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COVERING $\mathbb{R}^d$ WITH POISSON RANDOM BALLS

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Stoch. Geom. Lille - April 2011

A covering problem

Setting: Consider a Poisson point process Φ in $\mathbb{R}^d \times (0, +\infty)$

with intensity measure $dx\, \mu(dr)$

dx: Lebesgue measure on $\mathbb{R}^d$
$\mu(dr)$: a σ-finite non-negative measure on $(0, +\infty)$

Consider the open random subset of $\mathbb{R}^d$

$$\Sigma = \bigcup_{(x,r)\in\Phi} B(x,r)$$

Question 1: $\Sigma = \mathbb{R}^d$ *a.s.*? if the case, say "coverage occurs"
Question 2: if not the case, how "big" is $\mathbb{R}^d \setminus \Sigma$?

1 Basic facts

- $\mathbf{P}(\Sigma = \mathbb{R}^d) = 0$ or 1 (due to ergodicity)
- $\mathbf{P}(B(0,1) \subset \Sigma) > 0 \;\Rightarrow\; \mathbf{P}(\Sigma = \mathbb{R}^d) = 1$ (due to stationarity)
- $\mathbf{P}(0 \notin \Sigma) = \mathbf{P}\left(\{(x,r) \in \Phi \; ; \; 0 \in B(x,r)\} = \emptyset\right)$

$$= \exp\left(-v_d \int_0^{+\infty} r^d \mu(dr)\right) \quad \text{(due to Poisson property)}$$

<u>Csq 1</u> : $\Sigma = \mathbb{R}^d \; a.s. \;\Rightarrow\; \int_0^{+\infty} r^d \mu(dr) = +\infty$

<u>Csq 2</u> : $\int_0^{+\infty} r^d \mu(dr) = +\infty \;\Rightarrow\; Leb\left(\mathbb{R}^d \setminus \Sigma\right) = 0 \; a.s.$

case where μ is finite ("germ-grain" or Boolean model)
or
case where $supp(\mu) \subset [a, +\infty)$ for some $a > 0$ (only "large" balls)

$$\Sigma \; = \; \mathbb{R}^d \; a.s. \; \Leftrightarrow \; \int_0^{+\infty} r^d \, \mu(dr) = +\infty$$

Rmk: coverage never occurs with balls of equal radius

<u>one-dimensional case</u> (Mandelbrot '72, Shepp '72)

$$\Sigma \;=\; \mathbb{R} \; a.s.$$

$$\Leftrightarrow$$

$$\int_0^1 \exp\left(2\int_u^{+\infty}(r-u)\mu(dr)\right)du = +\infty$$

$\Rightarrow$: still valid for $d > 1$ (needs to be adapted, see later)

$\Leftarrow$: based on stopping time argument, not valid anymore for $d > 1$

2 High and low frequency coverage

Notations:

$$\mu_H(dr) = \mathbf{1}_{(0,1]}(r)\mu(dr) \quad \text{and} \quad \Sigma_H = \bigcup_{(x,r)\in\Phi\,;\,r\leq 1} B(x,r)$$

$$\mu_L(dr) = \mathbf{1}_{(1,+\infty)}(r)\mu(dr) \quad \text{and} \quad \Sigma_L = \bigcup_{(x,r)\in\Phi\,;\,r>1} B(x,r)$$

then $\Sigma = \Sigma_L \cup \Sigma_H$ (disjoint and independent sets)

Theorem:

$$\Sigma = \mathbb{R}^d \ a.s. \ \Leftrightarrow \ \Sigma_L = \mathbb{R}^d \ a.s. \ \text{or} \ \Sigma_H = \mathbb{R}^d \ a.s.$$

Low frequency coverage

Let $\Sigma_{(>n)} = \bigcup_{(x,r)\in\Phi\,;\,r>n} B(x,r)$ and $\Sigma_L^\delta = \bigcup_{(x,r)\in\Phi\,;\,r>1} B(x,\delta r)$

$$\begin{aligned}
\mathbf{P}\left(\Sigma_L = \mathbb{R}^d\right) = 1 \quad &\Leftrightarrow \quad \int_1^{+\infty} r^d\,\mu(dr) = +\infty \\
&\Leftrightarrow \quad \forall n \geq 1\,,\ \int_n^{+\infty} r^d\,\mu(dr) = +\infty \\
&\Leftrightarrow \quad \mathbf{P}\left(\cap_{n\geq 1}\Sigma_{(>n)} = \mathbb{R}^d\right) = 1
\end{aligned}$$

and also

$$\mathbf{P}\left(\Sigma_L = \mathbb{R}^d\right) = 1 \quad \Leftrightarrow \quad \forall\delta > 0\,,\ \mathbf{P}\left(\Sigma_L^\delta = \mathbb{R}^d\right) = 1$$

So, <u>if</u> low coverage occurs, <u>then</u> a.s. any point of $\mathbb{R}^d$ is covered by an infinite number of arbitrarily large balls.

