The Poisson storm process
Extremal coefficients and simulation

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Storm process

Introduced by Smith (1990) and generalized by Schlather (2002), the storm process is a prototype of max-stable processes with unit Fréchet margins:

$$F(z) = \exp\left(-\frac{\alpha}{z}\right) \qquad z > 0$$

Basic ingredients:

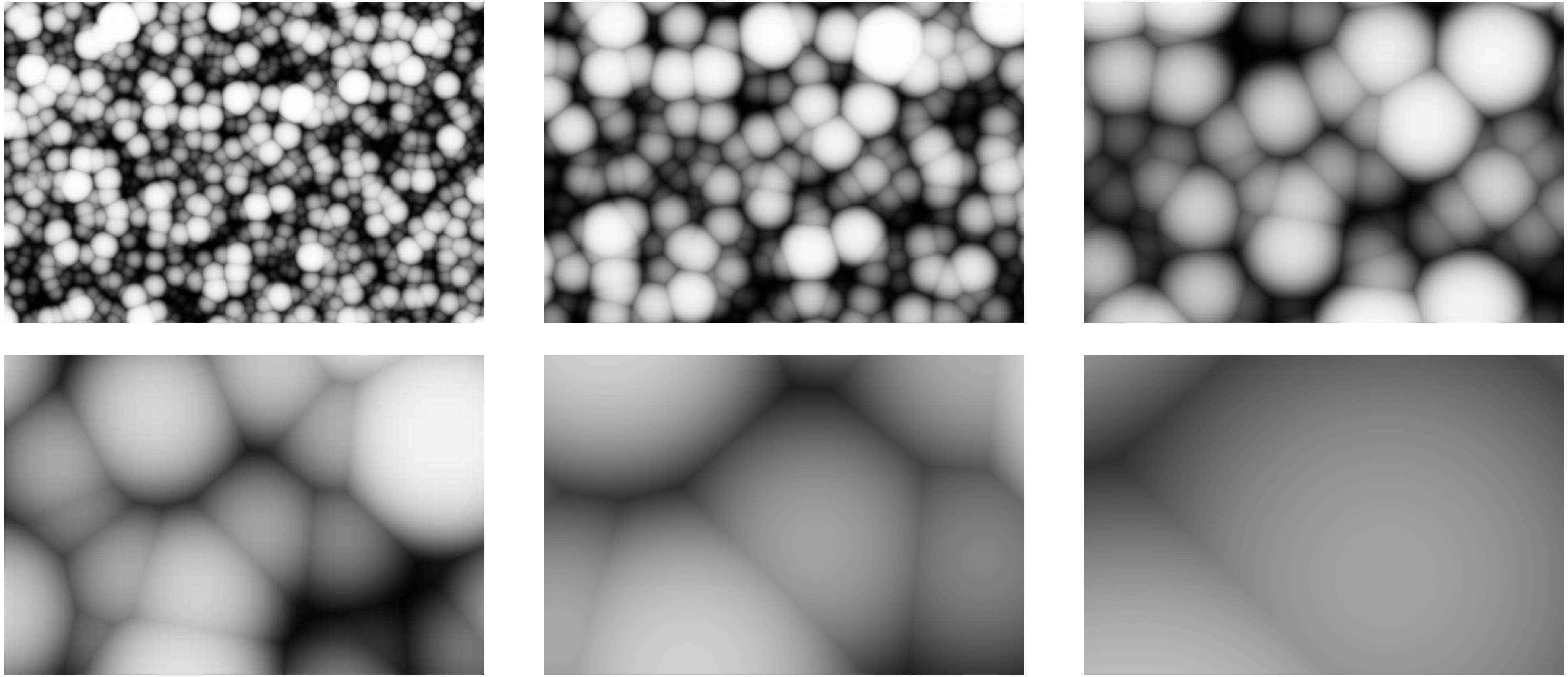
– Π homogenous Poisson point process (intensity μ) in $\mathbb{R}^d \times \mathbb{R}_+$;

– $\left(Y_{x,t},\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+\right)$ independent copies of a random function Y, defined in $\mathbb{R}^d$, positive and integrable.

Definition:

$$Z(s) = \sup_{(x,t)\in\Pi} \frac{Y_{x,t}(s-x)}{t} \qquad s \in \mathbb{R}^d$$

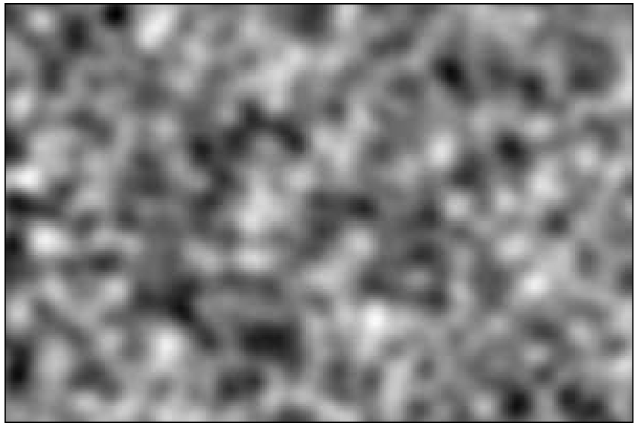
Gaussian storm processes at six different scales

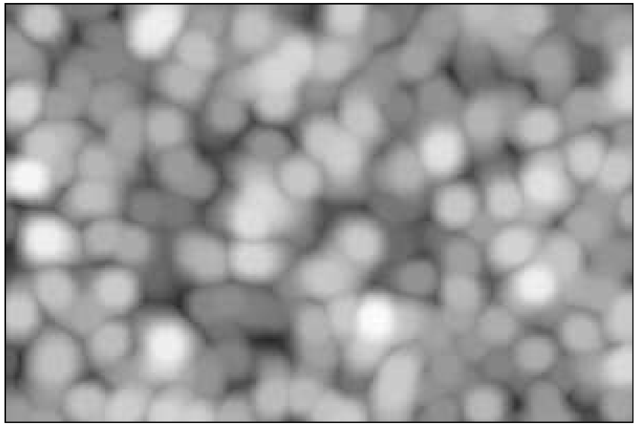


$$Y(s) = \exp\left(-\frac{|s|^2}{\sigma^2}\right)$$

Maximum of a storm process in a domain

Let K be a nonempty compact subset of $\mathbb{R}^d$, and let $Z^K(s) = \sup_{s' \in K} Z(s+s')$ be the sup-convolution of Z w.r.t. K





Example of a function and its sup-convolution by a disk

Provided that Y^K is integrable, the distribution of Z^K is also unit Fréchet:

$$P\left\{Z^K(s) < z\right\} = \exp\left(-\frac{\mu}{z}\int_{\mathbb{R}^d} E\Big\{Y^K(s)\Big\}\, ds\right)$$

Extremal coefficients

Assume Y^K to be integrable for each compact subset K of $\mathbb{R}^d$. Then

$$P\{Z^K(s) < z\} = \exp\left(-\frac{\theta(K)}{z}\right) \quad \text{with} \quad \theta(K) = \mu \int_{\mathbb{R}^d} E\left\{Y^K(s)\right\} ds$$

The coefficients $\theta(K)$ are called extremal coefficients (Smith, 1990).

