Realisability problems

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Expected functionals

- ξ is a random element in space $\mathcal{X}$
- Distribution of ξ is uniquely specified by

$$\mathbf{E}g(\xi)$$

for all bounded continuous functions g

- $g \mapsto \Phi(g) = \mathbf{E}g(\xi)$ is a linear functional
- What if $\Phi(g)$ is given only for some functions g?

- Then the distribution of ξ is not uniquely specified.
- But does there exist at least one ξ?

Positivity

- If $g \geq 0$, then $\Phi(g) = \mathbf{E}g(\xi) \geq 0$.

- Idea: positively extend Φ from functions $g \in \mathsf{G}$ to a wider family of functions $v \in \mathsf{E}$ such that E is a vector lattice.

- Daniell's theorem says that a positive linear upper semicontinuous normalised functional on vector lattice E is the Lebesgue integral w.r.t. probability measure, i.e. the expectation.

Trivial example

- ξ in a random variable in $\mathbb{R}$
- Known $m_i = \Phi(g_i) = \mathbf{E} g_i(\xi) = \mathbf{E}\xi^i$ for $i = 1, 2$
- By linearity, Φ is defined on $g(x) = c + 2bx + x^2$ and $g \geq 0$ if $c \geq b^2$. Thus

$$\Phi(g) = c + 2bm_1 + m_2 \geq (b + m_1)^2 + m_2 - m_1^2 \geq 0$$

only in case $m_2 - m_1^2 \geq 0$.

Classical example (Kellerer 1964)

- Let $\mathcal{I}$ be a certain family of subsets $I \subset \{1, \dots, d\}$

- Given probability distributions $\mathbf{P}_I$ on $\mathbb{R}^{\mathrm{card}(I)}$, does there exist a probability distribution on $\mathbb{R}^d$ whose marginals are $\mathbf{P}_I$, $I \in \mathcal{I}$.

Solution

- Need existence of a probability measure $\mathbf{P}$ on $\mathbb{R}^d$ such that

$$\Phi(v_I) = \int v_I(x) d\mathbf{P}(x) = \int v_I(x) d\mathbf{P}_I(x)$$

for all functions v_I that depend on the coordinates with numbers from I.

- Extend Φ from functions of the type v_I and their linear combinations to all functions v on $\mathbb{R}^d$.

- The functional Φ is positive on G iff

$$v(x) = \sum_{I \in \mathcal{I}} a_i v_I(x) \geq 0 \text{ for all } x \quad \Rightarrow \quad \Phi(v) = \sum_{I \in \mathcal{I}} a_i \int v_I(x) d\mathbf{P}_I(x) \geq 0 \,.$$

Random closed sets

- X is a random closed set in space $\mathbb{X} = \mathbb{R}^d$

- The distribution is defined by the capacity functional

$$T(K) = \mathbf{P}\{X \cap K \neq \emptyset\}, \quad K \in \mathcal{K}.$$

- A functional T on the family $\mathcal{K}$ of compact sets is the capacity functional of a random closed set X if and only if T is upper semicontinuous and completely alternating.

- Now assume that T is defined only on some compact sets.

- If the family of test sets is sufficiently rich (separating), no problems.

All finite sets

Theorem. *A completely alternating functional* T *defined on the family of all finite sets is the capacity functional of a random closed set if and only if* T *is upper semicontinuous on finite sets, i.e.*

$$\limsup T(K_n) \leq T(K)$$

for each sequence K_n *of finite sets that converges to a finite set* K *in the Hausdorff metric.*

Singletons

One-point covering function

$$p_x = T(\{x\}) = \mathbf{P}\{x \in X\}, \quad x \in \mathbb{X}\,.$$

Theorem. *A function p_x, $x \in \mathbb{X}$, with values in $[0,1]$ is the one-point covering function of a random closed set if and only if p is upper semicontinuous.*

Proof. Let $x_n \to x$ and let $\bar{U}_n$ be the closure of an open neighbourhood U_n of x that shrinks to x as $n \to \infty$. The upper semicontinuity and monotonicity of T yield that

$$\limsup p_{x_n} \le \limsup T(\bar{U}_n) \le T(\{x\}) = p_x \,,$$

that is p_x, $x \in \mathbb{X}$, is upper semicontinuous.

