HILBERTIAN JAMISON SEQUENCES AND RIGID DYNAMICAL SYSTEMS

by

Tanja Eisner & Sophie Grivaux

Abstract. — A strictly increasing sequence $(n_k)_{k\geq 0}$ of positive integers is said to be a *Hilbertian Jamison sequence* if for any bounded operator T on a separable Hilbert space such that $\sup_{k\geq 0}||T^{n_k}|| < +\infty$, the set of eigenvalues of modulus 1 of T is at most countable. We first give a complete characterization of such sequences. We then turn to the study of rigidity sequences $(n_k)_{k\geq 0}$ for weakly mixing dynamical systems on measure spaces, and give various conditions, some of which are closely related to the Jamison condition, for a sequence to be a rigidity sequence. We obtain on our way a complete characterization of topological rigidity and uniform rigidity sequences for linear dynamical systems, and we construct in this framework examples of dynamical systems which are both weakly mixing in the measure-theoretic sense and uniformly rigid.

1. Introduction and main results

We are concerned in this paper with the study of certain dynamical systems, in particular linear dynamical systems. Our main aim is the study of *rigidity sequences* $(n_k)_{k\geq 0}$ for weakly mixing dynamical systems on measure spaces, and we present tractable conditions on the sequence $(n_k)_{k\geq 0}$ which imply that it is (or not) a rigidity sequence. Our conditions on the sequence $(n_k)_{k\geq 0}$ come in part from the study of the so-called *Jamison sequences*, which appear in the description of the relationship between partial power-boundedness of an operator on a separable Banach space and the size of its unimodular point spectrum.

Let us now describe our results more precisely.

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1.1. A characterization of Hilbertian Jamison sequences. — Let X be a separable infinite-dimensional complex Banach space, and let $T \in \mathcal{B}(X)$ be a bounded operator on X. We are first going to study here the relationship between the behavior of the sequence $||T^n||$ of the norms of the powers of T, and the size of the unimodular point spectrum $\sigma_p(T) \cap \mathbb{T}$, i.e. the set of eigenvalues of T of modulus 1. It is known since an old result of Jamison [**19**] that a slow growth of $||T^n||$ makes $\sigma_p(T) \cap \mathbb{T}$ small, and vice-versa. More precisely, the result of [**19**] states that if T is power-bounded, i.e. $\sup_{n\geq 0} ||T^n|| < +\infty$, then $\sigma_p(T) \cap \mathbb{T}$ is at most countable. For a sample of the kind of results which can be obtained in the other direction, let us mention the following result of Nikolskii [**28**]: if T is a bounded operator on a separable Hilbert space such that $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure, then the series $\sum_{n\geq 0} ||T^n||^{-2}$ is convergent. This has been genera lized by Ransford in the paper [**33**], which renewed the interest in these matters. In particular Ransford started to investigate in [**33**] the influence of partial power-boundedness of an operator on the size of its unimodular point spectrum. Let us recall the following definition:

Definition 1.1. — Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers, and T a bounded linear operator on the space X. We say that T is *partially power-bounded with respect to* (n_k) if $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$.

In view of the result of Jamison, it was natural to investigate whether the partial power-boundedness of T with respect to (n_k) implies that $\sigma_p(T) \cap \mathbb{T}$ is at most countable. It was shown in [**34**] by Ransford and Roginskaya that it is not the case: if $n_k = 2^{2^k}$ for instance, there exists a separable Banach space X and $T \in \mathcal{B}(X)$ such that $\sup_{k\geq 0} ||T^{n_k}||$ is finite while $\sigma_p(T) \cap \mathbb{T}$ is uncountable. This question was investigated further in [**1**] and [**2**], where the following definition was introduced:

Definition 1.2. — Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers. We say that $(n_k)_{k\geq 0}$ is a *Jamison sequence* if for any separable Banach space X and any bounded operator T on X, $\sigma_p(T) \cap \mathbb{T}$ is at most countable as soon as T is partially power-bounded with respect to (n_k).

Whether $(n_k)_{k\geq 0}$ is a Jamison sequence or not depends of course on features of the sequence such as its growth, its arithmetical properties, etc. A complete characterization of Jamison sequences was obtained in [**2**]. It is formulated as follows:

Theorem 1.3. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers with $n_0 = 1$. The following assertions are equivalent:*

(1) *$(n_k)_{k\geq 0}$ is a Jamison sequence;*
(2) *there exists a positive real number ε such that for every $\lambda \in \mathbb{T} \setminus \{1\}$,*

$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon.$$

Many examples of Jamison and non-Jamison sequences were obtained in [**1**] and [**2**]. Among the examples of non-Jamison sequences, let us mention the sequences $(n_k)_{k\geq 0}$ such that n_{k+1}/n_k tends to infinity, or such that n_k divides n_{k+1} for each $k \geq 0$ and $\limsup n_{k+1}/n_k = +\infty$. Saying that $(n_k)_{k\geq 0}$ is not a Jamison sequence means that there exists a separable Banach space X and $T \in \mathcal{B}(X)$ such that $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$ and

$\sigma_p(T) \cap \mathbb{T}$ is uncountable. But the space X may well be extremely complicated: in the proof of Theorem 1.3, the space is obtained by a rather involved renorming of a classical space such as ℓ_2 for instance. This is a drawback in applications, and this is why it was investigated in [**1**] under which conditions on the sequence $(n_k)_{k\geq 0}$ it was possible to construct partially power-bounded operators with respect to $(n_k)_{k\geq 0}$ with uncountable unimodular point spectrum on a Hilbert space. It was proved in [**1**] that if the series $\sum_{k\geq 0}(n_k/n_{k+1})^2$ is convergent, there exists a bounded operator T on a separable Hilbert space H such that $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$ and $\sigma_p(T) \cap \mathbb{T}$ is uncountable. But this left open the characterization of Hilbertian Jamison sequences.

Definition 1.4. — We say that $(n_k)_{k\geq 1}$ is a *Hilbertian Jamison sequence* if for any bounded operator T on a separable infinite-dimensional complex Hilbert space which is partially power-bounded with respect to (n_k), $\sigma_p(T) \cap \mathbb{T}$ is at most countable.

Obviously a Jamison sequence is a Hilbertian Jamison sequence. Our first goal in this paper is to prove the somewhat surprising fact that the converse is true:

Theorem 1.5. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers. Then $(n_k)_{k\geq 0}$ is a Hilbertian Jamison sequence if and only if it is a Jamison sequence.*

Contrary to the proofs of [**1**] and [**2**], the proof of Theorem 1.5 is completely explicit: the operators with $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$ and $\sigma_p(T) \cap \mathbb{T}$ uncountable which we construct are perturbations by a weighted backward shift on ℓ_2 of a diagonal operator with unimodular diagonal coefficients. The construction here bears some similarities with a construction carried out in a different context in [**10**] in order to obtain frequently hypercyclic operators on certain Banach spaces.

1.2. Ergodic theory and rigidity sequences. — Let $(X, \mathcal{F}, \mu)$ be a finite measure space where μ is a positive regular finite Borel measure, and let φ be a measurable transformation of $(X, \mathcal{F}, \mu)$. We recall here that φ is said to *preserve the measure* μ if $\mu(\varphi^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{F}$, and that φ is said to be *ergodic* with respect to μ if for any $A, B \in \mathcal{F}$ with $\mu(A) > 0$ and $\mu(B) > 0$, there exists an $n \geq 0$ such that $\mu(\varphi^{-n}(A) \cap B) > 0$, where φ^n denotes the n^{th} iterate of φ. Equivalently, φ is ergodic with respect to μ if and only if

$$\frac{1}{N}\sum_{n=1}^{N} \mu(\varphi^{-n}(A) \cap B) \to \mu(A)\mu(B) \quad \text{as } N \to +\infty \quad \text{for every } A, B \in \mathcal{F}.$$

This leads to the notion of weakly mixing measure-preserving transformation of $(X, \mathcal{F}, \mu)$: φ is *weakly mixing* if

$$\frac{1}{N}\sum_{n=1}^{N} |\mu(\varphi^{-n}(A) \cap B) - \mu(A)\mu(B)| \to 0 \quad \text{as } N \to +\infty \quad \text{for every } A, B \in \mathcal{F}.$$

It is well-know that φ is weakly mixing if and only if $\varphi \times \varphi$ is an ergodic transformation of $X \times X$ endowed with the product measure $\mu \times \mu$. We refer the reader to [**9**], [**30**] or [**38**] for instance for more about ergodic theory of dynamical systems and various examples.

Our interest in this paper lies in weakly mixing *rigid* dynamical systems:

Definition 1.6. — A measure-preserving transformation of $(X, \mathcal{F}, \mu)$ is said to be *rigid* if there exists a sequence $(n_k)_{k\geq 0}$ of integers such that for any $A \in \mathcal{F}$, $\mu(\varphi^{-n_k}(A) \triangle A) \to 0$ as $k \to +\infty$.

If U_φ denotes the isometry on $L^2(X, \mathcal{F}, \mu)$ defined by $U_\varphi f := f \circ \varphi$ for any $f \in L^2(X, \mathcal{F}, \mu)$, it is not difficult to see that φ is rigid with respect to the sequence $(n_k)_{k\geq 0}$ if and only if $||U_\varphi^{n_k} f - f|| \to 0$ as $k \to +\infty$ for any $f \in L^2(X, \mathcal{F}, \mu)$. The function f itself is said to be rigid with respect to $(n_k)_{k\geq 0}$ if $||U_\varphi^{n_k} f - f|| \to 0$. Rigid functions play a major role in the study of mildly mixing dynamical systems, as introduced by Furstenberg and Weiss in [**12**], and rigid weakly mixing systems are intensively studied, see for instance the works [**15**], [**26**], [**16**] or [**36**] as well as the references therein for some examples of results and methods. Let us just mention here the fact that weakly mixing rigid transformations of $(X, \mathcal{F}, \mu)$ form a residual subset of the set of all measure-preserving transformations of $(X, \mathcal{F}, \mu)$ for the weak topology [**23**]. A *rigidity sequence* is defined as follows:

Definition 1.7. — Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of positive integers. We say that $(n_k)_{k\geq 0}$ is a *rigidity sequence* if there exists a measure space $(X, \mathcal{F}, \mu)$ and a measure-preserving transformation φ of $(X, \mathcal{F}, \mu)$ which is weakly mixing and rigid with respect to $(n_k)_{k\geq 0}$.

Remark 1.8. — In the literature one often defines rigidity sequences as sequences for which there exists an *invertible* measure-preserving transformation which is weakly mixing and rigid with respect to $(n_k)_{k\geq 0}$. In fact, these two definitions are equivalent since every rigid measure-preserving transformation φ is invertible (in the measure-theoretic sense). An easy way to see it is to consider the induced isometry U_φ defined above. Since φ is invertible if and only if U_φ is so, it suffices to show that U_φ is invertible. By the decomposition theorem for contractions due to Sz.-Nagy, Foiaş [**37**], U can be decomposed into a direct sum of a unitary operator and a weakly stable operator. Since $\lim_{k\to\infty} U_\varphi^{n_k} = I$ in the weak operator topology (see Fact 3.2 below), the weakly stable part cannot be present and thus U is a unitary operator and φ is invertible.

Rigidity sequences are in a sense already characterized: $(n_k)_{k\geq 0}$ is a rigidity sequence if and only if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\sigma(\lambda) \to 0 \quad \text{ as } k \to +\infty$$

(see Section 3.1 for more details). Still, there is a lack of practical conditions which would enable us to check easily whether a given sequence $(n_k)_{k\geq 0}$ is a rigidity sequence. It is the second aim of this paper to provide such conditions. Some of them can be initially found in the papers [**1**] and [**2**] which study Jamison sequences in the Banach space setting, and they turn out to be relevant for the study of rigidity. We show for instance that if n_{k+1}/n_k tends to infinity as k tends to infinity, $(n_k)_{k\geq 0}$ is a rigidity sequence (see Example 3.4 and Proposition 3.5). If $(n_k)_{k\geq 0}$ is any sequence such that n_k divides n_{k+1} for any $k \geq 0$, $(n_k)_{k\geq 0}$ is a rigidity sequence (Propositions 3.8 and 3.9). We also give some examples involving the denominators of the partial quotients in the continuous fraction expansion of some irrational numbers (Examples 3.15 and 3.16), as well as an example of a rigidity sequence such that $n_{k+1}/n_k \to 1$ (Example 3.17). In the other direction, it is not difficult to show that if $n_k = p(k)$ for some polynomial $p \in \mathbb{Z}[X]$ with $p(k) \geq 0$ for any k, $(n_k)_{k\geq 0}$ cannot be a rigidity sequence (Example 3.12), or that the sequence of prime numbers cannot be a rigidity sequence (Example 3.14). Other examples of non-rigidity sequences can be given (Example 3.13) when the sequences $(n_k x)_{k\geq 0}$, $x \in [0,1]$, have suitable equirepartition properties.

1.3. Ergodic theory and rigidity for linear dynamical systems. — If T is a bounded operator on a separable Banach space X, it is sometimes possible to endow the space X with a suitable probability measure m, and to consider $(X, \mathcal{B}, m, T)$ as a measurable dynamical system. This was first done in the seminal work [**11**] of Flytzanis, and the study was continued in the papers [**4**] and [**5**]. If X is a separable complex Hilbert space which we denote by H, $T \in \mathcal{B}(H)$ admits a non-degenerate invariant Gaussian measure if and only if its eigenvectors associated to eigenvalues of modulus 1 span a dense subspace of H, and it is ergodic (or here, equivalently, weakly mixing) with respect to such a measure if and only if it has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues (see Section 2.1 for the definitions) – this condition very roughly means that T has "plenty" of such eigenvectors, "plenty" being quantified by a continuous probability measure on the unit circle.

It comes as a natural question to describe rigidity sequences in the framework of linear dynamics, and it is not difficult to show that if $(n_k)_{k\geq 0}$ is a rigidity sequence, there exists a bounded operator on H which is weakly mixing and rigid with respect to $(n_k)_{k\geq 0}$ (see Section 4.1). Thus, every rigidity sequence can be realized in a linear Hilbertian measure-preserving dynamical system. However, the answer is not so simple when one considers topological and uniform rigidity, which are topological analogues of the (measurable) notion of rigidity. These notions were introduced by Glasner and Maon in the paper [**14**] for continuous dynamical systems on compact spaces:

Definition 1.9. — Let (X, d) be a compact metric space, and let φ be a continuous self-map of X. We say that φ is *topologically rigid* with respect to the sequence $(n_k)_{k\geq 0}$ if $\varphi^{n_k}(x) \to x$ as $k \to +\infty$ for any $x \in X$, and that φ is *uniformly rigid* with respect to $(n_k)_{k\geq 0}$ if

$$\sup_{x\in X} d(\varphi^{n_k}(x), x) \to 0 \quad \text{as } k \to +\infty.$$

Uniform rigidity is studied in [**14**], where in particular uniformly rigid and topologically weakly mixing transformations are constructed, see also [**8**], [**24**] and [**20**] for instance. Recall that φ is said to be *topologically weakly mixing* if for any non-empty open subsets U_1, U_2, V_1, V_2 of X, there exists an integer n such that $\varphi^{-n}(U_1) \cap V_1$ and $\varphi^{-n}(U_2) \cap V_2$ are both non-empty (topological weak mixing is the topological analogue of the notion of measurable weak mixing). Uniform rigidity sequences are defined in [**20**]:

Definition 1.10. — Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of integers. We say that $(n_k)_{k\geq 0}$ is a *uniform rigidity sequence* if there exists a compact dynamical system (X, d, φ) with φ a continuous self-map of X, which is topologically weakly mixing and uniformly rigid with respect to $(n_k)_{k\geq 0}$.

The question of characterizing uniform rigidity sequences is still open, as well as the question [**20**] whether there exists a compact dynamical system (X, d, φ) with φ continuous, which would be both weakly mixing with respect to a certain φ-invariant measure μ on X and uniformly rigid.

We investigate these two questions in the framework of linear dynamics. Of course we have to adapt the definition of uniform rigidity to this setting, as a Banach space is never compact.

