1, x_2, y) = x_1(1 + |x_2|^2 + |y|^2)\beta(\frac{x_1}{\varepsilon}, x_2,$

(ii) *for any*
$K^\varepsilon \in \Big\{V^\varepsilon, D_{x_2}V^\varepsilon, D^2_{x_2}V^\varepsilon, D_{x_1}D_{x_2}V^\varepsilon, D_yV^\varepsilon, D^2_yV^\varepsilon, D_{x_1}$
it holds,

$$K^\varepsilon(x_1, x_2, y) = x_1^2(1 + |x_2|^2 + |y|^2)\beta(\frac{x_1}{\varepsilon}, x_2$$

where $\beta(x_1, x_2, y)$ *is some bounded function which*

$$\lim_{|x_1|\longrightarrow\infty} \sup_{(x_2,y)\in\mathbb{R}^{d+1}} |\beta(x_1,x_2,y)| = 0$$

The point 2) is proved as follows,

Let $(t_n, x_n) \to (t, x)$. We assume that $t > t_n > 0$. W

$$Y_s^{t_n, x_n} = H(X_{t_n}^{x_n}) + \int_s^{t_n} \overline{f}(X_u^{x_n}, Y_u^{t_n, x_n})du - \int_s^{t_n} Z_u^{t}$$

where $X^{x_n} \overset{law}{\Rightarrow} X^x$.
Since H is a bounded continuous function and $\overline{f}$ sa
one can easily show that the sequence $\{(Y^{t_n, x_n}, \int_0^{\cdot} 1_{[s}$
is tight in $\mathcal{D}([0, t] \times \mathbb{R} \times \mathbb{R})$ endowed with the **S**-to

We rewrite equation (11) as follows,

$$\begin{aligned} Y_s^{t_n,x_n} &= H(X_{t_n}^{x_n}) + \int_s^t \overline{f}(X_u^{x_n}, Y_u^{t_n,x_n})du - \int_s^t 1_{[s,t_n]}(\\ &- \int_{t_n}^t \overline{f}(X_u^{x_n}, Y_u^{t_n,x_n})du. \\ &= A_n^1 + A_n^2 \end{aligned}$$

• *Convergence of* A_n^2

One has $\mathbb{E}\left|\int_{t_n}^{t} \overline{f}(X_u^{x_n}, Y_u^{t_n,x_n})du\right| \leq K|t - t_n|$. Hence

zero in probability.

• *Convergence of* A_n^1

Denote by (Y', M') the weak limit of $\{(Y^{t_n,x_n}, \int_0^{\cdot} 1_{[s,t_n}$

Arguing as previously, we show that $\int_s^t \overline{f}(X_u^{x_n}, Y_u^{t_n,x_n})$

Passing to the limit in (12), we obtain that

$$Y'_s = H(X^x_t) + \int_s^t \overline{f}(X^x_u, Y'_u)du - (M'_t - M'_s),\ s \in$$

The uniqueness of the considered BSDE ensures that $Y^{t,x}_s$ $\mathbb{P}$-ps. Hence $Y^{t_n,x_n} \overset{law}{\Rightarrow} Y^{t,x}$. As in (i), one $Y^{t_n,x_n}_0 \overset{law}{\Rightarrow} Y^{t,x}_0$ which yields to the continuity of $Y^{t,x}_0$

3. Proof of L^p viscosity solution We assume th
continuous. We only prove that v is L^p- viscosity su

Note that the definition of L^p- viscosity sub-solution
to the following : for every $\varepsilon > 0$, $r > 0$, there exis
$B_r(t_0, x_0)$ of positive measure such that, $\forall\,(s, x_1, x_2$

$$-\frac{\partial\varphi}{\partial s}(s, x_1, x_2) - \overline{L}(x_1, x_2)\varphi(s, x_1, x_2) - f(s, x_1, x_2, v(s$$

Since $\varphi \in W^{1,2}_{p,loc}\Big([0, T] \times \mathbb{R}^{d+1}, \mathbb{R}\Big)$ and $p > d + 2$, φ
continuous version which will be considered below.
Let $(t_0, x_0) \in [0, T] \times \mathbb{R}^{d+1}$ be a local maximum o
assume that $v(t_0, x_0) = \varphi(t_0, x_0)$.