Objectives of the presentation:

– to establish the consistency relationships that exist between extremal coefficients at various supports;

– to present an example of a storm process for which the extremal coefficients are analytically tractable;

– to give an algorithm for simulating this storm process.

Consistency relationships between extremal coefficients

Related work

Let $Z = (Z_1, ..., Z_d)$ be a max-stable vector with the same unit Fréchet margins. Following Pickands (1981), its multivariate distribution can be written as

$$P\left\{\bigvee_{i=1}^{d} \frac{Z_i}{z_i} < 1\right\} = \exp\left(-\int_S \bigvee_{i=1}^{d} \frac{t_i}{z_i}\, dH(t)\right)$$

where S is the unit simplex ($t \in S$ iff $t_1, ..., t_d \geq 0$ and $t_1 + \cdots + t_d = 1$), and H is a positive measure on S (spectral measure). This implies

$$\theta(K) = \int_S \bigvee_{i \in K} t_i\, dH(t) \qquad K \subset \{1, ..., d\}$$

From this formula, a set of inequalities relating the different $\theta(K)$'s was derived by Schlather and Tawn (2002).

Related work (2)

Molchanov (2008) derived another expression for the multivariate distribution of Z using the support function of a max-zonoid. Based on this expression, he arrived to the following corollary:

A set of coefficients $\big(\theta(K), K \subset \{1, ..., d\}\big)$ is a set of extremal coefficients for a simple max-stable distribution if and only if $\theta(\emptyset) = 0$ and $\theta(K)$ is a union-completely alternating function of K:

$$\sum_{J \subset I} (-1)^{|J|} \theta(K \cup K_J) \leq 0$$

for any subset K and for any family $(K_i, i \in I)$ of subsets of $\{1, ..., d\}$. In the formula, K_J is a short notation for $\cup_{j \in J} K_j$.

These inequalities complete the set of consistency relationships established by Schlather and Tawn.

Related work (3)

Regarding the bivariate distributions of a stationary max-stable process with unit Fréchet margins, Schlather and Tawn (2003) have obtained the following result:

If $K = \{x, x+h\}$, then $\theta(K) = \theta(o)[1+\gamma(h)]$, where γ is a function of conditionally negative type, satisfying $0 \leq \gamma \leq 1$ and the triangular inequality $\gamma(h+h') \leq \gamma(h) + \gamma(h')$.

In the case of a storm process, we have

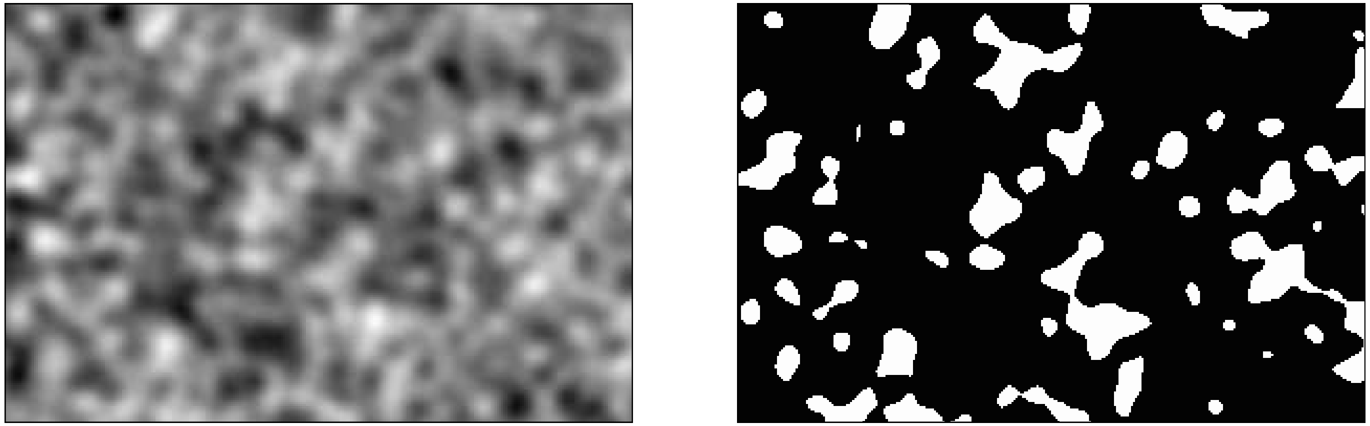
$$\gamma(h) = \frac{1}{2} \frac{\displaystyle\int_{I\!R^d} E\{|Y(s+h) - Y(s)|\} \, ds}{\displaystyle\int_{I\!R^d} E\{Y(s)\} \, ds}$$

γ is a function of conditionally negative type if $\sum_{i,j} \lambda_i \lambda_j \gamma(s_i - s_j) \leq 0$ whenever $\sum_{i \in I} \lambda_i = 0$.

Thresholds of a storm process

Definition:

These are the random sets $X_z = \{s \in I\!R^d : Z(s) \geq z\}$ for each $z > 0$.



Example of a function and one of its thresholds

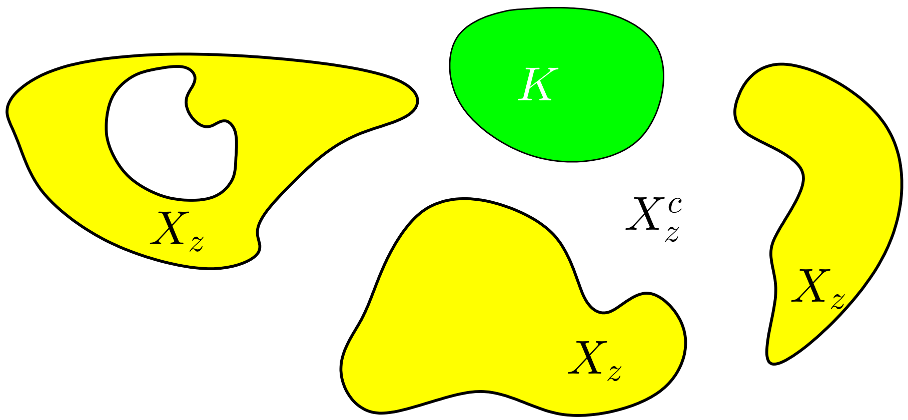
If Z is u.s.c., these random sets are topologically closed.

Statistical description of a random closed set

Avoiding functional:

The random set theory by Matheron (1975) shows that the statistical properties of X_z are given by its avoiding functional:

$$Q_z(K) = P\{X_z \cap K = \emptyset\} \qquad K \in \mathcal{K}$$



Remark:

An avoiding functional acts for a random closed set exactly as the complementary distribution function for a random variable.