In the other direction, consider a random variable υ uniformly distributed on $[0, 1]$. Then

$$X = \{x : \; p_x \ge \upsilon\}$$

is closed by the upper semicontinuity of p and $\mathbf{P}\{x \in X\} = \mathbf{P}\{\upsilon \le \Phi(x)\} = p_x$ for all x. □

Two-point covering function

- Two-point covering function of a random closed set X is

$$p_{x_1,x_2} = \mathbf{P}\{x_1, x_2 \in X\}, \quad x_1, x_2 \in \mathbb{X}.$$

- If X is stationary, then p_{x_1,x_2} depends only on $x_1 - x_2$.

- Problem: Characterise two-point covering functions of random closed sets, i.e. covariance functions of (stationary) upper semicontinuous indicator functions.

- Linear space G generated by constants and $g_{x,y}(F) = \mathbb{I}_{x,y\in F}$, $x, y \in \mathbb{X}$. Note that $\min(g_{x,y}, g_{z,w})$ does not belong to G. $\Phi : \mathsf{G} \mapsto \mathbb{R}$ is determined by $\Phi(g_{x,y}) = p_{x,y}$, i.e. two-point covering probabilities.

- Φ is positive if and only if

$$\sum_{ij=1}^{n} a_{ij}\, \mathbb{I}_{x_i,x_j\in F} \quad \Rightarrow \quad \sum_{ij=1}^{n} a_{ij} p_{x_i,x_j} \geq 0\,.$$

- Equivalently, $\sum_{ij=1}^{n} a_{ij} p_{x_i,x_j} \geq 0$ for all corner-positive matrices (a_{ij}) (nonnegative sums of all main minors), McMillan (1955), Shepp (1967)

- If $a_{ij} = a_i a_j$, we recover the positive-definiteness of $p_{x,y}$ (only necessary condition).

Example (closedness condition)

- Let $p_{x,y} = \frac{1}{4}$ and let $p_x = \frac{1}{2}$ for all points $x, y \in \mathbb{R}$.

- While this two-point covering function corresponds, e.g., to the indicator field with independent values, it cannot be obtained as the covering function of a random closed set.

Covariances in the product form

Theorem. *Assume that* $\mathbb{X}$ *is a separable space. A function*

$$p_{x_1,x_2} = \begin{cases} p_{x_1}p_{x_2} & \text{if } x_1 \neq x_2\,, \\ p_{x_1} & \text{if } x_1 = x_2 \end{cases}$$

is a two-point covering function of a random closed set if and only if p_x*,* $x \in \mathbb{X}$*, is an upper semicontinuous function with values in* $[0,1]$ *such that each point* x *with* $p_x \in (0,1)$ *has an open neighbourhood* U *such that* $p_y > 0$ *only for at most a countable number of* $y \in U$ *and*

$$\sum_{y \in U} p_y < \infty\,.$$

Correlation measures of point processes

- ξ is a point process in $\mathbb{X}$

- Which measures ρ on $\mathbb{X} \times \mathbb{X}$ appear as the second-order factorial moment measure (correlation measure) of a point process.

- Kuna, Lebowitz and Speer (2011) characterised correlation measures for point processes with finite third moments or under hard-core exclusion condition with fixed exclusion range.

Plan

- Tool: extension of a positive linear functional, i.e. $\Phi(g) \geq 0$ if $g \geq 0$ (w.r.t. the chosen order).
- Positivity condition: difficult computational problem.
- How to ensure that the extension is sufficiently regular and so the corresponding linear functional corresponds to a probability measure?
- Note: linear functionals on functions are not always representable as Lebesgue integral (Daniell's theory)!

Positive functionals and their extensions

- E is a vector lattice, G is a vector sub-space of E (e.g. E is generated by $\mathbb{I}_{x_1,\dots,x_k \in F}$ for all $k \geq 1$ and G is a linear space generated by $\mathbb{I}_{x \in F}$)

- G majorises E if each $v \in$ E satisfies $|v| \leq g$ for some $g \in$ G

Theorem (Kantorovich)**.** *Assume that* G *is a majorising vector subspace of a vector lattice* E*. Then each positive linear functional on* G *admits a positive extension on the whole* E*.*

- Regularity (e.g. continuity) of the extension is not guaranteed!