Definition 1.11. — Let X be complex separable Banach space, and let φ be a continuous transformation of X. We say that φ is *uniformly rigid* with respect to $(n_k)_{k\geq 0}$ if for any bounded subset A of X,

$$\sup_{x\in A} ||\varphi^{n_k}(x) - x|| \to 0 \quad \text{as } k \to +\infty.$$

When T is a bounded linear operator on X, T is uniformly rigid with respect to $(n_k)_{k\geq 0}$ if and only if $||T^{n_k} - I|| \to 0$ as $k \to +\infty$. We prove the following theorems:

Theorem 1.12. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers with $n_0 = 1$. The following assertions are equivalent:*

1. *for any $\varepsilon > 0$ there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that*
$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon \quad \textit{and} \quad |\lambda^{n_k} - 1| \to 0 \quad \textit{as} \quad k \to +\infty;$$
2. *there exists a bounded linear operator T on a separable Banach space X such that $\sigma_p(T) \cap \mathbb{T}$ is uncountable and $T^{n_k}x \to x$ as $k \to \infty$ for every $x \in X$;*
3. *there exists a bounded linear operator T on a separable Hilbert space H such that T admits a non-degenerate invariant Gaussian measure with respect to which T is weakly mixing and $T^{n_k}x \to x$ as $k \to +\infty$ for every $x \in H$, i.e. T is topologically rigid with respect to $(n_k)_{k\geq 0}$.*

We also have a characterization for uniform rigidity in the linear setting:

Theorem 1.13. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers with $n_0 = 1$. The following assertions are equivalent:*

1. *there exists an uncountable subset K of $\mathbb{T}$ such that λ^{n_k} tends to 1 uniformly on K;*
2. *there exists a bounded linear operator T on a separable Banach space X such that $\sigma_p(T) \cap \mathbb{T}$ is uncountable and $||T^{n_k} - I|| \to 0$ as $k \to \infty$;*
3. *there exists a bounded linear operator T on a separable Hilbert space H such that T admits a non-degenerate invariant Gaussian measure with respect to which T is weakly mixing and $||T^{n_k} - I|| \to 0$ as $k \to \infty$, i.e. T is uniformly rigid with respect to $(n_k)_{k\geq 0}$.*

In particular we get a positive answer to a question of [**20**] in the context of linear dynamics:

Corollary 1.14. — *Any sequence $(n_k)_{k\geq 0}$ such that n_{k+1}/n_k tends to infinity, or such that n_k divides n_{k+1} for each k and $\limsup n_{k+1}/n_k = +\infty$ is a uniform rigidity sequence for linear dynamical systems, and measure-theoretically weakly mixing uniformly rigid systems do exist in this setting.*

After this paper was submitted for publication, V. Bergelson, A. Del Junco, M. Lemańczyk and J. Rosenblatt sent us a preprint "Rigidity and non recurrence along sequences" [**7**], in which they independently investigated for which sequences there exists a weakly mixing transformation which is rigid with respect to this sequence. A substantial part of the results of Section 3 of the present paper is also obtained in [**7**], often with different methods. We are very grateful to V. Bergelson, A. Del Junco, M. Lemańczyk and J. Rosenblatt for making their preprint available to us.

2. Hilbertian Jamison sequences

Our aim in this section is to prove Theorem 1.5. Clearly, if $(n_k)_{k\geq 0}$ is a Jamison sequence, it is automatically a Hilbertian Jamison sequence, and the difficulty lies in the converse direction: using Theorem 1.3, we start from a sequence $(n_k)_{k\geq 0}$ such that for any $\varepsilon > 0$ there is a $\lambda \in \mathbb{T} \setminus \{1\}$ such that $\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon$, and we have to construct out of this a bounded operator on a *Hilbert space* which is partially power-bounded with respect to $(n_k)_{k\geq 0}$ and which has uncountably many eigenvalues on the unit circle. We are going to prove a stronger theorem, giving a more precise description of the eigenvectors of the operator:

Theorem 2.1. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers with $n_0 = 1$ such that for any $\varepsilon > 0$ there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that*

$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon.$$

Let $\delta > 0$ be any positive number. There exists a bounded linear operator T on the Hilbert space $\ell_2(\mathbb{N})$ such that T has perfectly spanning unimodular eigenvectors and

$$\sup_{k\geq 0} ||T^{n_k}|| \leq 1 + \delta.$$

In particular the unimodular point spectrum of T is uncountable.

Before embarking on the proof, we need to define precisely the notion of perfectly spanning unimodular eigenvectors and explain its relevance here.

2.1. A criterion for ergodicity of linear dynamical systems. — Let H be a complex separable infinite-dimensional Hilbert space.

Definition 2.2. — We say that a bounded linear operator T on H has a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for any Borel subset B of $\mathbb{T}$ with $\sigma(B) = 1$, we have $\overline{\text{sp}}[\ker(T - \lambda I) \; ; \; \lambda \in B] = H$.

When $T \in \mathcal{B}(H)$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, there exists a Gaussian probability measure m on H such that:
– m is T-invariant;
– m is non-degenerate, i.e. $m(U) > 0$ for any non-empty open subset U of H;
– T is weakly mixing with respect to m.

See [**5**] for extensions to the Banach space setting, and the book [**6**, Ch. 5]. In the Hilbert space case, the converse of the assertion above is also true: if $T \in \mathcal{B}(H)$ defines a weakly mixing measure-preserving transformation with respect to a non-degenerate Gaussian measure, T has perfectly spanning unimodular eigenvectors.

A way to check this spanning property of the eigenvectors is to use the following criterion, which was proved in [**17**, Th. 4.2]:

Theorem 2.3. — *Let X be a complex separable infinite-dimensional Banach space, and let T be a bounded operator on X. Suppose that there exists a sequence $(u_i)_{i\ge1}$ of vectors of X having the following properties:*

(i) *for each $i \ge 1$, u_i is an eigenvector of T associated to an eigenvalue μ_i of T where $|\mu_i| = 1$ and the μ_i's are all distinct;*

(ii) *$sp[u_i\ ;\ i \ge 1]$ is dense in X;*

(iii) *for any $i \ge 1$ and any $\varepsilon > 0$, there exists an $n \ne i$ such that $||u_n - u_i|| < \varepsilon$.*

Then T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues.

2.2. Proof of Theorem 2.1: the easy part. — We are first going to define the operator T, and show that it is bounded. We will then describe the unimodular eigenvectors of T, and show that T satisfies the assumptions of Theorem 2.3.

▶ **Construction of the operator T.** Let $(e_n)_{n\ge1}$ denote the canonical basis of the space $\ell_2(\mathbb{N})$ of complex square summable sequences. We denote by H the space $\ell_2(\mathbb{N})$. The construction depends on two sequences $(\lambda_n)_{n\ge1}$ and $(\omega_n)_{n\ge1}$ which will be suitably chosen further on in the proof: $(\lambda_n)_{n\ge1}$ is a sequence of unimodular complex numbers which are all distinct, and $(\omega_n)_{n\ge1}$ is a sequence of positive weights.

Let $j : \{2, +\infty\} \to \{1, +\infty\}$ be a function having the following two properties:
• for any $n \ge 2$, $j(n) < n$;
• for any $k \ge 1$, the set $\{n \ge 2\ ;\ j(n) = k\}$ is infinite (i.e. j takes every value k infinitely often).

Let D be the diagonal operator on H defined by $De_n = \lambda_n e_n$ for $n \ge 1$, and let B be the weighted backward shift defined by $Be_1 = 0$ and $Be_n = \alpha_{n-1}e_{n-1}$ for $n \ge 2$, where the weights α_n, $n \ge 1$, are defined by

$$\alpha_1 = \omega_1\, |\lambda_2 - \lambda_{j(2)}|$$

and

$$\alpha_n = \omega_n \left| \frac{\lambda_{n+1} - \lambda_{j(n+1)}}{\lambda_n - \lambda_{j(n)}} \right| \quad \text{for any } n \ge 2.$$

This definition of α_n makes sense because $j(n) < n$, so that $\lambda_n \ne \lambda_{j(n)}$. The operators D and B being thus defined, we set $T = D + B$.

▶ **Boundedness of the operator T.** The first thing to do is to prove that T is indeed a bounded operator on H, provided some conditions on the λ_n's and ω_n's are imposed. The diagonal operator D being obviously bounded, we have to figure out conditions for B to be bounded. If $\gamma > 0$ is fixed, we have $||B|| \leq \gamma$ provided

$$\omega_1 \, |\lambda_2 - \lambda_{j(2)}| \leq \gamma \quad \text{and} \quad \omega_{n-1} \left| \frac{\lambda_n - \lambda_{j(n)}}{\lambda_{n-1} - \lambda_{j(n-1)}} \right| \leq \gamma \quad \text{for any } n \geq 3.$$

If the weights $\omega_n > 0$ are arbitrary, the λ_n's can be chosen in such a way that these conditions are satisfied:

• $\omega_1 > 0$ is arbitrary, we take $\lambda_1 = 1$ for instance (we could start here from any $\lambda_1 \in \mathbb{T}$);
• we have $j(2) = 1$: take λ_2 such that $|\lambda_2 - \lambda_1| \leq \gamma/\omega_1$ with $\lambda_2 \neq \lambda_1$;
• take $\omega_2 > 0$ arbitrary;
• $j(3) \in \{1, 2\}$: take λ_3 so close to $\lambda_{j(3)}$, $\lambda_3 \notin \{\lambda_1, \lambda_2\}$, that

$$|\lambda_3 - \lambda_{j(3)}| \leq \frac{\gamma}{\omega_2} \, |\lambda_2 - \lambda_{j(2)}|$$

• take $\omega_3 > 0$ arbitrary, etc.

Thus $||B|| \leq \gamma$ provided λ_n is so close to $\lambda_{j(n)}$ for every $n \geq 2$ that

$$|\lambda_n - \lambda_{j(n)}| \leq \frac{\gamma}{\omega_{n-1}} \, |\lambda_{n-1} - \lambda_{j(n-1)}|.$$

No condition on the ω_n's needs to be imposed there.

▶ **Unimodular eigenvectors of the operator T.** The algebraic equation $Tx = \lambda x$ with $x = \sum_{k \geq 1} x_k e_k$ is equivalent to the equations $\lambda_k x_k + \alpha_k x_{k+1} = \lambda x_k$, i.e. $x_{k+1} = \frac{\lambda - \lambda_k}{\alpha_k} x_k$ for any $k \geq 1$, i.e.

$$x_k = \frac{(\lambda - \lambda_{k-1}) \ldots (\lambda - \lambda_1)}{\alpha_{k-1} \ldots \alpha_1} \, x_1.$$

Hence for any $n \geq 1$, the eigenspace $\ker(T - \lambda_n)$ is 1-dimensional and $\ker(T - \lambda_n) = \mathrm{sp}[u^{(n)}]$, where

$$u^{(n)} = e_1 + \sum_{k=2}^{n} \frac{(\lambda_n - \lambda_{k-1}) \ldots (\lambda_n - \lambda_1)}{\alpha_{k-1} \ldots \alpha_1} e_k.$$

Our aim is now to show the following lemma:

Lemma 2.4. — *By choosing in a suitable way the coefficients ω_n and λ_n, it is possible to ensure that for any $n \geq 2$,*

$$||u^{(n)} - u^{(j(n))}|| \leq 2^{-n}$$

(the sequence $(2^{-n})_{n \geq 2}$ could be replaced by any sequence $(\gamma_n)_{n \geq 2}$ going to zero with n).

Proof of Lemma 2.4. — We have:

$$\begin{aligned} u^{(n)} - u^{(j(n))} \quad &= \quad \sum_{k=2}^{j(n)} \left(\frac{(\lambda_n - \lambda_{k-1}) \ldots (\lambda_n - \lambda_1)}{\alpha_{k-1} \ldots \alpha_1} - \frac{(\lambda_{j(n)} - \lambda_{k-1}) \ldots (\lambda_{j(n)} - \lambda_1)}{\alpha_{k-1} \ldots \alpha_1} \right) e_k \\ &+ \quad \sum_{k=j(n)+1}^{n} \frac{(\lambda_n - \lambda_{k-1}) \ldots (\lambda_n - \lambda_1)}{\alpha_{k-1} \ldots \alpha_1} \, e_k := v^{(n)} + w^{(n)}. \end{aligned}$$

We denote the first sum by $v^{(n)}$ and the second one by $w^{(n)}$. If $\varepsilon_n > 0$ is any positive number, we can ensure that $||v^{(n)}|| < \varepsilon_n$ by choosing λ_n such that $|\lambda_n - \lambda_{j(n)}|$ is sufficiently small, because the quantities $\alpha_{k-1}\dots\alpha_1$ for $k \le j(n)$ do not depend on λ_n. Let us now estimate

$$\begin{aligned} ||w^{(n)}||^2 &= \sum_{k=j(n)+1}^{n} \left| \frac{(\lambda_n - \lambda_{k-1})\dots(\lambda_n - \lambda_1)}{\alpha_{k-1}\dots\alpha_1} \right|^2 \\ &= \sum_{k=j(n)+1}^{n} \frac{1}{\omega_{k-1}^2 \dots \omega_1^2} \cdot \left| \frac{(\lambda_n - \lambda_{k-1})\dots(\lambda_n - \lambda_1)}{\lambda_k - \lambda_{j(k)}} \right|^2 \end{aligned}$$

since

$$\alpha_{k-1}\dots\alpha_1 = \omega_{k-1}\dots\omega_1\,|\lambda_k - \lambda_{j(k)}|.$$

We estimate now each term in this sum. We can suppose that $|\lambda_p - \lambda_q| \le 1$ for any p and q (this is no restriction), so $|(\lambda_n - \lambda_{k-1})\dots(\lambda_n - \lambda_1)| \le |\lambda_n - \lambda_{j(n)}|$ since $j(n) \in \{1, \dots, k-1\}$. Thus for $k = j(n)+1, \dots, n$,

$$\frac{1}{\omega_{k-1}^2 \dots \omega_1^2} \cdot \left| \frac{(\lambda_n - \lambda_{k-1})\dots(\lambda_n - \lambda_1)}{\lambda_k - \lambda_{j(k)}} \right|^2 \le \frac{1}{\omega_{k-1}^2 \dots \omega_1^2} \cdot \left| \frac{\lambda_n - \lambda_{j(n)}}{\lambda_k - \lambda_{j(k)}} \right|^2.$$

If $k \in \{j(n)+1, \dots, n-1\}$, the term on the right-hand side can be made arbitrarily small provided that we choose λ_n in such a way that $|\lambda_n - \lambda_{j(n)}|$ is very small with respect to the quantities $|\lambda_k - \lambda_{j(k)}| \,.\, \omega_{k-1}\dots\omega_1$, $k < n$. However for $k = n$, we only get the bound $\omega_{n-1}^{-2}\dots\omega_1^{-2}$, which has to be made small if we want $||w^{(n)}||$ to be small. So we have to impose a condition the weights ω_n: we take ω_{n-1} so large with respect to $\omega_1, \dots, \omega_{n-2}$ that $\omega_{n-1}^{-2}\dots\omega_1^{-2}$ is extremely small.

All the conditions on the λ_n's and the ω_n's needed until now can indeed be satisfied simultaneously: at stage n of the construction, we take ω_{n-1} very large. After this we take λ_n extremely close to $\lambda_{j(n)}$. Thus we can ensure that $||w^{(n)}|| < \varepsilon_n$, hence that $||u^{(n)} - u^{(j(n))}|| < 2\varepsilon_n$. Taking $\varepsilon_n = 2^{-(n+1)}$ gives our statement. □

Thanks to Lemma 2.4, it is easy to see that T satisfies the assumptions of Theorem 2.3:

Proposition 2.5. — *The operator T satisfies the assumptions of Theorem 2.3. Hence it has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, and in particular its unimodular point spectrum is uncountable.*

Proof of Proposition 2.5. — It suffices to show that the sequence $(u^{(n)})_{n\ge1}$ satisfies properties (i), (ii) and (iii). That (i) is satisfied is clear, since the vectors $u^{(n)}$ are eigenvectors of T associated to the eigenvalues λ_n which are all distinct. Since for each $n \ge 1$ the vector $u^{(n)}$ belongs to the span of the first n basis vectors $e_1, \dots, e_n$ and $\langle u^{(n)}, e_n \rangle \ne 0$, the linear span of the vectors $u^{(n)}$, $n \ge 1$, contains all finitely supported vectors of $\ell_2(\mathbb{N})$, and thus (ii) holds true. It remains to prove (iii). As the function j takes every value in $\{1, +\infty\}$ infinitely often, it follows from Lemma 2.4 that for any $n \ge 1$ there exists a strictly increasing sequence $(p_s^{(n)})_{s\ge1}$ of integers such that

$$||u^{(p_s^{(n)})} - u^{(n)}|| \text{ tends to } 0 \text{ as } s \text{ tends to } +\infty.$$

Hence (iii) is true. □

In order to conclude the proof of Theorem 1.5, it remains to show that T is partially power-bounded with respect to $(n_k)_{k\geq 0}$, with $\sup_{k\geq 0} ||T^{n_k}|| \leq 1+\delta$. This is the most difficult part of the proof, which uses the assumption that $(n_k)_{k\geq 0}$ is not a Jamison sequence, and it is the object of the next section.

2.3. Proof of Theorem 2.1: the hard part. — In order to estimate the norms $||T^{n_k}||$, we will show that provided the ω_n's and λ_n's are suitably chosen, $||T^{n_k} - D^{n_k}|| \leq \delta$ for every $k \geq 1$. Since $||D^{n_k}|| = 1$, this will prove that $||T^{n_k}|| \leq 1+\delta$ for every $k \geq 1$.

► **An expression of** $(T^n - D^n)$**.** We first have to compute $(T^n - D^n)x$ for $n \geq 1$ and $x \in H$. For $k, l \geq 1$, let $t_{k,l}^{(n)} = \langle T^n e_l, e_k\rangle$ be the coefficient in row k and column l of the matrix representation of T^n. If $k > l$, $t_{k,l}^{(n)} = 0$ (all coefficients below the diagonal are zero), and if $l - k > n$, $t_{k,l}^{(n)} = 0$ (all coefficients which are not in one of the first n upper diagonals of the matrix are zero). We have $t_{k,k}^{(n)} = \lambda_k^n$ for any $k \geq 1$.

Lemma 2.6. — *For any $k, l \geq 1$ such that $1 \leq l - k \leq n$,*

$$t_{k,l}^{(n)} = \alpha_{l-1}\alpha_{l-2}\ldots\alpha_k \sum_{j_k+\ldots+j_l=n-(l-k)} \lambda_k^{j_k}\ldots\lambda_l^{j_l}.$$

Proof. — The proof is done by induction on $n \geq 1$.

