STIT Tessellations via Martingales

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Overview

- The model
- Martingales and STIT tessellations
- First applications
- Exact variance of the total surface area
- Exact variances for other intrinsic volumes
- Asymptotic expressions
- Limit theorems

Construction and Key Properties

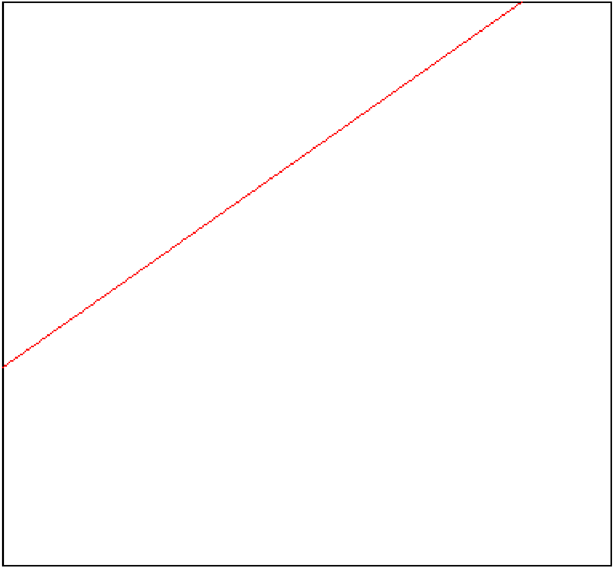
- We interpret a random tessellation as a special random closed set in $\mathbb{R}^d$.
- STIT tessellations $Y(t)$ formally arise as limits of rescaled iterations of tessellations.
- In bounded windows W there is an explicit construction of $Y(t, W)$ in terms of waiting times.
- There is also an 'explicit' global construction of STIT tessellations $Y(t)$ (Poisson point process on a complicated state space).
- Important for later considerations: These constructions allow an interpretation of $Y(t, W)$ (or $Y(t)$) as a Markov process on $(0, \infty)$ with values in the space of tessellations.

Waiting Time Construction

- Let $W \subset \mathbb{R}^d$ be a compact convex polytope and Λ a non-degenerate locally finite hyperplane measure.
- Assign to W an exponentially distributed lifetime with parameter $\Lambda([W])$.
- Upon expiry of this life time, a hyperplane is chosen according to $\Lambda([W])^{-1}\Lambda(\cdot \cap [W])$, is introduced in W and splits W into two polyhedral sub-cells W^+ and W^-.
- The construction is now continued recursively and independently in W^+ and W^- until some deterministic time threshold $t > 0$ is reached.
- Until time $t > 0$ there was constructed a random tessellation $Y(t, W)$ inside W with polyhedral cells.

Waiting Time Construction

Waiting Time Construction



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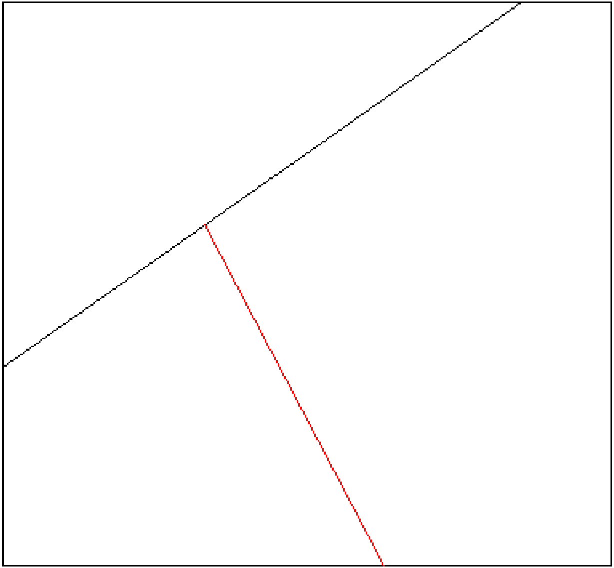

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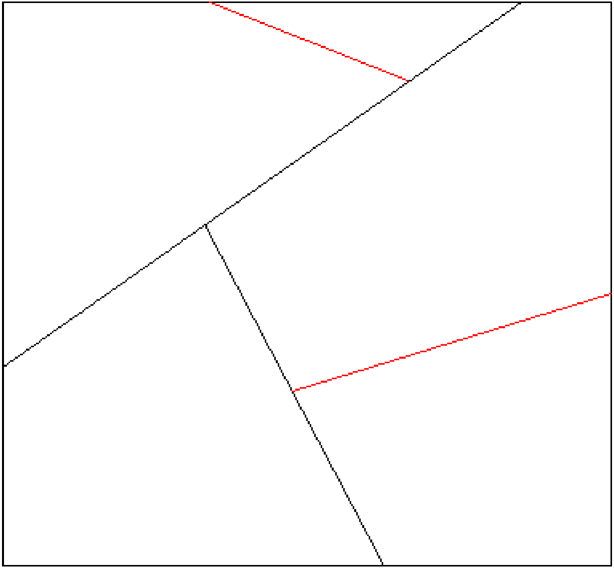

