Visibility estimates in Euclidean and hyperbolic germ-grain models and line tessellations

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Outline

Visibility in the vacancy of the Boolean model in $\mathbb{R}^d$

Visibility in a hyperbolic Boolean model

Visibility in a Poisson line tessellation

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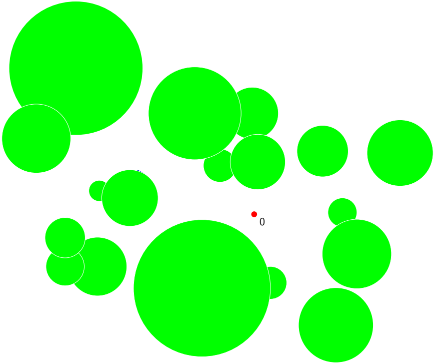
Outline

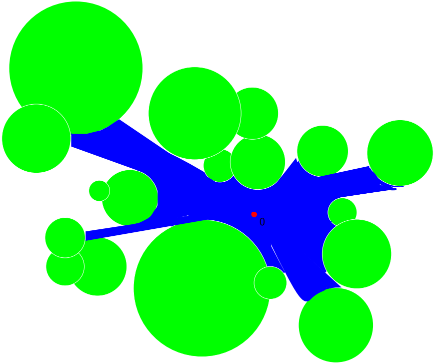
Boolean model in $\mathbb{R}^d$

- $\mathbf{X}_\lambda :=$ homogeneous Poisson point process of intensity λ in $\mathbb{R}^d$
- $\mathbf{K} :=$ random convex compact set containing the origin and with an a.s. bounded diameter
- K_x, $x \in \mathbf{X} :=$ collection of i.i.d. random convex grains distributed as $\mathbf{K}$
- $\mathcal{O} := \bigcup_{x \in \mathbf{X}} (x \oplus K_x)$ occupied phase
- Process conditioned on the event $A = \{O \notin \mathcal{O}\}$, $\mathbb{P}[A] = \exp(-\lambda \mathbb{E}[V_d(\mathbf{K})])$
- $O :=$ observer at the origin

Visibility star and maximal visibility

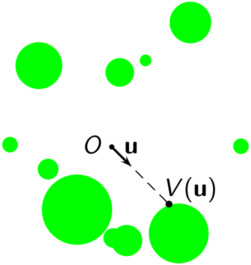


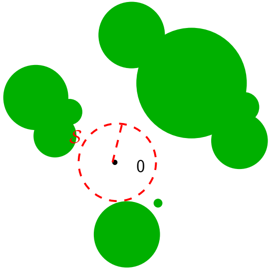
Visibility star and maximal visibility



$\mathcal{V} :=$ maximal visibility, i.e. distance to the furthest visible point

Visibility in one direction and spherical contact length





Visibility in direction $u \in \mathbb{S}^{d-1}$

$V(u) := \sup\{r > 0 : [0, r]u \subset \mathcal{O}^c\}$

$\mathbb{P}[V(u) \geq r] = \exp(-\lambda \frac{s_{d-1}}{(d-1)s_d} \mathbb{E}[V_{d-1}(\mathbf{K})] r)$

Spherical contact length

$\mathcal{S} := \inf_{u \in \mathbb{S}^{d-1}} V(u)$

$\mathbb{P}[\mathcal{S} \geq r] = \exp(-\lambda \mathbb{E}[V_d(B(0, r) \oplus \mathbf{K}) - V_d(\mathbf{K})])$

Distribution tail of the maximal visibility

$$\rho_{\max}(\mathbf{K}) := \sup\{r > 0 : \exists\ (d-1)\text{-dimensional ball } B_{d-1}(x,r) \subset \mathbf{K}\}$$

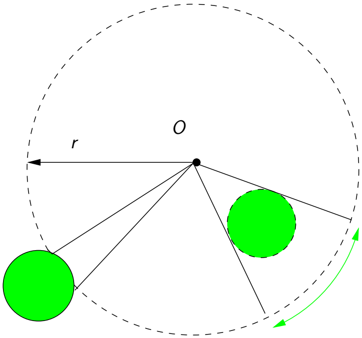
If $(d = 2)$ or
if $(d \geq 3$ and $\mathbb{P}[\rho_{\max}(\mathbf{K}) \leq \varepsilon^{d-2}] = O(\varepsilon)$ when $\varepsilon \to 0)$,
then

$$\log \mathbb{P}[\mathcal{V} \geq r] = \log \mathbb{P}[V(u) \geq r] + d(d-1)\log(r) + O(1).$$