• $n = 1$: in this case $l = k+1$, and the formula above gives $t_{k,k+1}^{(1)} = \alpha_k$, which is true.

• Suppose that the formulas above are true for any $m \leq n$. Let k and l be such that $1 \leq l - k \leq n+1$ (in particular $l \geq 2$). We have

$$t_{k,l}^{(n+1)} = t_{k,l-1}^{(n)} t_{l-1,l}^{(1)} + t_{k,l}^{(n)} t_{l,l}^{(1)} = \alpha_{l-1} t_{k,l-1}^{(n)} + \lambda_l t_{k,l}^{(n)}.$$

If $2 \leq l - k \leq n$, we can apply the induction assumption to the two quantities $t_{k,l-1}^{(n)}$ and $t_{k,l}^{(n)}$, and we get

$$\begin{aligned} t_{k,l}^{(n+1)} &= \alpha_{l-1}\alpha_{l-2}\ldots\alpha_k \sum_{j_k+\ldots+j_{l-1}=n-(l-1-k)} \lambda_k^{j_k}\ldots\lambda_{l-1}^{j_{l-1}} \\ &+ \alpha_{l-1}\alpha_{l-2}\ldots\alpha_k \sum_{j_k+\ldots+j_l=n-(l-k)} \lambda_k^{j_k}\ldots\lambda_{l-1}^{j_{l-1}}\lambda_l^{j_l+1} \\ &= \sum_{j_k+\ldots+j_{l-1}+j_l=n+1-(l-k),\, j_l=0} \lambda_k^{j_k}\ldots\lambda_{l-1}^{j_{l-1}}\lambda_l^{j_l} \\ &+ \sum_{j_k+\ldots+j_{l-1}+j_l=n+1-(l-k),\, j_l\geq 1} \lambda_k^{j_k}\ldots\lambda_{l-1}^{j_{l-1}}\lambda_l^{j_l} \\ &= \sum_{j_k+\ldots+j_{l-1}+j_l=n+1-(l-k)} \lambda_k^{j_k}\ldots\lambda_{l-1}^{j_{l-1}}\lambda_l^{j_l} \end{aligned}$$

and the formula is proved for $1 \leq l-k \leq n$. It remains to treat the cases where $l - k = 1$ and where $l-k = n+1$. If $l-k = 1$, we have $t_{k,k+1}^{(n+1)} = \alpha_k\lambda_k^n + \lambda_{k+1}t_{k,k+1}^{(n)}$. By the induction assumption

$$t_{k,k+1}^{(n)} = \alpha_k \sum_{j_k+j_{k+1}=n-1} \lambda_k^{j_k}\lambda_{k+1}^{j_{k+1}} = \alpha_k \frac{\lambda_{k+1}^n - \lambda_k^n}{\lambda_{k+1} - \lambda_k}$$

so that

$$t_{k,k+1}^{(n+1)} = \alpha_k(\lambda_k^n + \lambda_{k+1}\frac{\lambda_{k+1}^n - \lambda_k^n}{\lambda_{k+1} - \lambda_k}) = \alpha_k\,\frac{\lambda_{k+1}^{n+1} - \lambda_k^{n+1}}{\lambda_{k+1} - \lambda_k} = \alpha_k \sum_{j_k + j_{k+1} = n} \lambda_k^{j_k}\lambda_{k+1}^{j_{k+1}}$$

which is the formula we were looking for. Lastly, when $l-k = n+1$, $t_{k,n+1+k}^{(n+1)} = \alpha_{n+k}t_{k,n+k}^{(n)}$. By the induction assumption $t_{k,n+k}^{(n)} = \alpha_{n+k-1}\ldots\alpha_k$, thus $t_{k,n+1+k}^{(n+1)} = \alpha_{n+k}\ldots\alpha_k$ and the formula is proved in this case too. □

► **A first estimate on the norms** $||(T^n - D^n)||$**.** For $x = \sum_{l\geq 1} x_l e_l$, let us estimate $||(T^n - D^n)x||^2$: we have

$$(T^n - D^n)x = \sum_{l\geq 1} x_l \left(\sum_{k\geq 1} t_{k,l}^{(n)} e_k\right) - \sum_{l\geq 1} x_l t_{l,l}^{(n)} e_l = \sum_{l\geq 2} x_l \left(\sum_{k=\max(1,l-n)}^{l-1} t_{k,l}^{(n)} e_k\right)$$

so that by the Cauchy-Schwarz inequality

$$||(T^n - D^n)x||^2 \leq ||x||^2 \sum_{l\geq 2} \left|\left|\sum_{k=\max(1,l-n)}^{l-1} t_{k,l}^{(n)} e_k\right|\right|^2 \leq ||x||^2 \sum_{l\geq 2}\sum_{k=\max(1,l-n)}^{l-1} |t_{k,l}^{(n)}|^2.$$

We thus have to estimate for each $l \geq 2$ and $p \geq 1$ the quantities

$$\sum_{k=\max(1,l-n_p)}^{l-1} |t_{k,l}^{(n_p)}|^2.$$

For $k, l \geq 1$, $1 \leq l-k \leq n$, let

$$s_{k,l}^{(n)} = \sum_{j_k+\ldots+j_l = n-(l-k)} \lambda_k^{j_k}\ldots\lambda_l^{j_l}.$$

We have

$$t_{k,l}^{(n)} = \alpha_{l-1}\ldots\alpha_k\, s_{k,l}^{(n)} = \omega_{l-1}\ldots\omega_k\,\frac{|\lambda_l - \lambda_{j(l)}|}{|\lambda_k - \lambda_{j(k)}|}\, s_{k,l}^{(n)}$$

so that we have to estimate

$$\sum_{k=\max(1,l-n)}^{l-1} \omega_{l-1}^2\ldots\omega_k^2\,\frac{|\lambda_l - \lambda_{j(l)}|^2}{|\lambda_k - \lambda_{j(k)}|^2}\,|s_{k,l}^{(n)}|^2.$$

We are going to show that the following property holds true:

Lemma 2.7. — *For any* $1 \leq k \leq l-1$*, there exists for each* $k \leq j \leq l-1$ *a complex number* $c_j^{(k,l)}$ *depending only on* $\lambda_1, \ldots, \lambda_{l-1}$ *(and* k *and* l *of course), but on* λ_l *nor on* n*, such that for any* $n \geq l-k$*,*

$$s_{k,l}^{(n)} = \sum_{j=k}^{l-1} c_j^{(k,l)}\,\frac{\lambda_l^{n+1-(l-k)} - \lambda_j^{n+1-(l-k)}}{\lambda_l - \lambda_j}.$$

Proof. — The proof is again done by induction on $l \geq 2$.

• Let us first treat the case $l = 2$: we have to show that there exists $c_1^{(1,2)}$ such that for any $n \geq 2$,

$$s_{1,2}^{(n)} = c_1^{(1,2)} \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1}.$$

But

$$s_{1,2}^{(n)} = \sum_{j_1+j_2=n-1} \lambda_1^{j_1} \lambda_2^{j_2} = \sum_{j_1=0}^{n-1} \lambda_1^{j_1} \lambda_2^{n-1-j_1} = \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1}$$

so this holds true with $c_1^{(1,2)} = 1$.

• Suppose that the property is true for some $l \geq 2$, and consider for $1 \leq k \leq l$ and $n \geq l+1-k$ the quantities

$$\begin{aligned} s_{k,l+1}^{(n)} &= \sum_{j_k+\ldots+j_{l+1}=n-(l+1-k)} \lambda_k^{j_k} \ldots \lambda_{l+1}^{j_{l+1}} \\ &= \sum_{j_{l+1}=0}^{n-(l+1-k)} \left(\sum_{j_k+\ldots+j_l=n-(l+1+j_{l+1}-k)} \lambda_k^{j_k} \ldots \lambda_l^{j_l} \right) \lambda_{l+1}^{j_{l+1}}. \end{aligned}$$

If $1 \leq k \leq l-1$, we can apply the induction assumption and we get that

$$\begin{aligned} s_{k,l+1}^{(n)} &= \sum_{j_{l+1}=0}^{n-(l+1-k)} \lambda_{l+1}^{j_{l+1}} s_{k,l}^{(n-1-j_{l+1})} \\ &= \sum_{j_{l+1}=0}^{n-(l+1-k)} \lambda_{l+1}^{j_{l+1}} \sum_{j=k}^{l-1} c_j^{(k,l)} \left(\frac{\lambda_l^{n-j_{l+1}-(l-k)} - \lambda_j^{n-j_{l+1}-(l-k)}}{\lambda_l - \lambda_j} \right) \end{aligned}$$

where $c_j^{(k,l)}$ depends only on $\lambda_1, \ldots, \lambda_{l-1}$ (the equality in the third line of the display above comes from the fact that $j_{l+1} \leq n-1-l+k$, i.e. $l-k \leq n-1-j_{l+1}$). Thus

$$\begin{aligned} s_{k,l+1}^{(n)} &= \sum_{j=k}^{l-1} \frac{c_j^{(k,l)}}{\lambda_l - \lambda_j} \left(\sum_{j_{l+1}=0}^{n-(l+1-k)} \lambda_{l+1}^{j_{l+1}} \lambda_l^{n-j_{l+1}-(l-k)} - \lambda_{l+1}^{j_{l+1}} \lambda_j^{n-j_{l+1}-(l-k)} \right) \\ &= \sum_{j=k}^{l-1} \frac{c_j^{(k,l)}}{\lambda_l - \lambda_j} \left(\lambda_l^{n-(l-k)} \frac{1-(\lambda_{l+1}\overline{\lambda}_l)^{n-(l-k)}}{1-(\lambda_{l+1}\overline{\lambda}_l)} - \lambda_j^{n-(l-k)} \frac{1-(\lambda_{l+1}\overline{\lambda}_j)^{n-(l-k)}}{1-(\lambda_{l+1}\overline{\lambda}_j)} \right) \\ &= \sum_{j=k}^{l-1} \frac{c_j^{(k,l)}}{\lambda_l - \lambda_j} \left(\lambda_l \frac{\lambda_{l+1}^{n-(l-k)} - \lambda_l^{n-(l-k)}}{\lambda_{l+1} - \lambda_l} - \lambda_j \frac{\lambda_{l+1}^{n-(l-k)} - \lambda_j^{n-(l-k)}}{\lambda_{l+1} - \lambda_j} \right) \\ &= \sum_{j=k}^{l-1} \left(-\frac{\lambda_j c_j^{(k,l)}}{\lambda_l - \lambda_j} \right) \left(\frac{\lambda_{l+1}^{n-(l-k)} - \lambda_j^{n-(l-k)}}{\lambda_{l+1} - \lambda_j} \right) \\ &+ \left(\sum_{j=k}^{l-1} \frac{c_j^{(k,l)}}{\lambda_l - \lambda_j} \lambda_l \right) \left(\frac{\lambda_{l+1}^{n-(l-k)} - \lambda_l^{n-(l-k)}}{\lambda_{l+1} - \lambda_l} \right) \end{aligned}$$

i.e

$$s_{k,l+1}^{(n)} = \sum_{j=k}^{l} c_j^{(k,l+1)} \frac{\lambda_{l+1}^{n+1-(l+1-k)} - \lambda_j^{n+1-(l+1-k)}}{\lambda_{l+1} - \lambda_j}$$

where

$$c_j^{(k,l+1)} = -\frac{\lambda_j c_j^{(k,l)}}{\lambda_l - \lambda_j} \quad \text{for } k \le j \le l-1 \quad \text{and} \quad c_l^{(k,l+1)} = \sum_{j=k}^{l-1} \frac{c_j^{(k,l)}}{\lambda_l - \lambda_j} \lambda_l$$

depend only on $\lambda_1, \ldots, \lambda_l$. This settles the case where $1 \le k \le l-1$. When $k = l$, we get

$$s_{l,l+1}^{(n)} = \sum_{j_l + j_{l+1} = n-1} \lambda_l^{j_l} \lambda_{l+1}^{j_{l+1}} = \frac{\lambda_{l+1}^n - \lambda_l^n}{\lambda_{l+1} - \lambda_l},$$

and the statement is true with $c_l^{(l,l+1)} = 1$. □

Let us now go back to the estimate on $||(T^{n_p} - D^{n_p})x||^2$, $p \ge 0$: we want to show that if the coefficients λ_l are suitably chosen, the following holds true: for any $p \ge 0$ we have

- for any $2 \le l \le n_p$,

$$\sum_{k=1}^{l-1} |t_{k,l}^{(n_p)}|^2 \le \delta^2\, 2^{-l},$$

- for any $l \ge n_p + 1$,

$$\sum_{k=l-n_p}^{l-1} |t_{k,l}^{(n_p)}|^2 \le \delta^2\, 2^{-l}.$$

Let us first consider the case $2 \le l \le n_p$.

► **The "easy" estimate on** $||(T^{n_p} - D^{n_p})||$**.** Let us write

$$\begin{aligned}\sum_{k=1}^{l-1} |t_{k,l}^{(n_p)}|^2 &= \sum_{k=1}^{l-1} \omega_{l-1}^2 \ldots \omega_k^2 \left| \frac{\lambda_l - \lambda_{j(l)}}{\lambda_k - \lambda_{j(k)}} \right|^2 |s_{k,l}^{(n)}|^2 \\ &\le \sum_{k=1}^{l-1} \omega_{l-1}^2 \ldots \omega_k^2 \left| \frac{\lambda_l - \lambda_{j(l)}}{\lambda_k - \lambda_{j(k)}} \right|^2 \left(\sum_{j=k}^{l-1} |c_j^{(k,l)}| \left| \frac{\lambda_l^{n_p+1-(l-k)} - \lambda_j^{n_p+1-(l-k)}}{\lambda_l - \lambda_j} \right| \right)^2.\end{aligned}$$

In the sum indexed by j, we have two different cases to consider: either $j = j(l)$ or $j \ne j(l)$. The case $j = j(l)$ can happen only when $j(l) \ge k$. Thus the sum can be decomposed as

$$\begin{aligned}&\sum_{k=1}^{j(l)} \omega_{l-1}^2 \ldots \omega_k^2 \left| \frac{\lambda_l - \lambda_{j(l)}}{\lambda_k - \lambda_{j(k)}} \right|^2 \left(\sum_{j=k, j \ne j(l)}^{l-1} |c_j^{(k,l)}| \left| \frac{\lambda_l^{n_p+1-(l-k)} - \lambda_j^{n_p+1-(l-k)}}{\lambda_l - \lambda_j} \right| \right. \\ &\quad \left. + |c_{j(l)}^{(k,l)}| \left| \frac{\lambda_l^{n_p+1-(l-k)} - \lambda_{j(l)}^{n_p+1-(l-k)}}{\lambda_l - \lambda_{j(l)}} \right| \right)^2 \\ &\quad + \sum_{k=j(l)+1}^{l-1} \omega_{l-1}^2 \ldots \omega_k^2 \left| \frac{\lambda_l - \lambda_{j(l)}}{\lambda_k - \lambda_{j(k)}} \right|^2 \left(\sum_{j=k, j \ne j(l)}^{l-1} |c_j^{(k,l)}| \left| \frac{\lambda_l^{n_p+1-(l-k)} - \lambda_j^{n_p+1-(l-k)}}{\lambda_l - \lambda_j} \right| \right)^2\end{aligned}$$

which is less than

$$2\sum_{k=1}^{l-1}\omega_{l-1}^2\ldots\omega_k^2\left|\frac{\lambda_l-\lambda_{j(l)}}{\lambda_k-\lambda_{j(k)}}\right|^2\left(\sum_{j=k,j\neq j(l)}^{l-1}|c_j^{(k,l)}|\left|\frac{\lambda_l^{n_p+1-(l-k)}-\lambda_j^{n_p+1-(l-k)}}{\lambda_l-\lambda_j}\right|\right)^2$$
$$+2\sum_{k=1}^{j(l)}\omega_{l-1}^2\ldots\omega_k^2\cdot\frac{1}{|\lambda_k-\lambda_{j(k)}|^2}\cdot|c_{j(l)}^{(k,l)}|^2\cdot\left|\lambda_l^{n_p+1-(l-k)}-\lambda_{j(l)}^{n_p+1-(l-k)}\right|^2$$

and this in turn is less than

$$(1)\qquad |\lambda_l-\lambda_{j(l)}|^2\left(8\sum_{k=1}^{l-1}\omega_{l-1}^2\ldots\omega_k^2\cdot\frac{1}{|\lambda_k-\lambda_{j(k)}|^2}\cdot\left(\sum_{j=k,j\neq j(l)}^{l-1}|c_j^{(k,l)}|\frac{1}{|\lambda_l-\lambda_j|}\right)^2\right)$$
$$+2\sum_{k=1}^{j(l)}\omega_{l-1}^2\ldots\omega_k^2\cdot\frac{1}{|\lambda_k-\lambda_{j(k)}|^2}\cdot|c_{j(l)}^{(k,l)}|^2\cdot\left|\lambda_l^{n_p+1-(l-k)}-\lambda_{j(l)}^{n_p+1-(l-k)}\right|^2.$$

Suppose (as we may) that λ_l is so close to $\lambda_{j(l)}$ that

$$|\lambda_l-\lambda_{j(l)}|\le\frac{1}{2}\min_{j\le l-1,j\neq j(l)}|\lambda_j-\lambda_{j(l)}|.$$

Then for any $j\le l-1$ with $j\neq j(l)$ we have $|\lambda_l-\lambda_j|\ge|\lambda_j-\lambda_{j(l)}|-|\lambda_l-\lambda_{j(l)}|\ge\frac{1}{2}|\lambda_j-\lambda_{j(l)}|$. Thus the first term in the expression (1) above is less than

$$32\,|\lambda_l-\lambda_{j(l)}|^2\sum_{k=1}^{l-1}\omega_{l-1}^2\ldots\omega_k^2\cdot\frac{1}{|\lambda_k-\lambda_{j(k)}|^2}\cdot\left(\sum_{j=k,j\neq j(l)}^{l-1}|c_j^{(k,l)}|\frac{1}{|\lambda_{j(l)}-\lambda_j|}\right)^2.$$

Since the quantity between the brackets depends only on $\lambda_1,\ldots,\lambda_{l-1},\omega_1,\ldots,\omega_{l-1}$ but not on λ_l, the expression in the display above can be made arbitrarily small if $|\lambda_l-\lambda_{j(l)}|$ is small enough. So we take, for any $l\ge 2$, λ_l with $|\lambda_l-\lambda_{j(l)}|$ so small that

$$32\,|\lambda_l-\lambda_{j(l)}|^2\sum_{k=1}^{l-1}\omega_{l-1}^2\ldots\omega_k^2\cdot\frac{1}{|\lambda_k-\lambda_{j(k)}|^2}\cdot\left(\sum_{j=k,j\neq j(l)}^{l-1}|c_j^{(k,l)}|\frac{1}{|\lambda_{j(l)}-\lambda_j|}\right)^2\le\delta^2 2^{-(l+1)}.$$

Observe that the estimate we get here does not depend on n_p (it is valid for any n in fact). This takes care of the first term in the sum (1).