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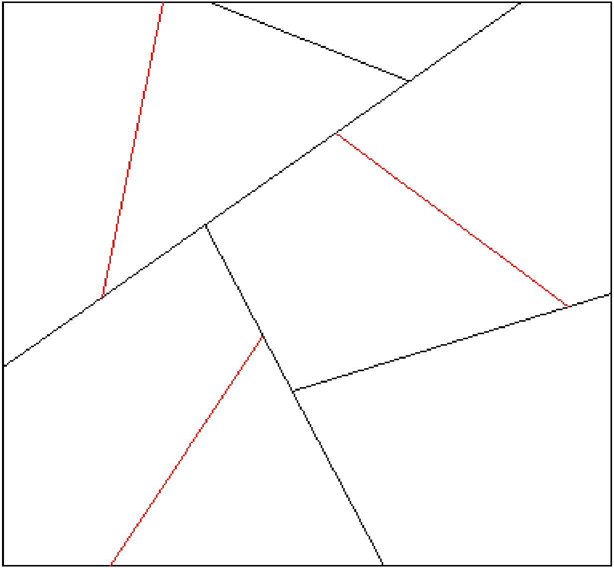

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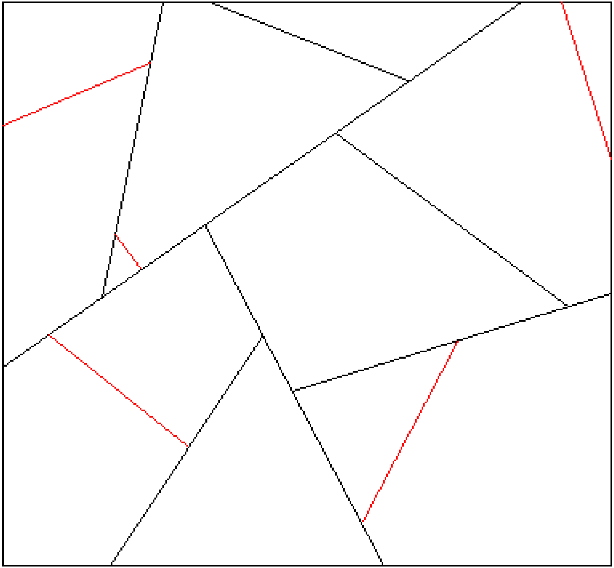

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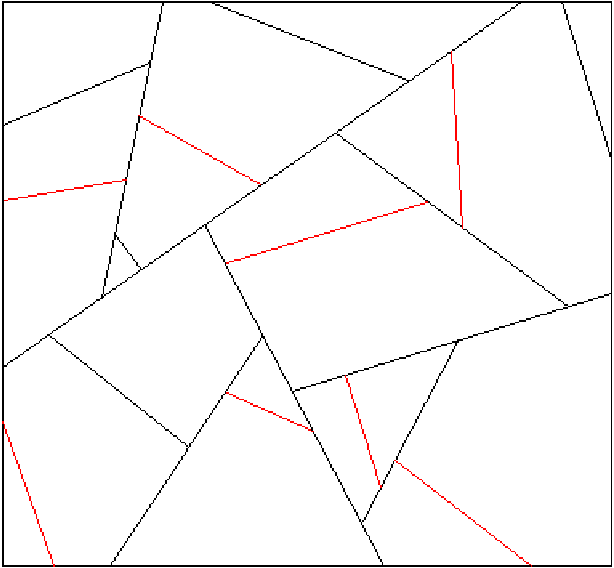
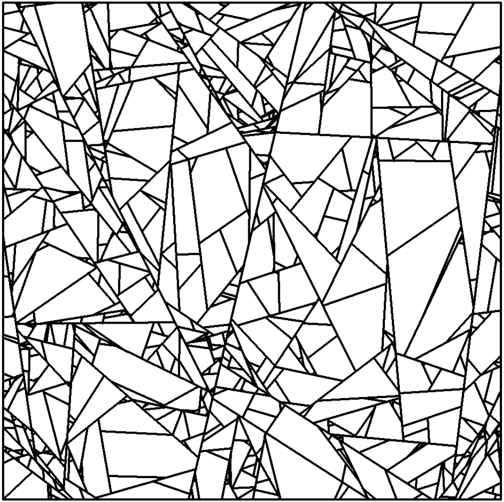