More on the thresholds of a storm process

Avoiding functional of X_z:

$$Q_z(K) = P\{X_z \cap K = \emptyset\} = P\{Z^K(o) < z\} = \exp\left(-\frac{\theta(K)}{z}\right)$$

X_z is infinitely divisible for the union:

X_z can be expressed as the union of n independent copies of X_{nz}:

$$Q_z(K) = \exp\left(-\frac{\theta(K)}{z}\right) = \exp\left(-n\frac{\theta(K)}{nz}\right) = \left[Q_{nz}(K)\right]^n$$

X_z has no fixed point:

$$P\{s \in X_z\} = P\{Z(s) \geq z\} = 1 - \exp\left(-\frac{\theta(s)}{z}\right) < 1$$

Characterization of random closed sets infinitely divisible and without a fixed point

Theorem (Matheron, 1975):

(i) There exists a positive σ-finite measure ζ_z on the set $\mathcal{F}'$ of the nonempty closed subsets of $\mathbb{R}^d$ such that $\zeta_z(\mathcal{F}_K) = -\ln Q_z(K)$ for each compact subset K of $\mathbb{R}^d$;

(ii) X_z has he same distribution as the union of a Poisson process locally finite with intensity ζ_z in $\mathcal{F}'$.

Comments:

– $\mathcal{F}_K = \{F \in \mathcal{F}' : \, F \cap K \neq \emptyset\}$

– a Poisson process in $\mathcal{F}'$ is locally finite if the number of elements of each $\mathcal{F}_K$ is almost surely finite.

Relationship between extremal coefficients and the hitting measures

$$\zeta_z(\mathcal{F}_K) = -\ln Q_z(K) = \frac{\theta(K)}{z} \qquad K \in \mathcal{K}$$

This implies $\theta(K) = \zeta_1(\mathcal{F}_K)$. From now onwards, we note ζ instead of ζ_1. Therefore

$$\theta(K) = \zeta(\mathcal{F}_K)$$

Note also the consistency relationship between the different ζ_z:

$$\zeta_z(\mathcal{F}_K) = \frac{1}{z}\zeta(\mathcal{F}_K) \qquad z > 0 \quad K \in \mathcal{K}$$

Consistency relationships

$$\theta(K) = \zeta(\mathcal{F}_K) \qquad K \in \mathcal{K}$$

– $\theta(K) \geq 0$ and $\theta(\emptyset) = \zeta(\mathcal{F}_\emptyset) = 0$;

– $K \subset K' \implies \theta(K) = \zeta(\mathcal{F}_K) \leq \zeta(\mathcal{F}_{K'}) = \theta(K')$;

– Let $\mathcal{F}^K_{(K_i, i\in I)}$ the family of closed sets hitting each K_i and avoiding K. Then

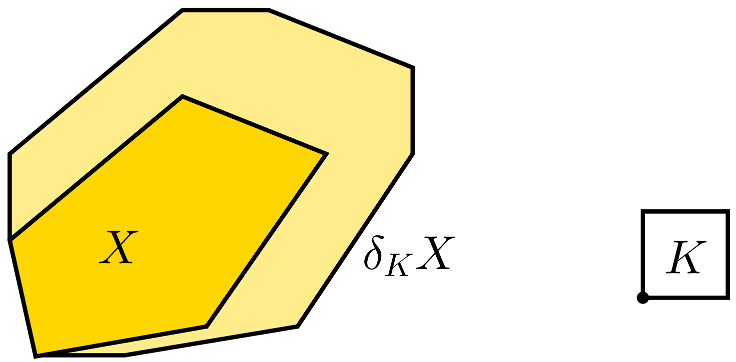
$$\begin{aligned} 0 &\leq \zeta\big(\mathcal{F}^K_{(K_i,i\in I)}\big) \\ &= \zeta\big(\mathcal{F}_{(K_i,i\in I)}\big) - \zeta\big(\mathcal{F}_{(K_i,i\in I),K}\big) \\ &= \sum_{J\subset I} (-1)^{|J|-1} \zeta\big(\mathcal{F}_{K_J \cup K}\big) \qquad \big(K_J = \cup_{j\in J} K_j\big) \\ &= \sum_{J\subset I} (-1)^{|J|-1} \theta(K_J \cup K) \end{aligned}$$

Construction of a storm process with explicit extremal coefficients

Case where the storms are indicator functions

$$\theta(K) = \mu \int_{\mathbb{R}^d} E\{Y^K(s)\}\, ds$$

If $Y(s) = 1_{s \in X}$ where X is a random compact subset of $\mathbb{R}^2$, then $Y^K(s) = 1_{s \in \delta_K X}$ where $\delta_K X$ is the dilation of X by K.



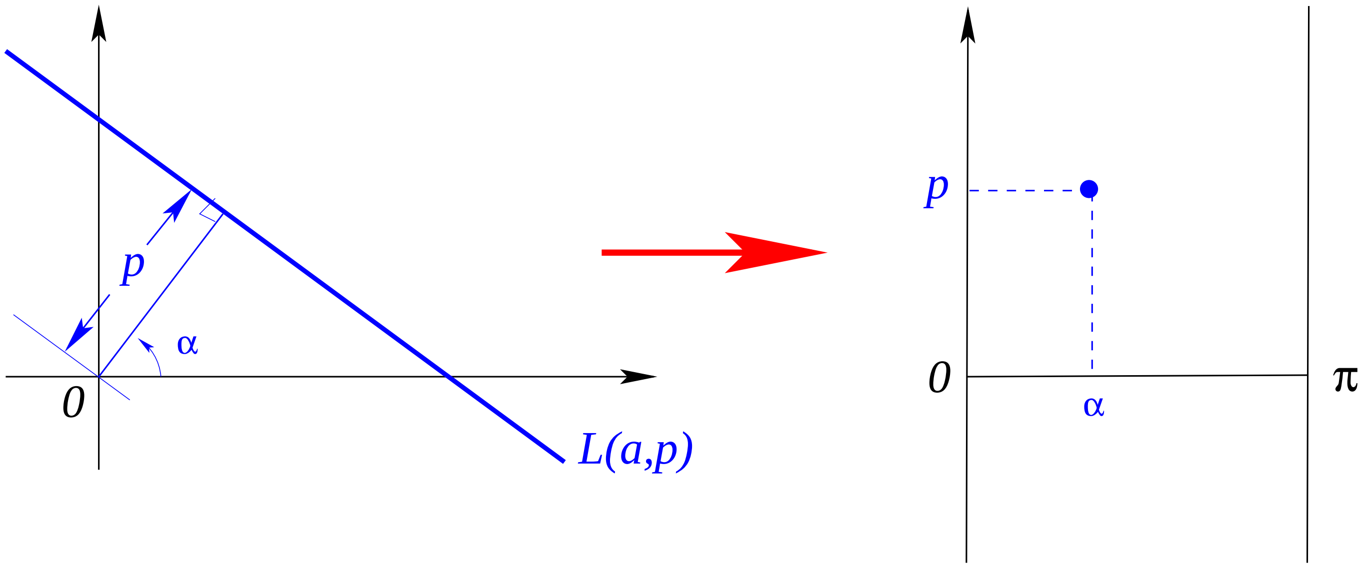
Accordingly

$$\theta(K) = \mu E\{a(\delta_K X)\}$$

How to choose X so as to have $E\{a(\delta_K X)\}$ analytically tractable?