▶ **The "hard" estimate on $||T^{n_p}-D^{n_p}||$.** We have now to estimate the term

$$(2)\qquad 2\sum_{k=1}^{j(l)}\omega_{l-1}^2\ldots\omega_k^2\cdot\frac{1}{|\lambda_k-\lambda_{j(k)}|^2}\cdot|c_{j(l)}^{(k,l)}|^2\cdot\left|\lambda_l^{n_p+1-(l-k)}-\lambda_{j(l)}^{n_p+1-(l-k)}\right|^2.$$

We have

$$\begin{aligned}|\lambda_l^{n_p+1-(l-k)}-\lambda_{j(l)}^{n_p+1-(l-k)}| &\le |\lambda_l^{n_p}-\lambda_{j(l)}^{n_p}|+|\lambda_l^{l-k-1}-\lambda_{j(l)}^{l-k-1}|\\ &\le |\lambda_l^{n_p}-\lambda_{j(l)}^{n_p}|+(l-k-1)|\lambda_l-\lambda_{j(l)}|\\ &\le |\lambda_l^{n_p}-\lambda_{j(l)}^{n_p}|+(l-2)|\lambda_l-\lambda_{j(l)}|\end{aligned}$$

so that the quantity in (2) is less than

$$4|\lambda_l^{n_p}-\lambda_{j(l)}^{n_p}|^2\sum_{k=1}^{j(l)}\omega_{l-1}^2\dots\omega_k^2\cdot\frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k-\lambda_{j(k)}|^2} \tag{3}$$
$$+4(l-2)^2|\lambda_l-\lambda_{j(l)}|^2\sum_{k=1}^{j(l)}\omega_{l-1}^2\dots\omega_k^2\cdot\frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k-\lambda_{j(k)}|^2}.$$

As previously the second term in this sum can be made arbitrarily small for any $l\geq 2$ provided $|\lambda_l-\lambda_{j(l)}|$ is sufficiently small, and we can ensure that it is less than $\delta^2 2^{-(l+2)}$. The difficult term to handle is the first one, and it is here that we use our assumption on the sequence $(n_p)_{p\geq 0}$ (which was never used in the proof until this point). Our assumption is that for any $\varepsilon>0$ there exists a $\lambda\in\mathbb{T}\setminus\{1\}$ such that $\sup_{p\geq 0}|\lambda^{n_p}-1|\leq\varepsilon$. This can be rewritten using the distance on $\mathbb{T}$ defined by

$$d_{(n_p)}(\lambda,\mu)=\sup_{p\geq 0}|\lambda^{n_p}-\mu^{n_p}|$$

as: for any $\varepsilon>0$ there exists a $\lambda\in\mathbb{T}\setminus\{1\}$ such that $d_{(n_p)}(\lambda,1)\leq\varepsilon$. This means (see [**2**]) that there exists an uncountable subset K of $\mathbb{T}$ such that $(K,d_{(n_p)})$ is a separable metric space. Thus K contains a subset K' which is uncountable and perfect for the distance $d_{(n_p)}$. This means that for every $\varepsilon>0$ and every $\lambda\in K'$, there exists a $\lambda'\in K'$, $\lambda'\neq\lambda$ such that $d_{(n_p)}(\lambda,\lambda')<\varepsilon$. Observe that since $n_0=1$, $|\lambda-\lambda'|\leq d_{(n_p)}(\lambda,\lambda')<\varepsilon$.

In the construction of the λ_l's, $l\geq 1$, we start by taking $\lambda_1\in K'$. Then we take λ_2 in K' such that $d_{(n_p)}(\lambda_2,\lambda_{j(2)})$ is extremely small, which is possible since $\lambda_{j(2)}=\lambda_1$ is not isolated in K'. In the same way we can take the λ_l's for $l\geq 2$ to be elements of K' such that $d_{(n_p)}(\lambda_l,\lambda_{(l)})$ is arbitrarily small, $\lambda_l\neq\lambda_{j(l)}$. Then $|\lambda_l-\lambda_{j(l)}|$ is also arbitrarily small.

With this suitable choice of the λ_l's, we can estimate the remaining term in (3):

$$4\,|\lambda_l^{n_p}-\lambda_{j(l)}^{n_p}|^2\sum_{k=1}^{j(l)}\omega_{l-1}^2\dots\omega_k^2\cdot\frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k-\lambda_{j(k)}|^2} \tag{4}$$
$$\leq 4\,d_{(n_p)}(\lambda_l,\lambda_{j(l)})^2\sum_{k=1}^{j(l)}\omega_{l-1}^2\dots\omega_k^2\cdot\frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k-\lambda_{j(k)}|^2}.$$

Since the sum in k depends only on $\lambda_1,\dots,\lambda_{l-1},\omega_1,\dots,\omega_{l-1}$, but not on λ_l, by taking λ_l such that $d_{(n_p)}(\lambda_l,\lambda_{(l)})$ is extremely small, we ensure that the righthand term in (4) is less than $\delta^2 2^{-(l+2)}$ for instance, for every $l\geq 2$.

Let us stress that the restrictions on the size of $|\lambda_l-\lambda_{j(l)}|$ and $d_{(n_p)}(\lambda_l,\lambda_{(l)})$ are imposed for any $l\geq 2$, and do not depend on a particular n_p. Thus we have proved that for any $p\geq 1$ and any $2\leq l\leq n_p$, we have with these choices of λ_l

$$\sum_{k=1}^{l-1}|t_{k,l}^{(n_p)}|^2\leq\delta^2 2^{-l}.$$

It remains to treat the case where $l \geq n_p + 1$, where we have to estimate the quantity

$$\sum_{k=l-n_p}^{l-1} |t_{k,l}^{(n_p)}|^2$$

which is less than

$$\sum_{k=l-n_p}^{l-1} \omega_{l-1}^2 \dots \omega_k^2 \cdot \left| \frac{\lambda_l - \lambda_{j(l)}}{\lambda_k - \lambda_{j(k)}} \right|^2 \left(\sum_{j=k}^{l-1} |c_j^{(k,l)}| \left| \frac{\lambda_l^{n_p+1-(l-k)} - \lambda_j^{n_p+1-(l-k)}}{\lambda_l - \lambda_j} \right| \right)^2 . \tag{5}$$

We make the same decomposition as above in the sum in j, by separating the cases $j = j(l)$ and $j \neq j(l)$. The case $j = j(l)$ can only happen when $j(l) \geq k$, so when $j(l) \geq l - n_p$, i.e. $n_p \geq l - j(l)$. The estimates on the term not involving the index $j = j(l)$ are worked out exactly as previously, and this term can be made arbitrarily small provided $|\lambda_l - \lambda_{j(l)}|$ is very small. The other term appears when $n_p \geq l - j(l)$, and is equal to

$$\begin{aligned} 2 \sum_{k=l-n_p}^{j(l)} |\lambda_l^{n_p+1-(l-k)} - \lambda_{j(l)}^{n_p+1-(l-k)}|^2 \omega_{l-1}^2 \dots \omega_k^2 \cdot \frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k - \lambda_{j(k)}|^2} \\ \leq 4\, d_{(n_p)}(\lambda_l, \lambda_{j(l)})^2 \sum_{k=1}^{j(l)} \omega_{l-1}^2 \dots \omega_k^2 \cdot \frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k - \lambda_{j(k)}|^2} \\ +4\,(l-2)^2 |\lambda_l - \lambda_{j(l)}|^2 \sum_{k=1}^{j(l)} \omega_{l-1}^2 \dots \omega_k^2 \cdot \frac{|c_{j(l)}^{(k,l)}|^2}{|\lambda_k - \lambda_{j(k)}|^2} . \end{aligned} \tag{6}$$

The reasoning is then exactly the same as previously, and if for each $l \geq 2$ the quantity $d_{(n_p)}(\lambda_l, \lambda_{j(l)})$ is sufficiently small we have for any $p \geq 1$ and any $l \geq n_p + 1$ that

$$\sum_{k=l-n_p}^{l-1} |t_{k,l}^{(n_p)}|^2 \leq \delta^2 2^{-l}.$$

Hence $||T^{n_p} - D^{n_p}|| \leq \delta$ for any $p \geq 1$. For $p = 0$, $||T - D|| = ||B|| < \delta$, so that

$$\sup_{p \geq 0} ||T^{n_p} - D^{n_p}|| \leq \delta.$$

This proves that T is partially power-bounded with respect to $(n_p)_{p\geq 0}$, with the estimate $\sup_{p\geq 0} ||T^{n_p}|| \leq 1 + \delta$, and this finishes the proof of Theorem 2.1.

Remark 2.8. — It is not difficult to see that the operators constructed in Theorem 2.1 are invertible: they are of the form $T = D + B$, where D is invertible with $||D|| = 1$, and $||B||$ can be made arbitrarily small in the construction.

3. Rigidity sequences

3.1. An abstract characterization of rigidity sequences. — As mentioned already in the introduction, it is in a sense not difficult to characterize rigidity sequences via measures on the unit circle although such a characterization is rather abstract and not so easy to handle in concrete situations. This characterization is well-know, and hinted at for instance in [**12**] or [**36**], but since we have been unable to locate it precisely in the literature, we give below a quick proof of it. A proof is also given in the preprint [**7**]. Here and later we denote by $\hat{\sigma}(n)$ the n-th Fourier coefficient of a measure σ.

Proposition 3.1. — *Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of positive integers. The following assertions are equivalent:*

(1) *there exists a dynamical system φ on a measure space $(X, \mathcal{F}, \mu)$ which is weakly mixing and rigid with respect to $(n_k)_{k\geq 0}$;*
(2) *there exists a continuous probability measure σ on $\mathbb{T}$ such that $\hat{\sigma}(n_k) \to 1$ as $n_k \to +\infty$.*

Recall that a probability measure σ on $\mathbb{T}$ is said to be *symmetric* if $\sigma(\overline{A}) = \sigma(A)$ for any Borel subset A of $\mathbb{T}$ (where $\overline{A}$ denotes the conjugate set of A).

In order to prove Proposition 3.1, we are going to use the following well-known fact:

Fact 3.2. — *Let φ be a measure-preserving transformation of the space $(X, \mathcal{F}, \mu)$. The following assertions are equivalent:*

(a) *φ is rigid with respect to $(n_k)_{k\geq 0}$;*
(b) *$U_\varphi^{n_k} \to I$ in the weak operator topology (WOT) of $L^2(X, \mathcal{F}, \mu)$;*
(c) *$U_\varphi^{n_k} \to I$ in the strong operator topology (SOT) of $L^2(X, \mathcal{F}, \mu)$.*

Proof of Fact 3.2. — The implication $(c) \Rightarrow (b)$ is obvious. To prove $(b) \Rightarrow (a)$ it suffices to apply (b) to the characteristic functions χ_A of sets $A \in \mathcal{F}$:

$$\langle U_\varphi^{n_k} \chi_A, \chi_A \rangle = \int_X \chi_A(\varphi^{n_k}(x))\chi_A(x) d\mu(x) \to \mu(A) \quad \text{as } n_k \to +\infty.$$

Now $\chi_{A \triangle \varphi^{-n_k}(A)} = \chi_A + \chi_{\varphi^{-n_k}(A)} - 2\chi_A \chi_{\varphi^{-n_k}(A)}$ so that

$$\mu(A \triangle \varphi^{-n_k}(A)) = 2\mu(A) - 2\int_X \chi_A(\varphi^{n_k}(x))\chi_A(x) d\mu(x) \to 0 \quad \text{as } n_k \to +\infty.$$

Hence φ is rigid with respect to $(n_k)_{k\geq 0}$. The proof of $(a) \Rightarrow (c)$ is done using the same kind of argument: thanks to the expression for $\chi_{A \triangle \varphi^{-n_k}(A)}$, we get that $||U_\varphi^{n_k} \chi_A - \chi_A||_{L^2} \to 0$ as $n_k \to +\infty$ for any $A \in \mathcal{F}$. Hence $||U_\varphi^{n_k} f - f||_{L^2} \to 0$ for any $f \in L^2(X, \mathcal{F}, \mu)$, which is assertion (c). □

Proof of Proposition 3.1. — Suppose first that (1) holds true, and let σ_0 be the reduced maximal spectral type of U_φ, i.e. the maximal spectral type of the unitary operator U induced by U_φ on the subspace $H_0 = \{f \in L^2(X, \mathcal{F}, \mu) \ ; \ \int_X f(x) d\mu(x) = 0\}$ (which is invariant by U_φ). Note that U_φ is unitary by Remark 1.8. For the definition and basic properties of the spectral type of a unitary operator see e.g.[**27**]. Let $f_0 \in H_0$ with $||f_0|| = 1$ be such that the spectral measure σ_{f_0} of f_0 is a representant of the class σ_0. Since φ is weakly mixing, σ_{f_0} is continuous, and it is a probability measure since $||f_0|| = 1$. For any $n \in \mathbb{Z}$ we have

$$\langle U^n f_0, f_0 \rangle = \int_{\mathbb{T}} \lambda^n d\sigma_{f_0}(\lambda) = \hat{\sigma}_{f_0}(n).$$

Since $||U_\varphi^{n_k} f - f||_{L^2} \to 0$ for any $f \in H_0$, we get in particular that $\hat{\sigma}_{f_0}(n_k) \to ||f_0||^2 = 1$ as $n_k \to +\infty$, so (2) holds true.