Isotropic random STIT Tessellation in 2D



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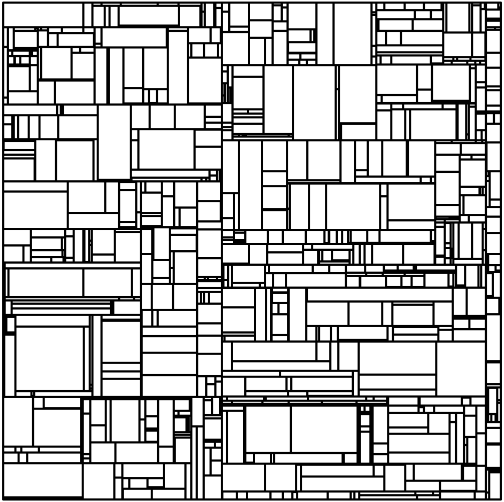
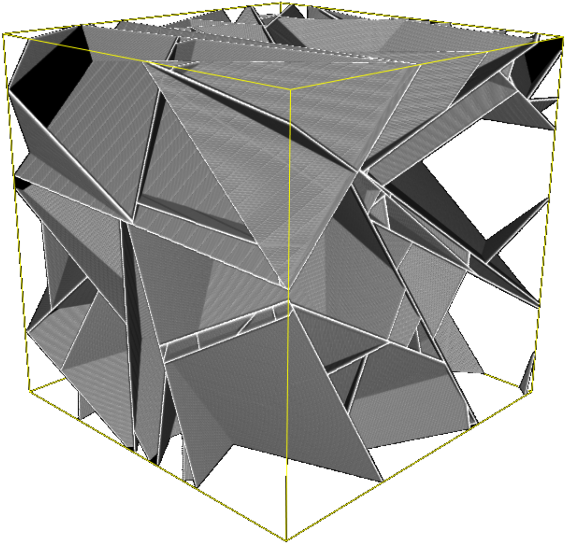
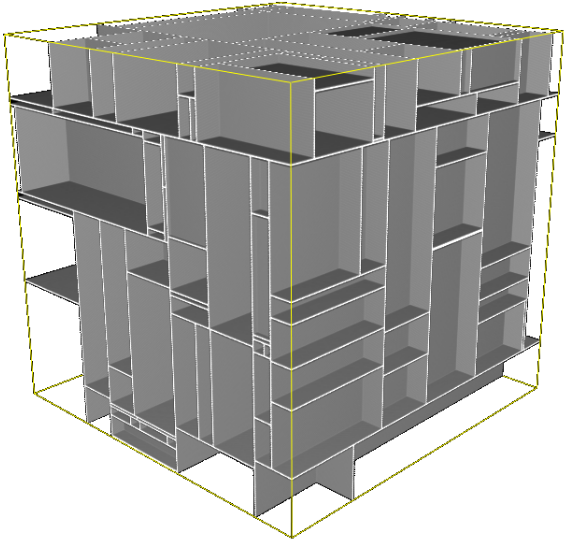


Isotropic random STIT Tessellation in 3D



Anisotropic random STIT Tessellation in 3D



Extension to $\mathbb{R}^d$

- We cannot start with a single hyperplane at time 0!
- But, Nagel und Weiß (2005) have shown that $Y(t, W)$ is spatially consistent, i.e for $V \subset W$ convex, we have $Y(t, V) \overset{D}{=} Y(t, W) \cap V$.
- Thus, by a consistency theorem there exists a random closed set $Y(t)$ in $\mathbb{R}^d$ with $Y(t) \cap W \overset{D}{=} Y(t, W)$.

Parameter and special cases

- The law of $Y(t)$ is characterized by t and Λ.
- In general, $Y(t)$ is neither stationary nor isotropic.
- If Λ is translation-invariant, then $Y(t)$ is stationary and thus a STIT tessellation.
- If Λ is the the isometry-invariant measure Λ_{iso}, then $Y(t)$ is stationary and isotropic.

Generator of $Y(t, W)$

The random process $Y(t, W)$ is a pure jump Markov process on the space of tessellations in W and its generator $\mathbb{L} := \mathbb{L}_{\Lambda;W}$ is given by:

$$\mathbb{L}F(Y) = \int_{[W]} \sum_{f \in Cells(Y \cap H)} \underbrace{[F(Y \cup \{f\}) - F(Y)]}_{\text{'add-one cost'}} \Lambda(dH),$$

with F measurable on the space of tessellation in W for which the integral is well defined.

A derived Martingale

From the standard theory (Dynkin's formula) it follows that the following real-valued process is a martingale with respect to the natural filtration of $Y(t, W)$:

$$F(Y(t, W)) - \int_0^t \mathbb{L}F(Y(s, W))ds$$

for $F \in \mathcal{D}(\mathbb{L})$.

A more special case

Denote by $\text{MaxFacets}(Y)$ the set of maximal facets of the tessellation Y. We regard now the special function

$$\Sigma_\phi(Y) := \sum_{f \in \text{MaxFacets}(Y)} \phi(f)$$

with ϕ measurable and bounded on the space of $(d-1)$-polytopes in W. Applications of the general result from the last slide shows that

$$\Sigma_\phi(Y(t,W)) - \int_0^t \int_{[W]} \sum_{f \in \text{Cells}(Y \cap H)} \phi(f) \Lambda(dH) ds$$

is a martingale.

First Consequences

Consider a STIT tessellation $Y(t,W)=Y(t\Lambda,W)$ and the measures

$$\mathcal{M}^{Y(t,W)}:=\sum_{c\in\mathsf{Cells}(Y(t,W))}\delta_c,\ \ \mathbb{M}^{Y(t,W)}:=\mathbb{E}\mathcal{M}^{Y(t,W)}$$

and likewise $\mathcal{M}^{\mathsf{PHT}(t\Lambda,W)}$ and $\mathbb{M}^{\mathsf{PHT}(t\Lambda,W)}$.

