A brief introduction to spatial point processes

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Notes

Preliminary

Files which can de downloaded
http://www-ljk.imag.fr/membres/Jean-Francois.Coeurjolly/documents/Lille/
or more simply on the workshop webpage, program page
http://math.univ-lille1.fr/ heinrich/geostoch2014/

- introductionSPP_cours.pdf : pdf file of the slides. Beamer version.
- introductionSPP_print.pdf : pdf file of the printed version.
- Short R code used to illustrate the talks.
- The code is using the **excellent** R package spatstat which can be downloaded from the R CRAN website.

Notes

Notes

Spatial data . . .

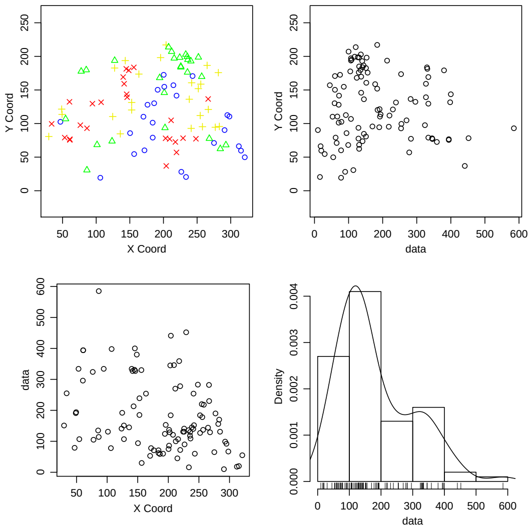
. . . can be roughly and mainly classified into three categories :

1. Geostatistical data.
2. Lattice data.
3. Spatial point pattern

Notes

Geostatistical data

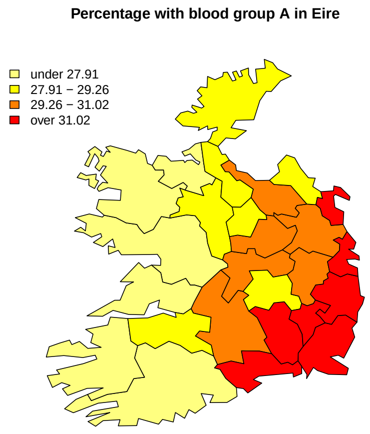
- sic.100 dataset (R package geoR)
- Cumulative rainfall in Switzerlan the 8th May.
- The observation consists in the **discretization** of a random field, $X = (X_u, u \in \mathbb{R}^2)$



Notes

Lattice data (1)

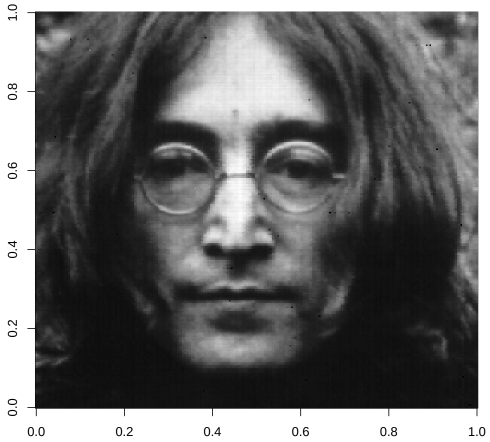
- Eire dataset (R package spdep)
- % of people with group A in eire, observed in 26 regions.
- The data are aggregated on the region ⇒ **random field on a network**.



Notes

Lattice data (2)

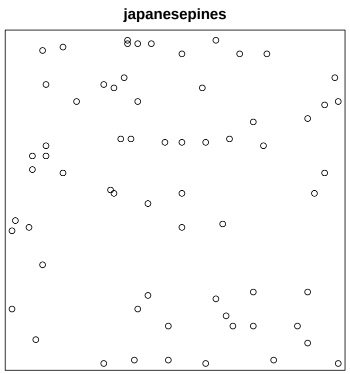
- Lennon dataset (`R` package `fields`)
- Real-valued random field (gray scale image with values in $[0, 1]$).
- Defined on the network $\{1, \dots, 256\}^2$.



Notes

Spatial point pattern (1)

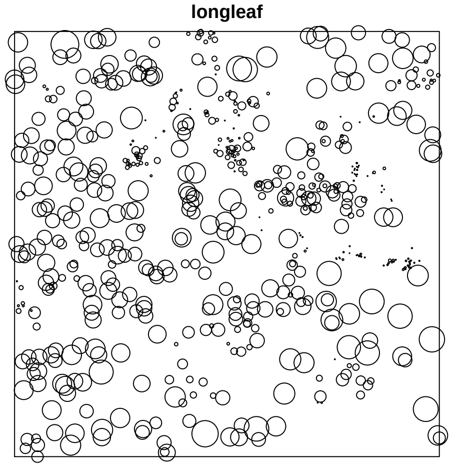
- Japanesepines dataset (`R` package `spatstat`)
- Locations of 65 trees on a bounded domain.
- $S = \mathbb{R}^2$ (equipped with $\|\cdot\|$).



Notes

Spatial point pattern (2)

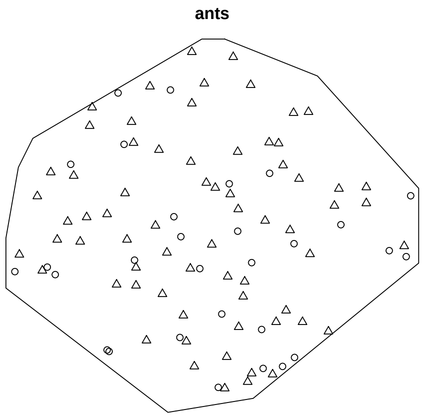
- Longleaf dataset (R package `spatstat`)
- Locations of 584 trees observed with their diameter at breast height.
- $\mathcal{S} = \mathbb{R}^2 \times \mathbb{R}^+$ (equipped with $\max(\|\cdot\|, |\cdot|)$).



Notes

Spatial point pattern (3)

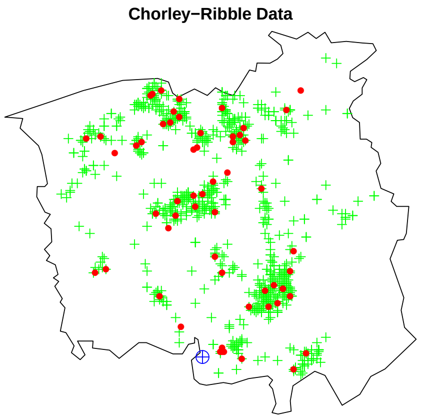
- Ants dataset (R package `spatstat`)
- Locations of 97 ants categorised into two species.
- $S = \mathbb{R}^2 \times \{0, 1\}$ (equipped with the metric $\max(\|\cdot\|, d_M)$ for any distance d_M on the mark space).



Notes

Spatial point pattern (3)

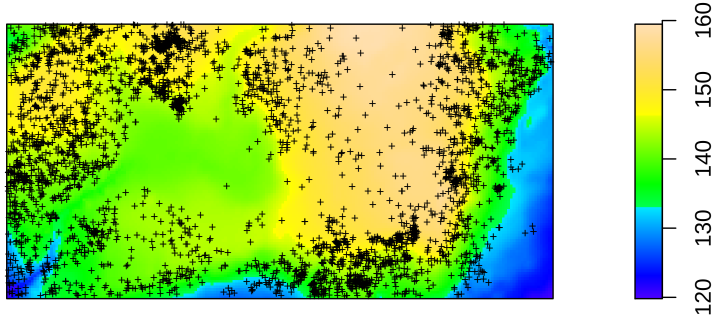
- chorley dataset (R package spatstat)
- Cases of larynx and lung cancers and position of an industrial incinerator.
- $S = \mathbb{R}^2 \times \{0,1\}$ (equipped with the metric $\max(\|\cdot\|, d_M)$ for any distance d_M on the mark space).



Notes

Spatial point pattern (4)

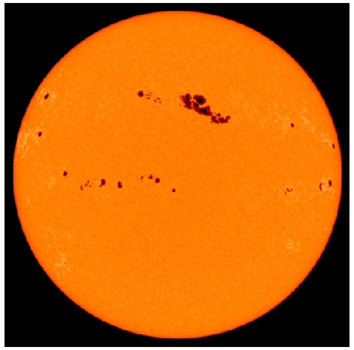
- Beischmedia dataset (R package spatstat)
- 3604 locations of trees observed with spatial covariates (here the elevation field).
- $S = \mathbb{R}^2$ (equipped with the metric $\|\cdot\|$), $z(\cdot) \in \mathbb{R}^2$.

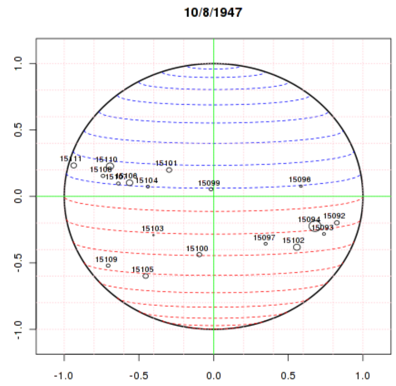


Notes

Spatial point pattern (5)

- Spatio-temporal point process on a complex space
- Daily observation of sunspots at the surface of the sun.
- can be viewed as the realization of a marked spatio-temporal point process on the sphere.
- $S = S_2 \times \mathbb{R}^+ \times \mathbb{R}^+$ (state, time, and mark)

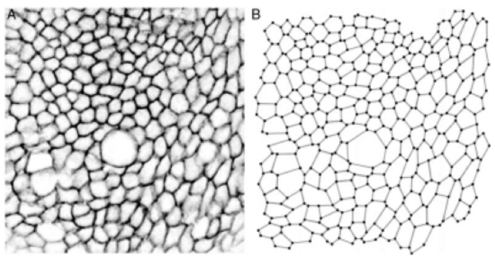


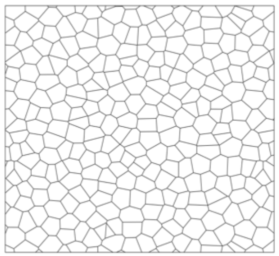


Notes

Spatial point pattern (6)

- Towards stochastic geometry ...
- Planar section of the pseudo-stratified epithelium of a drosophila wing marked with antibodies to highlight cell borders.
- The centers form of the tessellation form a point process.





Notes

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Notes

Mathematical definition of a spatial point process ?

- S : Polish state space of the point process (equipped with the σ-algebra of Borel sets $\mathcal{B}$).
- A configuration of points is denoted $x = \{x_1, \ldots, x_n, \ldots\}$. For $B \subseteq S$: $x_B = x \cap B$.
- N_{lf} : space of **locally finite configurations**, i.e.

$$\{x, n(x_B) = |x_B| < \infty, \forall B \text{ bounded } \subseteq S\}$$

equipped with $\mathcal{N}_{lf} = \sigma\left(\{x \in N_{lf}, n(x_B) = m\}, B \in \mathcal{B}, B \text{ bounded}, m \geq 1\right)$.

Definition

A point process X defined on S is a measurable application defined on some probability space $(\Omega, \mathcal{F}, P)$ with values on N_{lf}.