Conversely, let σ be a continuous probability measure on $\mathbb{T}$ such that $\hat{\sigma}(n_k) \to 1$. Then $\int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 d\sigma(\lambda) \to 0$ as $n_k \to +\infty$, so that in particular we have $\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\sigma(\lambda) \to 0$. Indeed $|\lambda^{n_k} - 1|^2 = 2(1 - \Re e(\lambda^{n_k}))$, hence $\int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 d\sigma(\lambda) = 2\Re e(1 - \hat{\sigma}(n_k)) \to 0$. Let now $\overline{\sigma}$ be the probability measure on $\mathbb{T}$ defined by $\overline{\sigma}(A) = \sigma(\overline{A})$ for any $A \in \mathcal{F}$. Then $\overline{\sigma}$ is a continuous measure on $\mathbb{T}$, and $\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\overline{\sigma}(\lambda) = \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\sigma(\lambda)$ so that in particular $\hat{\overline{\sigma}}(n_k) \to 1$ as $n_k \to +\infty$. Setting $\rho := (\sigma + \overline{\sigma})/2$, we obtain a continuous *symmetric* probability measure on $\mathbb{T}$ such that $\hat{\rho}(n_k) \to 1$. So we can assume without loss of generality that the measure σ given by (2) is symmetric, and we have to construct out of this a weakly mixing measure-preserving transformation of a probability space which is rigid with respect to (n_k). The class of transformations which we use for this is the class of stationary Gaussian processes. We refer the reader to one of the references [**9**], [**31**] or [**29**, Ch. 8] for details about this, and in the forthcoming proof we use the notations of [**29**, Ch. 8]. Since σ is a symmetric probability measure on $\mathbb{T}$, there exists a real-valued stationary Gaussian process $(X_n)_{n\in\mathbb{Z}}$ whose spectral measure is σ. This Gaussian process lives on a probability space $(\Omega, \Sigma, \mathbb{P})$ which can be realized as a sequence space: $\Omega = \mathbb{R}^{\mathbb{Z}}$, Σ is the σ-algebra generated by the sets $\Theta_{m,A} = \{(\omega_n)_{n\in\mathbb{Z}} \,;\, (\omega_{-m}, \dots, \omega_m) \in A\}$, $m \geq 0$, A is a Borel subset of $\mathbb{R}^{2m+1}$, and $\mathbb{P}$ is the probability that $(X_{-m}, \dots, X_m)$ belongs to A: there exists a $\mathbb{P}$-preserving automorphism τ of Ω such that $X_{n+1} = X_n \circ \tau$ for any $n \in \mathbb{Z}$. The automorphism τ defines a weakly mixing transformation of $(\Omega, \Sigma, \mathbb{P})$ (see [**31**] for instance), and we have to see that it is rigid with respect to the sequence (n_k). Using the same argument as in the proof of [**29**, Ch. 8, Th. 3.2], it suffices to show that for any functions f, g belonging to $\mathcal{G}_c$, the complexification of the Gaussian subspace of $L^2(\Omega, \Sigma, \mathbb{P})$ spanned by $X_n, n \in \mathbb{Z}$, we have $\langle U_\tau^{n_k} f - f, g\rangle \to 0$ as $n_k \to +\infty$. If $\Phi : \mathcal{G}_c \to L^2(\mathbb{T}, \sigma)$ denotes the map defined on the linear span of the X_n's by $\Phi(\sum c_n X_n) := \sum c_n \lambda^n$, then Φ extends to a surjective isometry of $\mathcal{G}_c$ onto $L^2(\mathbb{T}, \sigma)$, and we have for any $f \in \mathcal{G}_c$ that $U_\tau f = (\Phi^{-1} \circ M_\lambda \circ \Phi) f$, where M_λ denotes multiplication by the independent variable λ on $L^2(\mathbb{T}, \sigma)$. Thus

$$\langle U_\tau^{n_k} f - f, g\rangle = \langle M_\lambda^{n_k} \Phi f - \Phi f, \Phi g\rangle = \int_{\mathbb{T}} (\lambda^{n_k} - 1)(\Phi f)(\lambda)\overline{(\Phi g(\lambda))} d\sigma(\lambda).$$

Now if h is any function in $L^1(\mathbb{T}, \sigma)$, we have that $\int_{\mathbb{T}} |\lambda^{n_k} - 1|\, |h(\lambda)| d\sigma(\lambda) \to 0$ as $n_k \to +\infty$ (it suffices to approximate h by functions $h' \in L^\infty(\mathbb{T}, \sigma)$ in $L^1(\mathbb{T}, \sigma)$). Since $(\Phi f)\overline{(\Phi g)}$ belongs to $L^1(\mathbb{T}, \sigma)$, we get that $\langle U_\tau^{n_k} f - f, g\rangle \to 0$. It follows that $U_\tau^{n_k} \to I$ in the WOT of $L^2(\Omega, \Sigma, \mathbb{P})$ and τ is rigid with respect to (n_k) by Fact 3.2. □

Remark 3.3. — The Gaussian dynamical systems considered in the proof of Proposition 3.1 live on the space of sequences $\mathbb{R}^{\mathbb{Z}}$, which is not compact. But by the Jewett-Krieger Theorem (see for instance [**30**]), such a system is metrically isomorphic to a homeomorphism of the Cantor set.

3.2. Examples of rigidity and non-rigidity sequences. — Our first example of rigidity sequences (obtained also in [**7**]) is the following:

Example 3.4. — Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of positive numbers such that n_{k+1}/n_k tends to infinity. Then $(n_k)_{k\geq 0}$ is a rigidity sequence.

This fact follows from the following stronger Proposition 3.5 below, which will allow us to show later on in the paper that any such sequence is a uniform rigidity sequence in the linear framework. The proof of Proposition 3.5 uses ideas from [**2**].

Proposition 3.5. — *Let $(n_k)_{k\ge 0}$ be a strictly increasing sequence of positive numbers such that n_{k+1}/n_k tends to infinity. There exists a compact perfect subset K of $\mathbb{T}$ having the following two properties:*

(i) *for any $\varepsilon > 0$ there exists a compact perfect subset K_ε of K such that for any $\lambda \in K_\varepsilon$,*

$$\sup_{k\ge 0} |\lambda^{n_k} - 1| \le \varepsilon;$$

(ii) *λ^{n_k} tends to 1 uniformly on K.*

Note that the existence of a compact perfect subset K of $\mathbb{T}$ satisfying (ii) above implies that $(n_k)_{k\ge 0}$ is a rigidity sequence. Indeed, any continuous probability measure σ supported on K satisfies assertion (2) in Proposition 3.1.

Proof. — For any $k \ge 1$, let $\gamma_k = 5\pi \sup_{j\ge k}(n_{j-1}/n_j)$: γ_k decreases to 0 as k tends to infinity, and let k_0 be such that $\gamma_k \le \frac{1}{2}$ for any $k \ge k_0$. Let $\theta_k \in]0, \frac{\pi}{2}[$ be such that $\gamma_k = \sin\theta_k$ for $k \ge k_0$. The sequence (θ_k) decreases to 0, and $\theta_k \sim \gamma_k$ as k tends to infinity. Thus there exists a $k_1 \ge k_0$ such that for any $k \ge k_1$, $\theta_k \ge 4\pi \sup_{j\ge k}(n_{j-1}/n_j) \ge 4\pi\,(n_k/n_{k+1})$, so that $(n_{k+1}/n_k)\,.\,\theta_k \ge 4\pi$. Let

$$K_0 = \{\lambda \in \mathbb{T}\ ;\ \forall k \ge k_1 \quad |\lambda^{n_k} - 1| \le 2\gamma_k\}.$$

If we write $\lambda \in \mathbb{T}$ as $\lambda = e^{2i\theta}$, $\theta \in [0, \pi[$, λ belongs to K_0 if and only if $|\sin(n_k\theta)| \le \gamma_k$ for any $k \ge k_1$. Let $F_k = \{\theta \in [0, \pi[\ ;\ |\sin(n_k\theta)| \le \sin\theta_k\}$: F_k consists of intervals of the form $[-\frac{\theta_k}{n_k} + \frac{l\pi}{n_k}, \frac{\theta_k}{n_k} + \frac{l\pi}{n_k}]$, $l \in \mathbb{Z}$. We will construct a Cantor subset K of K_0 as $K = \bigcap_{k\ge k_1} \bigcup_{j\in I_k} J_j^{(k)}$ where the arcs $J_j^{(k)}$ have the form

$$J_j^{(k)} = \left\{ e^{i\theta}\ ;\ \theta \in \left[-\frac{\theta_k}{n_k} + \frac{l_j^{(k)}\pi}{n_k}, \frac{\theta_k}{n_k} + \frac{l_j^{(k)}\pi}{n_k} \right] \right\} \tag{7}$$

for some $l_j^{(k)} \in \mathbb{Z}$. Observe that such arcs are disjoint as soon as $\frac{2\theta_k}{n_k} < \frac{\pi}{n_k}$, i.e. $\theta_k < \frac{\pi}{2}$, which is indeed the case, and that the arc corresponding to $l_j^{(k)} = 0$ contains the point 1 in its interior. There are $2n_k$ such intervals. We are going to construct by induction on k a collection $(J_j^{(k)})_{j\in I_k}$ in such a way that each $J_j^{(k)}$ has the form given in (7) and is contained in an arc of the collection $(J_j^{(k-1)})_{j\in I_{k-1}}$ constructed at step $k-1$, and the collection $(J_j^{(k)})_{j\in I_k}$ contains the arc $[-\frac{\theta_k}{n_k}, \frac{\theta_k}{n_k}]$ corresponding to the case $l = 0$. We start for $k = k_1$ with the collection of all the $2n_1$ arcs above. Suppose that the arcs at step k are constructed, and write one of them as

$$J_j^{(k)} = \left[-\frac{\theta_k}{n_k} + \frac{l_j^{(k)}\pi}{n_k}, \frac{\theta_k}{n_k} + \frac{l_j^{(k)}\pi}{n_k} \right].$$

Let us look for arcs of the form

$$\left[-\frac{\theta_{k+1}}{n_{k+1}} + \frac{r\pi}{n_{k+1}}, \frac{\theta_{k+1}}{n_{k+1}} + \frac{r\pi}{n_{k+1}} \right],\ r \in \mathbb{Z},$$

contained in $J_j^{(k)}$. There are $\lfloor \frac{1}{\pi}(\frac{n_{k+1}}{n_k}\theta_k - \theta_{k+1})\rfloor = p_{k+1}$ such intervals contained in $J_j^{(k)}$. By construction $p_{k+1} \geq \frac{1}{\pi}(4\pi - \frac{\pi}{2}) - 1 \geq 2$. Remark that in the case where $J_j^{(k)}$ is the arc $\{e^{i\theta}\ ;\ \theta \in [-\frac{\theta_k}{n_k}, \frac{\theta_k}{n_k}]\}$ we have in the collection $(J_j^{(k+1)})$ the arc $\{e^{i\theta}\ ;\ \theta \in [-\frac{\theta_{k+1}}{n_{k+1}}, \frac{\theta_{k+1}}{n_{k+1}}]\}$ (which is indeed contained in the arc $\{e^{i\theta}\ ;\ \theta \in [-\frac{\theta_k}{n_k}, \frac{\theta_k}{n_k}]\}$). We obtain in this fashion a perfect Cantor set K, which contains the point 1 by construction, such that λ^{n_k} tends to 1 uniformly on K (as $|\lambda^{n_k} - 1| \leq 2\gamma_k$ for any $\lambda \in K$ and any $k \geq k_1$). Let $\varepsilon > 0$. There exists an integer κ such that for any $k \geq \kappa$ and any $\lambda \in K$, $|\lambda^{n_k} - 1| \leq \varepsilon$. Since 1 belongs to K, the set $K_\varepsilon = \{\lambda \in K\ ;\ |\lambda - 1| \leq \varepsilon/n_{\kappa-1}\}$ is a compact perfect subset of $\mathbb{T}$, and for any $\lambda \in K_\varepsilon$ and any $0 \leq k \leq \kappa - 1$,

$$|\lambda^{n_k} - 1| \leq \frac{\varepsilon}{n_{\kappa-1}} n_k \leq \varepsilon.$$

Hence $\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon$ for any $\lambda \in K_\varepsilon$, and Proposition 3.5 is proved. □

Remark 3.6. — We have shown at the end of the proof of Proposition 3.5 that if K is a compact perfect subset of $\mathbb{T}$ such that λ^{n_k} tends to 1 uniformly on K, and if K contains the point 1, then for any $\varepsilon > 0$ there exists a $\lambda \in K \setminus \{1\}$ such that $\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon$. If we do not suppose that K contains the point 1, the set $\tilde{K} = \{\lambda\overline{\mu}\ ;\ \lambda, \mu \in K\}$ is compact, perfect, contains the point 1, and λ^{n_k} still tends to 1 uniformly on $\tilde{K}$. We thus have the following fact, which we record here for further use:

Fact 3.7. — *The following assertions are equivalent:*

(i) *there exists a compact perfect subset K of $\mathbb{T}$ such that λ^{n_k} tends to* 1 *uniformly on K;*

(ii) *there exists a compact perfect subset K of $\mathbb{T}$ such that λ^{n_k} tends to* 1 *uniformly on K, and for any $\varepsilon > 0$ there exists a $\lambda \in K \setminus \{1\}$ such that* $\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon$.

Our next examples concern sequences $(n_k)_{k\geq 0}$ such that n_k divides n_{k+1} for any $k \geq 0$ (we write this as $n_k | n_{k+1}$). We begin with the case where $\limsup_{k\to\infty} n_{k+1}/n_k = +\infty$, since in this case we can derive a stronger conclusion. Recall [**1**] that such sequences are Jamison sequences.

Proposition 3.8. — *Let $(n_k)_{k\geq 0}$ be a sequence such that $n_k | n_{k+1}$ for any $k \geq 0$ and* $\limsup_{k\to\infty} n_{k+1}/n_k = +\infty$. *There exists a compact perfect subset K of $\mathbb{T}$ containing the point* 1 *such that $\lambda^{n_k} \to 1$ uniformly on K.*

Proof. — Since $n_k | n_{k+1}$ for any $k \geq 0$, we have $n_{k+1} \geq 2n_k$, so that

$$\sum_{k\geq 1} \frac{1}{n_k} \leq 1 \quad \text{and} \quad \sum_{j\geq k+1} \frac{1}{n_j} \leq \frac{2}{n_{k+1}} \quad \text{for any} \quad k \geq 1.$$

Let $(k_p)_{p\geq 1}$ be a strictly increasing sequence of integers such that $\frac{n_{k_p}}{n_{k_p-1}} \to +\infty$ as $k_p \to \infty$. For any sequence $\varepsilon \in \{0,1\}^{\mathbb{N}}$ of zeroes and ones, $\varepsilon = (\varepsilon_p)_{p\geq 1}$, consider the real number of $[0,1]$

$$\theta_\varepsilon = \sum_{p\geq 1} \frac{\varepsilon_p}{n_{k_p}}, \qquad \text{and} \qquad \lambda_\varepsilon = e^{2i\pi\theta_\varepsilon} \in \mathbb{T}.$$

The set $K = \{\lambda_\varepsilon \ ; \ \varepsilon \in \{0,1\}^{\mathbb{N}}\}$ is compact, perfect, and contains the point 1. Let us now show that $\lambda_\varepsilon^{n_k}$ tends to 1 uniformly in $\varepsilon \in \{0,1\}^{\mathbb{N}}$. Fix $\delta > 0$, and let $p_0 \geq 1$ be such that for any $p \geq p_0$, $\frac{n_{k_p}-1}{n_{k_p}} < \frac{\delta}{4\pi}$. Let $k \geq k_{p_0}$, and $\varepsilon \in \{0,1\}^{\mathbb{N}}$. There exists a $p \geq p_0$ such that $n_{k_p} \leq n_k \leq n_{k_{p+1}} - 1$. We have

$$n_k \theta_\varepsilon = n_k \sum_{j=1}^{p} \frac{\varepsilon_j}{n_{k_j}} + n_k \sum_{j \geq p+1} \frac{\varepsilon_j}{n_{k_j}} \cdot$$

Since $n_{k_j} | n_k$ for any $j = 1, \ldots, p$, $n_k \sum_{j=1}^{p} \frac{\varepsilon_j}{n_{k_j}}$ belongs to $\mathbb{Z}$. Hence

$$|e^{2i\pi n_k \theta_\varepsilon} - 1| \leq 2\pi \, n_k \sum_{j \geq p+1} \frac{1}{n_{k_j}} \leq 2\pi \, n_k \sum_{j \geq k_{p+1}} \frac{1}{n_j} \leq 4\pi \, \frac{n_k}{n_{k_{p+1}}} \leq 4\pi \, \frac{n_{k_{p+1}} - 1}{n_{k_{p+1}}} < \delta,$$

so $|\lambda_\varepsilon^{n_k} - 1| < \delta$ for any $k \geq k_{p_0}$ and $\varepsilon \in \{0,1\}^{\mathbb{N}}$. This proves our statement. □

Let us now move over to the case where $n_k | n_{k+1}$ for every $k \geq 0$, but where n_{k+1}/n_k is possibly bounded: for instance $n_k = 2^k$ for any $k \geq 0$. Is $(n_k)_{k \geq 0}$ a rigidity sequence? Somewhat surprisingly, the answer is yes. This was kindly shown to us by Jean-Pierre Kahane, who proved the following proposition:

Proposition 3.9. — *Let $(n_k)_{k \geq 0}$ be a sequence such that $n_k | n_{k+1}$ for every $k \geq 0$. Then $(n_k)_{k \geq 0}$ is a rigidity sequence.*

This proposition is also proved in the preprint [**7**].

Proof. — Let $(a_k)_{k \geq 1}$ be a decreasing sequence of positive numbers going to 0 as k goes to infinity, with $a_k < 1$ for every $k \geq 1$, such that the series $\sum_{k \geq 1} a_k$ is divergent. Consider the infinite convolution of Bernoulli measures defined on $[0, 2\pi]$ by

$$\mu = *_{j \geq 1}((1 - a_j)\delta_0 + a_j \delta_{\frac{1}{n_j}}),$$

where δ_a denotes the Dirac measure at the point a for any $a \in [0, 2\pi]$. Clearly μ is a probability measure on $[0, 2\pi]$ which is continuous. Indeed μ is the distribution of the random variable

$$\xi = \sum_{j \geq 1} \frac{\varepsilon_j}{n_j},$$

where $(\varepsilon_j)_{j \geq 1}$ is a sequence of independent Bernoulli random variables taking values 0 and 1 with probabilities $p_{0j} = 1 - a_j$ and $p_{1j} = a_j$ respectively. Since $\sum a_j = +\infty$, the measure μ is continuous by a result of Lévy (see [**13**] for a simple proof). It thus remains to prove that $\hat{\mu}(n_k) \to 1$ as $n_k \to +\infty$. Since $n_j | n_{j+1}$ for each $j \geq 0$,

$$\hat{\mu}(n_k) = \prod_{j \geq k+1} (1 - a_j + a_j e^{2i\pi \frac{n_k}{n_j}}) = \prod_{j \geq k+1} (1 - a_j(1 - e^{2i\pi \frac{n_k}{n_j}})).$$

Recall now the following easy fact: for any $N \geq 1$ and any complex numbers x_j with $|x_j| \leq 1$ for every $j = 1, \ldots, N$, we have $|\prod_{j=1}^{N} x_j - 1| \leq \sum_{j=1}^{N} |x_j - 1|$. Since for any $j \geq k+1$,

$$|1 - a_j(1 - e^{2i\pi \frac{n_k}{n_j}})| = |1 - a_j + a_j e^{2i\pi \frac{n_k}{n_j}}| \leq 1 - a_j + a_j = 1,$$

we get that

$$|\hat{\mu}(n_k)-1| \leq \sum_{j\geq k+1} a_j |1-e^{2i\pi \frac{n_k}{n_j}}| \leq 2\pi\, a_{k+1} \sum_{j\geq k+1} \frac{n_k}{n_j} \leq 4\pi\, a_{k+1}$$

since the sequence $(a_j)_{j\geq 1}$ is decreasing and $n_k \sum_{j\geq k+1} \frac{1}{n_j} \leq n_{k+1} \sum_{j\geq k+1} \frac{1}{n_j} \leq 2$, as seen in Proposition 3.8 above. Hence $\hat{\mu}(n_k) \to 1$, and this proves Proposition 3.9. □

Remark 3.10. — Remark that if $n_k = 2^k$ for instance, the only λ's in $\mathbb{T}$ such that λ^{n_k} tends to 1 are the $2^{k^{th}}$ roots of 1. More generally, it is not difficult to see (using an argument of [**1**]) that if $(n_k)_{k\geq 0}$ is a Jamison sequence (which is the case as soon as $\sup n_{k+1}/n_k$ is finite), $\lambda^{n_k} \to 1$ if and only if there exists a k_0 such that $\lambda^{n_{k_0}} = 1$.