Proof of: $\Sigma = \mathbb{R}^d \; a.s. \;\Leftrightarrow\; \Sigma_L = \mathbb{R}^d \; a.s. \;$ or $\; \Sigma_H = \mathbb{R}^d \; a.s.$

To prove: coverage but not LF coverage $\Rightarrow$ HF coverage

- $\Sigma = \mathbb{R}^d \; a.s. \Rightarrow \; B(0,1) \subset \Sigma \; a.s.$
- $\mathbf{P}(\Sigma_L = \mathbb{R}^d) = 0 \;\Rightarrow\; \mathbf{P}(B(0,1) \cap \Sigma_L = \emptyset) > 0$

Then

$$\begin{aligned} p &:= \mathbf{P}(B(0,1) \cap \Sigma_L = \emptyset) \\ &= \mathbf{P}(B(0,1) \cap \Sigma_L = \emptyset \text{ and } B(0,1) \subset \Sigma) \\ &= \mathbf{P}(B(0,1) \cap \Sigma_L = \emptyset \text{ and } B(0,1) \subset \Sigma_H) \\ &= \mathbf{P}(B(0,1) \cap \Sigma_L = \emptyset) \times \mathbf{P}(B(0,1) \subset \Sigma_H) \\ &= p \times \mathbf{P}(B(0,1) \subset \Sigma_H) \end{aligned}$$

hence $\mathbf{P}(B(0,1) \subset \Sigma_H) = 1$ and so $\mathbf{P}(\Sigma_H = \mathbb{R}^d) = 1$

High frequency coverage

Theorem:

$$\limsup_{u\to 0} u^d \exp\left(v_d \int_u^1 (r-u)^d \mu(dr)\right) = +\infty$$

$$\Rightarrow$$

$$\Sigma_H = \mathbb{R}^d \ a.s.$$

$$\Rightarrow$$

$$\int_0^1 u^{d-1} \exp\left(v_d \int_u^1 r^{d-1}(r-u)\mu(dr)\right) du = +\infty$$

Remark: The necessary condition is the same as

- Shepp's ($d = 1$)

- Kahane's (convex sets with a “flat” boundary instead of balls)

Sketch of proof

Sufficient condition: discretize $[0,1]^d$ in order to prove that the suff. condition implies $\mathbf{P}([0,1]^d \subset \Sigma_{H}) > 0$

Necessary condition: for $\varepsilon > 0$, let

$$m_\varepsilon = Leb\left([0,1]^d \cap \Sigma^c_{H,\geq\varepsilon}\right) = \int_{[0,1]^d} \mathbf{1}_{y \notin \Sigma_{H,\geq\varepsilon}} \, dy$$

Then $\mathbf{E}(m_\varepsilon) = e^{-\kappa_\varepsilon}$ with $\kappa_\varepsilon = v_d \int_\varepsilon^1 r^d \mu(dr)$
$\mathbf{E}(m_\varepsilon^2) \leq ... \leq I\, e^{-2\kappa_\varepsilon}$ with $I \in (0, +\infty)$ if necessary cond. not true
By Cauchy-Schwartz,

$$\mathbf{P}(m_\varepsilon > 0) \geq \frac{\mathbf{E}(m_\varepsilon)^2}{\mathbf{E}(m_\varepsilon^2)} \geq I^{-1}$$

Hence, $\mathbf{P}([0,1]^d \not\subset \Sigma_{H,\geq\varepsilon}) \geq I^{-1}$ and so $\mathbf{P}([0,1]^d \not\subset \Sigma_{H}) > 0$

3 Critical intensity

notation: $\psi(\mu) = \mathbf{P}\left(\Sigma = \mathbb{R}^d\right)$ (recall that Σ depends on μ)

clear:

- $\psi(\mu) = 0$ or 1
- for fixed μ, $\lambda \mapsto \psi(\lambda\mu)$ is non decreasing
- $\psi(\mu) = \max\left(\psi(\mu_L), \psi(\mu_H)\right)$
- $\psi(\mu_L) = 1 \;\Leftrightarrow\; \int_1^{+\infty} r^d \, \mu(dr) = +\infty$
- $\psi(\mu_L) = \psi(\lambda\mu_L)\;,\;\forall\lambda > 0$

Exact value of the critical intensity

Theorem: let $\psi(\mu_{_H}) = \mathbf{P}\left(\Sigma_{_H} = \mathbb{R}^d\right)$

There exists $\lambda_c(\mu) \in [0, +\infty]$ s.t.