Measurability of $X \Leftrightarrow N(B) = |X_B|$ is a r.v. for any bounded $B \in \mathcal{B}$.

Notes

Theoretical characterization of the distribution of X

Proposition

The distribution of a point process X

1. is determined by the finite dimensional distributions of its counting function, i.e. the joint distribution of $N(B_1), \dots, N(B_m)$ for any bounded $B_1, \dots, B_m \in \mathcal{B}$ and any $m \geq 1$.
2. is uniquely determined by its void probabilities, i.e. by

$$P(N(B) = 0), \qquad \text{for bounded } B \in \mathcal{B}.$$

- From now on, we assume that $S = \mathbb{R}^d$ (and even $d = 2$) or a bounded domain of $\mathbb{R}^2$.
- Everything can de extended to marked spatial point processes and/or to more complex domains.

Notes

Moment measures

- Moments play an important role in the modelling of classical inference.
- For point processes = moments of counting variables.

Definition : for $n \geq 1$ we define

- the n-th order moment measure (defined on S^n) by

$$\mu^{(n)} = \mathrm{E} \sum_{u_1,\dots,u_n} \mathbf{1}(\{u_1, \dots, u_n\} \in D), \quad D \subseteq S^n.$$

- the n-th order reduced moment measure (defined on S^n) by

$$\alpha^{(n)}(D) = \mathrm{E} \sum_{u_1,\dots,u_n}^{\neq} \mathbf{1}(\{u_1, \dots, u_n\} \in D), \quad D \subseteq S^n.$$

where the $\neq$ sign means that the n points are pairwise distinct.

Notes

Intensity functions

Assume $\mu^{(1)}$ and $\alpha^{(2)}$ are absolutely continuous w.r.t. Lebesgue measure, and denote by ρ and $\rho^{(2)}$ the densities.

Campbell Theorems

1. For any measurable function $h : S \to \mathbb{R}$

$$\mathrm{E} \sum_{u \in X} h(u) = \int_S h(u)\rho(u)\mathrm{d}u.$$

2. For any measurable function $h : S \times S \to \mathbb{R}$

$$\mathrm{E} \sum_{u,v \in X}^{\neq} h(u,v) = \int_S \int_S h(u,v)\rho^{(2)}(u,v)\mathrm{d}u\mathrm{d}v.$$

$\rho(u)\mathrm{d}u \simeq$ Probability of the occurence of u in $B(u, \mathrm{d}u)$

$\rho^{(2)}(u,v) \simeq$ Probability of the occurence of u in $B(u, \mathrm{d}u)$ and v in $B(v, \mathrm{d}v)$.

Notes

Poisson point processes

Classical definition : $X \sim$Poisson(S, ρ)

- $\forall m \geq 1$, $\forall$ bounded and disjoint $B_1, \ldots, B_m \subset S$, the r.v. $X_{B_1}, \ldots, X_{B_m}$ are **independent**.
- $N(B) \sim \mathcal{P}\left(\int_B \rho(u)\mathrm{d}u\right)$ for any bounded $A \subset S$.
- $\forall B \subset S$, $\forall F \in N_{lf}$

$$P(X_B \in F) = \sum_{n \geq 0} \frac{e^{-\int_B \rho(u)\mathrm{d}u}}{n!} \int_B \cdots \int_B \mathbf{1}(\{x_1, \ldots, x_n\} \in F) \prod_{i=1}^{n} \rho(x_i)\mathrm{d}x_i.$$

- If $\rho(\cdot) = \rho$, X is said to be homogeneous which implies

$$\mathrm{E}N(B) = \rho|B|, \quad \mathrm{Var}N(B) = \rho|B|.$$

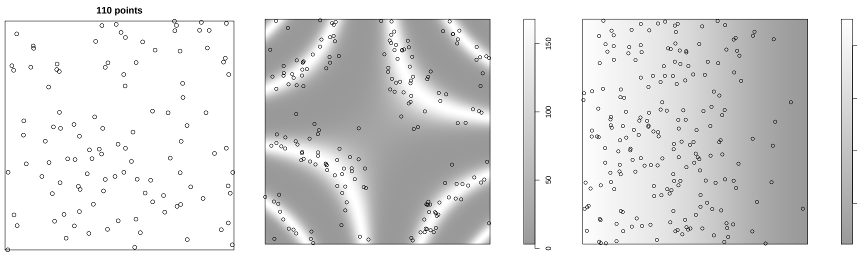
- and if $S = \mathbb{R}^d$, X is stationary and isotropic.

Notes

A few realizations on $S=[-1,1]^2$

- $\rho(u)=\beta e^{-u_1-u_1^2-.5u_1^3}$.
- $\rho=200$.
- $\rho(u)=\beta e^{2\sin(4\pi u_1 u_2)}$.

(β is adjusted s.t. the mean number of points in S, $\int_S \rho(u)\mathrm{d}u=200$.)



Notes

A few properties of Poisson point processes

Proposition : if $X\sim$Poisson(S,ρ)

- Void probabilities : $v(B)=P(N(B)=0)=e^{-\int_B(\rho(u)\mathrm{d}u)}$.
- For any $u,v\in S$, $\rho^{(2)}(u,v)=\rho(u)\rho(v)$ (also valid for $\rho^{(k)}$, $k\geq 1$)
- and if $|S|<\infty$, X admits a density w.r.t. Poisson$(S,1)$ given by

$$f(x)=e^{|S|-\int_S\rho(u)\mathrm{d}u}\prod_{u\in x}\rho(u).$$

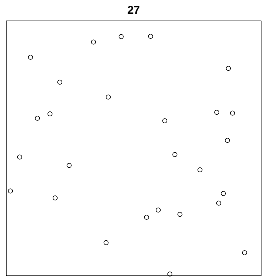
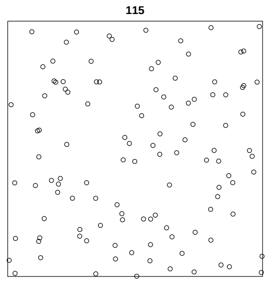
- Slivnyak-Mecke Theorem : for any non-negative function $h:S\times N_{lf}\to\mathbb{R}^+$, then

$$\mathrm{E}\sum_{u\in X}h(u,X\setminus u)=\int_S \mathrm{E}h(u,X)\rho(u)\mathrm{d}u.$$

<u>Example</u> : if $\rho(\cdot)=\rho$, $\mathrm{E}\sum_{u\in X\cap[0,1]^2}\mathbf{1}(d(u,X\setminus u)\leq R)=\rho\left(1-e^{-\rho\pi R^2}\right)$

Notes

Simulation

	
$\rho = 20,\ R = 0.1$	$\rho = 100,\ R = 0.05$
$\sum_{u \in x} \mathbf{1}(d(u, x \setminus u) \le R) = 9$	$\sum_{u \in x} \mathbf{1}(d(u, x \setminus u) \le R) = 60$
$\rho(1 - \exp(-\rho \pi R^2)) \simeq 9.33$	$\rho(1 - \exp(-\rho \pi R^2)) \simeq 54.41$

Notes

Statistical inference for a Poisson point process

- Simulation :
  - homogeneous case : very simple
  - inhomogeneous case : a **thinning** procedure can be efficiently done if $\rho(u) \le c$: simulate Poisson(c,W) and delete a point u with prob. $1 - \rho(u)/c$.
- Inference :
  - consists in estimating ρ, $\rho(\cdot;\theta)$ or $\rho(u)$ depending on the context.
  - All these estimates can be used even if the spatial point process is not Poisson (wait for a few slides)
  - Asymptotic properties very simple to derive under the Poisson assumption.
- Goodness-of-fit tests : tests based on quadrats counting, based on the void probability,. . .

Notes

Homogeneous case

- We consider here the problem of estimating the parameter ρ of a homogeneous Poisson point process defined on S and observed on a window $W \subseteq S$.
- Since $N(W) \sim \mathcal{P}(\rho|W|)$, the natural estimator of ρ is

$$\widehat{\rho} = N(W)/|W|$$

Properties

- (i) $\widehat{\rho}$ corresponds to the maximum likelihood estimate.
- (ii) $\widehat{\rho}$ is unbiased.
- (iii)$\mathrm{Var}\,\widehat{\rho} = \frac{\rho}{|W|}$.

Proof : (i) follows from the definition of the density (ii-iii) can be checked using the Campbell formulae.

Notes

Homogeneous case (2)

Asymptotic results

- For large $N(W)$, $\widehat{\rho}|W| \simeq \mathcal{N}(\rho|W|, \rho|W|)$ and so

$$|W|^{1/2}(\widehat{\rho} - \rho) \simeq \mathcal{N}(0, \rho).$$

  (the approximation is actually a convergence as $W \to \mathbb{R}^d$)
- Variance stabilizing transform :

$$2|W|^{1/2}(\sqrt{\widehat{\rho}} - \sqrt{\rho}) \simeq \mathcal{N}(0,1)$$

- We deduce a $1-\alpha$ ($\alpha \in (0,1)$) confidence interval for ρ

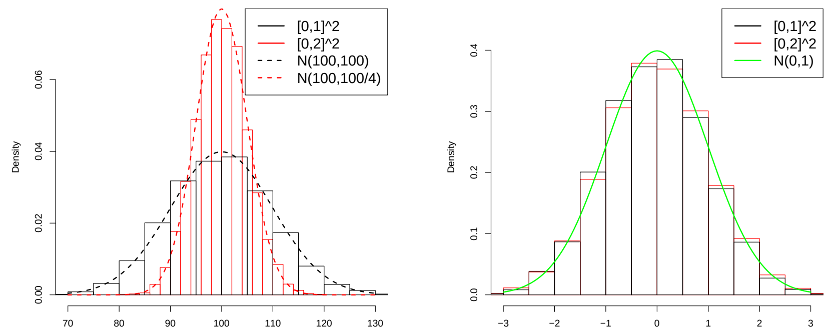
$$\mathrm{IC}_{1-\alpha}(\rho) = \left(\sqrt{\widehat{\rho}} \pm \frac{z_{\alpha/2}}{2|W|^{1/2}} \right)^2.$$

Notes

A simulation example

We generated $m = 10000$ replications of homogeneous Poisson point processes with intensity $\rho = 100$ on $[0,1]^2$ (blcak plots) and on $[0,2]^2$ (red plots).

Histograms of $\widehat{\rho}$	Histograms of $\widehat{2}|W|^{1/2}(\sqrt{\widehat{\rho}} - \sqrt{\rho})$



Notes

Examples | Definitions, Poisson | Summary statistics | Modelling and inference

A simulation example

We generated $m = 10000$ replications of homogeneous Poisson point processes with intensity $\rho = 100$ on $[0,1]^2$ (black plots) and on $[0,2]^2$ (red plots).