Remark 3.11. — The proof of Proposition 3.9 yields a bit more, namely that given any sequence $(a_k)_{k\geq 0}$ of positive numbers decreasing to zero and such that the series $\sum a_k$ diverges, there exists a continuous probability measure σ on $\mathbb{T}$ such that $|\hat{\sigma}(n_k) - 1| \leq a_k$ for every $k \geq 0$. This will turn out to be crucial in the proof of the statement of Example 3.17. In general one cannot obtain such a measure σ with $\sum |\hat{\sigma}(n_k)-1| < +\infty$: this would imply that the series $\sum |\lambda^{n_k} - 1|$ converges σ-a.e., so that $|\lambda^{n_k} - 1| \to 0$ σ-a.e., and we have seen in Remark 3.10 above that this is impossible if n_{k+1}/n_k is bounded for instance.

The proof of Proposition 3.9 uses in a crucial way the divisibility assumption on the n_k's, and it comes as a natural question to ask whether it can be dispensed with: if there exists an $a > 1$ such that $n_{k+1}/n_k \geq a$ for any $k \geq 0$, must $(n_k)_{k\geq 0}$ be a rigidity sequence? We were not able to settle this question, but it is answered in [**7**] in the negative: the sequence $(n_k)_{k\geq 0}$ with $n_k = 2^k + 1$ cannot be a rigidity sequence. Indeed we have $2n_k = n_{k+1} + 1$, so that if $(n_k)_{k\geq 0}$ were a rigidity sequence, with φ an associated weakly mixing measure-preserving transformation on $(X, \mathcal{F}, \mu)$, we should have both $U_\varphi^{2n_k} \to I$ (SOT) and $U_\varphi^{n_k+1} \to I$ (SOT), so that $U_\varphi = I$ which is impossible.

Obviously a rigidity sequence must have density 0 (this is pointed out already in [**20**]). Some of the simplest examples of non-rigidity sequences $(n_k)_{k\geq 0}$ satisfy $n_{k+1}/n_k \to 1$. Our three Examples 3.12, 3.13 and 3.14 overlap with examples of [**7**].

Example 3.12. — Let $p \in \mathbb{Z}[X]$ be a polynomial with nonnegative coefficients, $p \neq 0$. Then the sequence $(n_k)_{k\geq 0}$ with $n_k = p(k)$ cannot be a rigidity sequence.

This follows directly from Weyl's polynomial equidistribution theorem (see for instance [**25**, p. 27]): for any irrational number $\theta \in [0,1]$, the sequence $(p(k)\theta)_{k\geq 0}$ is uniformly equidistributed. Hence

$$\frac{1}{N}\sum_{k=1}^{N} e^{2i\pi p(k)\theta} \to 0 \quad \text{as } N \to +\infty$$

for every $\theta \in [0,1] \setminus \mathbb{Q}$. Hence if σ is any continuous probability measure on $\mathbb{T}$,

$$\frac{1}{N}\sum_{k=1}^{N} \hat{\sigma}(n_k) \to 0,$$

and this forbids $\hat{\sigma}(n_k)$ to tend to 1. We have proved in fact:

Example 3.13. — If there exists a countable subset Q of $[0,1]$ and a $\delta > 0$ such that for any $\theta \in [0,1] \setminus Q$,

$$\liminf_{N\to+\infty} \Big|\frac{1}{N}\sum_{k=1}^{N} e^{2i\pi n_k\theta}\Big| \le 1-\delta,$$

then $(n_k)_{k\ge0}$ is not a rigidity sequence.

See [**1**] for some examples of such sequences. Let us point out that (contrary to what happens for Jamison sequences), it is obvious to exhibit non-rigidity sequences $(n_k)_{k\ge0}$ with $\liminf n_{k+1}/n_k = 1$ and $\limsup n_{k+1}/n_k = +\infty$: take any sequence $(n_{2k})_{k\ge0}$ such that $n_{2k+2}/n_{2k} \to +\infty$, and set $n_{2k+1} = n_{2k}+1$. If $(n_k)_{k\ge0}$ were a rigidity sequence, with φ an associated weakly mixing measure-preserving transformation on $(X,\mathcal{F},\mu)$, we should have $U_\varphi^{n_k} \to I$ (SOT), so that $U_\varphi = I$, a contradiction. A similar type of argument yields

Example 3.14. — If $(n_k)_{k\ge0}$ denotes the sequence of prime numbers, then $(n_k)_{k\ge0}$ is not a rigidity sequence.

Proof. — This follows from a result of Vinogradov that any sufficiently large odd number can be written as a sum of three primes. Suppose by contradiction that $(n_k)_{k\ge0}$ is a rigidity sequence with φ an associated weakly mixing measure-preserving transformation on $(X,\mathcal{F},\mu)$. Then $U_\varphi^{n_k} \to I$ (SOT). Let $f \ne 0$ be a function in $L^2(X,\mathcal{F},\mu)$ with $\int_X f d\mu = 0$. If $\varepsilon > 0$ is any positive number, let k_0 be such that for any $k \ge k_0$, $||U_\varphi^{n_k} f - f|| < \varepsilon$ and every odd integer greater than or equal to k_0 can be written as a sum of three primes. Consider the finite set of integers $A = \{0, n_{k_1}, n_{k_1}+n_{k_2}, n_{k_1}+n_{k_2}+n_{k_3}\ ;\ 0 \le k_i \le k_0 \text{ for } i = 1,2,3\}$. We claim that for any odd integer $2n+1 \ge k_0$, there exists an $m \in A$ such that $||U_\varphi^{2n+1} f - U_\varphi^m f|| < 3\varepsilon$. Indeed, let us write $2n+1$ as $2n+1 = n_{k_1}+n_{k_2}+n_{k_3}$ with $0 \le k_1 \le k_2 \le k_3$, and consider separately four cases:

– if $k_1 > k_0$, then $||U_\varphi^{n_{k_1}+n_{k_2}+n_{k_3}} f - f|| \le ||U_\varphi^{n_{k_1}} f - f|| + ||U_\varphi^{n_{k_2}} f - f|| + ||U_\varphi^{n_{k_3}} f - f|| < 3\varepsilon$;
– if $k_1 \le k_0$ and $k_2 > k_0$, $||U_\varphi^{n_{k_1}+n_{k_2}+n_{k_3}} f - U_\varphi^{n_{k_1}} f|| \le ||U_\varphi^{n_{k_2}} f - f|| + ||U_\varphi^{n_{k_3}} f - f|| < 2\varepsilon$;
– if $k_2 \le k_0$ and $k_3 > k_0$, $||U_\varphi^{n_{k_1}+n_{k_2}+n_{k_3}} f - U_\varphi^{n_{k_1}+n_{k_2}} f|| \le ||U_\varphi^{n_{k_3}} f - f|| < \varepsilon$;
– if $k_3 \le k_0$, there is nothing to prove.

Now since φ is weakly mixing, $U_\varphi^{2n+1} f \to 0$ (WOT) along a set D which is of density 1 in the set of odd integers. Since A is finite, it follows that there exists some $m \in A$ such that $\|U_\varphi^{l_j} f - U_\varphi^m f\| < 3\varepsilon$ for an increasing sequence $(l_j)_{j\ge0} \subset D$. Thus, for every $g \in L^2(X,\mathcal{F},\mu)$ we have

$$|\langle U_\varphi^m f, g\rangle| \le |\langle U_\varphi^m f - U_\varphi^{l_j}, g\rangle| + |\langle U_\varphi^{l_j} f, g\rangle| \le 3\varepsilon\|g\| + |\langle U_\varphi^{l_j} f, g\rangle|.$$

Taking the weak limit as $j \to \infty$ of the above expression implies $\|U_\varphi^m f\| \le 3\varepsilon$. Thus $f = 0$, a contradiction. □

The proof of Example 3.14 actually shows that if there exists an integer $r \ge 2$ such that any sufficiently large integer in a set of positive density can be written as a sum of r elements of the set $\{n_k\ ;\ k \ge 0\}$, then $(n_k)_{k\ge0}$ cannot be a rigidity sequence. As pointed out in [**7**], the statement of Example 3.14 can also be deduced from the fact that $(n_k x)_{k\ge0}$ is uniformly distributed for all but a countable set of values of $x \in [0,1]$.

We finish this section with some more examples of rigidity sequences. We consider the sequence $(q_n)_{n\geq 1}$ of quotients of the convergents of some irrational numbers $\alpha \in]0,1[$. Let α be such a number, and let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

with the a_n's positive integers, be its continued fraction expansion. The convergents of α are the rational numbers $\frac{p_n}{q_n}$ defined recursively by the equations

$$\begin{cases} p_0 = 0,\ p_1 = 1,\ p_{n+1} = a_n p_n + p_{n-1} \text{ for } n \geq 2 \\ q_0 = 1,\ q_1 = a_1,\ q_{n+1} = a_n q_n + q_{n-1} \text{ for } n \geq 2. \end{cases}$$

See for instance [**18**] for more about continued fraction expansions and approximations of irrational numbers by rationals. We have

$$\frac{1}{2q_n q_{n+1}} \leq \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \tag{8}$$

for any $n \geq 1$. It follows that $|e^{2i\pi q_n \alpha} - 1| \to 0$ as $n \to +\infty$. Hence there exist infinitely many numbers $\lambda \in \mathbb{T} \setminus \{1\}$ such that $|\lambda^{q_n} - 1| \to 0$ as $n \to +\infty$, and the sequence $(q_n)_{n\geq 1}$ is a possible candidate for a rigidity sequence. We begin by recalling a particular case of a result of Katok and Stepin [**22**], see also [**32**]:

Example 3.15. — If, with the notation above,

$$|\alpha - \frac{p_n}{q_n}| = o(\frac{1}{q_n^2}),$$

then $(q_n)_{n\geq 0}$ is a rigidity sequence.

This can also be seen as a direct consequence of our Example 3.4: by the lower bound in (8), the assumption is equivalent to $q_{n+1}/q_n \to +\infty$ (i.e. $a_n \to +\infty$). It is also possible to show that $(q_n)_{n\geq 0}$ is a rigidity sequence (and even more) for some irrational numbers α with $\liminf a_n < +\infty$. For instance:

Example 3.16. — Let $m \geq 2$ be an integer, and let α_m be the Liouville number

$$\alpha_m = \sum_{k\geq 0} m^{-(k+1)!}.$$

If $(q_n)_{n\geq 1}$ denotes the sequence of denominators of the convergents of α_m, then there exists a perfect compact subset of $\mathbb{T}$ on which λ^{q_n} tends uniformly to 1. In particular $(q_n)_{n\geq 1}$ is a rigidity sequence.

Proof. — The proof relies on a paper of Shallit [**35**] where the continued fraction expansion of α_m is determined: if $[a_0, a_1, \dots, a_{N_v}]$ is the continued fraction expansion of $\sum_{k=0}^{v} m^{-(k+1)!}$, v a nonnegative integer, then the continued fraction expansion of the next partial sum $\sum_{k=0}^{v+1} m^{-(k+1)!}$ is given by

$$[a_0, a_1, \dots, a_{N_{v+1}}] = [a_0, a_1, \dots, a_{N_v}, m^{v(v+1)!} - 1, 1, a_{N_v} - 1, a_{N_v - 1}, \dots, a_2, a_1]$$

as soon as N_v is even. One has $N_{v+1} = 2N_v + 2$ so that N_{v+1} is indeed even. This yields that the continued fraction expansion of α_m is

$$[0, m-1, m+1, m^2-1, 1, m, m-1, m^{12}-1, 1, m-2, m, 1, m^2-1, m+1, m-1, m^{72}-1, 1, \ldots].$$

We have $a_{N_v+1} = m^{(v-1)v!} - 1$. For any $v \geq 0$,

$$\frac{q_{N_v} + 2}{q_{N_v} + 1} = m^{(v-1)v!} - 1 + \frac{q_{N_v}}{q_{N_v} + 1} \geq m^{(v-1)v!} - 1 \geq \frac{1}{2} m^{(v-1)v!} \quad \text{for} \quad v \geq 2.$$

Applying the proof of Proposition 3.5 to the sequence $(n_v)_{v \geq 0} = (q_{N_v+1})_{v \geq 0}$, we get that there exists a perfect compact subset K of $\mathbb{T}$ containing the point 1 such that for any $\lambda \in K$ and any $v \geq 1$,

$$|\lambda^{q_{N_v+1}} - 1| \leq 10\pi \sup_{j \geq v} \frac{q_{N_{j-1}+1}}{q_{N_j+1}} \leq 10\pi \frac{q_{N_v+1}}{q_{N_{v+1}+1}} \leq 10\pi \frac{q_{N_v+2}}{q_{N_v+1}} \leq 20\pi\, m^{-(v-1)v!}.$$

Let now p be an integer such that $N_{v-1} + 2 \leq p \leq N_v$ for some $v \geq 0$, and $\lambda \in K$. We need to estimate $|\lambda^p - 1|$. If $p = N_{v-1} + 2$, we have $q_{N_{v-1}+2} = a_{N_{v-1}+2}\, q_{N_{v-1}+1} + q_{N_{v-1}} \leq (a_{N_{v-1}+2} + 1) q_{N_{v-1}+1}$. In the same way $q_{N_{v-1}+3} \leq (a_{N_{v-1}+2} + 1)(a_{N_{v-1}+3} + 1) q_{N_{v-1}+1}$ etc., and

$$q_{N_{v-1}+j} \leq \prod_{i=2}^{j} (a_{N_{v-1}+i} + 1)\, q_{N_{v-1}+1} \qquad \text{for any } 2 \leq j \leq N_v - N_{v-1} = N_{v-1} + 2.$$

So for $N_{v-1} + 2 \leq p \leq N_v$ we have

$$q_p \leq \prod_{i=2}^{p-N_{v-1}} (a_{N_{v-1}+i} + 1)\, q_{N_{v-1}+1}$$

so that

$$|\lambda^{q_p} - 1| \leq \prod_{i=2}^{N_{v-1}+2} (a_{N_{v-1}+i} + 1)\, |\lambda^{q_{N_{v-1}+1}} - 1|.$$

It remains to estimate the quantity $\prod_{i=2}^{N_{v-1}+2} (a_{N_{v-1}+i} + 1)$. We have $\{a_{N_{v-1}+2}, \ldots, a_{N_v}\} = \{1, a_{N_{v-1}} - 1, a_{N_{v-1}-1}, \ldots, a_2, a_1\}$ so that $\prod_{i=2}^{N_{v-1}+2} (a_{N_{v-1}+i} + 1) \leq 2 \prod_{i=2}^{N_{v-1}} (a_i + 1)$. Let us write $R_{v-1} = \prod_{i=2}^{N_{v-1}} (a_i + 1)$. We have $R_v \leq R_{v-1}\, m^{(v-2)(v-1)!}\, 2\, R_{v-1}$ by the inequality above, i.e.

$$\begin{aligned} R_v \leq 2\, R_{v-1}^2 m^{(v-2)(v-1)!} &\leq 2^{1+2} R_{v-2}^4 m^{(v-2)(v-1)! + 2(v-3)(v-2)!} \\ &\leq \ldots \leq 2^{2^{v+1}} m^{\sum_{k=1}^{v-1} 2^{k-1}(v-(k+1))(v-k)!}. \end{aligned}$$

Now $(v-1)! \geq 2^{k-1}(v-k)!$, so

$$R_v \leq 2^{2^{v+1}} m^{(v-1)(v-2)(v-1)!}.$$

Hence

$$|\lambda^{q_p} - 1| \leq 2^{2^{v+1}} m^{(v-1)(v-2)(v-1)!} |\lambda^{q_{N_{v-1}+1}} - 1|$$

for any $N_{v-1} + 2 \leq p \leq N_v$. Now we have

$$|\lambda^{q_{N_{v-1}+1}} - 1| \leq 20\pi\, m^{-(v-1)v!}$$

so that

$$|\lambda^{q_p} - 1| \leq 20\pi\, 2^{2^{v+2}} m^{(v-1)(v-1)!(v-2-v)} = 20\pi\, 2^{2^{v+2}} m^{-2(v-1)(v-1)!}$$

for $N_{v-1}+2 \leq p \leq N_v$. Since the quantity $2^{2^{v+2}} m^{-2(v-1)(v-1)!}$ tends to 0 as v tends to infinity, it follows that λ^{q_n} tends to 1 uniformly on K as n tends to infinity. □

A stronger result is proved in [**7**]: actually if α is any irrational number in $]0,1[$, the sequence $(q_n)_{n\geq 0}$ of denominators of the convergents of α is always a rigidity sequence.