Remark. Non-asymptotic upper and lower bounds in dimension two

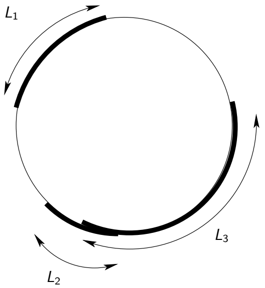
Proof: connection with random coverings of the sphere

Each obstacle creates a shadow on the sphere of radius r:



$\mathbb{P}[\mathcal{V} \geq r] =$ probability that the sphere of radius r is not covered

Proof: covering probability of $\mathbb{S}^{d-1}$



- n i.i.d. random geodesic balls in the unit-sphere $\mathbb{S}^{d-1}$ with uniformly distributed centers and ν-distributed random (normalised) radii

  (where ν is a probability measure on $[0, 1/2]$).
- Probability to cover $\mathbb{S}^{d-1}$?

Proof: asymptotic estimation of the covering probability

$$\mathbb{P}[u_0 \text{ not covered}] = \left(1 - \int \varphi_d(z)\mathrm{d}\nu(z)\right)^n \ (u_0 \in \mathbb{S}^{d-1} \text{ fixed})$$

where $\varphi_d(t) = \frac{s_{d-1}}{s_d} \int_0^{\pi t} \sin^{d-2}(\theta)\mathrm{d}\theta$, $0 \leq t \leq 1$ and s_d is the area of $\mathbb{S}^{d-1}$.

If $\nu([0,\varepsilon]) = O(\varepsilon)$ when $\varepsilon \to 0$, then

$$\log \mathbb{P}[\mathbb{S}^{d-1} \text{ uncovered}] = \log \mathbb{P}[u_0 \text{ uncovered}] + (d-1)\log(n) + O(1).$$

The expansion only depends on the *mean* $\varphi_d^{-1}\left(\int \varphi_d(z)\mathrm{d}\nu(z)\right)$.

Percolation in visibility vs. continuum percolation

Continuum percolation:
if the vacant set has an unbounded component with probability > 0
Percolation in visibility:
if the visibility is not finite with probability > 0

Roy (1990), **Meester & Roy** (1994), **Sarkar** (1997)
In $\mathbb{R}^2$ with balls, $\exists\ 0 < \lambda_c < \infty$ s.t. with probability 1

λ	# unbounded c.c. of $\mathcal{O}$	# unbounded c.c. of $\mathbb{H}^2 \setminus \mathcal{O}$
$[0, \lambda_c)$	0	1
λ_c	0	0
(λ_c, ∞)	1	0

But there is no percolation in visibility in $\mathbb{R}^d$!

Convergence with small obstacles

Context: deterministic radii all equal to $R \to 0$.
Question: behaviour of the associated visibility $\mathcal{V}_R$?

$$\mathcal{V}_R = -c_d^{(1)} \frac{\log(R)}{R^{d-1}} + c_d^{(2)} \frac{\log|\log(R)|}{R^{d-1}} + \frac{c_d^{(3)} + c_d^{(4)} \xi_R}{R^{d-1}}$$

where ξ_R converges in distribution to the Gumbel law when $R \to 0$.

Proof. $\mathbb{P}[\mathcal{V}_R \geq f(R)] =$ probability to cover the unit-sphere with a large number of small spherical caps.

Janson (1986): Poisson number of mean Λ of spherical caps with radius εU (U bounded variable). If

$$K_1 \varepsilon^{d-1} \Lambda + (d-1)\log(\varepsilon) - (d-1)\log(-\log(\varepsilon)) + K_2 \underset{\varepsilon \to 0}{\longrightarrow} u,$$

then the covering probability goes to $\exp(-e^{-u})$.

Visibility to infinity in the Euclidean space

Shrinking obstacles when the distance to the origin increases:

$$\mathcal{O}_\beta := \bigcup_{x \in \mathbf{X}} B(x, \|x\|^\beta)$$

Visibility to infinity with probability > 0 iff $\beta < -1/(d-1)$

Rarefaction of the Boolean model at large distances:

$\mathbf{Y}_\alpha :=$ Poisson point process of intensity measure $\|x\|^{\alpha - d} \mathrm{d}x$

Visibility to infinity with probability > 0 iff $\alpha < 1$

Outline

Poincaré disc model

- Model Unit disc $\mathbb{D} = \mathbb{H}^2$ equipped with the hyperbolic metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