Poisson polygons

Parametrization of a line in two dimensions



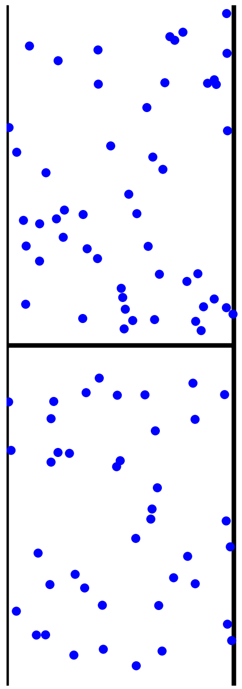
Equation of a line:

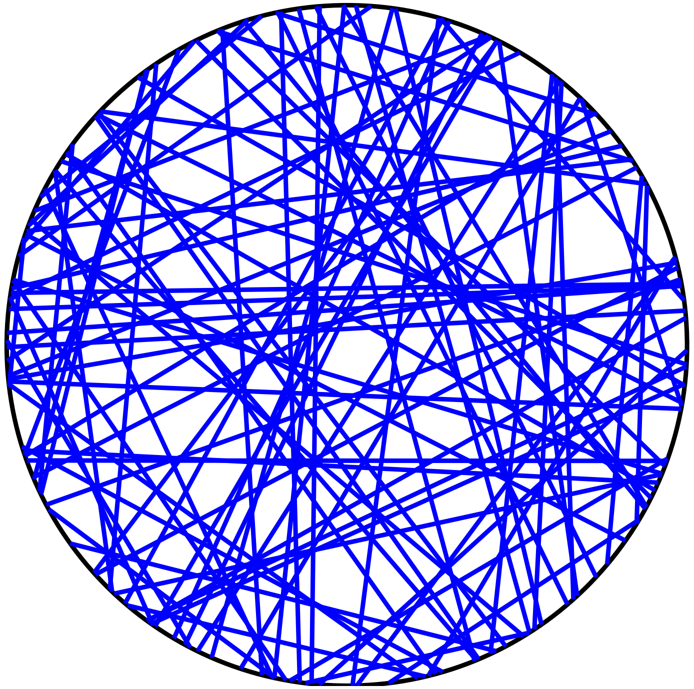
$$x \cos \alpha + y \sin \alpha = p$$

$0 \leq \alpha < \pi$ direction $\qquad -\infty < p < +\infty$ location

Poisson polygons

Poisson line process

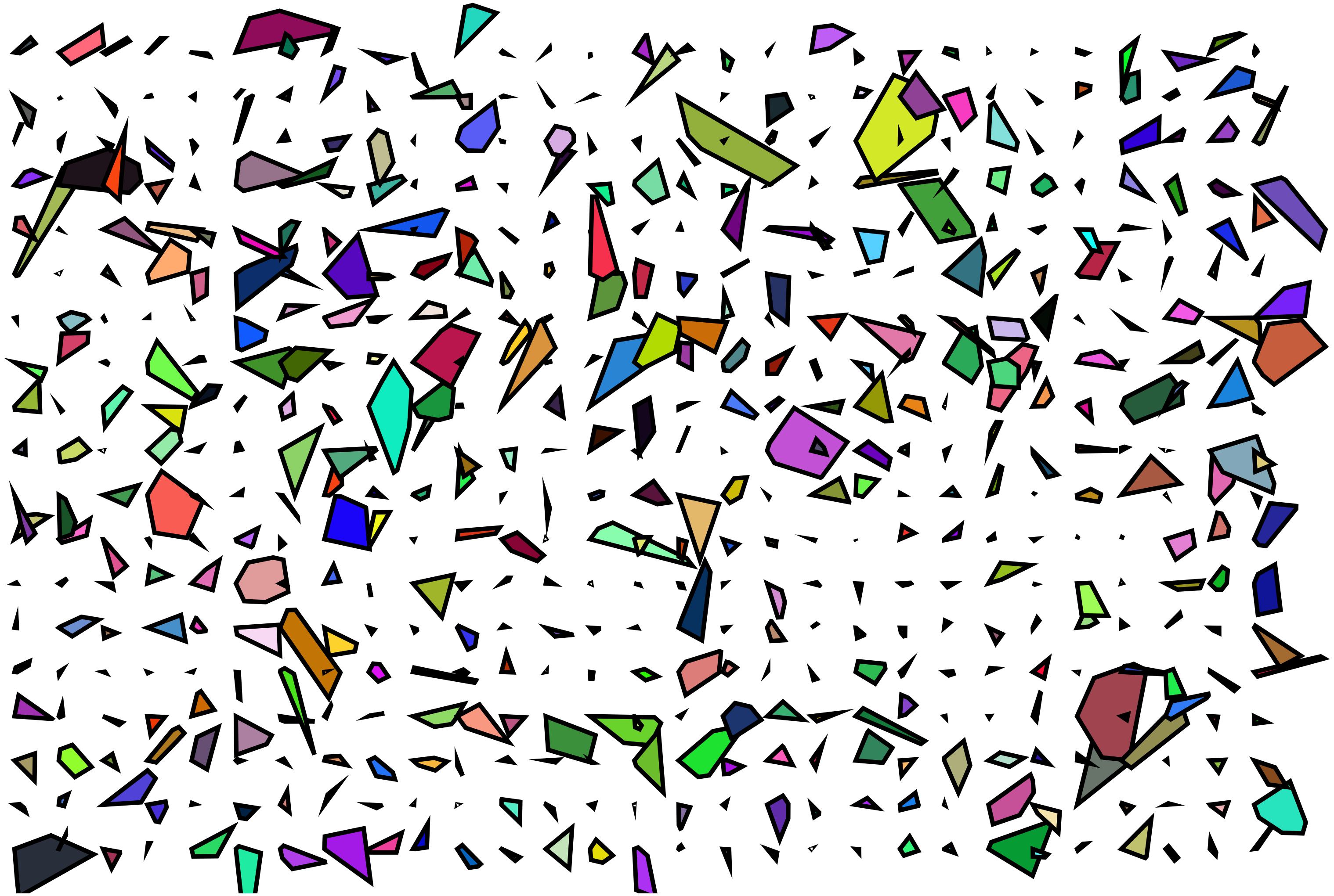




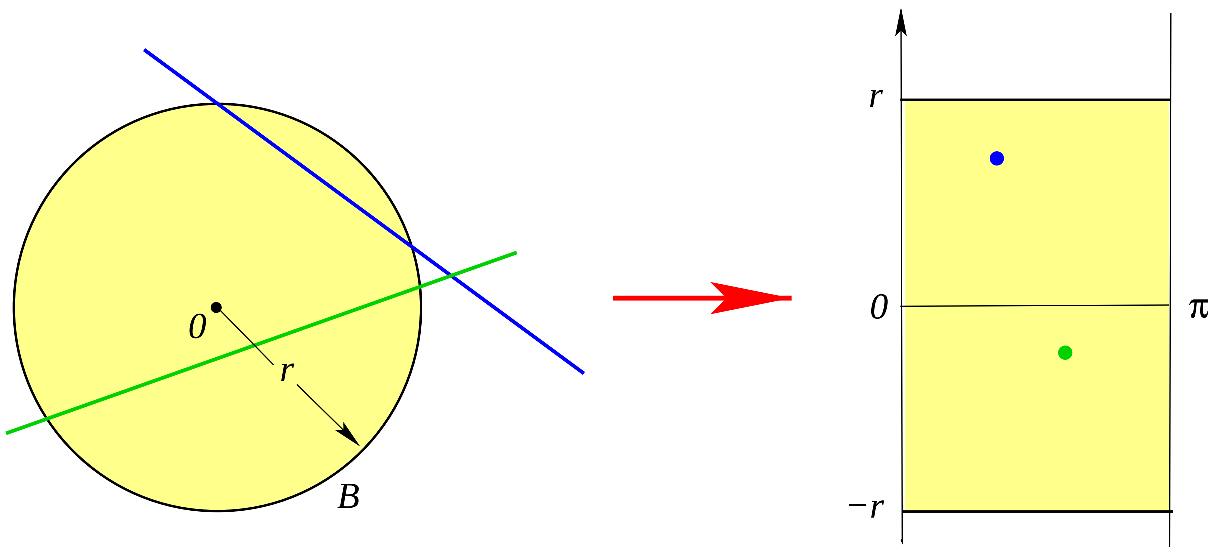
A Poisson line process is parametrized by a homogeneous Poisson point process with intensity λ on $[0, \pi[\times \mathbb{R}$.

Poisson lines delimit Poisson polygons

Realizations of Poisson polygons



Number of Poisson lines hitting a convex domain



The number of lines hitting a convex domain B is Poisson distributed with mean $\lambda\, p(B)$

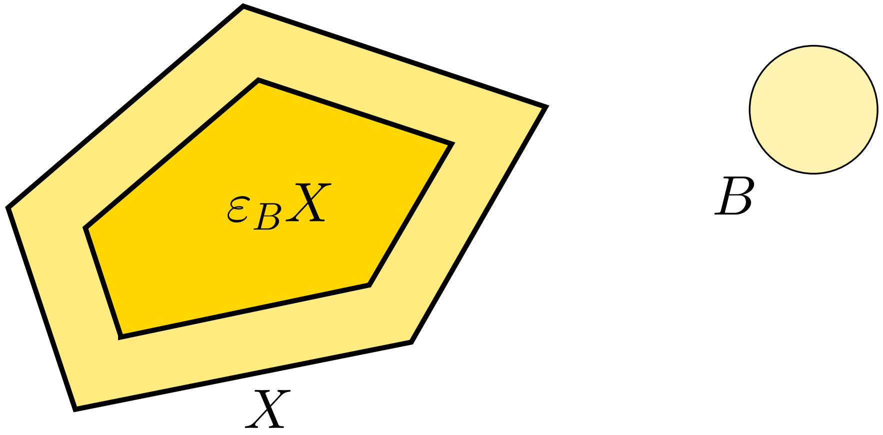
– λ Poisson line intensity

– $p(B)$ perimeter of B

Conditional invariance by erosion

Property:

Poisson polygons are conditionally invariant by erosion (Matheron, 1975)



$$E\{\varphi(\varepsilon_B X) \,|\, \varepsilon_B X \neq \emptyset\} = E\{\varphi(X)\}$$

Remark:

This property is a generalization in more than one dimension of the lack of memory of the exponential distribution.

Extremal coefficients of a Poisson storm process
Case where K is convex

$$\theta(K) = \mu E\{a(\delta_K X)\}$$

By Steiner's formula (that applies because X is isotropic)

$$\theta(K) = \mu \left[E\{a(X)\} + \frac{1}{2\pi} E\{p(X)\} p(K) + a(K) \right]$$