	$W = [0,1]^2$	$W = [0,2]^2$
Emp. Mean of $\widehat{\rho}$	100.17	100.07
Emp. Var. of $\widehat{\rho}$	98.57	25.69
Emp. Coverage rate of 95% confidence intervals	95.31%	94.78%

Notes

Application : pines datasets

- We consider three unmarked datasets : japanesepines, swedishpines, finpines.
- Plot the data, estimate the intensity parameter.
- Construct a confidence interval for each of them. Which one is significantly more abundant ?
- Judge the assumption of the Poisson model using a GoF test based on quadrats.

Notes

Inhomogeneous case : parametric estimation

- Assume that ρ is parametrized by a vector $\theta \in \mathbb{R}^p$ ($p \geq 1$). The most well-known model is the log-linear one :
$$\rho(u) = \rho(u;\theta) = \exp(\theta^\top z(u))$$
where $z(u) = (z_1(u), z_2(u), \ldots, z_p(u))$ correspond to known spatial functions or spatial covariates.
- θ can be estimated by maximizing the log-likelihood on W
$$l_W(X,\theta) = \sum_{u \in X_W} \log \rho(u;\theta) + \int_W (1-\rho(u;\theta))\mathrm{d}u$$
$$= |W| + \underbrace{\sum_{u \in X_W} \theta^\top z(u) - \int_W \exp(\theta^\top z(u))\mathrm{d}u}_{:=\ell_W(X,\theta)}.$$

In other words
$$\widehat{\theta} = \mathrm{Argmax}_\theta \;\; \ell_W(X,\theta).$$

Notes

Inhomogeneous case : parametric estimation (2)

- Why would $\widehat{\theta}$ be a good estimate ?
  Compute the score function

$$s_W(X,\theta) = \nabla \ell_W(X,\theta) = \sum_{u \in X_W} z(u) - \int_W z(u) \underbrace{\exp(\theta^\top z(u))}_{:=\rho(u)} \mathrm{d}u.$$

  The true parameter θ_0 (i.e. $X \sim P_{\theta_0}$) minimizes the expectation of the score function. Indeed from Campbell formula

$$\mathrm{E} s_W(X,\theta) = \int_W z(u) \left(\exp(\theta_0^\top z(u)) - \exp(\theta^\top z(u)) \right) \mathrm{d}u = 0$$

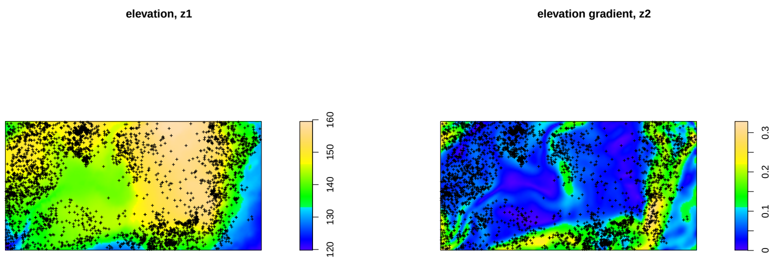
  when $\theta = \theta_0$.

- Rathbun and Cressie (1994) showed the **strong consistency** and the **asymptotic normality** of $\widehat{\theta}$ as $W \to \mathbb{R}^d$.

Notes

Data example : dataset `bei`

A point pattern giving the locations of 3605 trees in a tropical rain forest. Accompanied by covariate data giving the elevation (altitude) (z_1) and slope of elevation (z_2) in the study region.



Assume an inhomogeneous Poisson point process (which is not true, see the next chapter) with intensity

$$\log \rho(u) = \beta + \theta_1 z_1(u) + \theta_2 z_2(u).$$

<u>Question</u> : how can we prove that each covariate has a significant influence ?

Notes

Inhomogeneous case : nonparametric estimation

(Diggle 2003)

- Idea is to mimic the kernel density estimation to define a nonparametric estimator of the spatial function ρ.
- Let $k : \mathbb{R}^d \to \mathbb{R}^+$ a symmetric kernel with intensity one. Examples of kernels
  - Gaussian kernel : $(2\pi)^{-d/2}\exp(-\|y\|^2/2)$.
  - Cylindric kernel : $\frac{1}{\pi}\mathbf{1}(\|y\| \le 1)$.
  - Epanecnikov kernel : $\frac{3}{4}\mathbf{1}(|y| < 1)(1-|y|^2)$.
- Let h be a positive real number (which will play the role of a bandwidth window), then the nonparametric estimate (with border correction) at the location v is defined as

$$\widehat{\rho}_h(v) = K_h(v)^{-1} \sum_{u \in X_W} \frac{1}{h^d} k\left(\frac{\|v-u\|}{h}\right)$$

Notes

Intuitively, this works . . .

Indeed, using the Campbell formula and a change of variables we can obtain

$$\begin{aligned}
\mathrm{E}\,\widehat{\rho}_h(v) &= K_h(v)^{-1}\mathrm{E}\sum_{u \in X_W} \frac{1}{h^d} k\left(\frac{\|v-u\|}{h}\right) \\
&= K_h(v)^{-1}\int_W \frac{1}{h^d} k\left(\frac{\|v-u\|}{h}\right)\rho(u)\mathrm{d}u \\
&= K_h(v)^{-1}\int_{\frac{W-v}{h}} k\left(\|\omega\|\right)\rho(\omega h + v)\mathrm{d}\omega \\
&\overset{\text{h small}}{\simeq} K_h(v)^{-1}\int_{\frac{W-v}{h}} k\left(\|\omega\|\right)\rho(v)\mathrm{d}\omega \\
&\simeq \rho(v).
\end{aligned}$$

More theoretical justifications and properties and a discussion on the bandwidth parameter and edge corrections can be found in Diggle (2003).

Notes

Objective and classification

Objective :

- Define some descriptive statistics for s.p.p. (independently on any model so).
- Measure the abundance of points, the clustering or the repulsiveness of a spatial point pattern w.r.t. the Poisson point process.

Classification :

- First-order type based on the intensity function.
- Second-order type statistics : pair correlation function, Ripley's K function.
- Statistics based on distances : empy space function F, nearest-neigbour G, J function.

(We assume that ρ and $\rho^{(2)}$ exist in the rest of the talk)

Notes

Summary statistics based on the intensity function

Thanks to **Campbell formulae**, the estimates of the intensity for a Poisson point process can be used to estimate the intensity of a general spatial point process X. In particular

1. if X is stationary $\widehat{\rho} = N(W)/|W|$ is an estimate of ρ.
2. Non-stationary, parametric estimation of the intensity : if $\rho(u) = \rho(u;\theta)$ can be used using the "Poisson likelihood", i.e.

$$l_W(X,\theta) = \sum_{u \in X_W} \log \rho(u;\theta) - \int_W \rho(u;\theta)\mathrm{d}u.$$

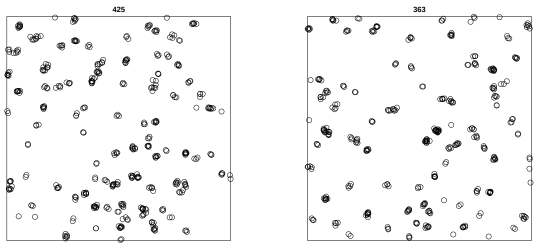
3. Non stationary, non-parametric estimation of the intensity (see previous chapter for notation) :

$$\widehat{\rho}_h(u) = K_h(u)^{-1} \sum_{v \in X_W} \frac{1}{h^d} k\left(\frac{\|v-u\|}{h}\right).$$

Notes

A simulation example in the stationary case

We generated $m = 10000$ replications of a stationary log-Gaussian Cox processes (Thomas process, $\kappa = 50$, $\sigma = .005$) with intensity $\rho = 400$.



	$W = [0,1]^2$	$W = [0,2]^2$
Emp. Mean of $\widehat{\rho}$	400.4	399.5
Emp. Var. of $\widehat{\rho}$	1741.4	507.4

- A survey of the estimation of the asymptotic variance of $\widehat{\rho}$ can be found in Prokesova and Heinrich (2010) and references therein.

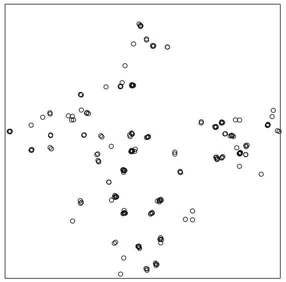
Notes

Parametric intensity estimation for non Poisson models

We generated $B = 1000$ replications of Thomas process with parameters $\kappa = 50$, $\sigma = .005$ and with intensity function

$$\rho(u) = \exp(\beta - \theta u_1^2 u_2^2)$$

with $\theta = -2$ and β adjusted s.t. $\mathrm{E}N(W) = 200$ for $W = [0,1]^2$ and 800 for $W = [0,2]^2$.



Then for each replication, θ is estimated using the "Poisson likelihood"

	$W = [0,1]^2$	$W = [0,2]^2$
Emp. Mean of $\widehat{\theta}$	-2.03	-2.01
Emp. Var. of $\widehat{\theta}$	0.13	0.03

- Asymptotic results are more awkward to derive and depend on mixing coefficients of the spatial point process X.
- See Guan (2006), Guan and Loh (2008), Waagepetersen, Guan and Jalilian (2012) and Coeurjolly and Møller (2012) for details and refinements.

Notes

Ripley's K function

We assume (for simplicity) the stationarity and isotropy of X.

Definition

The Ripley's K function is literally defined for $r \geq 0$ by

$$K(r) = \frac{1}{\rho}\,\mathrm{E}\Big(\text{number of extra events within distance r of a randomly chosen event}\Big)$$
$$= \frac{1}{\rho}\,\mathrm{E}\Big(N(B(0,r)\setminus 0)\mid 0\in X\Big)$$

We define the L function as $L(r) = (K(r)/\pi)^{1/2}$.

Properties :

- Under the Poisson case, $K(r) = \pi r^2$; $L(r) = r$.
- If $K(r) > \pi r^2$ or $L(r) > r$ (resp. $K(r) < \pi r^2$ or $L(r) < r$) we suspect clustering (regularity) at distances lower than r.

Notes

Examples Definitions, Poisson Summary statistics Modelling and inference

Pair correlation function

Definition

If ρ and $\rho^{(2)}$ exist, then the pair correlation function is defined by

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)}$$

where we set for convention $a/0 = 0$ for $a \geq 0$.

$$g(u,v)\begin{cases} = 1 & \text{if } X \sim \text{Poisson}(S,\rho). \\ > 1 & \text{for attractive point pattern.} \\ < 1 & \text{for repulsive point pattern.}\end{cases}$$

If $S = \mathbb{R}^d$ and X is stationary and isotropic, then

$$g(u,v) = \frac{\rho^{(2)}(\|v-u\|)}{\rho^2} = \overline{g}(\|v-u\|.$$

Notes

Particular case for stationary and isotropic processes

Theorem

For stationary and isotropic processes in $S = \mathbb{R}^d$

$$g(r) = \frac{K'(r)}{\sigma_d r^{d-1}}$$

where $\sigma_d = d\omega_d$ is the surface area of unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

Proof : Using polar decomposition we obtain

$$K(r) = \int_{B(0,r)} g(\|u\|)\mathrm{d}u = \int_0^r \int_{S^{d-1}} t^{d-1} g(t)\mathrm{d}t = \sigma_d \int_0^r t^{d-1} g(t)\mathrm{d}t.$$

Notes

Edge corrected estimation of the K function

Definition

We define

- the border-corrected estimate as

$$\widehat{K}_{BC}(r) = \frac{1}{\widehat{\rho}} \sum_{u \in X_{W_{\ominus r}}, v \in X_W}^{\neq} \frac{\mathbf{1}(v \in B(u,R))}{N(W_{\ominus r})}$$

  where $W_{\ominus r} = \{u \in W : B(u,r) \subseteq W\}$ is the erosion of W by r.