We will argue by contradiction. Assume that there ex
0 such that

$\frac{\partial \varphi}{\partial s}(s, x_1, x_2) + \overline{L}(x_1, x_2)\varphi(s, x_1, x_2) + \bar{f}(s, x_1, x_2, v($
$-\varepsilon_0,$
λ-$a.s$ in $B_{r_0}(t_0, x_0)$

Define

$$\tau = \inf \left\{ s \geq t_0; \quad |X_s^{t_0,x_0} - x_0| > r \right\} \wedge (t_0 + r_0$$

Since X is a Markov diffusion, $Y_s^{t_0,x_0} = v(s, X_s^{t_0,x_0}$ an
process
$(\bar{Y}_s,\, \bar{Z}_s) := ((Y_{s\wedge\tau}^{t_0,x_0}),\, 1_{[0,\,\tau]}(s)(Z_s^{t_0,x_0}))_{s\in[t_0,\,t_0+r_0]}$ solv

$$\overline{Y}_s \quad = \quad v(\tau,\, X_\tau^{t_0,x_0}) + \int_s^{t_0+r_0} 1_{[0,\,\tau]}(u)\bar{f}(u,\, X_u^{t_0,x_0}$$

$$- \int_s^{t_0+r_0} \bar{Z}_u dM_u^{X^{t_0,x_0}},\, s \in [t_0,\, t_0 + r_0].$$

On other hand, by Itô-Krylov formula, the process
$(\widehat{Y}_s,\, \widehat{Z}_s) = \Big(\varphi(s,\, X_{s\wedge\tau}^{t_0,x_0}),\; 1_{[0,\,\tau]}(s)\nabla\varphi(s,\, X_s^{t_0,x_0})\Big)_{s\in[t_0,}$
the BSDE

$$\widehat{Y}_s = \varphi(\tau, X_\tau^{t_0,x_0}) - \int_s^{t_0+r_0} 1_{[0,\tau]}(u)[(\frac{\partial\varphi}{\partial u} + \bar{L}\varphi)$$

$$- \int_s^{t_0+r_0} \widehat{Z}_u dM_u^{X^{t_0,x_0}}.$$

From the choice of τ, $(\tau, X_\tau^{t_0,x_0}) \in B_{r_0}(t_0, x_0)$. Therefore, $v(\tau, X_\tau^{t,x}) \leq \psi(\tau, X_\tau^{t_0,x_0})$ and hence, the strict comparison theorem $\Longrightarrow \overline{Y}_{t_0} < \widehat{Y}_{t_0}$, i.e $\varphi(t_0, x_0)$, which contradicts our assumptions.

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Academic Publishers., 1999.

Averaged coefts. For function $g \in \{b_i, a_{ij}, f\}$, we de
$g^+(x_2) := \lim_{x_1 \to +\infty} \frac{1}{x_1} \int_0^{x_1} g(t, x_2) dt$,
$g^-(x_2) := \lim_{x_1 \to -\infty} \frac{1}{-x_1} \int_0^{x_1} g(t, x_2) dt$

$g^\pm$ is called the Ċesaro limit (or mean) of g.

We put, $\rho(x_1, x_2) := a_{00}(x_1, x_2)^{-1} = [\frac{1}{2}\varphi^2(x_1, x_2)]^-$

$g^\pm(x_1, x_2) := g^+(x_2) 1_{\{x_1 > 0\}} + g^-(x_2) 1_{\{x_1 \leq 0\}}$
and

$$\overline{g}(x_1, x_2) \;:=\; \frac{(\rho g)^\pm(x_1, x_2)}{\rho^\pm(x_1, x_2)}$$

We have,

$$\overline{g}(x_1, x_2) = \frac{\lim_{x_1 \to +\infty} \frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) g(t, x_2)}{\lim_{x_1 \to +\infty} \frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) dt}$$

$$+ \frac{\lim_{x_1 \to -\infty} \frac{1}{-x_1} \int_0^{x_1} \rho(t, x_2) g(t, x_2}{\lim_{x_1 \to -\infty} \frac{1}{-x_1} \int_0^{x_1} \rho(t, x_2) dt}$$

$\overline{b}$, $\overline{a}$ and $\overline{f}$ may have discontinuity at $x_1 = 0$.