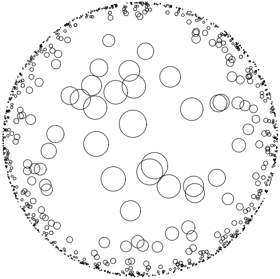
- Length $L(\gamma) = 2\int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2}dt$
- Isometries $\mathrm{Aut}(\mathbb{D}) = \{z \longmapsto \frac{az+\overline{c}}{cz+\overline{a}} : |a|^2 - |c|^2 = 1\}$
- Area

$$\mu_{\mathbb{H}}(dr, d\theta) = \frac{4r}{(1 - r^2)^2}\mathbf{1}_{]0,1[}(r)drd\theta$$

- Balls $R > 0$, $\overline{R} := \tanh(R/2)$

$$B_{\mathbb{H}}(z, R) = B_{\mathbb{R}^2}(z\frac{1 - \overline{R}^2}{1 - |z|^2\overline{R}^2}, \overline{R}\frac{1 - |z|^2}{1 - |z|^2\overline{R}^2})$$

Boolean model in the hyperbolic plane



$\mathbf{X}_\lambda :=$ Poisson point process of intensity measure $\lambda\mu_{\mathbb{H}}(\mathrm{d}r)$

$$\mathcal{O} := \bigcup_{z\in\mathbf{X}_\lambda} B_{\mathbb{H}}(z,R_z) = \bigcup_{z\in\mathbf{X}_\lambda} B_{\mathbb{R}^2}\Big(z\frac{1-\overline{R_z}^2}{1-|z|^2\overline{R_z}^2}, \overline{R_z}\frac{1-|z|^2}{1-|z|^2\overline{R_z}^2}\Big)$$

where R_x, $x\in\mathbf{X}_\lambda$, bounded i.i.d. r.v.

Continuum percolation and visibility percolation in $\mathbb{H}^2$

Tykesson (2005)

$\exists\ 0 < \lambda_o < \lambda_v < \infty$ s.t. with probability 1

λ	# unbounded c.c. of $\mathcal{O}$	# unbounded c.c. of $\mathbb{H}^2 \setminus \mathcal{O}$
$[0, \lambda_o]$	0	1
(λ_o, λ_v)	∞	∞
$[\lambda_v, \infty)$	1	0

Benjamini, Jonasson, Schramm, Tykesson (2009)

(deterministic R)

- Visibility to infinity with probability > 0 iff $2\lambda\sinh(R) < 1$
- Visibility to infinity with probability > 0 inside balls iff $\lambda > \lambda'_c$

Visibility in one direction in $\mathbb{H}^2$

Deterministic R

Benjamini, Jonasson, Schramm, Tykesson (2009)

$V(u) := \sup\{r > 0 : [0, ru] \subset \mathbb{H}^2 \setminus \mathcal{O}\}$

$$\mathbb{P}[V(u) \geq r] = \Theta(e^{-\alpha r}) \quad \text{where } \alpha = 2\lambda \sinh(R)$$

Proof.

- *Positive correlations* $\mathbb{E}[\varphi(\mathcal{O})\psi(\mathcal{O})] \geq \mathbb{E}[\varphi(\mathcal{O})]\mathbb{E}[\psi(\mathcal{O})]$ for all bounded, increasing and measurable functions φ and ψ

$$\mathbb{P}[V(u) \geq r + s] \geq \mathbb{P}[V(u) \geq r]\mathbb{P}[V(u) \geq s], \quad r, s > 0$$

- Calculation of $\mu_{\mathbb{H}^2}(\{z \in \mathbb{H}^2 : d_{\mathbb{H}^2}(z, [0, ru]) < R\})$

Random R: if $\mathbb{E}(e^R) < \infty$, $\alpha = 2\lambda\mathbb{E}[\sinh(R)]$

Distribution tail of the maximal visibility in $\mathbb{H}^2$

$$\mathcal{V} := \sup\{r > 0 : \exists u \in \mathbb{S}^1 \text{ s.t. } ru \in \mathbb{H}^2 \setminus \mathcal{O}\}$$

$$\mathbb{P}[\mathcal{V} \geq r] = \begin{cases} \Omega(1)e^{-(\alpha-1)r} & \text{if } \alpha > 1 \\ \Omega(1)\frac{1}{r} & \text{if } \alpha = 1 \\ C + o(1) & \text{if } \alpha < 1 \end{cases}$$

- Different behaviours of $\mathcal{V}$ and $V(u)$
- Polynomial decay in the critical case