- the translation-corrected estimate as

$$\widehat{K}_{TC}(r) = \frac{1}{\widehat{\rho^2}} \sum_{u,v \in X_W}^{\neq} \frac{\mathbf{1}(v - u \in B(0,r))}{|W \cap W_{v-u}|}$$

  where $W_u = W + u = \{u + v : v \in W\}$.

Remark : everything extends to 2nd-order reweighted stationary point processes ; asymptotic properties depend on mixing conditions, . . .

Notes

Estimation of the pair correlation function

For convenience, we consider only stationary and isotropic point processes.

- Then, the pair correlation function $g(u,v) = g(\|u - v\|)$ can be estimated using the following edge corrected kernel estimate

$$\widehat{g}(r) = \frac{1}{\widehat{\rho^2}} \sum_{u,v \in X_W}^{\neq} \frac{k_h(\|v-u\| - r)}{\sigma_d r^{d-1} |W \cap W_{v-u}|}$$

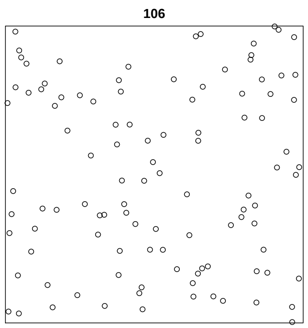
where $k_h(t) = h^{-d} k(t/h)$.

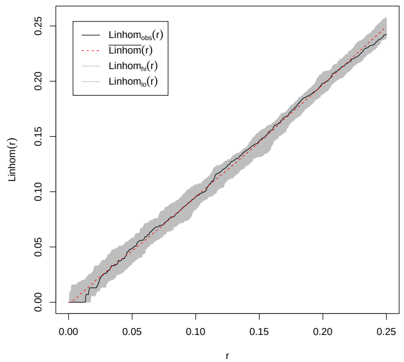
- Alternatively, we can estimate estimate the derivative of the K function (after smooting using e.g. spline techniques) and define

$$\widehat{g}(r) = \frac{\widehat{K'}(r)}{\sigma_d r^{d-1}}.$$

Notes

Example of L function for a Poisson point pattern

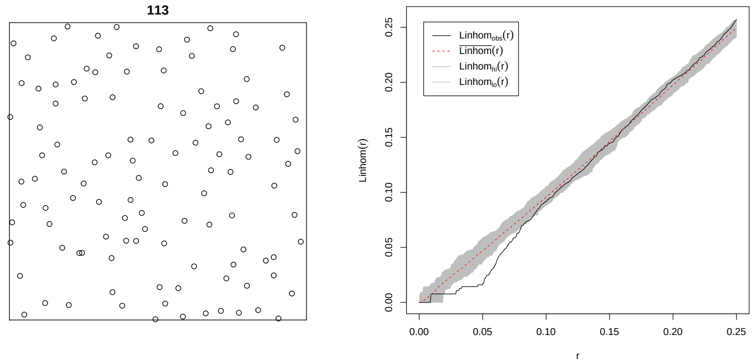




- The enveloppes are constructed using a Monte-Carlo approach under the Poisson assumption.
- $\Rightarrow$ we don't reject the Poisson assumption.

Notes

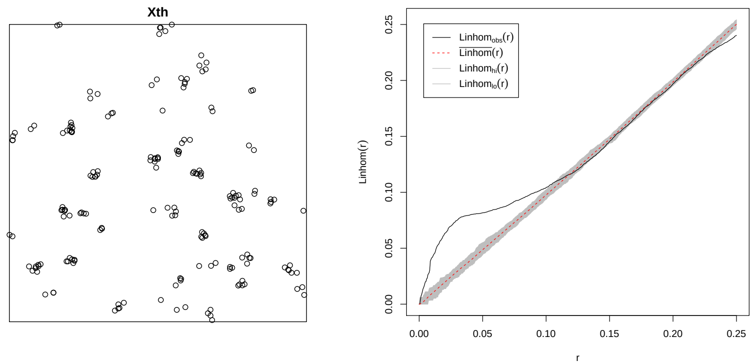
Example of L function for a repulsive point pattern



- ⇒ the point pattern does not come from the realization of a homogeneous Poisson point process.
- exhibits repulsion at short distances ($r \leq .05$)

Notes

Example of L function for a clustered point pattern



- ⇒ the point pattern does not come from the realization of a homogeneous Poisson point process.
- exhibits attraction at short distances ($r \leq .08$).

Notes

Statistics based on distances : F, G and J functions

Assume X is stationary (definitions can be extended in the general case)

Definition

- The empty space function is defined by

$$F(r) = P(d(0, X) \leq r) = P(N(B(0, r)) > 0), \qquad r > 0.$$

- The nearest-neighbour distribution function is

$$G(r) = P(d(0, X \setminus 0) \leq r | 0 \in X)$$

- J-function : $J(r) = (1 - G(r))/(1 - F(r)), \qquad r > 0.$

- Poisson case : $\forall r > 0$, $F(r) = G(r) = 1 - e^{-\pi r^2}$, $J(r) = 1$.
- $F(r) < F_{pois}(r)$, $G(r) > G_{pois}(r)$, $J(r) < 1$: attraction at dist. $< r$.
- $F(r) > F_{pois}(r)$, $G(r) < G_{pois}(r)$, $J(r) > 1$: repulsion at dist. $< r$.

Notes

Non-parametric estimation of F, G and J

As for the K and L functions, several edge corrections exist. We focus here only on the border correction. We assume that X is observed on a bounded window W with positive volume.

Definition

- Let $I \subseteq W$ be a finite regular grid of points and $n(I)$ its cardinality. Then, the (border corrected) estimator of F is

$$\widehat{F}(r) = \frac{1}{n(I_r)} \sum_{u \in I_r} \mathbf{1}(d(u, X) \leq r)$$

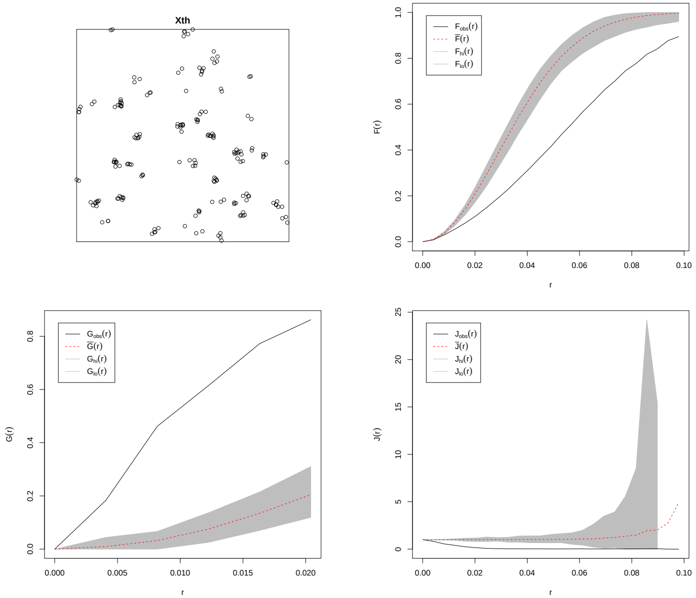
where $I_r = I \cap W_{\ominus r}$.

- The (border corrected) estimator of G is

$$\widehat{G}(r) = \frac{1}{N(W_{\ominus r})} \sum_{u \in X \cap W_{\ominus r}} \mathbf{1}(d(u, X \setminus u) \leq r)$$

Notes

Application to a clustered point pattern data



Notes

Objective

The main objectives of this section are

- to present more realistic models than the too simple Poisson point process to take into account the spatial dependence between points.
- to present statistical methodologies to infer these models.

We can distinguish several classes of models for spatial point processes

1. point processes based on the thinning of a Poisson point processes, on the superimposition of Poisson point processes. [sometimes hard to relate the stochastic process producing the realization and the physical phenomenon producing the data]
2. Cox point processes (which include Cluster point processes,. . .).
3. Gibbs point processes.
4. Determinental point processes.

Notes

An attempt to classify these models . . .

Model	Allows to model	Are moments expressible in a closed form ?	Density w.r.t. Poisson ?
Cox	**attraction**	**yes**	**no**
Gibbs	**repulsion** but also attraction	**no**	**yes**
Determinental	**repulsion**	**yes**	**yes**

This course only focuses on the two first classes of point processes, i.e. on Cox and Gibbs point processes.

Notes

Definition

We let $S \subseteq \mathbb{R}^d$ throughout this section. B denotes any bounded domain $\subseteq S$.

Definition

Suppose that $Z = \{Z(u) : u \in S\}$ is a nonnegative random field so that with probability one, $u \to Z(u)$ is a locally integrable function. If the conditional distribution of X given Z is a Poisson process on S with intensity function Z, then X is said to be a *Cox process* driven by Z.

Remarks :

- Z is a random field means that $Z(u)$ is a random variable $\forall u \in S$.
- if $\mathrm{E}Z(u)$ exists and is locally integrable then w.p. 1, $Z(u)$ is a locally integrable function.

Notes

Basic properties

Proposition

1. Provided $Z(u)$ has finite expectation and variance for any $u \in S$

$$\rho(u) = \mathrm{E}Z(u),\ \rho^{(2)}(u,v) = \mathrm{E}[Z(u)Z(v)],\ g(u,v) = \frac{\mathrm{E}[Z(u)Z(v)]}{\rho(u)\rho(v)}.$$

2. The void probabilities are given by

$$v(B) = \mathrm{E}\exp\left(-\int_B Z(u)\mathrm{d}u\right)$$

for bounded $B \subseteq S$.

Proof : direct consequence of the fact that $X|Z$ is a Poisson point process with intensity function Z.

Notes

Over-dispersion of Cox processes

Proposition

Let A, B bounded sets of S, then

$$\mathrm{Cov}(N(A), N(B)) = \int_A \int_B \mathrm{Cov}(Z(u), Z(v))\mathrm{d}u\mathrm{d}v + \int_{A\cap B} \mathrm{E}Z(u)\mathrm{d}u$$

Consequence :

- In particular, $\mathrm{Var}N(A) \geq \mathrm{E}N(A)$ with equality only when X is a Poisson process.
- $\Rightarrow$ over-dispersion of the counting variables.

Other remarks :

- Most of models have pcf such that $g \geq 1$ (but a few exceptions $\exists$).
- If $S = \mathbb{R}^d$ and X is stationary and/or isotropic then X is stationary and/or isotropic.
- Explicit expressions of the F, G and J functions in the stationary case are in general difficult to derive.