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and likewise $\mathcal{M}^{\mathrm{PHT}(t\Lambda,W)}$ and $\mathbb{M}^{\mathrm{PHT}(t\Lambda,W)}$. Define further

$$\mathcal{F}_k^{Y(t,W)} := \sum_{f \in \mathrm{MaxFaces}_k(Y(t,W))} \delta_f, \quad \mathbb{F}_k^{Y(t,W)} := \mathbb{E}\mathcal{F}_k^{Y(t,W)},$$

$k = 1, \ldots, d-1$, and

$$\mathcal{F}_k^{\mathrm{PHT}(t\Lambda,W)} := \sum_{f \in \mathrm{Faces}_k(\mathrm{PHT}(t\Lambda,W))} \delta_f, \quad \mathbb{F}_k^{\mathrm{PHT}(t\Lambda,W)} := \mathbb{E}\mathcal{F}_k^{\mathrm{PHT}(t\Lambda,W)}$$

for $k = 1, \ldots, d-1,$.

First Consequences

Theorem: It holds

$$\mathbb{M}^{Y(t,W)} = \mathbb{M}^{\mathrm{PHT}(t\Lambda,W)}$$

and

$$\mathbb{F}_k^{Y(t,W)} = (d-k)2^{d-k-1}\int_0^t \frac{1}{s}\mathbb{F}_k^{\mathrm{PHT}(s\Lambda,W)}ds$$

for $k = 1, \dots, d-1$.

First Consequences

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for $k = 1,\dots,d-1$.

- This means that STITs and PHTs have the same typical cell distribution, but
- the spatial arrangement of the cells is different in both tessellation models.

First Consequences

Theorem: It holds

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for $k = 1, \ldots, d-1$.

Idea of the proof:

(1) uses a uniqueness theorem for the solution of a certain operator differential equation on a space of measures together with a derived martingale.

(2) uses a derived martingale, part (1), Slivnyak's theory for Poisson point process, scaling properties and adjacency relationships for Poisson hyperplane tessellations.

Application: Typical k-dimensional maximal faces

- Let us consider the isotropic case, i.e. $\Lambda = \Lambda_{iso}$ and $Y(t) = Y(t\Lambda_{iso})$.
- Fix a measurable and translation-invariant $\varphi_k : \mathrm{MaxFaces}_k \to \mathbb{R}$, $\varphi \geq 0$.
- Define the φ_k-density of $Y(t)$:

$$\overline{\varphi}_k(Y(t)) = \lim_{r\to\infty} \frac{1}{r^d \,\mathrm{Vol}_d(W)} \mathbb{E} \sum_{f \in \mathrm{MaxFaces}_k(Y(t,rW))} \varphi_k(f).$$

- Apply now the theory from above.

Application: Typical k-dimensional maximal faces

- We have

$$\overline{\varphi}_k(Y(t)) = (d-k)2^{d-k-1}\int_0^t \frac{1}{s}\overline{\varphi}_k(\mathrm{PHT}(s))ds.$$

- For $\varphi_k \equiv 1$ this yields

$$\begin{aligned} N_k &= (d-k)2^{d-k-1}\int_0^t \frac{1}{s}\binom{d}{k}\kappa_d\left(\frac{\kappa_{d-1}}{d\kappa_d}\right)^d s^d ds \\ &= (d-k)2^{d-k-1}\frac{\kappa_d}{d}\binom{d}{k}\left(\frac{\kappa_{d-1}}{d\kappa_d}\right)^d t^d. \end{aligned}$$

Application: Typical k-dimensional maximal faces

- Now the same procedure for $\varphi_k(f) := \mathbf{1}[\cdot](f - c(f))$ with some center function c, $f \in \mathrm{MaxFaces}_k(Y(t))$.
- We get

$$N_k \mathbb{Q}_k^{Y(t)} = (d-k)2^{d-k-1} \int_0^t \frac{1}{s} N_k^{\mathrm{PHT}(s)} \mathbb{Q}_k^{\mathrm{PHT}(s)} ds$$

(the distribution of the 'naked' polytopes).

- Inserting the values for N_k and $N_k^{\mathrm{PHT}(s)}$ gives

$$\mathbb{Q}_k^{Y(t)} = \int_0^t \frac{ds^{d-1}}{t^d} \mathbb{Q}_k^{\mathrm{PHT}(s)} ds.$$

The mixing distribution is a beta-distribution on $(0, t)$ with parameters d and 1.

Application: Typical maximal segment

Let $p_l^{(d)}(x)$ be the length density of the typical maximal Segment in $\mathbb{R}^d$. Then

- $p_l^{(d)}(x) = \int_0^t \lambda_1 s e^{-\lambda_1 s x} \frac{ds^{d-1}}{t^d} ds = \frac{d}{(\lambda_1 t)^d x^{d+1}} \gamma(d+1, \lambda_1 t x)$ with

$$\lambda_1 = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}.$$

- $d = 2$: $\frac{1}{t^2 x^3}\left(\pi^2 - (\pi^2 + 2\pi t x + 2t^2 x^2) e^{-\frac{2}{\pi} t x}\right)$, mean $\frac{\pi}{t}$, no higher moments exist.
- $d = 3$: $\frac{3}{t^3 x^4}\left(48 - (48 + 24 t x + 6 t^2 x^2 + t^3 x^3) e^{-\frac{1}{2} t x}\right)$, mean $\frac{3}{t}$, variance $\frac{24}{t^2}$, no higher moments exist.