We finish our study of rigidity sequences by giving an example of a rigidity sequence such that $n_{k+1}/n_k \to 1$. This answers a question of [**7**].

Example 3.17. — There exists a sequence $(n_k)_{k\geq 0}$ with $n_{k+1}/n_k \to 1$ as $k \to +\infty$ which is a rigidity sequence.

Proof. — Let $(k_p)_{p\geq 2}$ be a very quickly increasing sequence of integers with $k_1 = 1$ which will be determined later on in the proof. For $p \geq 0$, let $N_p = 2^{2^p}$, and consider the set

$$A_{N_p} = \bigcup_{k=k_{p+1}}^{2k_{p+2}-1} A_{p,k}$$

where

$$A_{N_p,k} = \{N_p^k, N_p^k + N_p^{k-1}, N_p^k + 2N_p^{k-1}, N_p^k + 3N_p^{k-1}, \ldots, N_p^k + ((N_p-1)N_p - 1)N_p^{k-1}\}.$$

For instance,

$$A_2 = \bigcup_{k=2}^{2k_2-1} \{2^k, 2^k + 2^{k-1}\}, \quad A_4 = \bigcup_{k=k_2}^{2k_3-1} \{4^k, 4^k + 4^{k-1}, 4^k + 2\,4^{k-1}, \ldots, 4^k + 11\,4^{k-1}\}, \text{ etc.}$$

As the last element of A_{N_p} is $N_p^{2k_{p+2}} - N_p^{2k_{p+2}-2}$ which is less than the first element of $A_{N_{p+1}}$, $N_{p+1}^{k_{p+2}} = N_p^{2k_{p+2}}$, these sets are successive and disjoint. Let $(n_j)_{j\geq 0}$ be the strictly increasing sequence such that $A = \bigcup_{p\geq 0} A_{N_p} = \{n_j \ ; \ j \geq 0\}$. Let us first check that $n_{j+1}/n_j \to 1$: first of all, if n_j and n_{j+1} belong to the same set $A_{N_p,k}$,

$$\frac{n_{j+1}}{n_j} = \frac{N_p^k + l\,N_p^{k-1}}{N_p^k + (l-1)\,N_p^{k-1}} = 1 + \frac{N_p^{k-1}}{N_p^k + (l-1)\,N_p^{k-1}} \leq 1 + \frac{1}{N_p}\cdot$$

If n_j is in some set $A_{N_p,k}$ and n_{j+1} is in $A_{N_p,k+1}$,

$$\frac{n_{j+1}}{n_j} = \frac{N_p^{k+1}}{N_p^{k+1} - N_p^{k-1}} = \frac{1}{1 - \frac{1}{N_p^2}} = \frac{N_{p+1}}{N_{p+1} - 1}\cdot$$

Lastly, if n_j is the last integer of A_{N_p} and n_{j+1} is the first integer of $A_{N_{p+1}}$,

$$\frac{n_{j+1}}{n_j} = \frac{N_{p+1}^{k_{p+2}}}{N_p^{2k_{p+2}} - N_p^{2k_{p+2}-2}} = \frac{N_p^{2k_{p+2}}}{N_p^{2k_{p+2}} - N_p^{2k_{p+2}-2}} = \frac{N_{p+1}}{N_{p+1} - 1}\cdot$$

Thus $n_{j+1}/n_j \to 1$. Let now σ be a continuous probability measure on $\mathbb{T}$ such that

– for any $0 \leq k \leq 2k_2 - 1$, $|\hat{\sigma}(2^k) - 1| \leq a_k$
– for any $p \geq 1$ and $k_{p+1} \leq k \leq 2k_{p+2} - 1$,

$$|\hat{\sigma}(N_p^k) - 1| \leq \frac{a_{2k_{p+1}-1}}{a_{k_{p+1}}} a_k$$

where $a_0 = a_1 = 1$ and $a_k = \frac{1}{k \log k}$ for $k \geq 2$. Such a measure does exist by Proposition 3.9 and Remark 3.11. Indeed the successive terms of the sequence

$$(1, 2, 4, \ldots, 2^{2k_2-1}, 4^{k_2}, 4^{k_2+1}, \ldots) = (m_j)_{j\geq 0}$$

divide each other. The sequence $(a_0, a_1, \ldots, a_{2k_2-1}, \frac{a_{2k_2-1}}{a_{k_2}} a_{k_2}, \frac{a_{2k_2-1}}{a_{k_2}} a_{k_2+1}, \ldots) = (b_j)_{j\geq 0}$ is decreasing to zero, and $\sum b_j$ is divergent: if the sequence (k_p) grows fast enough,

$$\sum_{j\geq 0} b_j \geq \sum_{k=2}^{2k_2-1} a_k + \frac{a_{2k_2-1}}{a_{k_2}} \sum_{k=k_2}^{2k_3-1} a_k + \ldots$$

and since the series $\sum a_k$ is divergent, it is possible to choose k_{p+1} so large with respect to k_p that

$$\frac{a_{2k_p-1}}{a_{k_p}} \sum_{k=k_p}^{2k_{p+1}-1} a_k \geq 1$$

for instance for each p. So we have a probability measure σ on $\mathbb{T}$ such that $|\hat{\sigma}(m_j)-1| \leq b_j$ for each $j \geq 0$. It remains to show that $|\hat{\sigma}(n_k) - 1| \to 0$. For $k_{p+1} \leq k \leq 2k_{p+2} - 1$ and $0 \leq l \leq (N_p - 1)N_p - 1$, we have

$$\begin{aligned} |\hat{\sigma}(N_p^k + l\, N_p^{k-1}) - 1| &\leq |\hat{\sigma}(N_p^k) - 1| + l\, |\hat{\sigma}(N_p^{k-1}) - 1| \\ &\leq |\hat{\sigma}(N_p^k) - 1| + ((N_p - 1)N_p - 1)|\hat{\sigma}(N_p^{k-1}) - 1|. \end{aligned}$$

If $k_{p+1} + 1 \leq k \leq 2k_{p+2} - 1$, this is less than

$$\begin{aligned} \frac{a_{2k_{p+1}-1}}{a_{k_{p+1}}}(a_k + ((N_p - 1)N_p - 1)a_{k-1}) &\leq (N_p - 1)N_p \frac{a_{2k_{p+1}-1}}{a_{k_{p+1}}} a_{k_{p+1}} \\ &\leq (N_p - 1)N_p\, a_{2k_{p+1}-1}. \end{aligned}$$

If k_{p+1} is large enough compared to N_p, this quantity is less than 2^{-p}. If $k = k_{p+1}$, $N_p^{k_{p+1}-1} = N_{p-1}^{2k_{p+1}-2}$. Hence

$$\begin{aligned} |\hat{\sigma}(N_p^{k_{p+1}} + l\, N_p^{k_{p+1}-1}) - 1| &\leq a_{2k_{p+1}-1} + ((N_p - 1)N_p - 1) \frac{a_{2k_p-2}}{a_{k_p}} a_{2k_{p+1}-2} \\ &\leq (1 + ((N_p - 1)N_p - 1) \frac{a_{2k_p-2}}{a_{k_p}}) a_{2k_{p+1}-2} \end{aligned}$$

and this again can be made less than 2^{-p} provided k_{p+1} is sufficiently large with respect to N_p and k_p. Hence $\hat{\sigma}(n_k) \to 1$ as $k \to +\infty$, and this proves that $(n_k)_{k\geq 0}$ is a rigidity sequence. □

4. Topologically and uniformly rigid linear dynamical systems

4.1. Back to rigidity in the linear framework. — Before moving over to topological versions of rigidity for linear dynamical systems, we have to settle the following natural question: which sequences $(n_k)_{k\geq 0}$ appear as rigidity sequences (in the measure-theoretic sense) for linear dynamical systems? Here is the answer:

Theorem 4.1. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers. The following assertions are equivalent:*

(1) *there exists a continuous probability measure* σ *on* $\mathbb{T}$ *such that* $\hat{\sigma}(n_k) \to 1$ *as* $k \to +\infty$, *i.e.,* $(n_k)_{k\geq 0}$ *is a rigidity sequence;*

(2) *there exists a bounded linear operator* T *on a separable complex infinite-dimensional Hilbert space* H *which admits a non-degenerate Gaussian measure* m *with respect to which* T *defines a weakly mixing measure-preserving transformation which is rigid with respect to* $(n_k)_{k\geq 0}$.

Proof. — The implication (2) ⇒ (1) is an immediate consequence of Proposition 3.1, so let us prove that (1) ⇒ (2). Let σ be a continuous probability measure σ on $\mathbb{T}$ such that $\hat{\sigma}(n_k) \to 1$ as $n_k \to +\infty$, and let $L \subseteq \mathbb{T}$ be the support of the measure σ. It is a compact perfect subset of $\mathbb{T}$, and $\sigma(\Omega) > 0$ for any non-empty open subset Ω of L. Kalish constructed in [**21**] an example of a bounded operator on a Hilbert space whose point spectrum is equal to L, and, as in [**3**], we use this example for our purposes: let T_0 be the operator defined on $L^2(\mathbb{T})$ by $T_0 = M - J$, where $Mf(\lambda) = \lambda f(\lambda)$ and $Jf(\lambda) = \int_{(1,\lambda)} f(\zeta)d\zeta$ for any $f \in L^2(\mathbb{T})$ and $\lambda \in \mathbb{T}$. For $\lambda \in \mathbb{T}$, $\lambda = e^{i\theta}$, $(1, \lambda)$ denotes the arc $\{e^{i\alpha} \; ; \; 0 \leq \alpha \leq \theta\}$, and $(\lambda, 1)$ the arc $\{e^{i\alpha} \; ; \; \theta \leq \alpha \leq 2\pi\}$. For every λ, the characteristic function χ_λ of the arc $(\lambda, 1)$ is an eigenvector of T_0 associated to the eigenvalue λ. Let T be the operator induced by T_0 on the space $H = \overline{\text{sp}}[\chi_\lambda \; ; \; \lambda \in L]$. It is proved in [**21**] that $\sigma(T) = \sigma_p(T) = L$, and it is not difficult to see that $E : \lambda \mapsto \chi_\lambda$ is a continuous eigenvector field for T on L which is spanning. Hence it is a perfectly spanning unimodular eigenvector field with respect to the measure σ (see [**3**] for details), and there exists a non-degenerate Gaussian measure m on H whose covariance operator S is given by $S = KK^*$, where $K : L^2(\mathbb{T}, \sigma) \to H$ is the operator defined by $K\varphi = \int_{\mathbb{T}} \varphi(\lambda)E(\lambda)d\sigma(\lambda)$ for $\varphi \in L^2(\mathbb{T}, \sigma)$, with respect to which T defines a weakly mixing measure-preserving transformation. It remains to prove that T is rigid with respect to $(n_k)_{k\geq 0}$, i.e. that $U_T^{n_k} f$ tends weakly to f in $L^2(H, \mathcal{B}, m)$. Using the same kind of arguments as in [**4**] or [**6**, Ch. 5], we see that it suffices to prove that for any elements x, y of H,

$$\int_H \langle x, T^{n_k} z\rangle \overline{\langle y, z\rangle} dm(z) \longrightarrow \int_H \langle x, z\rangle \overline{\langle y, z\rangle} dm(z) \qquad \text{as } n_k \to +\infty.$$

But this is clear: since $TK = KV$, where V is the multiplication operator by λ on $L^2(\mathbb{T}, \sigma)$, we have

$$\begin{aligned} \int_H \langle x, T^{n_k} z\rangle \overline{\langle y, z\rangle} dm(z) &= \langle KK^* T^{*n_k} x, y\rangle = \langle V^{*n_k} K^* x, K^* y\rangle \\ &= \int_{\mathbb{T}} \lambda^{-n_k} \langle x, E(\lambda)\rangle \overline{\langle y, E(\lambda)\rangle} d\sigma(\lambda). \end{aligned}$$

The function $h(\lambda) = \langle x, E(\lambda)\rangle \overline{\langle y, E(\lambda)\rangle}$ belongs to $L^1(\mathbb{T}, \sigma)$, and we have seen in the proof of Proposition 3.1 that $\int_{\mathbb{T}} |\lambda^{n_k} - 1| \, |h(\lambda)| d\sigma(\lambda) \to 0$. Hence

$$\int_{\mathbb{T}} \lambda^{-n_k} h(\lambda) d\sigma(\lambda) \to \int_{\mathbb{T}} h(\lambda) d\sigma(\lambda) = \int_H \langle x, z\rangle \overline{\langle y, z\rangle} dm(z),$$

and this proves our statement. □

Remark 4.2. — The Kalish-type operators which are used in the proof of Theorem 4.1 have no reason at all to be power-bounded with respect to (n_k), contrary to what happens when considering topological rigidity. We only know for instance, applying the rigidity assumption to the function $f(z) = ||z||$, that

$$\int_H ||(T^{n_k} - I)z||\, dm(z) \to 0 \qquad \text{as } n_k \to +\infty.$$

4.2. A characterization of topologically rigid sequences for linear dynamical systems. — Let us prove Theorem 1.12. First of all, (3) implies (2) since, as recalled in Section 2.1, (3) implies that T has perfectly spanning unimodular eigenvectors. We suppose next that (2) holds and show (1). Let X and T be as in (2). For any $\lambda \in \sigma_p(T) \cap \mathbb{T}$, let e_λ be an associated eigenvector with $||e_\lambda|| = 1$. Since $T^{n_k} e_\lambda \to e_\lambda$, $|\lambda^{n_k} - 1| \to 0$ for any $\lambda \in \sigma_p(T) \cap \mathbb{T}$. Moreover, by the uniform boundedness principle, $\sup_{k \geq 0} ||T^{n_k}|| = M$ is finite. Suppose by contradiction that there exists an $\varepsilon_0 > 0$ such that for any $\lambda, \mu \in \sigma_p(T) \cap \mathbb{T}$ with $\lambda \neq \mu$, $\sup_{k\geq 0} |\lambda^{n_k} - \mu^{n_k}| \geq \varepsilon_0$. Then for any $\lambda, \mu \in \sigma_p(T) \cap \mathbb{T}$,

$$|\lambda^{n_k} - \mu^{n_k}| - ||e_\lambda - e_\mu|| \leq ||\lambda^{n_k} e_\lambda - \mu^{n_k} e_\mu|| \leq M\, ||e_\lambda - e_\mu||$$

so that $|\lambda^{n_k} - \mu^{n_k}| \leq (M+1)||e_\lambda - e_\mu||$. Hence $\varepsilon_0 \leq (M+1)||e_\lambda - e_\mu||$, and the unimodular eigenvectors of T are $\varepsilon_0/(M+1)$-separated. Since X is separable there can only be countably many such eigenvectors, which contradicts the fact that $\sigma_p(T) \cap \mathbb{T}$ is uncountable. So for any $\varepsilon > 0$ there exist λ, μ in $\sigma_p(T) \cap \mathbb{T}$ with $\lambda \neq \mu$ such that $\sup_{k \geq 0} |(\lambda\overline{\mu})^{n_k} - 1| \leq \varepsilon$, and $|(\lambda\overline{\mu})^{n_k} - 1| \to 0$. So (1) holds true.
We state again what we have to prove in order to obtain that (1) implies (3):

Theorem 4.3. — *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers with $n_0 = 1$ such that for any $\varepsilon > 0$ there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ with*

$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon \quad \text{and} \quad |\lambda^{n_k} - 1| \to 0 \quad \text{as} \quad k \to \infty.$$

Then there exists a bounded linear operator T on a Hilbert space H such that T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues and for every $x \in H$, $T^{n_k} x \to x$ as $k \to \infty$.

Before starting the proof of Theorem 4.3, let us point out that the statement is not true anymore if we only suppose that there exists a $\lambda \in \mathbb{T}\setminus\{1\}$ such that $|\lambda^{n_k} - 1| \to 0$: if $(q_n)_{n\geq 0}$ is the sequence of denominators of the partial quotients in the continued fraction expansion of $\alpha = \sqrt{2}$ for instance, $\lambda = e^{2i\pi\alpha}$ is such that $|\lambda^{q_n} - 1| \to 0$. But the sequence $(\frac{q_{n+1}}{q_n})_{n\geq 0}$ is bounded (see for instance [**18**]), so that $(q_n)_{n\geq 0}$ is not even a Jamison sequence.