Notes

A first example

Definition

A *mixed Poisson process* is a Cox process where $Z(u) = Z_0$ is given by a positive random variable for any $u \in S$, i.e. $X|Z_0$ follows a homogeneous Poisson process with intensity Z_0.

- Limited interest ...
- X is stationary and (provided Z_0 has first two moments)

$$\rho = \mathrm{E}Z_0 \quad \text{and} \quad g(u,v) = \frac{\mathrm{E}[Z_0^2]}{\mathrm{E}[Z_0]^2} \geq 1.$$

- The K and L functions are given by

$$K(r) = \beta\omega_d r^d \quad \text{and} \quad L(r) = \beta^{1/d} r \geq r$$

where $\omega_d = |B(0,1)|$ and $\beta = \frac{\mathrm{E}[Z_0^2]}{\mathrm{E}[Z_0]^2}$.
(recall that $K'(r) = d\omega_d g(r) r^{d-1}$).

Notes

Neymann-Scott processes

Definition

Let C be a stationary Poisson process on $\mathbb{R}^d$ with intensity $\kappa > 0$. Conditional on C, let $X_c, c \in C$ be independent Poisson processes on $\mathbb{R}^d$ where X_c has intensity function

$$\rho_c(u) = \alpha k(u - c)$$

where $\alpha > 0$ is a parameter and k is a kernel (i.e. for all $c \in \mathbb{R}^d$, $u \to k(u-c)$ is a density function). Then $X = \cup_{c \in C} X_c$ is a Neymann-Scott process with cluster centres C and clusters $X_c, c \in C$.

- X is also a Cox process on $\mathbb{R}^d$ driven by $Z(u) = \sum_{c \in C} \alpha k(u - c)$.
- Simulating a Neymann-Scott process (on W) is very simple (if k has compact support $T < \infty$)
  1. Generate $C \sim \text{Poisson}(W \oplus T, \kappa)$.
  2. For each $c \in C$, generate $X_c \sim \text{Poisson}(W, \rho_c)$.
  3. Concatenate all the X_c's.
- If k has unbounded support, an exact simulation is still possible.

Notes

Two classical NS pp

We obtain specific models by choosing specific kernel densities.

1. the *Matérn cluster process* where

$$k(u) = \mathbf{1}(\|u\| \leq R)\frac{1}{\omega_d R^d}$$

   is the uniform density on the $B(0,R)$.

2. the *Thomas process* where

$$k(u) = \left(\frac{1}{2\pi\sigma^2}\right)^{d/2} \exp\left(-\frac{\|u\|^2}{2\sigma^2}\right)$$

   is the density of $\mathcal{N}(0, \sigma^2 I_d)$.

When R is small or when σ is small, then point pattern exhibit strong attraction.

Notes

Basic properties of NS pp

- κ is the mean number of cluster centres per unit square, α is the mean number of daughters points per cluster.
- X is stationary (since Z is stationary) and is isotropic if $k(u) = k(\|u\|)$.
- Intensity of X : $\rho(u) = \alpha\kappa$.
- The (stationary) pair correlation function is given by

$$g(u,v) = 1 + \frac{k * k(v-u)}{\kappa} \geq 1 \quad \text{where} \quad k*k(u) = \int k(c)k(v-u+c)\mathrm{d}c.$$

- The F, G and J functions are also expressible in terms of k. In particular

$$J(r) = \int k(u)\exp\left(-\alpha \int_{\|v\|\leq r} k(u+v)\mathrm{d}v\right)\mathrm{d}u$$

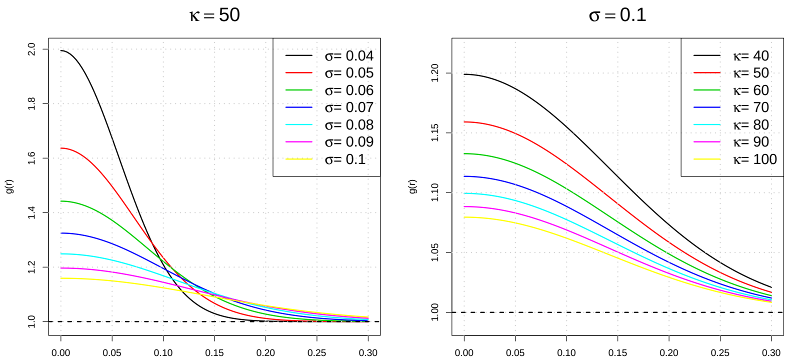
  whereby we deduce that $\exp(-\alpha) \leq J(r) \leq 1$.

Notes

Back to the Thomas process

Recall that k is the density of a $\mathcal{N}(0, \sigma^2 I_d)$. Applying the previous results, we get (for the pcf)

$$g(r) = 1 + \frac{1}{(4\pi\sigma^2)^{d/2}} \exp\left(-r^2/(4\sigma^2)\right)/\kappa$$

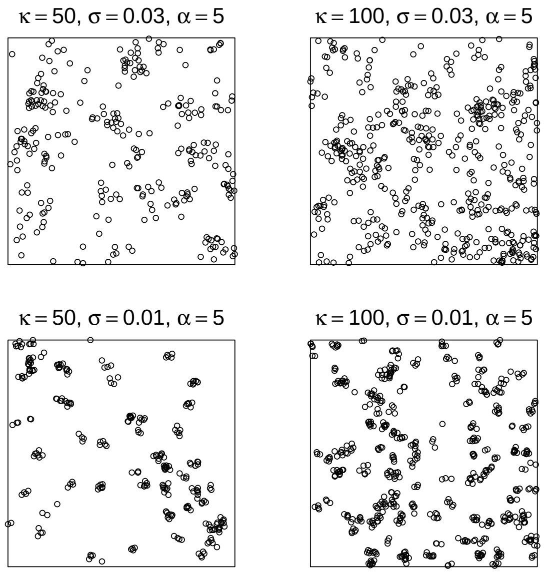


(similar developments can be done for the K, L, J functions and with more work for the Matérn process).

Notes

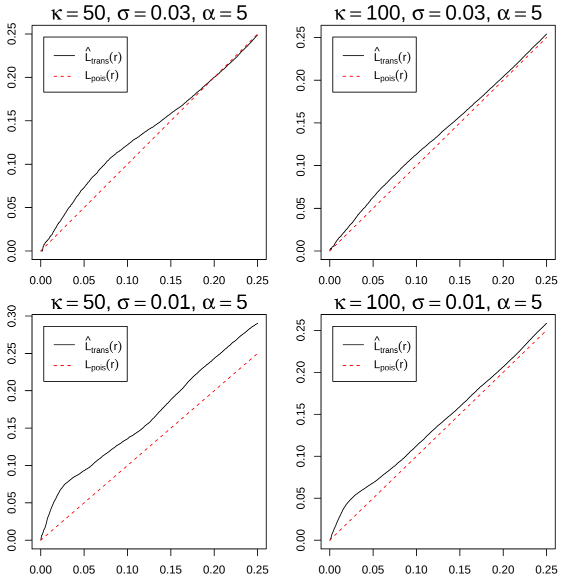
Examples Definitions, Poisson Summary statistics Modelling and inference

Four realizations of Thomas point processes



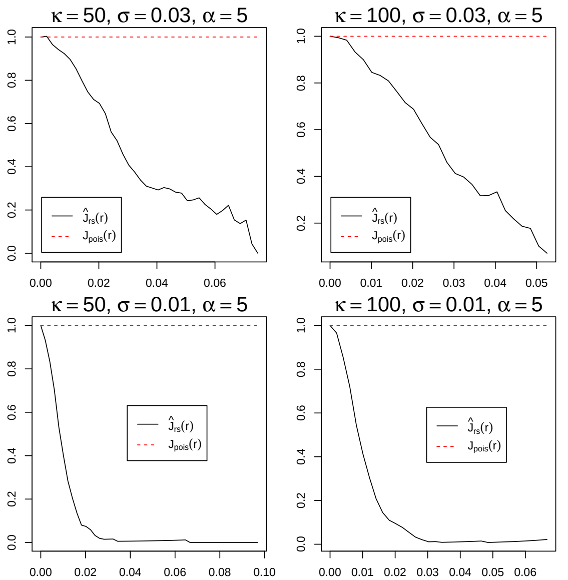
Notes

Correponding L estimates



Notes

Correponding J estimates



Notes

Complements

- Inhomogeneous Neymann-Scott processes can be obtained by replacing the intensity parameter κ by a spatial function $\kappa(u)$.
- The natural extension of NS processes is given by shot-noise Cox processes which is a Cox process driven by

$$Z(u) = \sum_{(c,\gamma)\in\Phi} \gamma k(c,u)$$

where $k(\cdot,\cdot)$ is a kernel and Φ is a Poisson point process on $\mathbb{R}^d \times (0,\infty)$ with a locally integrable intensity function ζ. (see e.g. Møller and Waagepetersen 2004 for complements).

Notes

Log-Gaussian Cox processes

Definition

Let X be a Cox process on $\mathbb{R}^d$ driven by $Z = \exp Y$ where Y is a Gaussian random field. Then, X is said to be a *log Gaussian Cox process* (LGCP).

Remarks :

- we could consider $Z = h(Y)$ for some non-negative function h, but the exp leads to tractable calculations.
- another possibility : using a χ^2 field, i.e. $Z(u) = Y_1(u)^2 + \ldots + Y_m(u)^2$ are the Y_i's are independent Gaussian fields with zero mean.
- LGCP are easy to simulate since the problem is transfered to generate a Gaussian field (which can be handled by several methods).
- The mean and covariance function of Y determine the distribution of X.

Notes

Particular cases

- In the following we let

$$m(u) = \mathrm{E}Y(u) \quad \text{and } c(u,v) = \mathrm{Cov}(Y(u), Y(v))$$

and we focus on the case where $c(u,v)$ depends only on $\|v - u\|$ (covariance function invariant by translation and by rotation).

- Conditions on c are needed to get a covariance function. Among functions satisfying these properties we find :
  - the *power exponential family* satisfies these conditions

$$c(u,v) = \sigma^2 r(\|v-u\|/\alpha) \text{ with } r(t) = \exp\left(-t^\delta\right), t \geq 0$$

  with $\alpha, \sigma > 0$. $\delta = 1$ is the exponential correlation function ; $\delta = 1/2$ is the stable correlation function ; $\delta = 2$ is the Gaussian correlation function.
  - the *cardinal sine correlation* :

$$c(u,v) = \sigma^2 r(\|v-u\|/\alpha) \text{ with } r(t) = \frac{\sin(t)}{t}, t \geq 0$$

Notes

Examples Definitions, Poisson Summary statistics Modelling and inference

Summary statistics for the LGCP

Proposition

Let X be a LGCP then under the previous notation

1. the intension function of X is

$$\rho(u) = \exp\left(m(u) + c(u,u)/2\right).$$

2. The pair correlation function g of X is

$$g(u,v) = \exp(c(u,v)).$$

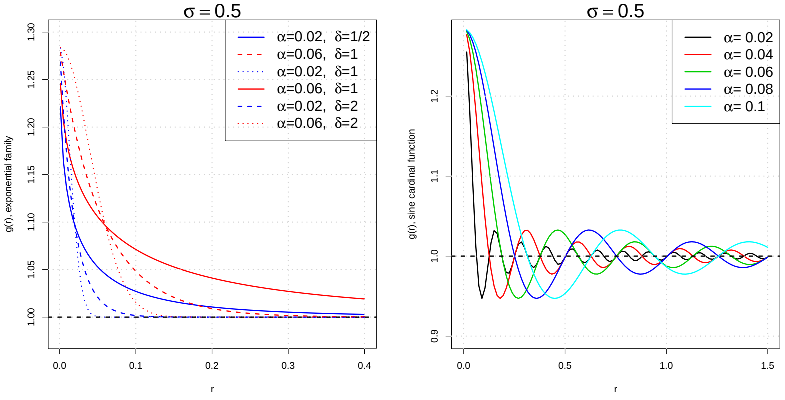
Proof : based on the fact that for $U \sim \mathcal{N}(\zeta, \sigma^2)$, the Laplace transform of U is $\mathrm{E}\exp(tU) = \exp(\zeta + \sigma^2 t/2)$.