Positivity

Theorem. *Assume that* G *can be represented as the direct sum of* $\mathbb{R}$ *(i.e. constant functions) and a vector space* G'*. Then a linear functional* $\Phi : \mathsf{G} \mapsto \mathbb{R}$ *with* $\Phi(1) = 1$ *is positive if and only if*

$$\Phi(g) \geq \inf_{x \in \mathcal{X}} g(x) , \quad g \in \mathsf{G}' .$$

Regularity modulus

- Assume that the vector space G of functions on $\mathcal{X}$ contains constants and, for each $g_1, g_2 \in$ G, there is $g \in$ G such that $(g_1 \vee g_2) \leq g$.

- A regularity modulus is a lower semicontinuous function $\chi : \mathcal{X} \mapsto [0, \infty]$ such that

$$H_g = \{x \in \mathcal{X} : \ \chi(x) \leq g\}$$

is relatively compact for each $g \in$ G.

- Assume that each $g \in$ G is χ-regular, i.e. g is continuous on $H_{g'}$ for each g' in G.

Regular extension (main theorem)

Theorem. *If* Φ *is a linear functional on* G *with* $\Phi(1) = 1$*, then there exists a Borel random element* ξ *in* $\mathcal{X}$ *such that*

$$\begin{cases} \mathbf{E}g(\xi) = \Phi(g) & \text{for all } g \in \mathsf{G}\,, \\ \mathbf{E}\chi(\xi) \le r, \end{cases}$$

for some real r *if and only if*

$$\sup_{g \in \mathsf{G},\ g \le \chi} \Phi(g) \le r\,.$$

- Equivalent condition (if $\mathsf{G} = \mathbb{R} + \mathsf{G}'$)

$$\inf_{x \in \mathcal{X}} [\chi(x) - g(x)] + \Phi(g) \le r\,, \quad g \in \mathsf{G}'\,.$$

Family of extensions

Theorem. *Assume that* G *consists of continuous functions. Let* Φ *be a linear positive functional on* G.

Then the family $\mathcal{M}$ *of all Borel random elements* ξ *that satisfy* $\mathbf{E}g(\xi) = \Phi(g)$ *for all* $g \in \mathsf{G}$ *and* $\mathbf{E}\chi(\xi) \leq r$ *is weak compact.*

Invariant extension

- $\theta \in \Theta$ are transformations acting on $\mathcal{X}$
- Φ is a Θ-invariant functional and χ is a regularity modulus

Theorem. *Assume that at least one of the following conditions holds*

1) G consists of continuous functions on $\mathcal{X}$ and χ is pointwisely approximated from below by a monotone sequence of functions $g_n \in \mathsf{G}$.

2) χ is Θ-invariant.

Then Φ is realisable by a Θ-stationary random element ξ with $\mathbf{E}\chi(\xi) \leq r$ if and only if

$$\sup_{g \in \mathsf{G},\ g \leq \chi} \Phi(g) \leq r\,.$$

Finite point processes

- $\mathcal{N}$ — locally finite counting measures with the vague topology
- $Y \in \mathcal{N}$ is a counting measure, ξ is a point process
- Functional

$$g_h(Y) = \sum_{x_i, x_j \in Y,\; i \neq j} h(x_i, x_j)\,, \quad Y \in \mathcal{N}\,,$$

$h \in \mathcal{C}_{\mathrm{o}}$ — symmetric continuous with compact support

- Correlation measure ρ of point process ξ

$$\int_{\mathbb{X}\times\mathbb{X}} h(x,y)\rho(dx,dy) = \mathbf{E} g_h(\xi)\,, \quad h \in \mathcal{C}_{\mathrm{o}}\,.$$

Positivity condition

- Functional Φ on $\mathsf{G} = \{g_h : \ h \in \mathcal{C}_o\}$ is positive if

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{X}} g_h(Y), \quad h \in \mathcal{C}_o,$$

where $\mathcal{X} \subseteq \mathcal{N}$.

- Need to ensure that the extension of Φ corresponds to a probability measure. Note that
  - each function g_h is vague-continuous, and so is χ-regular for any χ;
  - $\mathcal{N}$ is locally compact.

Bounded cardinality

- $\mathcal{X}_k$ consists of counting measures with total mass at most k on a compact space $\mathbb{X}$.

- $\mathcal{X}_k$ is compact, g_h is continuous and so Riesz–Markov is applicable

- Example. If $k = 2$, then $\inf_{Y \in \mathcal{X}_2} g_h(Y)$ is the minimum of zero (in case Y is empty or consists of a single point) or the minimum of h.