Proof: second moment method

$u_0 \in \mathbb{S}^1$ fixed

$$Y_r = Y_r(\varepsilon) := \{u \in \mathbb{S}^1 : \langle u, u_0 \rangle \in [0, \varepsilon) \text{ and } [0, ru] \subset \mathbb{H}^2 \setminus \mathcal{O}\}$$

$$y_r = y_r(\varepsilon) := \int_{Y_r(\varepsilon)} d\theta$$

$$\frac{\mathbb{E}[y_r]^2}{\mathbb{E}[y_r^2]} \leq \mathbb{P}[Y_r \neq \emptyset] = \mathbb{P}[\mathcal{V}(\varepsilon) \geq r] \leq 4 \frac{\mathbb{E}[y_r]^2}{\mathbb{E}[y_r^2]}$$

where

$$\mathbb{E}[y_r(\varepsilon)] = \varepsilon \mathbb{P}[u_0 \in Y_r(\varepsilon)]$$

$$\mathbb{E}[y_r(\varepsilon)^2] = \Theta\left(\varepsilon \int_0^\varepsilon \mathbb{P}[u_0, u_\theta \in Y_r(\varepsilon)] d\theta\right)$$

Proof

Upper bound of the second moment method

$$\mathbb{P}[Y_r \neq \emptyset] = \frac{\mathbb{E}[y_r]}{\mathbb{E}[y_r | Y_r \neq \emptyset]}$$
$$\leq 2\frac{\mathbb{E}[y_r]}{\mathbb{E}[y_r | u_0 \in Y_r]} = \frac{2E[y_r]^2}{\varepsilon \int_0^\varepsilon \mathbb{P}[u_0, u_\theta \in Y_r] d\theta}$$

$\leq$: discretization and sort of 'Markovianity' of $\mathbf{1}_{Y_r}(u_\theta)$

Calculation of $\mathbb{P}[u_0, u_\theta \in Y_r]$

Measure of the union of two hyperbolic cylinders
Use of FKG inequality

Further results

Near criticality: critical exponents

$\mathcal{E} := \{z \in \mathbb{H}^2 : [0,z] \in \mathbb{H}^2 \setminus \mathcal{O}\}$
When $\lambda \searrow \lambda_c = (2\mathbb{E}[\sinh(R)])^{-1}$,
$\mathbb{E}[\mu_{\mathbb{H}^2}(\mathcal{E})] = \Theta\left(\frac{1}{\alpha-1}\right)$ and $\mathbb{E}[\mathcal{V}] = \Theta\left(\frac{1}{\alpha-1}\right)$

Rarefaction of the Boolean model

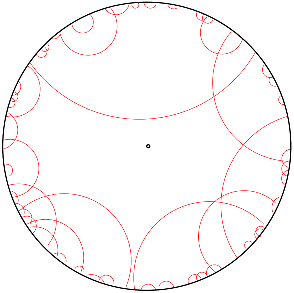
When $\lambda \searrow 0$, $\mathbb{P}[\mathcal{V} = \infty] \to 1$

Intensity as a functional of the radius

R deterministic
For $r > 0$ and $p \in (0,1)$, $\exists$ explicit $\lambda(R)$ s.t. $\lim_{R \to 0} \mathbb{P}[\mathcal{V} \leq r] = p$.

Outline

Poisson line tessellation



$\mathcal{P}_\lambda :=$ Poisson point process in $\mathbb{H}^2$ of intensity measure

$$\nu_\lambda(dr, d\theta) = 2\lambda \frac{1+r^2}{(1-r^2)^2} dr d\theta$$

$\mathcal{L} := \bigcup_{x \in \mathcal{P}_\lambda} G_x$ invariant by Aut($\mathbb{D}$)

where $G_x :=$ hyperbolic line containing x and orthogonal to $[0, x]$

Visibility in the Poisson line tessellation

$\mathcal{V} := \sup\{r > 0 : \exists\ u \in \mathbb{S}^1 \text{ s.t. } ru \cap \mathcal{L} = \emptyset\}$

$\mathcal{V}$ circumscribed radius of the zero-cell from the tessellation

$$\mathbb{P}[\mathcal{V} \geq r] = \begin{cases} \Omega(1)e^{-(2\lambda-1)r} & \text{if } \lambda > 1/2 \\ \Omega(1)\frac{1}{r} & \text{if } \lambda = 1/2 \\ C + o(1) & \text{if } \lambda < 1/2 \end{cases}$$

Euclidean case
No visibility percolation, explicit distribution in dimension two (2002)

Prospects

- Visibility star, study of its radius-vector function
- Number of visible obstacles
- Extension to other covering models with hard spheres
- Behaviour of the visibility near the criticality
- Calculation of a Hausdorff dimension
- Visibility inside balls

Thank you for your attention!