Assumptions. We consider the following conditions,

(A1) The function $b^{(1)}, \sigma^{(1)}, \varphi$ are uniformly Lips
variables (x_1, x_2),

(A2) for each x_1, their derivative in x_2 up to and inclu
order derivatives are bounded continuous functions

(A3) $a := (\sigma^{(1)}\sigma^{(1)\,*})$ is uniformly elliptic, $i.e$: $\exists\Lambda >$
$\mathbb{R}^d$, $\quad \xi^* a(x)\xi \geq \Lambda|\xi|^2$.

Moreover, there exists positive constants C_1, C_2, C_3

$$\begin{cases} (i)\ C_1 \leq a_{00}(x_1, x_2) \leq C_2 \\ \\ (ii)\ \sum_{i=1}^{d}[a_{ii}(x_1, x_2) + b_i^2(x_1, x_2)] \leq C_3(1 + \end{cases}$$

(B1) Let $D_{x_2}u$ and $D^2_{x_2}u$ denote respectively the gra
and the matrix of second derivatives of u with respec
following limits are uniform in x_2,

$$\frac{1}{x_1}\int_0^{x_1} \rho(t,\, x_2)dt \longrightarrow \rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm$$

$$\frac{1}{x_1}\int_0^{x_1} D_{x_2}\rho(t,\, x_2)dt \longrightarrow D_{x_2}\rho^{\pm}(x_2) \qquad \text{as} \quad x_1$$

$$\frac{1}{x_1}\int_0^{x_1} D^2_{x_2}\rho(t,\, x_2)dt \longrightarrow D^2_{x_2}\rho^{\pm}(x_2) \qquad \text{as} \quad x_1$$

(B2) For every i and j, the funct. ρb_i, $D_{x_2}(\rho b_i)$, $D^2_{x_2}(\rho$
$D^2_{x_2}(\rho a_{ij})$ have limits in Cèsaro sense.

(B3) For every $k \in \{\rho,\, \rho b_i,\, D_{x_2}(\rho b_i),\, D^2_{x_2}(\rho b_i),\, \rho a_{ij},\, D$
there
exists a bounded function α such that

$$\begin{cases} \frac{1}{x_1}\int_0^{x_1} k(t,\, x_2)dt - k_\pm(x_1,\, x_2) = (1 + |x_2|^2)\alpha(x_1, \\ \\ \lim_{|x_1|\longrightarrow\infty} \sup_{x_2\in\mathbb{R}^d} |\alpha(x_1,\, x_2)| = 0. \end{cases}$$

(C1) There are positive constants C_4, C_5 such th
$(x_1,\, x_2,\, y,\, y') \in \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^2$:

$$\begin{cases} (i) \quad |f(x_1,\, x_2,\, y) - f(x_1,\, x_2,\, y')| \le C_4|y - y'|, \\ \\ (ii)\ H \text{ is a continuous bounded function and } |f(x \end{cases}$$

(C2) ρf has a limit in Česaro sense and there exist
measurable function β such that

$$\begin{cases} \frac{1}{x_1}\int_0^{x_2} \rho(t, x_2) f(t, x_2, y)dt - (\rho f)_{\pm}(x_1, x_2, y) = (1 - \\ \\ \lim_{|x_1|\to\infty} \sup_{(x_2, y)\in \mathbb{R}^d\times\mathbb{R}} |\beta(x_1, x_2, y)| = 0, \end{cases}$$

where $(\rho f)_{\pm}(x_1, x_2, y) := (\rho f)^{+}(x_2, y)1_{\{x_1>0\}}+(\rho f)$

(C3) For each x_1, ρf has a derivatives up to a sec
x_2 uniformly in y and these derivatives are boundec
(C2).

Throughout the paper, **(A)** stands for conditions
(A3) ; **(B)** for conditions (B1), (B2), (B3) and **(C**
(C2), (C3).