- $\forall\lambda < \lambda_c(\mu)$, $\psi(\lambda\mu_{_H}) = 0$
- $\forall\lambda > \lambda_c(\mu)$, $\psi(\lambda\mu_{_H}) = 1$

Moreover $\lambda_c(\mu) = d/\ell(\mu)$ where

$$\ell(\mu) = \limsup_{u\to 0}\left(|\ln u|^{-1}\, v_d \int_u^1 r^d\, \mu(dr)\right) \in [0, +\infty]$$

Hausdorff dimensions (same ideas as El Helou's '78)

Theorem: Let A be a compact subset of $[0,1]^d$.

- If $\dim_{\mathcal{H}}(A) > \ell(\mu)$ then $\mathbf{P}(A \neq \Sigma_H) > 0$
- If $0 < \dim_{\mathcal{H}}(A) < \ell(\mu)$ then $A = \Sigma_H$ *a.s.*
- If $\ell(\mu) \in (0, d]$ then

$$\mathbf{P}\left(\dim_{\mathcal{H}}([0,1]^d \cap \Sigma^c) = d - \ell(\mu)\right) > 0$$

and

$$\dim_{\mathcal{H}}([0,1]^d \cap \Sigma^c) \leq d - \ell(\mu) \ a.s.$$

Link with continuum percolation

- **Occupancy percolation**
  question: is the connected component of Σ containing 0 unbounded?

$$\text{critical intensity}\,\lambda_{occup}(\mu) \le \lambda_c(\mu) = d/\ell(\mu)$$

- **Vacancy percolation**
  question: is there a connected component of Σ^c larger than one point?

$$\text{critical intensity}\,\lambda_{vac}(\mu) \le (d-1)/\ell(\mu)$$

4 Examples and discussion

Power law measure: $\mu^{\alpha}(dr) := r^{-\alpha}\, dr$

- low frequency

$$\Sigma_L^{\alpha} = \mathbb{R}^d \ a.s. \ \Leftrightarrow \ \alpha \le d+1$$

- high frequency
  - $\alpha > d+1 \ \Rightarrow \ \mathbf{P}(\Sigma_H^{\alpha} = \mathbb{R}^d) = 1$
  - $\alpha < d+1 \ \Rightarrow \ \mathbf{P}(\Sigma_H^{\alpha} = \mathbb{R}^d) = 0$
  - for $\alpha = d+1$, $\mathbf{P}(\Sigma_H^{d+1} = \mathbb{R}^d) = 1 \ \Leftrightarrow \ d \le 5$
    (μ_H^{d+1} is scale invariant and $\lambda_c(\mu_H^{d+1}) = d/v_d$)

Critical "power law" measures

Let us consider the high frequency measures

$$\mu_+(\mathrm{d}r) = r^{-(d+1)}\ d/v_d\ (1 + 2|\ln(r)|^{-1})\ \mathbf{1}_{(0,a)}(r)\mathrm{d}r$$

$$\mu_-(\mathrm{d}r) = r^{-(d+1)}\ d/v_d\ (1 - 2|\ln(r)|^{-1})\ \mathbf{1}_{(0,a)}(r)\mathrm{d}r$$

where $a > 0$ is s.t. $1 - 2|\ln(a)|^{-1} > 0$

Some calculus... $\ell(\mu_+) = \ell(\mu_-) = d$, so that $\lambda_c(\mu_+) = \lambda_c(\mu_-) = 1$

Applying the sufficient and the necessar conditions, we get that

$$\Sigma_+ = \mathbb{R}^d\ a.s. \quad \text{and} \quad \Sigma_- \neq \mathbb{R}^d\ a.s.$$

Multiscale measure: $\mu_\rho(dr) = \sum_n \rho^{nd} \delta_{\rho^{-n}}(dr)$ $(\rho > 0)$

(The intensity measure is scale invariant)

- case $\rho < 1$ (low frequency):

$$\Sigma = \mathbb{R}^d \ a.s. \ \text{ since } \int_1^\infty r^d \mu_\rho(dr) = +\infty$$

- case $\rho > 1$ (high frequency): $\ell(\mu_\rho) = v_d / \ln(\rho)$, then

  if $\ln(\rho) < v_d/d$ then $\Sigma = \mathbb{R}^d$ $a.s.$

  if $\ln(\rho) > v_d/d$ then $\Sigma \neq \mathbb{R}^d$ $a.s.$

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