Application: Typical k-dimensional maximal faces

For the mean j-th intrinsic volume of the typical k-dimensional maximal face, $0 \leq j \leq k$, we have

$$\mathbb{E}V_j(I_k) = \frac{d}{(d-j)\kappa_j}\binom{k}{j}\left(\frac{2\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^j \frac{1}{t^j}.$$

Next goal

The next goal is to use the martingale approach to study second-order properties of the STIT tessellations, such as variances and covariances.

Another derived Martingale - Notation

- Let ϕ be bounded and measurable on the space of $(d-1)$-polytopes.
- Define

$$\Sigma_\phi(Y(t,W)) := \sum_{f \in \mathrm{MaxFacets}(Y(t,W))} \phi(f)$$

and

$$A_\phi(Y(t,W)) := \int_{[W]} \sum_{f \in \mathrm{Cells}(Y(t,W) \cap H)} \phi(f) \Lambda(dH)$$

and put

$$\bar{\Sigma}_\phi(Y(t,W)) := \Sigma_\phi(Y(t,W)) - \mathbb{E}\Sigma_\phi(Y(t,W)),$$

$$\bar{A}_\phi(Y(t,W)) := A_\phi(t,W) - \mathbb{E}A_\phi(t,W).$$

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A 'second-order' martingale

The random process

$$\bar{\Sigma}^2_\phi(Y(t,W)) - \int_0^t A_{\phi^2}(Y(s,W))ds - 2\int_0^t \bar{A}_\phi(Y(s,W))\bar{\Sigma}_\phi(Y(s,W))ds$$

is a martingale wrt. filtration induced by $(Y(t,W))_{t>0}$.

Variance of the Total Surface Area

A 'second-order' martingale

The random process

$$\bar{\Sigma}_{\phi}^{2}(Y(t,W)) - \int_0^t A_{\phi^2}(Y(s,W))ds - 2\int_0^t \bar{A}_{\phi}(Y(s,W))\bar{\Sigma}_{\phi}(Y(s,W))ds$$

is a martingale wrt. filtration induced by $(Y(t,W))_{t>0}$.

- Take $\phi(f) := \mathrm{Vol}_{d-1}(f)$.
- Then, $A_{\mathrm{Vol}_{d-1}}(Y(\cdot,W))$ is constant, thus $\bar{A}_{\mathrm{Vol}_{d-1}} \equiv 0$.
- Taking expectations in the above formula, we get

$$\mathrm{Var}\,\mathrm{Vol}_{d-1}(Y(t,W)) = \int_0^t \mathbb{E}A_{\mathrm{Vol}_{d-1}^2}(Y(s,W))ds.$$

- It remains to calculate $\mathbb{E}A_{\mathrm{Vol}_{d-1}^2}(Y(s,W))$.

Calculation of $\mathbb{E}A_{\mathrm{Vol}_{d-1}^2}(Y(s,W))$

$$
\begin{aligned}
&\mathbb{E}A_{\mathrm{Vol}_{d-1}^2}(Y(s,W)) \\
= \; & \mathbb{E}\int_{[W]} \sum_{f\in\mathrm{Cells}(Y(s,W)\cap H)} \mathrm{Vol}_{d-1}^2(f)\Lambda(dH) \\
= \; & \mathbb{E}\int_{[W]}\int_{W\cap H}\int_{W\cap H} \mathbf{1}[x,y \text{ in the same cell of } Y(s,W)\cap H]dxdy\Lambda(dH) \\
= \; & \int_{[W]}\int_{W\cap H}\int_{W\cap H} e^{-s\Lambda([\overline{xy}])}dxdy\Lambda(dH),
\end{aligned}
$$

since STIT tessellations have Poisson typical cells and $Y(s)\cap H$ is also a STIT tessellation.

Variance of the Total Surface Area

Putting together

$$\mathrm{Var}\,\mathrm{Vol}_{d-1}(Y(t,W)) = \int_0^t \mathbb{E}A_{\mathrm{Vol}_{d-1}^2}(Y(s,W))ds$$

and

$$\mathbb{E}A_{\mathrm{Vol}_{d-1}^2}(Y(s,W)) = \int_{[W]}\int_{W\cap H}\int_{W\cap H} e^{-s\Lambda([\overline{xy}])}dxdy\Lambda(dH),$$

we get by integration

$$\mathrm{Var}\,\mathrm{Vol}_{d-1}(Y(t,W)) = \int_{[W]}\int_{W\cap H}\int_{W\cap H} \frac{1-e^{-t\Lambda([\overline{xy}])}}{\Lambda([\overline{xy}])}dxdy\Lambda(dH).$$