- one to one correspondendce between (m,c) and (ρ, g).
- If c is translation invariant then X is second order reweighted stationary (stationary if m is constant, and isotropic if in addition $c(u,v)$ depends only on $\|v-u\|$).

Notes

A few plots of pair correlation function

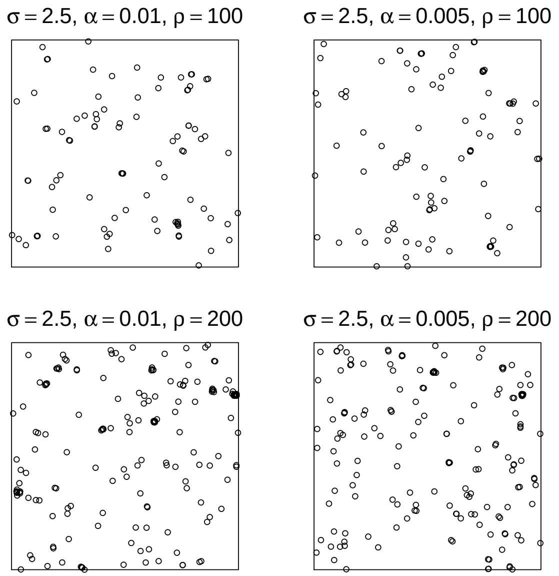
- pcf for the *power exponential family* : $\log g(r) = \sigma^2 \exp\left(-\left(\frac{r}{\alpha}\right)^{\delta}\right), \quad \alpha, \sigma, \delta > 0$
- pcf for the *cardinal sine correlation* : $\log g(r) = \sigma^2 \frac{\sin(r/\alpha)}{r/\alpha}, \quad \alpha, \sigma > 0$



Notes

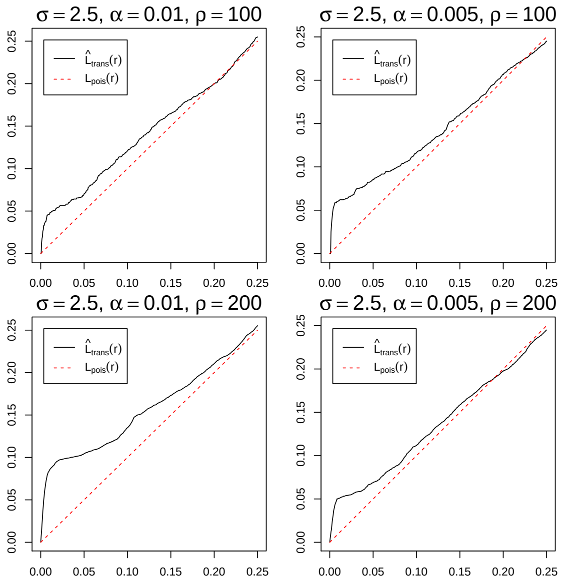
Four realizations of (stationary) LGCP point processes

- with exponential correlation function ($\delta = 1$).
- The mean m of the Gaussian process is such that $\rho = \exp(m + \sigma^2/2)$.



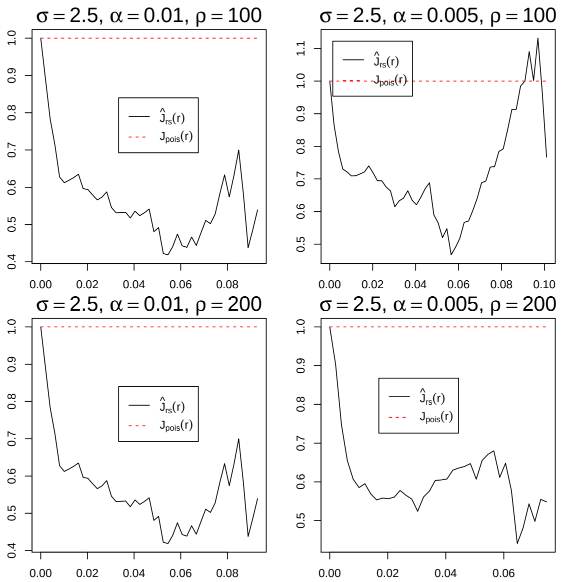
Notes

Correponding L estimates



Notes

Correponding J estimates



Notes

Is likelihood available ?

- Assume (only here) that S is a bounded domain, then the density of X_S w.r.t a Poisson processes with unit rate is given by

$$f(x) = \mathrm{E}\left[\exp\left(|S| - \int_S Z(u)\mathrm{d}u\right)\prod_{u\in x} Z(u)\right]$$

for finite point configurations $x \subset S$. Explicit expression of the expectation is usually unknown and the integral may be difficult to calculate.
$\Rightarrow$ MLE is usually impossible to calculate (approximations or Bayesian should be used)

- In most of applications, we only observe the realization of X.
$\Rightarrow$ Z should be considered as a latent process generating the point process, which is not observed.

Notes

General method based on minimum contrast estimation

- Assume we observe the realization of a stationary Cox point process which belongs to a parametric family with parameter θ (ex : $\theta = (\alpha, \kappa, \sigma^2)$ for the Thomas process, $\theta = (\mu, \alpha, \sigma^2)$ for a LGCP with exponential correlation function).
- For most of Cox point processes, $\rho = \rho_\theta$, $K = K_\theta$ or $g = g_\theta$ functions are expressible in a closed form, for instance :
  - for a planar ($d = 2$) **Thomas process** (NS process with Gaussian kernel) : $\rho = \alpha\kappa$ and

$$g_\theta(r) = 1 + \frac{1}{\sqrt{4\pi\sigma^2}}\exp\left(-r^2/(4\sigma^2)\right)/\kappa \quad \text{and} \quad K_\theta(r) = \pi r^2 + \left(1 - \exp\left(-r^2/(4\sigma^2)\right)\right)/\kappa$$

  - for a **LGCP with exponential correlation function**

$$\rho = \exp(m + \sigma^2/2) \quad \text{and} \quad \log g_\theta(r) = \sigma^2 \exp(-r/alpha).$$

Notes

General method based on minimum contrast estimation (2)

- Then the idea is then to estimate θ using a **minimum contrast approach** : i.e. define $\hat{\theta}$ as the minimizer of

$$\int_{r_1}^{r_2} \left|\widehat{K}(r)^q - K_\theta(r)^q\right|^2 \mathrm{d}r \quad \text{or} \quad \int_{r_1}^{r_2} \left|\widehat{g}(r)^q - g_\theta(r)^q\right|^2 \mathrm{d}r$$

  where
  - $\widehat{K}(r)$ and $\widehat{g}(r)$ are the nonparametric estimates of $K(r)$ and $g(r)$.
  - where $[r_1, r_2]$ is a set of r fixed values.
  - q is a power parameter (adviced in the literature to be set to $q = 1/4$ or $1/2$).

Notes

A short simulation

- we generated 200 replications of a Thomas process with parameters $\kappa = 100$, $\sigma^2 = 10^{-4}$ and $\alpha = 5$
- we estimated the parameters σ^2 and κ using the minimimum contrast estimat based on the K function.
- Then α is estimated using $\widehat{\alpha} = \widehat{\rho}/\widehat{\kappa}$

	Parameter κ	
	$W = [0,1]^2$	$W = [0,2]^2$
Emp. mean	98.9	102.4
Emp. var.	251.9	78.1

	Parameter α	
	$W = [0,1]^2$	$W = [0,2]^2$
Emp. mean	4.9	4.9
Emp. var.	40.1	6.1

	Parameter σ^2	
	$W = [0,1]^2$	$W = [0,2]^2$
Emp. mean	1.01×10^{-4}	9.7×10^{-5}
Emp. var.	1.5×10^{-5}	8.2×10^{-6}

Notes

Introduction

- the objective of this section is to introduce a new class of point processes : the class of Gibbs point processes.
- **Gibbs point process** :
  - are mainly used to model **repulsion** between point (but a few models allows also to produce **aggregated models**). That's why this kind of models are widely used in statistical physics to model particles systems.
  - are defined (in a bounded domain) by a **density** w.r.t. a Poisson point process
    $\Rightarrow$ **very easy** to interpret the model and the parameters.
  - their main drawback : **moments are not expressible** in a closed form and density known up to a scalar
    $\Rightarrow$ **specific inference methods** are required.

Notes

Important restriction of this section

- Throughout this chapter : we assume that the point process X is defined in a bounded domain $S \subset \mathbb{R}^d$ ($|S| < \infty$).
- Gibbs point processes defined on $\mathbb{R}^d$ are of particular interest :
  - in statistical physics because they can model **phase transition** .
  - in asymptotic statistics : if for instance we want to prove the convergence of an estimator as the window expands to $\mathbb{R}^d$

  However, the formalism is more complicated and technical and this is not considered here.

$\Rightarrow$ from now, X is a **finite point process in** S **(bounded)** taking values in N_f (space of finite configurations of points)

$$N_f = \{x \subset S : n(x) < \infty\} .$$

Most of the results presented here have an extension to $S = \mathbb{R}^d$.

Notes

Definition of Gibbs point processes

Definition

A finite point process X on a bounded domain S ($0 < |S| < \infty$) is said to be a Gibbs point process if it admits a density f w.r.t. a Poisson point process with unit rate, i.e. for any $F \subseteq N_f$

$$P(X \in F) = \sum_{n \geq 0} \frac{\exp(-|S|)}{n!} \times \int_S \cdots \int_S \mathbf{1}(\{x_1, \ldots, x_n\} \in F) f(\{x_1, \ldots, x_n\}) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

where the term $n = 0$ is read as $\exp(-|S|)\mathbf{1}(\emptyset \in F)f(\emptyset)$.

- Gpp can be viewed as a perturbation of a Poisson point process.
- f is easily interpretable since it is in some sense a weight w.r.t. a Poisson process.