$$\theta(K) = \mu \left[\frac{1}{\pi\lambda^2} + \frac{p(K)}{\pi\lambda} + a(K) \right]$$

Extremal coefficients of a Poisson storm process
Case where K is finite

$$\theta(K) = \mu E\{a(\delta_K X)\}$$

Apply the inclusion-exclusion formula

$$\theta(K) = \mu \sum_{\emptyset \neq L \subset K} (-1)^{|L|-1} E\{a(\varepsilon_L X)\}$$

where $a(X)$ is the area of X and $\varepsilon_B X$ is the erosion of X by B.

Extremal coefficients of a Poisson storm process
Case where K is finite (2)

$$E\{a(\varepsilon_L X)\} = E\{a(\varepsilon_L X) \mid \varepsilon_L X \neq \emptyset\} P\{\varepsilon_L X \neq \emptyset\}$$

By the conditional invariance by erosion, we have

$$E\{a(\varepsilon_L X) \mid \varepsilon_L X \neq \emptyset\} = E\{a(X)\} = \frac{1}{\pi\lambda^2}$$

On the other hand, L is contained in a Poisson polygon if and only if no Poisson line hits the convex hull $\widehat{L}$ of L. Accordingly

$$P\{\varepsilon_L X \neq \emptyset\} = \exp(-\lambda p(\widehat{L}))$$

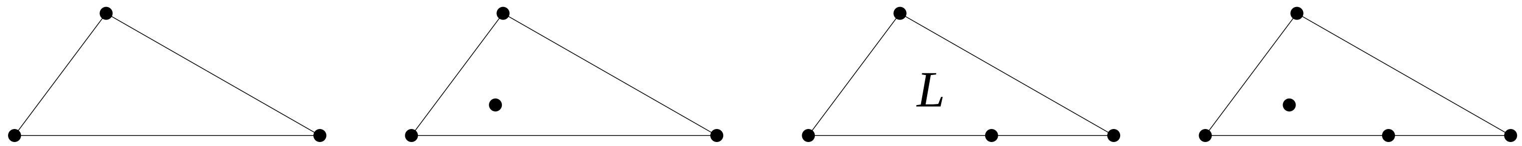
Hence

$$\theta(K) = \frac{\mu}{\pi\lambda^2} \sum_{\emptyset \neq L \subset K} (-1)^{|L|-1} \exp\big(-\lambda p(\widehat{L})\big)$$

$\mu =$ Poisson storm intensity $\qquad \lambda =$ Poisson line intensity

Extremal coefficients of a Poisson storm process
Case where K is finite (3)

Consider on 2^K the equivalence relation $L\,\mathcal{R}\,L'$ if and only if $\widehat{L} = \widehat{L}'$. Let $C(L)$ be the equivalence class of L.