Remarque 1 (i) *Whenever f does not depends on* $\tilde{Y}_0^{t,x}$ *is a L^p-viscosity solution of the PDE*

$$\begin{cases} \frac{\partial v}{\partial s}(s,x_1,x_2) = \overline{L}(x_1,x_2)v(s,x_1,x_2) + f(x_1,x_2,\, v(s, \\ \\ v(0,x_1,x_2) = H(x_1,x_2), \quad x = (x_1,x_2) \in \mathbb{R}^{d+1} \end{cases}$$

where $(X^x,\, \tilde{Y}_s^{t,x},\, \tilde{Z}_s^{t,x}; \quad 0 \le s \le t)$*, solves the decou*

$$\begin{cases} X_s^x = x + \int_0^s \overline{b}(X_u^x)du + \int_0^s \overline{\sigma}(X_u^x)dB_u,\, 0 \le s \le t. \\ \\ \tilde{Y}_s^{t,x} = H(X_t^x) + \int_s^t g(X_u^x,\, \tilde{Y}_u^{t,x})du - \int_s^t \tilde{Z}_u^{t,x} dM_u^{X^x}, 0 \end{cases}$$

(ii) Since f satisfies **(C)** and ρ is bounded, one can
that $\overline{f}$ satisfies $(C1)$. Therefore, for a fixed positiv
the BSDE with data $(H(X_t^x), \overline{f})$ admit a unique str
$(Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq t}$. Moreover, if the function $x \in \mathbb{R}^{d+1}$
$Y_0^{t,x}$ is continuous, it is a L^p-viscosity solution of PI

proof of the identification of $\int \overline{f}$

There exists a positive constant C which does not
such that

$$\sup_{\varepsilon}\left\{\mathbb{E}\left(\sup_{0\leq s\leq t}|Y_s^{\varepsilon}|^2+\int_0^t|Z_s^{\varepsilon}|^2\,d\langle M^{X^{\varepsilon}}\rangle_s\right)\right\}\leq C$$

Lemma 2 $\int_0^t \overline{f}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^{\varepsilon})du \stackrel{law}{\Longrightarrow} \int_0^t \overline{f}(X_u^1, X_u^2,$

The proof of this Lemma is based on the following,

Lemma 3 *Assume (A2-i), (B1).*
Let $X_s^1 = x_1 + \int_0^s \overline{\varphi}(X_u^1, X_u^2)dW_u, 0 \leq s \leq t$. *Then,*
sequence, the set
$D_{B(0,\frac{1}{n})} = \left\{ s : s \in [0, t] \ / \ X_s^1 \in B(0, \frac{1}{n}) \right\}$ *satisfies* $\lim_{n\to}$
where $|.|$ *stands for the Lebesgue's measure on* $[0,$

Lemma 4 $\sup_{0\leq s\leq t} \left| \int_0^s \left(f(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^{\varepsilon}) - \overline{f}(X_u^{1,\varepsilon}, X \right.\right.$
tends to 0 in probability as $\varepsilon \longrightarrow 0$.

Lemma 5 $\int_0^t \overline{f}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^{\varepsilon})du \overset{law}{\Longrightarrow} \int_0^t \overline{f}(X_u^1, X_u^2,$
as $\varepsilon \longrightarrow 0$*.*

The proof of this Lemma is based on the following,

Lemma 6 *Assume (A2-i), (B1).*
Let $X_s^1 = x_1 + \int_0^s \overline{\varphi}(X_u^1, X_u^2)dW_u$, $0 \leq s \leq t$*. Then,*
sequence, the set $D_{B(0,\frac{1}{n})} = \Big\{ s : s \in [0, t] \ / \ X_s^1 \in$
tisfies
$\lim_{n \to +\infty} |D_{B(0,\frac{1}{n})}| = 0 \quad \mathbb{P}$ *a.s,*

where $|.|$ *stands for the Lebesgue's measure on* $[0,$

Identification of the limits

Proposition 1 *Let* $(\overline{Y}, \overline{M})$*, the limit process defined*
??. *Then,*
(i) *For every* $s \in [0, t] - \mathsf{D}$,

$$\begin{cases} \overline{Y}_s = H(X_t) + \int_s^t \overline{f}(X_u^1,\ X_u^2, \overline{Y})du - (\overline{M}_t - \overline{M} \\ \\ \mathbb{E}\left(\sup_{0\leq s\leq t} |\overline{Y}_s|^2 + |X_s^1|^2 + |X_s^2|^2\right) \leq C. \end{cases}$$