A point process ξ realising ρ consists of coordinates of a random vector distributed according to the normalised ρ with probability $\rho(\mathbb{X} \times \mathbb{X})$ and otherwise letting $\xi = \emptyset$.

Moment condition

- For $\alpha > 2$

$$\chi_\alpha(Y) = Y(\mathbb{X})^\alpha, \quad Y \in \mathcal{N}.$$

is a regularity modulus, since

$$\{Y \in \mathcal{N} :\ \chi_\alpha(Y) \le c + g_h(Y)\} \subset \{Y \in \mathcal{N} :\ Y(\mathbb{X})^\alpha \le c + c' Y(\mathbb{X})^2\}$$

- The realisability condition

$$\sup_{g \in \mathsf{G},\ g \le \chi_\alpha} \Phi(g) < \infty$$

ensures the existence of a point process with finite α-moment, i.e. $\mathbf{E}\xi(\mathbb{X})^\alpha < \infty$.

Non-finite point processes

- Let $\beta : \mathbb{X} \mapsto \mathbb{R}$ be a lower semi-continuous strictly positive function.
- Then

$$\chi_{\alpha,\beta}(Y) = \left(\sum_{x\in Y} \beta(x)\right)^{\alpha}, \quad Y \in \mathcal{N},$$

is a regularity modulus for $\alpha > 2$.

- Realisability condition becomes

$$\inf_{Y\in\mathcal{X}} [\chi_{\alpha,\beta}(Y) - g_h(Y)] + \int_{\mathbb{X}\times\mathbb{X}} h(x,y)\rho(dx,dy) \le r, \quad h \in \mathcal{C}_o.$$

Hard-core point processes with a fixed exclusion distance

- $\mathbb{X}$ is a compact metric space with metric d

- $\mathcal{X}^{\varepsilon}$ be the family of ε-hard-core point sets and so is a subset of $\mathcal{N}_s$ (simple counting measures)

- $\mathcal{X}^{\varepsilon}$ is compact, so only positivity condition is required

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{X}^{\varepsilon}} g_h(Y)$$

This is a stronger condition than

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

Hard-core point process on a compact space

- Counting measures from $\cup_{\varepsilon>0}\mathcal{X}^{\varepsilon}$ (the exclusion distance is not specified)

- Define

$$\chi_{\psi}^{\mathrm{hc}}(Y) = \sum_{x_i, x_j \in Y,\ i \neq j} \psi(\mathsf{d}(x_i, x_j)), \quad Y \in \mathcal{N}_s\,,$$

where $\psi : (0, \infty) \mapsto [0, \infty]$ be a monotone decreasing right-continuous function, such that $\psi(t) \to \infty$ as $t \downarrow 0$.

- ψ should grow at zero sufficiently fast to ensure that $\chi_{\psi}^{\mathrm{hc}}$ is a regularity modulus.

A combinatorial lemma

$P_t(\mathbb{X})$ denotes the packing number of $\mathbb{X}$ with metric d, i.e. the maximum number of points in the space $\mathbb{X}$ with pairwise distance exceeding t.

Lemma 1. *If* $Y = \sum \delta_{x_i}$ *is a counting measure of total mass* n*, then for all* $t > 0$*,*

$$\sum_{i \neq j} \mathbb{1}_{\mathsf{d}(x_i, x_j) \leq t} \geq n\Big(\frac{n}{P_t(\mathbb{X})} - 1\Big)\,.$$

Proof. Successive transformations of Y that decrease the value of the left-hand sum and eventually bring Y to a set of multiple points with equal multiplicities. □

Asymptotics for the minimal number of close pairs

- $\gamma_t(n)$ is the minimal number of pairs (x_i, x_j) with d$(x_i, x_j) \le t$ and $x_i, x_j \in Y$ over all counting measures of total mass n