$$\sum_{L' \in C(L)} (-1)^{|L'|-1} \exp\big(-\lambda p(\widehat{L}')\big) = \exp\big(-\lambda p(\widehat{L})\big) \sum_{L' \in C(L)} (-1)^{|L'|-1}$$

$$= \exp\big(-\lambda p(\widehat{L})\big)\, 1_{|C(L)|=1}$$

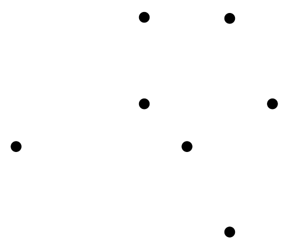
Finally

$$\theta(K) = \frac{\mu}{\pi\lambda^2} \sum_{\substack{\emptyset \neq L \subset K \\ |C(L)|=1}} (-1)^{|L|-1} \exp\big(-\lambda p(\widehat{L})\big)$$

Extremal coefficients of a Poisson storm process
Case where K is finite with points in general position

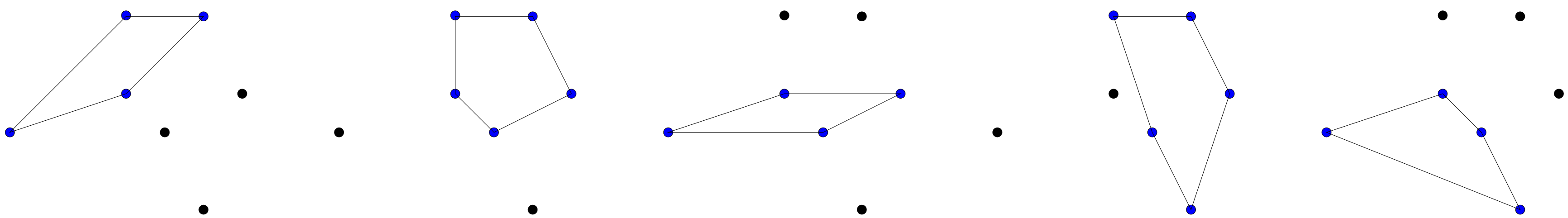
$$\theta(K) = \frac{\mu}{\pi\lambda^2} \sum_{\substack{\emptyset \neq L \subset K \\ int(\widehat{L}) \cap K = \emptyset}} (-1)^{|L|-1} \exp\big(-\lambda p(\widehat{L})\big)$$

Example:



The computation of $\theta(K)$ involves 67 terms (7 points, 21 segments, 25 triangles, 12 quadrilaterals and 2 pentagons).

They can be summarized in a list of 5 maximal configurations:



Extremal coefficients of a Poisson storm process
Case where K is a square grid

$$\theta(K) = \frac{\mu}{\pi\lambda^2} \sum_{\substack{\emptyset \neq L \subset K \\ |C(L)|=1}} (-1)^{|L|-1} \exp\big(-\lambda p(\widehat{L})\big)$$

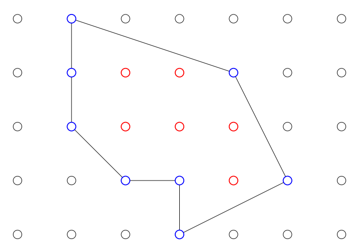
Results:

– $a(\widehat{L}) = |L|/2 - 1$ (consequence of Pick's formula);

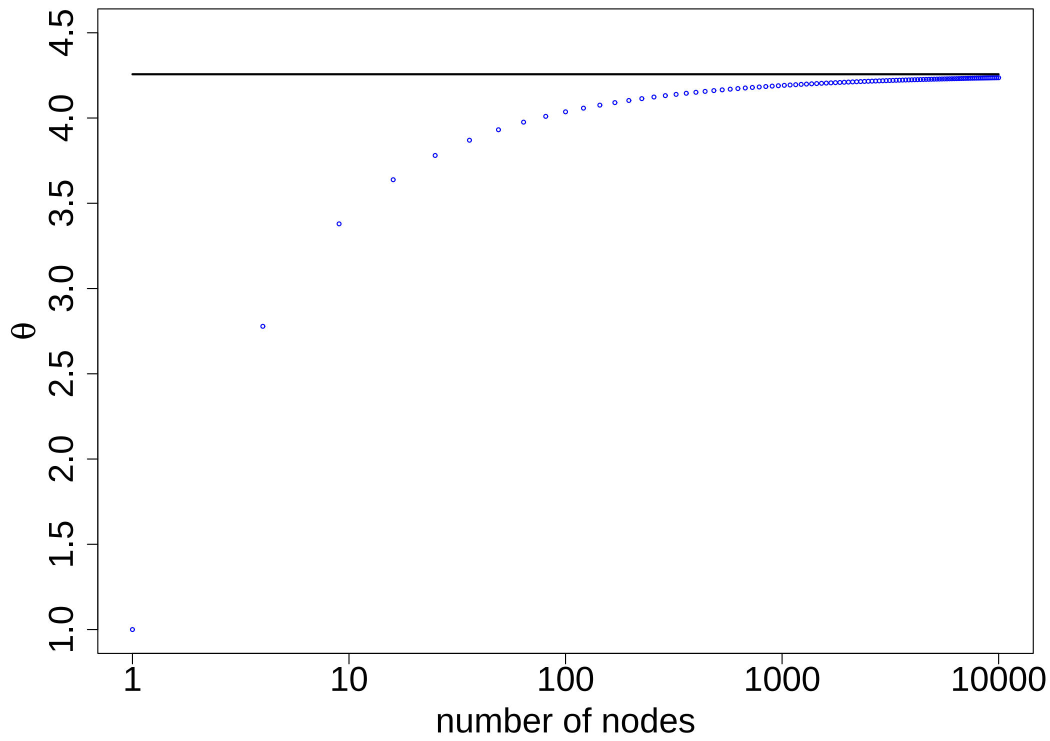
– $|L| \leq 4$ (application of the pigeonhole principle);

– quadrilaterals are parallelograms.

Pick's formula: if X is a simple polygon and its vertices are the nodes of a square grid (unit mesh), then $a(X) = \iota(X) + \beta(X)/2 - 1$. Here $a(X) = 6 + 8/2 - 1 = 9$.



Extremal coefficients of a Poisson storm process
Case where K is a square grid (2)



Poisson storm process with Poisson intensity $\mu = 0.01$ and Poisson line intensity $\lambda = 0.1/\sqrt{\pi} = 0.0564$. The domain under study, a 10×10 square, is sampled by different square grids, ranging from 1 to 10000 nodes.

Simulation

Presentation of the problem

Model:

$$Z(s) = \sup_{\substack{(x,t)\in\Pi \\ s-x\in X_{x,t}}} \frac{1}{t} \qquad s \in I\!R^2$$

where Π is a Poisson process with intensity μ on $I\!R^2 \times I\!R_+$, and the $X_{x,t}$'s are independent copies of Poisson polygons with line intensity λ.

objective:

Produce realizations of the model in a continuous domain D.