Variance of the Total Surface Area

- Assume that $\Lambda = \Lambda_{iso}$.
- Then, the affine Blaschke-Petkantschin formula implies

$$\int_{[W]} \int_{W \cap H} \int_{W \cap H} g(x, y) dx dy \Lambda_{iso}(dH)$$

$$= \frac{(d-1)\kappa_{d-1}}{d\kappa_d} \int_W \int_W \frac{g(x, y)}{\|x - y\|} dx dy.$$

- Take now

$$g(x, y) = \frac{1 - \exp(-t\Lambda_{iso}([\overline{xy}]))}{\Lambda_{iso}([\overline{xy}])} = \frac{1 - e^{-\frac{2\kappa_{d-1}}{d\kappa_d} t\|x-y\|}}{\frac{2\kappa_{d-1}}{d\kappa_d} \|x - y\|}.$$

Variance of the Total Surface Area

Theorem (Schreiber and T. 2010)

For a stationary and isotropic STIT tessellation $Y(t)$ we have

$$\begin{aligned}
& \mathrm{Var}(\mathrm{Vol}_{d-1}(Y(t,W))) \\
= & \frac{d-1}{2}\int_W\int_W \frac{1-e^{-\frac{2\kappa_{d-1}}{d\kappa_d}t\|x-y\|}}{\|x-y\|^2}dxdy \\
= & \frac{d(d-1)\kappa_d}{2}\int_0^\infty \overline{\gamma}_W(r)r^{d-3}\left(1-e^{-\frac{2\kappa_{d-1}}{d\kappa_d}tr}\right)dr.
\end{aligned}$$

Variance of the Total Surface Area

Theorem (Schreiber and T. 2010)

For a stationary and isotropic STIT tessellation $Y(t)$ we have

$$\mathrm{Var}(\mathrm{Vol}_{d-1}(Y(t,W))) = \frac{d(d-1)\kappa_d}{2}\int_0^\infty \overline{\gamma}_W(r) r^{d-3}\left(1 - e^{-\frac{2\kappa_{d-1}}{d\kappa_d}tr}\right) dr.$$

Corollary (Weiß, Ohser, Nagel, $d = 2$ and Schreiber and T. in general)

The pair-correlation function $g_d(r)$ of the random surface measure of a stationary and isotropic $Y(t)$ is given by

$$g_d(r) = 1 + \frac{d-1}{2t^2r^2}\left(1 - e^{-\frac{2\kappa_{d-1}}{d\kappa_d}tr}\right).$$

Variances for Intrinsic Volumes

- We consider only the stationary and isotropic case $\Lambda = \Lambda_{iso}$.
- Our aim is to calculate $\operatorname{Var} \Sigma_{V_j}(Y(t, W))$, where, recall,

$$\Sigma_{V_j}(Y(t, W)) = \sum_{f \in \text{MaxFacets}(Y(t,W))} V_j(f).$$

Variances for Intrinsic Volumes

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$$\Sigma_{V_j}(Y(t,W)) = \sum_{f \in \mathrm{MaxFacets}(Y(t,W))} V_j(f).$$

- However, we start with

$$F_j(Y(t,W)) := \sum_{c \in \mathrm{Cells}(Y(t,W))} V_j(c).$$

- If f splits c into c^+ and c^-, we have

$$F_j(Y(t,W) \cup \{f\}) - F_j(Y(t,W)) = V_j(c^+) + V_j(c^-) - V_j(c) = V_j(f),$$

thus $F_j(Y(t,W)) = \Sigma_{V_j}(Y(t,W)) + V_j(W)$.

Variances for Intrinsic Volumes

- Recall, that $A_{V_j}(Y(t,W)) = \int_{[W]} \sum_{f \in \text{Cells}(Y(t,W)\cap H)} V_j(f)\Lambda_{iso}(dH)$.
- Crofton's formula and $F_j(Y(t,W)) = \Sigma_{V_j}(Y(t,W)) + V_j(W)$ implies now

$$\begin{aligned} A_{V_j}(Y(t,W)) &= \sum_{c \in \text{Cells}(Y(t,W))} \int_{[W]} V_j(c \cap H)\Lambda_{iso}(dH) \\ &= \sum_{c \in \text{Cells}(Y(t,W))} \gamma_{j+1} V_{j+1}(c) \\ &= \gamma_{j+1} F_{j+1}(Y(t,W)) \\ &= \gamma_{j+1}(\Sigma_{V_{j+1}}(Y(t,W)) + V_{j+1}(W)), \end{aligned}$$

thus

$$\bar{A}_{V_j}(Y(t,W)) = \gamma_{j+1} \bar{\Sigma}_{V_{j+1}(Y(t,W))}.$$

Another Martingale

- Denote $\Sigma_{\phi;t} := \Sigma_{\phi}(Y(t,W))$.
- By considering once again the time-augmented process $(Y(t,W),t)$ and Dynkin's formula one shows that

$$\bar{\Sigma}_{V_i;t}\bar{\Sigma}_{V_j;t} - \int_0^t A_{V_iV_j;s}ds - \int_0^t [\bar{A}_{V_i;s}\bar{\Sigma}_{V_j;s} + \bar{A}_{V_j;s}\bar{\Sigma}_{V_i;s}]ds$$