Notes

The simplest example . . .

is the inhomogeneous Poisson point process. Indeed for $X \sim \text{Poisson}(S, \rho)$ (such that $\mu(S) < \infty$), we recall that X admits a density w.r.t. to a Poisson point process with unit rate given for any $x \in N_f$ by

$$f(x) = \exp(|S| - \mu(S)) \prod_{u \in x} \rho(u).$$

In most of cases, f is specified up to a proportionality $f = c^{-1}h$ where $h : N_f \to \mathbb{R}^+$ is a known function.
$\Rightarrow$ c is given by

$$c = \sum_{n \geq 0} \frac{\exp(-|S|)}{n!} \int_S \cdots \int_S h(\{x_1, \ldots, x_n\}) \mathrm{d}x_1 \ldots \mathrm{d}x_n = \mathrm{E}[h(Y)]$$

where $Y \sim \text{Poisson}(S, 1)$.

Notes

Papangelou conditional intensity

Definition

The Papangelou conditional intensity for a point process X with density f is defined by

$$\lambda(u, x) = \frac{f(x \cup u)}{f(x)}$$

for any $x \in N_f$ and $u \in S$ $(u \notin x)$, taking $a/0 = 0$ for $a \geq 0$.

- λ does not depend on c.
- for Poisson(S,ρ), $\lambda(u, x) = \rho(u)$ does not depend on x!
- $\lambda(u, x)\mathrm{d}u$ can be interpreted as the conditional probability of observing a point in an infinitesimal region containing u of size $\mathrm{d}u$ given the rest of X is x.

Notes

Attraction, repulsion, heredity

Definition

We often say that X (or f) is

- **attractive** if
$$\lambda(u, x) \leq \lambda(u, y) \text{ whenever } x \subset y.$$
- **repulsive** if
$$\lambda(u, x) \geq \lambda(u, y) \text{ whenever } x \subset y.$$
- **hereditary** if
$$f(x) > 0 \Rightarrow f(y) > 0 \text{ for any } y \subset x.$$

- if f is hereditary, then $f \Leftrightarrow \lambda$ (one-to-one correspondence).

Notes

Existence of a Gpp in S ($|S| < \infty$)

Proposition

Let $\phi^\star : S \to \mathbb{R}^+$ be a function so that $c^\star = \int_S \phi^\star(u)\mathrm{d}u < \infty$. Let $h = cf$, we say that X (or f) satisfies the

- local stability property if for any $x \in N_f$, $u \in S$

$$h(x \cup u) \le \phi^\star(u)h(x) \Leftrightarrow \lambda(u,x) \le \phi^\star(u).$$

- the Ruelle stability property if for any $x \in N_f$ and for $\alpha > 0$

$$h(x) \le \alpha \prod_{u \in x} \phi^\star(u).$$

local stability condition $\Rightarrow$ Ruelle stability condition (and that f is hereditary) $\Rightarrow$ existence of point process in S.

Proof : the first implication is obvious ; for the last one it consists in checking that $c < \infty$.

Notes

Pairwise interaction point processes

For simplicity, we focus on the isotropic case.

Definition

A istotropic parwise interaction point process (PIPP) has a density of the form (for any $x \in N_f$)

$$f(x) \propto \prod_{u \in x} \phi(u) \prod_{\{u,v\} \subseteq x} \phi_2(\|v-u\|)$$

where $\phi : S \to \mathbb{R}^+$ and $\phi_2 : \mathbb{R}^+_* \to \mathbb{R}+$.

- If ϕ is constant (equal to β) then the Gpp is said to be homogeneous (note that $\prod_{u \in x} \phi(u) = \beta^{n(x)}$).
- ϕ_2 is called the interaction function.
- this class of models is hereditary
- f is repulsive if $\phi_2 \le 1$, in which case the process is locally stable if $\int_S \phi(u)\mathrm{d}u$.

Notes

Strauss point process

Among the class of PIPP, the main example is the Strauss point process defined by

$$f(x) \propto \beta^{n(x)} \gamma^{s_R(x)} \qquad \lambda(u,x) = \beta \gamma^{t_R(u,x)}$$

where $\beta > 0, R < \infty$, where $s_R(x)$ is the number of R-close pairs of points in x and $t_R(u,x) = s_R(x \cup u) - s_R(x)$ is the number of R-close neighbours of u in x

$$s_R(x) = \sum_{\{u,v\} \in x} \mathbf{1}(\|v-u\| \le R) \text{ and } t_R(u,x) = \sum_{v \in x} \mathbf{1}(\|v-u\| \le R).$$

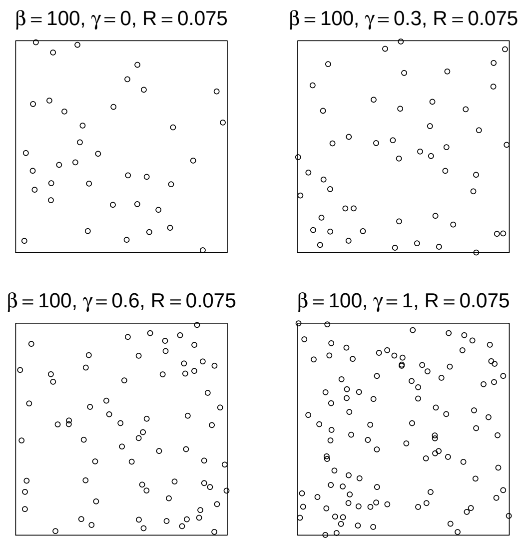
The parameter γ is called the **interaction parameter** :

- $\gamma = 1$: homogeneous Poisson point process with intensity β.
- $0 < \gamma < 1$: repulsive point process.
- $\gamma = 0$: hard-core process with hard-core R ; the points are prohibited from being closer han R.
- $\gamma > 1$: the model is not well-defined (if there exists a set $A \subset S$ with $|A| > 0$ and $diam(A) \le R$, then $c > \sum_{n \ge 0} \frac{(\beta |A|)^n}{n!} \gamma^{n(n-1)/2} = \infty$).

Notes

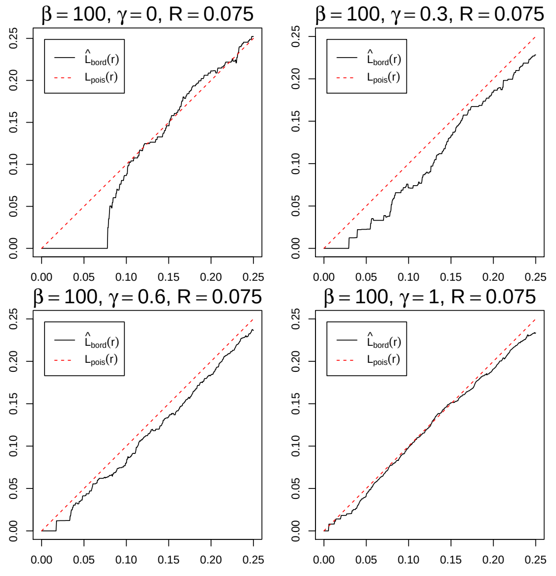
Realizations of Strauss point processes

(simulation of spatial Gibbs point processes can be done using spatial birth-and-death process or using MCMC with reversible jumps, see Møller and Waagepetersen for details)



Notes

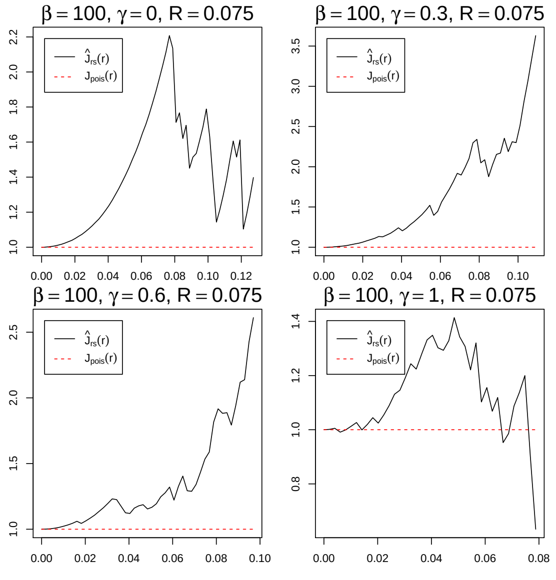
Corresponding L estimates



Notes

Examples Definitions, Poisson Summary statistics Modelling and inference

Corresponding J estimates



Notes

Finite range property (spatial Markov property)

Definition

A Gibbs point process X has a finite range R if the Papangelou conditional intensity satisfies

$$\lambda(u, x) = \lambda(u, x \cap B(u, R)).$$

- the probability to insert a point u into x depends only on some neighborhood of u.
- this definition is actually more general and leads to the definition of Markov point process (omitted here to save time).
- interesting property when we want to deal with edge effects.
- Finite range of the Strauss point process $= R$.

Notes

Other pairwise interaction point processes

- **Strauss** point process : $\phi_2(r) = \gamma^{\mathbf{1}(r \leq R)}$.
- **Piecewise Strauss** point process :

$$\phi_2(r) = \gamma_1^{\mathbf{1}(r \leq R_1)} \gamma_2^{\mathbf{1}(R_1 < r \leq R_2)} \dots \gamma_p^{\mathbf{1}(R_{p-1} < r \leq R)},$$

  with $\gamma_i \in [0, 1]$ and $0 \leq R_1 < \dots < R_p = R < \infty$ (finite range R) .
- **Overlap area** process :

$$\phi_2(r) = \gamma^{|B(u, R/2) \cap B(v, R/2)|},$$

  with $r = \|v - u\|$ with $\gamma \in [0, 1]$ (finite range R) .
- **Lennard-Jones** process :

$$\phi_2(r) = \exp(\alpha_1(\sigma/r)^6 - \alpha_2(\sigma/r)^{12}),$$

  with $\alpha \geq 0$, $\alpha_2 > 0$, $\sigma > 0$ (well-known example used in statistical physics, not locally stable but Ruelle stable) (infinite range) .

Notes

Non pairwise interaction point processes

- **Geyer's triplet** point process :

$$f(x) \propto \beta^{n(x)} \gamma^{s_R(x)} \delta^{u_R(x)}$$

$\beta > 0$, $s_R(x)$ is defined as in the Strauss case and

$$u_R(x) = \sum_{\{u,v,w\}} \mathbf{1}(\|v-u\| \leq R, \|w-v\| \leq R, \|w-u\| \leq R)$$

  - (i) $\gamma \in [0,1]$ and $\delta \in [0,1]$: locally stable, repulsive, finite range R.
  - (ii) $\gamma > 1$ and $\delta \in (0,1)$: locally stable, neither attractive nor repulsive, finite range R.

Notes

Non pairwise interaction point processes (2)

- **Area-interaction** point process :

$$f(x) \propto \beta^{n(x)} \gamma^{-|U_{x,R}|}$$

where $U_{x,R} = \cup_{u \in x} B(u,R)$, $\beta > 0$ and $\gamma > 0$. It is attractive for $\gamma \geq 1$ and repulsive for $0 < \gamma \leq 1$. In both cases, it is locally stable since

$$\lambda(u,x) = \beta \gamma^{-|B(u,R) \setminus \cup_{v \in x : \|v-u\| \leq 2R} B(v,R)|}$$

satisfies $\lambda(u,x) \leq \beta$ when $\gamma \geq 1$ and $\lambda(u,x) \leq \beta \gamma^{-\omega_d R^d}$ in the other case. (finite range $2R$)

Notes

GNZ formula

The following result is also a characterization of a Gibbs point process.