Proof of Theorem 4.3. — We take the same kind of operator as in the proof of Theorem 2.1, and show that under the assumptions of Theorem 1.12, such an operator $T = D + B$ is such that $||T^{n_p} - D^{n_p}||$ tends to 0 as n_p tends to infinity. Before starting on this, we take advantage of the assumption of the theorem to construct a particular perfect compact subset of $\mathbb{T}$, in which our coefficients λ_l will be chosen later in the proof:

Lemma 4.4. — *There exists a perfect compact subset K of $\mathbb{T}$ such that $(K, d_{(n_k)})$ is separable and for any $\lambda \in K$, $|\lambda^{n_k} - 1| \to 0$ as $n_k \to +\infty$.*

Proof of Lemma 4.4. — The proof proceeds along the same lines as in [**2**]: let $(\mu_n)_{n\geq 1}$ be a sequence of elements of $\mathbb{T}\setminus\{1\}$ such that

$$d_{(n_k)}(\mu_1,1)<4^{-1}\ ,\ d_{(n_k)}(\mu_n,1)<4^{-n}d_{(n_k)}(\mu_{n-1},\overline{\mu}_{n-1})\text{ for any }n\geq 2,$$

$d_{(n_k)}(\mu_n,\overline{\mu}_n)$ decreases with n, and moreover $|\mu_n^{n_k}-1|\to 0$ as $n_k\to+\infty$. If $(s_1,\ldots,s_n)$ is any finite sequence of zeros and ones, we associate to it an element $\lambda_{(s_1,\ldots,s_n)}$ of $\mathbb{T}$ in the following way: we start with $\lambda_{(0)}=\mu_1$ and $\lambda_{(1)}=\overline{\mu}_1$, and we have

$$d_{(n_k)}(\lambda_{(0)},\lambda_{(1)})=d_{(n_k)}(\mu_1,\overline{\mu_1})>0.$$

Then if $\lambda_{(s_1,\ldots,s_{n-1})}$ has already been defined, we set

$$\lambda_{(s_1,\ldots,s_{n-1},0)}=\lambda_{(s_1,\ldots,s_{n-1})}\mu_n\quad\text{and}\quad\lambda_{(s_1,\ldots,s_{n-1},1)}=\lambda_{(s_1,\ldots,s_{n-1})}\overline{\mu}_n.$$

We have

$$d_{(n_k)}(\lambda_{(s_1,\ldots,s_{n-1})},\lambda_{(s_1,\ldots,s_{n-1},s_n)})<4^{-n}d_{(n_k)}(\mu_{n-1},\overline{\mu}_{n-1})$$

and

$$d_{(n_k)}(\lambda_{(s_1,\ldots,s_{n-1},0)},\lambda_{(s_1,\ldots,s_{n-1},1)})=d_{(n_k)}(\mu_n,\overline{\mu}_n),$$

so that for any infinite sequence $s=(s_1,s_2,\ldots)$ of zeros and ones, we can define $\lambda_s\in\mathbb{T}$ as $\lambda_s=\lim_{n\to+\infty}\lambda_{(s_1,\ldots,s_n)}$. It is not difficult to check (see [**2**] for details) that the map $s\longmapsto\lambda_s$ from 2^ω into $\mathbb{T}$ is one-to-one, so that $K=\{\lambda_s\ ;\ s\in 2^\omega\}$ is homeomorphic to the Cantor set, hence compact and perfect, and that $(K,d_{(n_k)})$ is separable. It remains to see that for any $s\in 2^\omega$, $|\lambda_s^{n_k}-1|\to 0$ as $n_k\to+\infty$. We have for any $p\geq 1$

$$\lambda_s=\lambda_{(s_1,\ldots,s_p)}\prod_{j\geq p}\lambda_{(s_1,\ldots,s_{j+1})}\overline{\lambda}_{(s_1,\ldots,s_j)},$$

so that for any $p\geq 1$,

$$\begin{aligned}|\lambda_s^{n_k}-1| &= \left|\lambda_{(s_1,\ldots,s_p)}^{n_k}\prod_{j\geq p}\lambda_{(s_1,\ldots,s_{j+1})}^{n_k}\overline{\lambda}_{(s_1,\ldots,s_j)}^{n_k}-1\right|\\ &\leq |\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|+\left|\prod_{j\geq p}\lambda_{(s_1,\ldots,s_{j+1})}^{n_k}-\lambda_{(s_1,\ldots,s_j)}^{n_k}\right|.\end{aligned}$$

Hence

$$\begin{aligned}|\lambda_s^{n_k}-1| &\leq |\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|+\sum_{j\geq p}d_{(n_k)}(\lambda_{(s_1,\ldots,s_{j+1})},\lambda_{(s_1,\ldots,s_j)})\\ &\leq |\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|+2\sum_{j\geq p}4^{-(j+1)}d_{(n_k)}(\mu_j,\overline{\mu}_j)\\ &\leq |\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|+2\,d_{(n_k)}(\mu_p,\overline{\mu}_p)\sum_{j\geq p}4^{-(j+1)}\\ &= |\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|+\frac{2}{3}\,4^{-p}d_{(n_k)}(\mu_p,\overline{\mu}_p).\end{aligned}$$

Given any $\gamma>0$, take p such that the second term is less than $\gamma/2$. Since $|\mu_n^{n_k}-1|\to 0$ as $n_k\to+\infty$, $|\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|\to 0$ as $n_k\to+\infty$ for any finite sequence $(s_1,\ldots,s_p)$. Hence there exists an integer $k_0\geq 1$ such that for any $k\geq k_0$, $|\lambda_{(s_1,\ldots,s_p)}^{n_k}-1|\leq\gamma/2$. Thus for any $k \geq k_0$ and any $s \in 2^\omega$, we have $|\lambda_s^{n_k} - 1| < \gamma$. So we have proved that for any $\lambda \in K$, $|\lambda^{n_k} - 1| \to 0$ as $n_k \to +\infty$. □

Let us now go back to the proof of Theorem 1.12. We have seen that

$$||T^{n_p} - D^{n_p}||^2 \leq \sum_{l \geq 2} \sum_{k=\max(1, l-n_p)}^{l-1} |t_{k,l}^{(n_p)}|^2,$$

and that it is possible for each $l \geq 2$ to take λ_l with $d_{(n_p)}(\lambda_l, \lambda_{j(l)})$ so small that

$$\sum_{k=\max(1, l-n_p)}^{l-1} |t_{k,l}^{(n_p)}|^2 \leq 2^{-l}.$$

So we do the construction in this way with the additional requirement that for each $l \geq 1$, λ_l is such that $|\lambda_l^{n_p} - 1| \to 0$ as $n_p \to +\infty$ (this is possible by Lemma 4.4). Let now $\varepsilon > 0$ and $l_0 \geq 2$ be such that $\sum_{l \geq l_0+1} 2^{-l} < \frac{\varepsilon}{2}$. We have for any p such that $n_p \geq l_0 + 1$

$$||T^{n_p} - D^{n_p}||^2 \leq \sum_{l=2}^{l_0} \sum_{k=1}^{l-1} |t_{k,l}^{(n_p)}|^2 + \frac{\varepsilon}{2}.$$

The proof will be complete if we show that for any k, l with $1 \leq k \leq l-1$, $t_{k,l}^{(n_p)} \to 0$ as $n_p \to +\infty$, or, equivalently, that $s_{k,l}^{(n_p)} \to 0$. Recall that by Lemma 2.7, $s_{k,l}^{(n_p)}$ can be written as

$$s_{k,l}^{(n_p)} = \sum_{j=k}^{l-1} c_j^{(k,l)} \frac{\lambda_l^{n_p+1-(l-k)} - \lambda_j^{n_p+1-(l-k)}}{\lambda_l - \lambda_j}$$

as soon as $n_p \geq l-k$. Since $\lambda_j^{n_p} \to 1$ for any $j \geq 1$,

$$s_{k,l}^{(n_p)} \to s_{k,l} := \sum_{j=k}^{l-1} c_j^{(k,l)} \frac{\lambda_l^{1-(l-k)} - \lambda_j^{1-(l-k)}}{\lambda_l - \lambda_j} \quad \text{as } n_p \to +\infty.$$

Thus we have to show that $s_{k,l} = 0$ for any $1 \leq k \leq l-1$. This is a consequence of the following lemma:

Lemma 4.5. — *For any k, l with $1 \leq k \leq l-1$ and any p with $0 \leq p \leq l-k-1$, we have*

$$\sum_{j=k}^{l-1} c_j^{(k,l)} \frac{\lambda_l^{1-(l-k-p)} - \lambda_j^{1-(l-k-p)}}{\lambda_l - \lambda_j} = 0.$$

Proof. — The proof is done by induction on $l \geq 2$. For $l = 2$, we just have to check that

$$c_1^{(1,2)} \frac{\lambda_2^0 - \lambda_1^0}{\lambda_2 - \lambda_1} = 0,$$

which is obviously true. Supposing now that the induction assumption is true for some $l \geq 2$, consider k with $1 \leq k \leq l$ and p with $0 \leq p \leq l-k$. Then

$$\sum_{j=k}^{l} c_j^{(k,l+1)} \frac{\lambda_{l+1}^{-(l-k-p)} - \lambda_j^{-(l-k-p)}}{\lambda_{l+1} - \lambda_j}$$

is equal to

$$
\begin{aligned}
&-\lambda_{l+1}^{-(l-k-p)}\sum_{j=k}^{l} c_j^{(k,l+1)}\lambda_j^{-(l-k-p)}\,\frac{\lambda_{l+1}^{(l-k-p)}-\lambda_j^{(l-k-p)}}{\lambda_{l+1}-\lambda_j}\\
&= -\lambda_{l+1}^{-(l-k-p)}\sum_{j=k}^{l} c_j^{(k,l+1)}\lambda_j^{-(l-k-p)}\sum_{m=0}^{l-k-p-1}\lambda_j^{m}\lambda_{l+1}^{l-k-p-1-m}\\
&= -\lambda_{l+1}^{-1}\sum_{m=0}^{l-k-p-1}\lambda_{l+1}^{-m}\left(\sum_{j=k}^{l} c_j^{(k,l+1)}\lambda_j^{-(l-k-p-m)}\right).
\end{aligned}
$$

It suffices now to show that each sum

$$\sum_{j=k}^{l} c_j^{(k,l+1)}\lambda_j^{-(l-k-p-m)}$$

is equal to 0. We have seen in the proof of Lemma 2.7 that for $1 \le k \le l-1$,

$$c_j^{(k,l+1)} = -\frac{\lambda_j c_j^{(k,l)}}{\lambda_l-\lambda_j} \quad \text{for } k \le j \le l-1 \quad \text{and } c_l^{(k,l+1)} = \sum_{j=k}^{l-1}\frac{\lambda_l c_j^{(k,l)}}{\lambda_l-\lambda_j}$$

and that $c_l^{(l,l+1)} = 1$. Thus for $1 \le k \le l-1$

$$
\begin{aligned}
\sum_{j=k}^{l} c_j^{(k,l+1)}\lambda_j^{-(l-k-p-m)} &= -\sum_{j=k}^{l-1}\frac{c_j^{(k,l)}}{\lambda_l-\lambda_j}\lambda_j^{-(l-k-p-m-1)} + \sum_{j=k}^{l-1}\frac{c_j^{(k,l)}}{\lambda_l-\lambda_j}\lambda_l^{-(l-k-p-m-1)}\\
&= \sum_{j=k}^{l-1} c_j^{(k,l)}\,\frac{\lambda_l^{1-(l-k-p-m)}-\lambda_j^{1-(l-k-p-m)}}{\lambda_l-\lambda_j}.
\end{aligned}
$$

Since $p+m \le l-k-1$, this quantity vanishes by the induction assumption. For $k = l$, we only have to consider the case $p = 0$, and here

$$c_{l,l+1}^{(l)}\,\frac{\lambda_{l+1}^{0}-\lambda_l^{0}}{\lambda_{l+1}-\lambda_l} = 0.$$

This finishes the proof of Lemma 4.5. □

We have shown that $t_{k,l}^{(n_p)} \to 0$ for each $1 \le k \le l-1$, and it follows immediately that $||T^{n_p}-D^{n_p}|| \to 0$ as $n_p \to +\infty$. Now if $x \in H$ and ε is any fixed positive number, take l_0 such that $\sum_{l \ge l_0+1} |x_l|^2 \le \varepsilon/2$. Since

$$||D^{n_p}x - x||^2 \le \left(\sum_{l=1}^{l_0}|\lambda_l^{n_p}-1|^2\right)||x||^2 + 2\sum_{l \ge l_0+1}|x_l|^2$$

and $|\lambda_l^{n_p}-1|$ tends to 0 for each $l \ge 2$, it follows that $||D^{n_p}x - x|| \to 0$ as $n_p \to +\infty$ for any $x \in H$, hence $||T^{n_p}x - x|| \to 0$ which is the conclusion of Theorem 1.12. □

4.3. A characterization of uniformly rigid sequences for linear dynamical systems. — We now prove Theorem 1.13. Clearly (3) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) is obvious: using the notation of Section 4.2 above, $||T^{n_k}e_\lambda - e_\lambda||$ tends to 0 uniformly on $\sigma_p(T)\cap\mathbb{T} =: K$ which is uncountable, i.e. $|\lambda^{n_k}-1|$ tends to 0 uniformly on K.

The converse implication (1) $\Rightarrow$ (3) follows from Fact 3.7, the proof of Theorem 4.3 above and Lemma 4.6 below. First replacing K by a compact perfect subset of its closure, and then using Fact 3.7, we can suppose that K is such that $|\lambda^{n_k}-1|$ tends to 0 uniformly on K and for any $\varepsilon>0$ there exists a $\nu\in K\setminus\{1\}$ such that $\sup_{k\geq 0}|\nu^{n_k}-1|\leq\varepsilon$. Then

Lemma 4.6. — *Under the assumption above on K, there exists a perfect compact subset K' of $\mathbb{T}$ such that $(K', d_{(n_k)})$ is separable and λ^{n_k} tends to 1 uniformly on K'.*

Proof of Lemma 4.6. — The proof runs along the same lines as that of Lemma 4.4: we start from elements μ_n, $n\geq 1$, of $K\setminus\{1\}$ having the same properties as in Lemma 4.4 (which we know exist - this is why we had to use Fact 3.7), and we construct the unimodular numbers λ_s, $s\in 2^\omega$, as in Lemma 4.4, with

$$|\lambda_s^{n_k}-1|\leq|\lambda_{(s_1,\dots,s_p)}^{n_k}-1|+\frac{2}{3}4^{-p}d_{(n_k)}(\mu_p,\overline{\mu}_p)$$

for any $k\geq 0$, $p\geq 1$, and $s\in 2^\omega$. Let $\gamma>0$, and take p such that the second term is less than $\gamma/2$. Then for any $s\in 2^\omega$ and any $k\geq 0$ we have

$$|\lambda_s^{n_k}-1|\leq|\lambda_{(s_1,\dots,s_p)}^{n_k}-1|+\frac{\gamma}{2}\leq|\mu_1^{n_k}-1|+\ldots+|\mu_p^{n_k}-1|+\frac{\gamma}{2}\leq p\,||\lambda^{n_k}-1||_{\infty,K}+\frac{\gamma}{2}.$$

Take κ such that for any $k\geq\kappa$, $||\lambda^{n_k}-1||_{\infty,K}\leq\gamma/(2p)$: we have $|\lambda_s^{n_k}-1|\leq\gamma$ for any $s\in 2^\omega$ and $k\geq\kappa$, and this shows that λ^{n_k} tends to 1 uniformly on the set $K'=\{\lambda_s\,;\,s\in 2^\omega\}$. Since $(K', d_{(n_k)})$ is separable, Lemma 4.6 is proved. □

Now in the construction of the operator T, we choose the coefficients λ_l in the set K' given by Lemma 4.6. We have seen in the proof of Theorem 1.12 above that $||T^{n_k}-D^{n_k}||$ tends to 0 as n_k tends to infinity. So it suffices to prove that with the additional uniformity assumption of Theorem 1.13, $||D^{n_k}-I||=\sup_{l\geq 1}|\lambda_l^{n_k}-1|$ tends to 0 as n_k tends to infinity. But $||D^{n_k}-I||\leq||\lambda^{n_k}-1||_{\infty,K'}$ which tends to 0, so our claim is proved.

Proof of Corollary 1.14. — If $(n_k)_{k\geq 0}$ is any sequence with $n_{k+1}/n_k\to+\infty$, or if $n_k|n_{k+1}$ for any $k\geq 0$ and $\limsup n_{k+1}/n_k=+\infty$, we have seen in Propositions 3.5 and 3.8 that Theorem 1.13 applies, proving Corollary 1.14. Theorem 1.13 also applies to the sequences $(q_n)_{n\geq 0}$ considered in Example 3.16. We thus obtain examples, in the linear framework, of measure-preserving transformations on a Hilbert space which are both weakly mixing in the measure-theoretic sense and uniformly rigid. □

Remark 4.7. — If $(n_k)_{k\geq 0}$ is such that $n_k|n_{k+1}$ for any $k\geq 0$ and $\limsup n_{k+1}/n_k=+\infty$, the proof of Proposition 3.8 shows that the set $K=\{\lambda_\varepsilon\,;\,\varepsilon\in\{0,1\}^{\mathbb{N}}\}$ contains a dense subset of numbers λ which are N^{th} roots of 1 for some $N\geq 1$. Hence, in all the constructions of operators $T=D+B$ considered here, it is possible to choose the numbers λ_l, $l\geq 1$, as being N^{th} roots of 1. In this way the operator T becomes additionally chaotic (i.e. it is topologically transitive and has a dense set of periodic vectors). This gives further examples of chaotic operators which are not topologicaly mixing (the first examples of such operators were given in [**1**]), and shows in particular that there exist chaotic operators which are uniformly rigid.

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Tanja Eisner, Korteweg de Vries Institut voor Wiskunde, Universiteit van Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands • *E-mail :* `T.Eisner@uva.nl`

Sophie Grivaux, Laboratoire Paul Painlevé, UMR 8524, Université Lille 1, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France • *E-mail :* `grivaux@math.univ-lille1.fr`