(ii) *Moreover,* $\overline{M}$ *is* $\mathcal{F}_s$*-martingale, where* $\mathcal{F}_s = \sigma\Big\{X_u,$

Proposition 2 *Let* $(\tilde{Y}_s, \tilde{Z}_s, 0 \leq s \leq t)$ *be the uniqu*
the BSDE $(H(X_t), \overline{f})$. *Then, for every* $s \in [0, t]$, $\mathbb{E}$
$\frac{1}{2}\mathbb{E}\left([\overline{\mathsf{M}} - \int_0^{\cdot} \tilde{\mathsf{Z}}_{\mathsf{u}} d\mathsf{M}_{\mathsf{u}}^{\mathsf{X}}]_{\mathsf{t}} - [\overline{\mathsf{M}} - \int_0^{\cdot} \tilde{\mathsf{Z}}_{\mathsf{u}} d\mathsf{M}_{\mathsf{u}}^{\mathsf{X}}]_{\mathsf{s}}\right) = 0.$

Proposition 3 $\left(Y^{\varepsilon}, \int_0^{\cdot} Z_u^{\varepsilon} dM_u^{X^{\varepsilon}}\right) \overset{law}{\Longrightarrow} \left(\tilde{Y}, \int_0^{\cdot} \tilde{Z}_u dM_u^{X}\right.$

Application to PDE.

Proposition 4 *(Continuity in law of the flow $x \mapsto X$*
Assume **(A), (B)**. *Let X_s^x be the unique weak sol*
SDE (**??**), *and*
$X_s^n := x_n + \int_0^s \overline{b}(X_u^n)du + \int_0^s \overline{\sigma}(X_u^n)dB_u, \ 0 \leq s \leq t$
Assume that x_n converges towards $x = (x^1, x^2) \in \mathbb{I}$
$X^n \overset{law}{\Longrightarrow} X^x$.

Théorème 1 *Assume* **(A), (B), (C)**. *Let* $p >$
(i) $\lim_{\varepsilon \to 0} \mathbb{E}|\mathsf{Y}_0^\varepsilon - \mathsf{v}(\mathsf{t}, \mathsf{x})|^2 = 0$.
(ii) $Y_0^{t,x} := v(t, x) \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^{\mathsf{d}+1})$, *and it is a*
*solution of PDE (**??**).*

Proof (i) We shall prove that $\lim_{\varepsilon\to 0}\mathbb{E}|\mathsf{Y}_0^\varepsilon - \overline{\mathsf{Y}}_0|^2 =$

$$\begin{cases} Y_0^\varepsilon = H(X_t^\varepsilon) + \int_0^t f(\overline{X}_u^\varepsilon, X_u^{2,\varepsilon}, Y_u^\varepsilon)du - M_t^\varepsilon \\ \overline{Y}_0 = H(X_t) + \int_0^t \overline{f}(X_u, \overline{Y}_u)du - \overline{M}_t \end{cases}$$

>From Jakubowski (1997), the projection : $y \mapsto$
nuous in the $\mathbf{S}$-topology. We then deduce that Y
towards $\overline{Y}_0$ in distribution. Moreover, since Y_0^ε and Y
ministic and bounded, we have $\lim_{\varepsilon\to 0}\mathbb{E}|\mathsf{Y}_0^\varepsilon - \overline{\mathsf{Y}}_0|^2 =$
$\lim_{\varepsilon\to 0}\mathbb{E}|\mathsf{v}^\varepsilon(\mathsf{t},\mathsf{x}) - \mathsf{v}(\mathsf{t},\mathsf{x})|^2 = 0$.

(ii) $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1} \mapsto \mathsf{Y}^{t,x}$ is continuous in law a
we derive the result.