Georgii-Nguyen-Zeissin Formula

Let X be a finite and hereditary Gibbs point process defined on S. Then, for any function $h : S \times N_f \to \mathbb{R}^+$, we have

$$\mathrm{E}\Big[\sum_{u \in X} h(u, X \setminus u)\Big] = \int_S \mathrm{E}[h(u,X)\lambda(u,X)]\mathrm{d}u.$$

Proof : we know that $\mathrm{E}g(X) = \mathrm{E}[g(Y)f(Y)]$ where f is the density of a Poisson point process with unit rate Y. Apply this to the function $g(X) = \sum_{u\in X} h(u, X\setminus u)$

$$\begin{aligned}
\mathrm{E}[g(X)] &= \mathrm{E}\big[\sum_{u\in Y} h(u, Y\setminus u) f(Y)\big] \\
&= \int_S \mathrm{E}[h(u,Y)f(Y\cup u)]\mathrm{d}u \quad \text{from the Slivnyak-Mecke Theorem} \\
&= \int_S \mathrm{E}[h(u,Y)f(Y)\lambda(u,Y)]\mathrm{d}u \quad \text{since } X \text{ is hereditary} \\
&= \int_S \mathrm{E}[h(u,X)\lambda(u,X)]\mathrm{d}u.
\end{aligned}$$

Notes

First and second order intensities

Proposition

1. The intensity function is given by

$$\rho(u) = \mathrm{E}[\lambda(u,X)].$$

2. The second order intensity function is given by

$$\rho^{(2)}(u,v) = \mathrm{E}[\lambda(u,X)\lambda(v,X)]$$

- can be deduced from the GNZ formula.
- Except for the Poissonian case, moments are not expressible in a closed form, e.g.

$$\rho(u) = \frac{1}{c}\sum_{n\geq 0} \frac{\exp(-|S|)}{n!} \int_S \cdots \int_S \lambda(u, \{x_1,\ldots,x_n\}) h(\{x_1,\ldots,x_n\})\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

- Approximations can be obtained using a Monte-Carlo approach or using a saddle-point approximation (very recent).

Notes

Position of the problem

- we observe a realization of X on $W = S$ ($|S| < \infty$; edge effects occur when $W \subset S$) of a parametric Gibbs point process with density which belongs to a parametric family of densities $(f_\theta = h_\theta / c_\theta)_{\theta \in \Theta}$ for $\Theta \subset \mathbb{R}^p$.
- Problem : estimate the parameter θ based on a single realization.
- *MLE approach* : the log-likelihood is $\boxed{\ell_W(x;\theta) = \log h_\theta - \log c_\theta}$.
  Pbm : Given a model h_θ can be computed but c_θ cannot be evaluated even for a single value of θ ; asymptotic properties are only partial.
  $\Rightarrow$ several solutions exist
  1. Approximate c_θ using a Monte-Carlo approach.
  2. Bayesian approach, importance sampling method (to estimate a ratio of normalizing constants).
  3. Combine the MLE with the Ogata-Tanemura approximation.
  4. Find another method which does not involve c_θ.

Notes

Pseudo-likelihood

- To avoid the computation of the normalizing constant, the idea is to compute a likelihood based on conditional densities

$$PL_W(x;\theta) = \exp(-|W|) \lim_{i\to\infty} \prod_{j=1}^{m_i} f(x_{A_{ij}} | x_{W \setminus A_{ij}};\theta)$$

  where $\{A_{ij} : j = 1, \ldots, m_i\}$ $i = 1, 2, \ldots$ are nested subdivisions of W.
- By letting $m_i \to \infty$ and $m_i \max |A_{ij}|^2 \to 0$ as $i \to \infty$ and taking the log, Jensen and Møller (91) obtained

$$LPL_W(x;\theta) = \sum_{u \in x_W} \lambda(u, x \setminus u; \theta) - \int_W \lambda(u, x; \theta) \mathrm{d}u$$

Notes

Comments on the Pseudo-likelihood

The MPLE is the estimate $\widehat{\theta}$ maximizing

$$LPL_W(x;\theta) = \sum_{u \in x_W} \log \lambda(u, x \setminus u; \theta) - \int_W \lambda(u, x; \theta) \mathrm{d}u$$

1. **Independent on c_θ,** so the LPL is up to an integral discretization and up to edge effects very to compute.
2. If X has a finite range R, then since x is observed in W, we can replace W by $W_{\ominus R}$ so that for instance $\lambda(u, x; \theta)$ can always be computed for any $u \in W_{\ominus R}$ **(border correction)**.
3. If $\log \lambda(u, x; \theta) = \theta^\top v(u, x)$ (exponential family - class of all examples presented before), then LPL is a **concave** function of θ.
4. under suitable conditions $\widehat{\theta}$ is a **consistent** estimate and satisfies a **CLT** (and a fast covariance estimate is available) as the window W expands to $\mathbb{R}^d$. [Jensen and Künsch'94, Billiot Coeurjolly and Drouilhet'08-'10, Coeurjolly and Rubak'12].

Notes

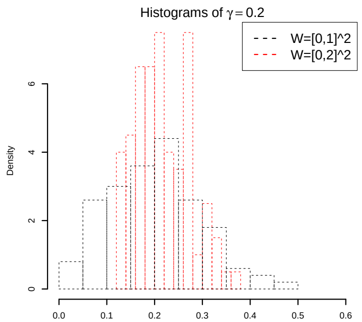
Simulation example

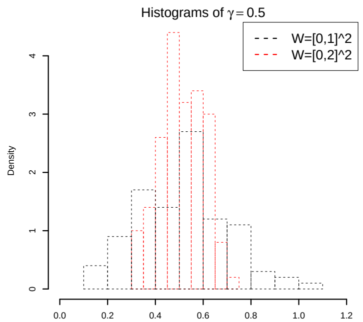
We generated 100 replications of Strauss point processes (a border correction was applied) :

1. mod1 : $\beta = 100$, $\gamma = 0.2$, $R = .05$.
2. mod2 : $\beta = 100$, $\gamma = 0.5$, $R = .05$.

	Estimates of β			
	$W = [0,1]^2$		$W = [0,2]^2$	
mod1	99.52	(17.84)	97.98	(9.24)
mod2	99.28	(20.48)	98.21	(8.53)

	Estimates of γ			
	$W = [0,1]^2$		$W = [0,2]^2$	
mod1	0.20	(0.09)	0.21	(0.06)
mod2	0.52	(0.19)	0.51	(0.09)





Notes

Takacs-Fiksel method

- Denote for any function h (eventually depending on θ)

$$L_W(X,h;\theta) = \sum_{u \in X_W} h(u, X \backslash u; \theta) \text{ and } R_W(X,h;\theta) = \int_W h(u,X;\theta)\lambda(u,X;\theta)\mathrm{d}u$$

- The GNZ formula states : $\mathrm{E}[L_W(X,h;\theta)] = \mathrm{E}[R_W(X,h;\theta)]$.
- **Idea** : if θ is a p-dimensional vector,
  1. choose p test function h_i and define the contrast

$$U_W(X,\theta) = \sum_{i=1}^{p} \left(L_W(X,h;\theta) - R_W(X,h;\theta)\right)^2.$$

  2. Define $\widehat{\theta}^{TF} = \mathrm{argmin}_\theta U_W(X,\theta)$.

Notes

Takacs-Fiksel (2)

General comments :

- **like the MPLE** :
  - independent of c_θ, border correction possible in case of X has a finite range
  - consistent and asymptotically Gaussian estimate (Coeurjolly et al.'12).
- **Another advantage** : interesting choices of test functions cal least to a decreasing of computation time.
  Ex : $h_i(u,X) = n(B(u,r_i))\lambda^{-1}(u,X;\theta) \Rightarrow R_W$ independent of θ.
- Actually : **MPLE = TFE** with $h = (h_1,\dots,h_p)^\top = \lambda^{(1)}(\cdot,\cdot;\theta)$.
  Indeed (assume $\log \lambda(u,X;\theta) = \theta^\top v(u,X)$ (for simplicity)

$$\nabla LPL_W(X;\theta) = \sum_{u \in X_W} v(u, X \setminus u) - \int_W v(u,X)\lambda(u,X;\theta)\mathrm{d}u.$$

Notes

A funny example for the Strauss point process

Recall that the Papangelou conditional intensity of a Strauss point process is

$$\lambda(u,X) = \beta\gamma^{t_R(u,X)} \text{ with } t_R(u,X) = \sum_{v\in X} \mathbf{1}(\|v-u\| \le R).$$

Choose $h_1(u,X) = \mathbf{1}(n(B(u,R)=0))$ and $h_2(u,X) = \mathbf{1}(n(B(u,R)=1))$, then

- $L_W(X,h_1) = L_1$ and $R_W(X,h_1) = \beta\int_W \mathbf{1}(n(B(u,R)=0)) = \beta I_1$.
- $L_W(X,h_2) = L_2$ and $R_W(X,h_2) = \beta\gamma\int_W \mathbf{1}(n(B(u,R)=1)) = \beta I_2$.

Then, the contrast function rewrites

$$U_W(X) = (L_1 - \beta I_1)^2 + (L_2 - \beta\gamma I_2)^2$$

which leads to the **explicit** solution

$$\widehat{\beta} = \frac{L_1}{I_1} \quad \text{and } \widehat{\gamma} = \frac{L_2}{I_2} \times \frac{I_1}{L_1}.$$

Notes

Complements

Other parametric approaches :

- Variational approach : (Baddeley and Dereudre'12).
- Method based on a logistic regression likelihood (Baddeley, Coeurjolly, Rubak, Waagepetersen'13).

Model fitting :

- Monte-Carlo approach : we can compare a summary statistic e.g. L with $L_{\widehat{\theta}}$.
  Pbm : L_θ not expressible in a closed form and must be approximated.
- We can still use the GNZ formula : given a test function h, we can construct

$$L_W(X,h;\widehat{\theta}) - R_W(X,h;\widehat{\theta}) =: \mathrm{Residuals(X,h)}.$$

If the model is correct, then $\mathrm{Residuals(X,h)}$ should be close to zero. (Baddeley et al.'05,08', Coeurjolly and Lavancier'12).

Notes

General Conclusion

The anaysis of **spatial point pattern**

- very large domain of research including probability, mathematical statistics, applied statistics
- own specific models, methodologies and software(s) to deal with.
- is involved in more and more applied fields : economy, biology, physics, hydrology, environmetrics,. . .

Still a lot of **challenges**

- Modelling : the “true model”, problems of existence, phase transition.
- Many classical statistical methodologies need to be adapted (and proved) to s.p.p. : robust methods, resampling techniques, multiple hypothesis testing.
- High-dimensional problems : $S = \mathbb{R}^d$ with d large, selection of variables, regularization methods,. . .
- Space-time point processes.

Notes

Notes