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  is a martingale wrt. filtration induced by $(Y(t,W))_{t>0}$.
- Using that $\bar{A}_{V_j}(Y(t,W)) = \gamma_{j+1}\bar{\Sigma}_{V_{j+1}(Y(t,W))}$, also

$$\bar{\Sigma}_{V_i;t}\bar{\Sigma}_{V_j;t} - \int_0^t A_{V_iV_j;s}ds - \int_0^t [\gamma_{i+1}\bar{\Sigma}_{V_{i+1};s}\bar{\Sigma}_{V_j;s} + \gamma_{j+1}\bar{\Sigma}_{V_i;s}\bar{\Sigma}_{V_{j+1};s}]ds$$

  is a martingale as well.

The Recursion Formula

- Using that $\bar{A}_{V_j}(Y(t,W)) = \gamma_{j+1}\bar{\Sigma}_{V_{j+1}(Y(t,W))}$, also

$$\bar{\Sigma}_{V_i;t}\bar{\Sigma}_{V_j;t} - \int_0^t A_{V_iV_j;s}ds - \int_0^t [\gamma_{i+1}\bar{\Sigma}_{V_{i+1};s}\bar{\Sigma}_{V_j;s} + \gamma_{j+1}\bar{\Sigma}_{V_i;s}\bar{\Sigma}_{V_{j+1};s}]ds$$

  is a martingale as well.
- Taking expectation, we get the recursion formula

$$\mathrm{Cov}(\Sigma_{V_i;t}, \Sigma_{V_j;t}) = \int_0^t \mathbb{E}A_{V_iV_j;s}ds$$

$$+ \int_0^t [\gamma_{i+1}\,\mathrm{Cov}(\Sigma_{V_{i+1};s}, \Sigma_{V_j;s}) + \gamma_{j+1}\,\mathrm{Cov}(\Sigma_{V_i;s}, \Sigma_{V_{j+1};s})]ds.$$

- The recursion terminates, since $\Sigma_{V_d;t} \equiv 0$, which allows an explicit expression for all $\mathrm{Cov}(V_{i;t}, V_{j;t})$.
- Recall that $\mathrm{Var}\,\Sigma_{V_{d-1};t} = \int_0^t \mathbb{E}A_{V_{d-1}^2;s}ds$.

Exact Expression

$$\mathcal{I}^n(f;t) := \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} f(s_n) ds_n ds_{n-1} \cdots ds_1$$
$$= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds.$$

Theorem (Schreiber and T. 2010)

The covariance between $\Sigma_{V_{d-1-k};t}$ and $\Sigma_{V_{d-1-l};t}$ for $k, l \in \{0, \ldots, d-1\}$ of a stationary and isotropic random STIT tessellation $Y(t, W)$ is given by

$$\mathrm{Cov}(\Sigma_{V_{d-1-k};t}, \Sigma_{V_{d-1-l};t}) = \sum_{m=0}^{k} \sum_{n=0}^{l} \binom{k+l-m-n}{k-m}$$

$$\left(\prod_{i=m+1}^{k} \gamma_{d-i} \right) \left(\prod_{j=n+1}^{l} \gamma_{d-j} \right) \mathcal{I}^{k+l-m-n+1}(\mathbb{E}A_{V_{d-1-m}V_{d-1-n};(\cdot)}; t).$$

Asyptotic Expressions: The Problem

- Take a convex body $W \subset \mathbb{R}^d$ and consider the sequence $W_R := R \cdot W$ as $R \to \infty$.
- How does $\operatorname{Var} \Sigma_{V_i}(Y(t, W_R))$ behave, as $R \to \infty$?
- Which terms dominate the expression?
- How is the formula influenced by the geometry of W?
- The answers for $d = 2$ and $d \geq 3$ are different!
- For simplicity we consider only $d \geq 3$.

Asymptotics

Proposition

For $k, l, m, n \in \{0, \ldots, d-1\}$, with t fixed we have with $p := k + l - m - n + 1$

$$\mathcal{I}^p(\mathbb{E}A^{W_R}_{V_{d-1-m}V_{d-1-n};(\cdot)}; t) = \begin{cases} O(R^{2(d-1)-m-n}), & \text{if } m+n \leq d-3, \\ O(R^d \log R), & \text{if } m+n = d-2, \\ O(R^{2d-1-m-n}), & \text{if } m+n \geq d-1. \end{cases}$$

Asymptotic picture

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Corollary

The asymptotic covariance $\mathrm{Cov}(\Sigma_{V_{d-1-k};t}, \Sigma_{V_{d-1-l};t})$ is dominated by the term $n = m = 0$, i.e. by

$$\mathcal{I}^{k+l+1}(\mathbb{E}A^{W_R}_{\mathrm{Vol}^2_{d-1}}; t) = \mathcal{I}^{k+l}(\mathrm{Var}\,\Sigma^{W_R}_{V_{d-1}}).$$

Influence of W

- The geometry of W is for $d \geq 3$ reflected by either one of the (non-additive) functionals

$$E_2(W) = \int_W \int_W \frac{dxdy}{\|x-y\|^2}$$

or

$$I_{d-1}(W) = \frac{d\kappa_d}{2} \int_{\mathcal{L}} \mathrm{Vol}_1^{d-1}(W \cap L) dL,$$

since

$$E_2(W) = \frac{2}{(d-1)(d-2)} I_{d-1}(W).$$

- For $d = 2$ only the area $\mathrm{Vol}_2(W)$ matters, but not the precise shape of W. But here some logarithm enters the discussion ...