- Then, for $t > 0$,

$$\lim_{n\to\infty} \frac{\gamma_t(n)}{n^2} = P_t(\mathbb{X})^{-1}\,,$$

Regularity modulus χ_ψ^{hc}

Lemma 2. *Function* χ_ψ^{hc} *is a regularity modulus on* $\mathcal{N}_s$ *if*

$$\psi(t)/P_t(\mathbb{X}) \to \infty \quad \text{as } t \downarrow 0\,.$$

Proof. Show that the total mass of any Y from

$$\begin{aligned} H_\lambda &= \{Y \in \mathcal{N}_s :\ \chi_\psi^{\mathrm{hc}}(Y) \le \lambda Y(\mathbb{X})^2\} \\ &\subset \{Y :\ n^{-2}\gamma_t(n)\psi(t) \le \lambda\} \end{aligned}$$

is bounded from above. □

Realisability condition

Theorem. *A locally finite measure* ρ *on* $\mathbb{X} \times \mathbb{X}$ *is the correlation measure of a simple point process* ξ *such that* $\mathbf{E}\chi_\psi^{\mathrm{hc}}(\xi) \le r$ *if and only if*

$$\Phi(g_h) \ge \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

and

$$\int_{\mathbb{X}\times\mathbb{X}} \psi(\mathsf{d}(x,y))\rho(dx,dy) \le r\,.$$

- Note that χ_ψ^{hc} can be approximated from below by functions g_h and so this yields the existence of invariant extensions.

A direct condition on ρ

Theorem. *A locally finite measure ρ on $\mathbb{X} \times \mathbb{X}$ is a correlation measure of a simple point process ξ if*

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

and

$$r = \int_{\mathbb{X}\times\mathbb{X}} P_{\mathsf{d}(x,y)}(\mathbb{X})\rho(dx, dy) < \infty\,.$$

In this case, for every $r' > r$, there exists ξ such that

$$\mathbf{E} \sum_{x_i, x_j \in \xi,\ i \neq j} P_{\mathsf{d}(x_i, x_j)}(\mathbb{X}) \leq r'\,.$$

for every $r' > r$.

If furthermore Θ is a group of continuous transformations on $\mathbb{X}$ that leave ρ invariant, then ξ can be chosen Θ-stationary.

Non-compact case ($\mathbb{R}^d$)

- Usual positivity condition

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

- Regularity condition

$$\int_{B_n \times B_n} \|x - y\|^{-d} \rho(dx, dy) < \infty$$

for each ball B_n of radius n.

Random binary functions

- $\mathcal{X}$ is the family of all subsets of $\mathbb{X}$ with the topology of pointwise convergence

- Random indicator function ξ

- Let G be a family of continuous (for pointwise convergence) functionals on $\mathcal{X}$.

- A linear functional Φ on G is realisable, i.e. $\Phi(g) = \mathbf{E}g(\xi)$ if and only if Φ is positive, i.e.

$$\Phi(g) \geq \inf_{F \subset \mathbb{X}} g(F)\,, \quad g \in \mathsf{G}\,.$$

Example: one-point covering function

- G generated by constants and $g_x(F) = \mathbb{1}_{x\in F}$
- $\Phi : \mathsf{G} \mapsto \mathbb{R}$ is determined by $\Phi(g_x) = p_x$.
- Positivity condition $p_x \in [0,1]$
- Note that ξ is the indicator of a not necessarily closed random set.

Two-point covering function

- G generated by constants and $g_{x,y}(F) = \mathbb{1}_{x,y\in F}$, $x, y \in \mathbb{X}$. $\Phi : \mathsf{G} \mapsto \mathbb{R}$ is determined by $\Phi(g_{x,y}) = p_{x,y}$, i.e. two-point covering probabilities.

- Φ is positive if and only if

$$\sum_{ij=1}^{n} a_{ij} p_{x_i,x_j} \geq \inf_{F\subset\mathbb{X}} \sum_{ij=1}^{n} a_{ij}\, \mathbb{1}_{x_i,x_j\in F} \ .$$

Functionals

- ν is a σ-finite reference measure on $\mathbb{X}$

- Define

$$g_h(F) = \int_{F\times F} h(x,y)\nu(dx)\nu(dy)$$

$$\Phi(g_h) = \int_{\mathbb{X}\times\mathbb{X}} p_{x,y} h(x,y)\nu(dx)\nu(dy)\,.$$

- $p_{x,y}$ is weakly realisable if there exists random set X such that $\mathbf{E} g_h(X) = \Phi(g_h)$ for all h, i.e. $p_{x,y} = \mathbf{P}\{x,y\in X\}$ for almost all x,y.

- $p_{x,y}$ is strongly realisable if there exists X with $p_{x,y} = \mathbf{P}\{x,y\in X\}$ for all x,y.