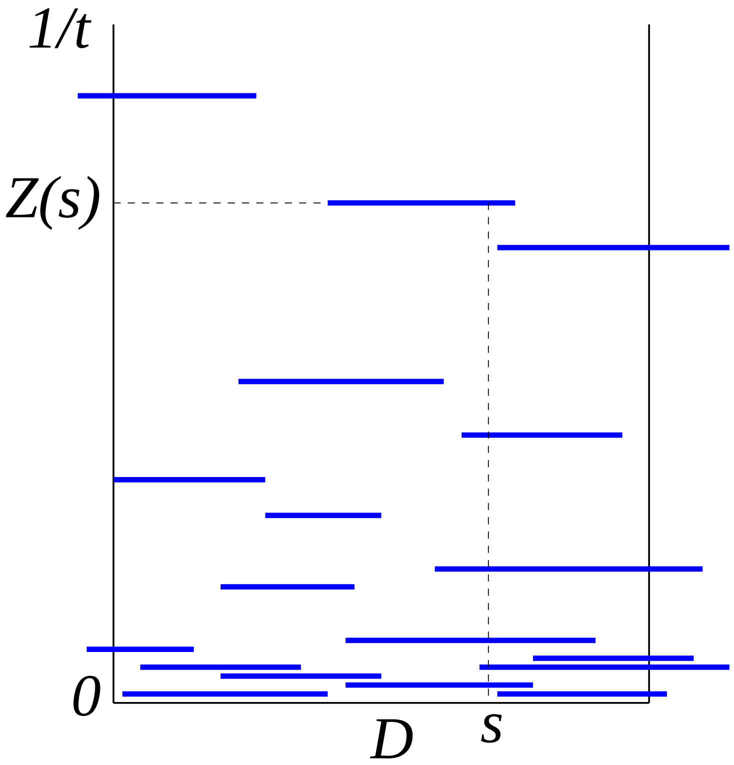
Remark:

There is no inconvenience to assume that D is a disk with center o and radius r.

First remark

The Poisson storms hitting D occur in time according to a Poisson process with intensity

$$\mu E\{a(\delta_D X)\} = \theta(D)$$



Second remark

A storm X hitting D has its distribution weighted by $a(\delta_D X)$:

$$dF_D(X) = \frac{dF(X)a(\delta_D X)}{E\{a(\delta_D X)\}}$$

Using Steiner's formula $a(\delta_D X) = a(X) + rp(X) + \pi r^2$, $dF_D(X)$ can be rewritten as

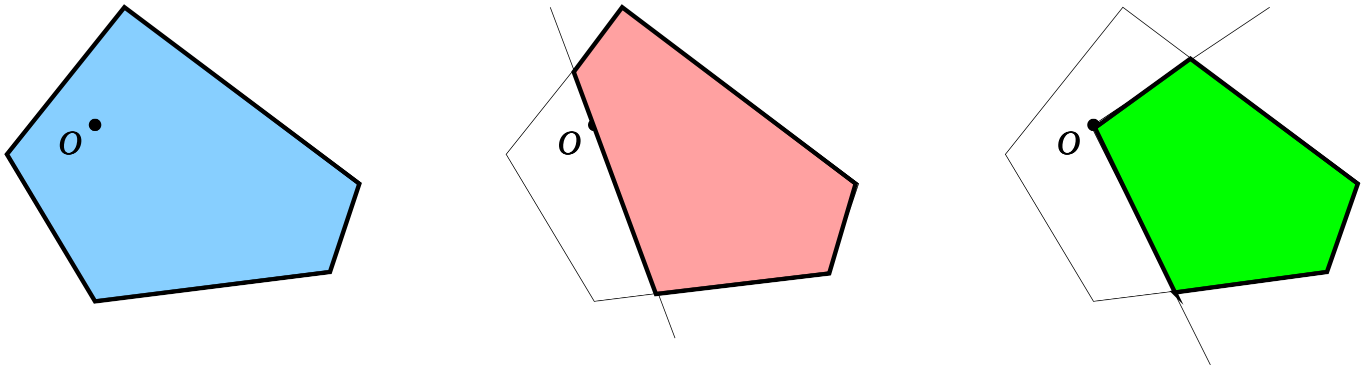
$$dF_D(X) = \frac{dF(X)a(X)}{E\{a(X)\}} \frac{E\{a(X)\}}{E\{a(\delta_D X)\}} + \frac{dF(X)p(X)}{E\{p(X)\}} \frac{rE\{p(X)\}}{E\{a(\delta_D X)\}} + dF(X) \frac{\pi r^2}{E\{a(\delta_D X)\}}$$

or equivalently

$$dF_D(X) = \frac{dF(X)a(X)}{E\{a(X)\}} \frac{1}{(1+\pi\lambda r)^2} + \frac{dF(X)p(X)}{E\{p(X)\}} \frac{2\pi\lambda r}{(1+\pi\lambda r)^2} + dF(X) \frac{\pi^2\lambda^2 r^2}{(1+\pi\lambda r)^2}$$

Accordingly, everything boils down to simulate polygons weighted in area, in perimeter and in number.

Simulation of weighted polygons



Area: Generate Poisson lines sequentially by increasing distance from the origin. Continue the procedure until the generation of additional lines no longer affects the polygon containing the origin.

Perimeter: Split an area-weigted polygon using a uniformly oriented line through the origin. Select at random one of two polygons thus delimited. (Thanks to P. Calka).

Number: Take the intersection between an area-weigted polygon and a cone delimited by two uniform rays emanating from the origin and separated by an angle with p.d.f. $f(\alpha) = \alpha \sin \alpha / \pi$ on $[0, \pi[$ (Miles, 1974).

Algorithm

Notation:

– $\mathcal{E}(a)$ exponential distribution with parameter a (mean $1/a$)

– $\mathcal{U}(A)$ uniform distribution over the domain A.

Algorithm:

(i) set $t = 0$*;*

(ii) generate $u \sim \mathcal{E}(\theta(D))$ *and put* $t = t + u$*;*

(iii) generate $X \sim dF_D$ *and* $x \sim \mathcal{U}(\delta_D X)$*;*

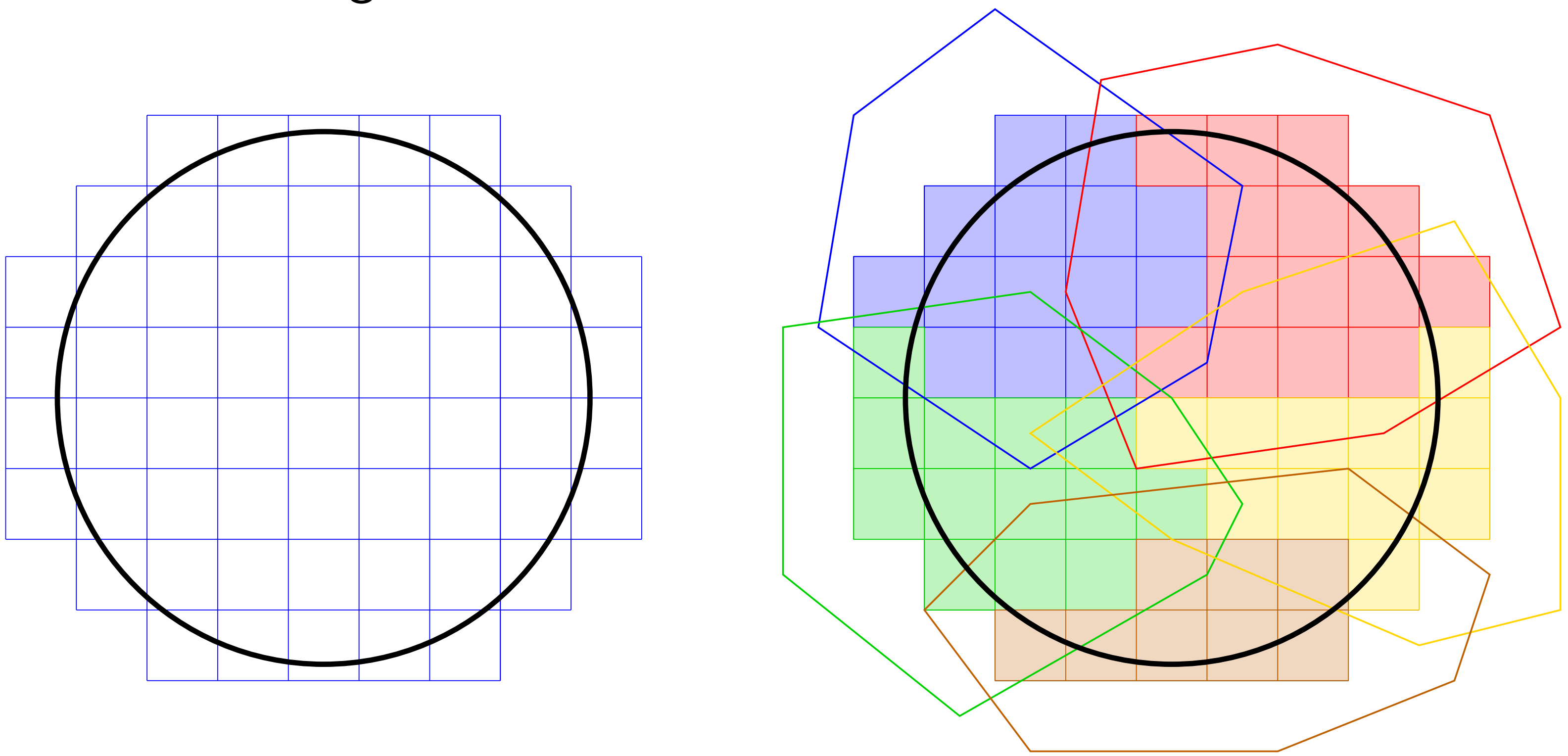
(iv) save (X, x, t) *and then goto (ii).*

Problem:

This algorithm does has no stopping criterion.

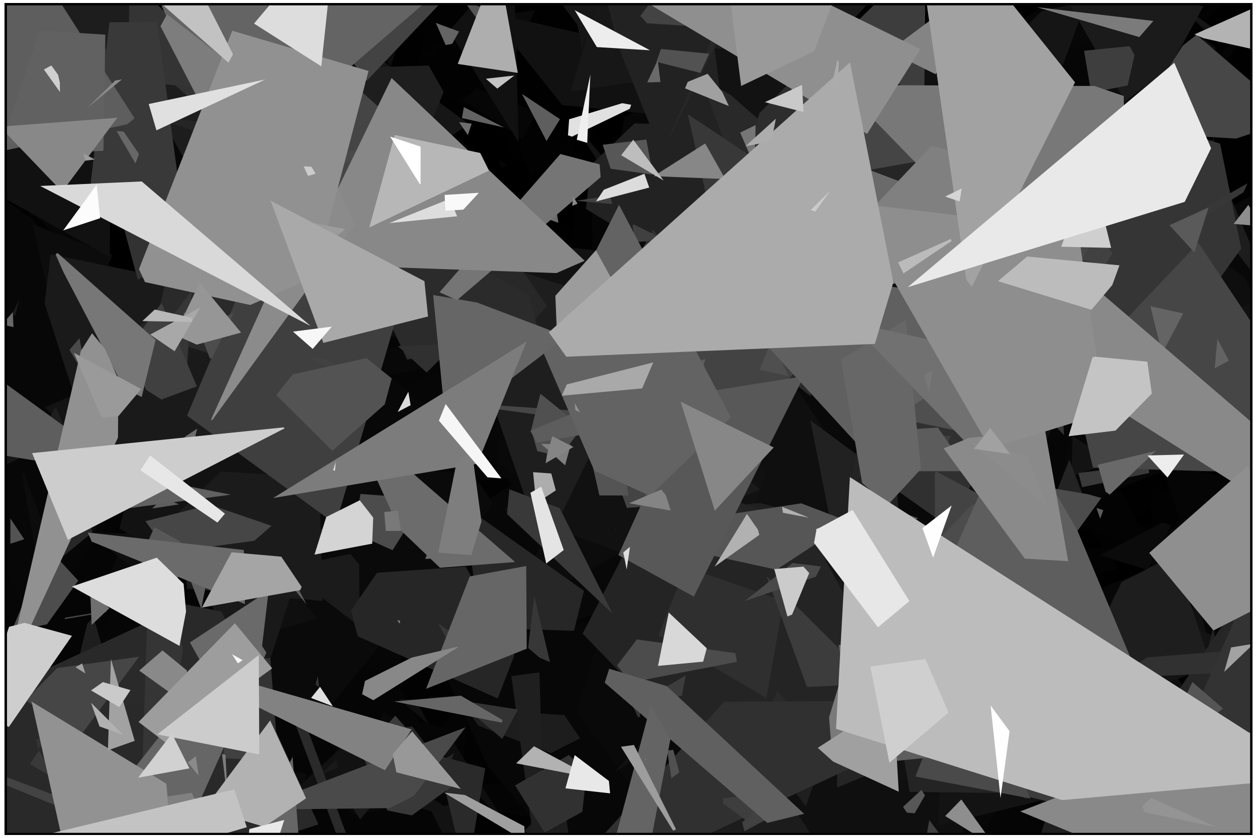
How does the algorithm terminate?

Consider a covering of the simulation domain D:



The algorithm terminates once all cells of the covering are contained in a Poisson storm.

Realization of a Poisson storm process



Simulation field 300×200 – mean polygon area $100\pi \approx 314$

Conclusions and perspectives

– All extremal coefficients are characterized by a hitting measure. This measure provides a physical interpretation of a number of consistency relationships between them;

– A planar storm process has been devised using the indicator function of Poisson polygons. Its extremal coefficients $\theta(K)$ are analytically tractable when K is convex or of limited cardinality;

– This spatial model can be extented to work in 3 dimensions (Poisson polyedra) and more than 3 dimensions (Poisson polytopes). However the question of how to simulate weighted polytopes is still open;

– It could be used as a benchmark to compare the performances of various estimators of extremal coefficients.