The Asymptotic result

Theorem (Schreiber and T. 2010)

Asymptotically, as $R \to \infty$, we have for $k, l \in \{0, \dots, d-1\}$

$$\mathrm{Cov}(\Sigma^{W_R}_{V_{d-1-k};t}, \Sigma^{W_R}_{V_{d-1-l};t}) =$$

$$\frac{1}{d-2}\left(\prod_{i=1}^{k}\gamma_{d-i}\right)\left(\prod_{j=1}^{l}\gamma_{d-j}\right)\frac{t^{k+l}}{k!l!} I_{d-1}(W) R^{2(d-1)} + O(R^{2d-3})$$

and for $k = l$

$$\mathrm{Var}(\Sigma^{W_R}_{V_{d-1-k};t}) = \frac{1}{d-2}\left(\prod_{i=1}^{k}\gamma_{d-i}\right)^2 \frac{t^{2k}}{(k!)^2} I_{d-1}(W) R^{2(d-1)} + O(R^{2d-3}).$$

Limit theorems

We restrict ourself from now on to the case of the total surface area, but limit theorems for other functionals are also available.
We approach the problem in two different (but closely related) settings:

- We can consider the surface area constructed by facets born after an initial time instant $s_0 > 0$.
- It is also natural to ask for the surface area constructed by all facets.

Interestingly both approaches lead to results of very different qualitative nature!

First limit theorem

Theorem (Schreiber and T. 2010)

For each $s_0 > 0$, the random variable

$$\frac{1}{R^{d/2}}[(\mathrm{Vol}_{d-1}(Y(1,W_R)) - \mathbb{E}\mathrm{Vol}_{d-1}(Y(1,W_R)))$$
$$-(\mathrm{Vol}_{d-1}(Y(s_0,W_R)) - \mathbb{E}\mathrm{Vol}_{d-1}(Y(s_0,W_R)))]$$

converges, as $R \to \infty$, in law to $\mathcal{N}(0, V_W(\mathrm{Vol}_{d-1},\Lambda)\int_{s_0}^1 s^{1-d}ds)$, a normal distribution with mean 0 and variance $V_W(\mathrm{Vol}_{d-1},\Lambda)\int_{s_0}^1 s^{1-d}ds$, where $V_W(\mathrm{Vol}_{d-1},\Lambda)$ is explicitly known.

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Idea of the proof:

- Use STIT scaling and ergodicity to calculate the asymptotic variance.
- Check the assumptions of the martingale CLT for cadlag martingales.

Planar CLT

Theorem (Schreiber and T. 2010)

We have for the stationary and isotropic STIT tessellation $Y(1)$ in the plane

$$\frac{1}{R\sqrt{\log R}}[\mathrm{Vol}_1(Y(1,W_R)) - \mathbb{E}\mathrm{Vol}_1(Y(1,W_R))] \Longrightarrow \mathcal{N}(0,\pi\,\mathrm{Vol}_2(W)),$$

where $\Longrightarrow$ means convergence in law.

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where $\Longrightarrow$ means convergence in law.

Idea of the proof:

- Use a space-time scaling to construct an asymptotically equivalent martingale with good properties.
- Check again the assumptions of the martingale CLT for cadlag martingales.

Higher dimensions

Theorem (Schreiber and T. 2010)

For $d > 2$, $\Lambda = \sum_{i=1}^{d} \int_{-\infty}^{+\infty} \delta_{re_i + e_i^{\perp}} dr$ and $W = [0,1]^d$,

$$R^{2(d-1)}[\mathrm{Vol}_{d-1}(1, W_R) - \mathbb{E}\mathrm{Vol}_{d-1}(1, W_R)]$$

converges, as $R \to \infty$, to a non-Gaussian square-integrable random variable $\Xi(W)$ with explicitly known variance.

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Idea of the proof:

- Show by a conditioning argument that $\Xi(W)$ satisfies

$$\mathbb{P}(\Xi(W) > \xi) = \exp(-\Theta(\xi \log \xi)),$$

whereas a Gaussian random variable has tails of order $\exp(-\Theta(\xi^2))$.

Why this difference?

- Overall variance is of order $R^{2(d-1)}$ for $d > 2$ and of order $R^2 \log R$ for $d = 2$.
- The variance contribution after the initial time instant $s_0 > 0$ is of order R^d for all d.
- Thus, for $d = 2$ the initial segments contribute a negligible amount to the total variance.
- But, in higher dimensions $d > 2$, the *big bang* phase is crucial and dominates the scenery.

Thank you for your attention!