Difficulties

- Aim to realise a random closed set
- Good: The family $\mathcal{F}$ of closed sets is compact in the Fell topology
- Bad: The functional g_h is not continuous
- Even worse: The finite point covering probabilities do not generate the Fell σ-algebra (only for the subfamily of regular closed sets).

ε-neighbourhoods

- $\mathcal{F}^\varepsilon$ is the family of ε-neighbourhoods of closed sets in $\mathbb{R}^d$

- G is generated by constants and the functions g_h, which are shown to be continuous on $\mathcal{F}^\varepsilon$.

Theorem. *A function $p_{x,y}$, $x, y \in \mathbb{R}^d$, is weakly realisable by a random closed set X with realisations in $\mathcal{F}^\varepsilon$ for some $\varepsilon > 0$ if and only if*

$$\Phi(g_h) \geq \inf_{F \in \mathcal{F}^\varepsilon} g_h(F)\,, \quad h \in \mathcal{C}_\text{o}\,.$$

ε-neighbourhoods of varying ε

Regularity modulus

$$\chi(F) = \inf\{\varepsilon > 0 : \; F \in \mathcal{F}^{1/\varepsilon}\}$$

Theorem. *A function $p_{x,y}$, $x, y \in \mathbb{R}^d$, is weakly realisable by a random closed set X such that $\mathbf{E}\chi(X) \le r$ if and only if*

$$\inf_{F \in \mathcal{F}^0} [\chi(F) - g_h(F)] + \Phi(g_h) \le r \,, \qquad h \in \mathcal{C}_{\mathrm{o}} \,.$$

If $p_{x,y} = S_{x-y}$ for $x, y \in \mathbb{R}^d$, with an even continuous function S, then $p_{x,y}$ is strongly realisable by a stationary random closed set X.

Convexity restriction

- $\mathcal{C}$ is the family for convex closed sets

- A functional $\Phi(g_h)$ is realisable as the covariance of a convex random closed set if and only if

$$\Phi(g_h) \geq \inf_{F \in \mathcal{C}} g_h(F)\,.$$

- For sets from the convex ring the regularity modulus is the smallest number of convex components.

One-dimensional case

- Assume $\mathbb{X} = [0, 1]$
- Let $\chi(F)$ be the number of convex components of F

Theorem. *If $p_{x,y}$ is a function of $x, y \in [0, 1]$ such that*

$$\sup_{\varphi \in \mathcal{C}^1_{\mathrm{o}}, 0 \le \varphi \le 1} \int_{\mathbb{X} \times \mathbb{X}} p_{x,y} \varphi'(x) \varphi'(y) dx dy = \infty \,,$$

then there is no random closed set X satisfying $\mathbf{E}\chi(X)^2 < \infty$ having $p_{x,y}$ as its two-point covering function.

Contact distribution function

$$T_X(B_r(x)) = \mathbf{P}\{X \cap B_r(x) \neq \emptyset\}$$

is related to the spherical contact distribution function

$$H_X(r;x) = \mathbf{P}\{\mathsf{d}(x,X) \leq r | x \notin X\}, \quad r \geq 0.$$

Realisability of capacity functionals on balls

Theorem. *A function* $\tau_x(r)$, $r \geq 0$, $x \in \mathbb{R}^d$, *is realisable as* $T_X(B_r(x))$ *for a random closed set* X *if and only if*

$$\Phi(g) = \sum_{i=1}^{q} a_i \tau_{x_i}(r_i) \geq 0$$

for each non-negative function

$$g(F) = \sum_{i=1}^{q} a_i \, \mathbb{1}_{B_{r_i}(x_i) \cap F \neq \emptyset} \geq 0\,, \quad F \in \mathcal{F}\,.$$

Example: two-points

Theorem. *Let* $x_1, x_2 \in \mathbb{R}^d$*, with* $l = \|x_1 - x_2\|$*, and let* $\tau_{x_1}(r)$ *and* $\tau_{x_2}(r)$ *be cumulative distribution functions of two sub-probability measures on* $\mathbb{R}_+$*. Then there exists a random closed set* X *such that* $\tau_{x_i}(r) = T_X(B_r(x_i))$ *for* $r \geq 0$ *and* $i = 1, 2$ *if and only if for all* $r \geq 0$

$$\tau_{x_1}(\max(r - l, 0)) \leq \tau_{x_2}(r) \leq \tau_{x_1}(r + l).$$

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