A HYPERCYCLIC RANK ONE PERTURBATION OF A UNITARY OPERATOR

by

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Abstract. — We prove that there exists a rank 1 perturbation of a unitary operator on a complex separable infinite dimensional Hilbert space which is hypercyclic.

1. Introduction

We are interested in this note in the construction of some special hypercyclic operators on Hilbert spaces. Our work fits into the framework of linear dynamics, which is the study of the properties of the iterates T^n, $n \geq 0$, of a bounded linear operator T acting on an infinite dimensional separable Banach space X. It is of particular interest to study the behavior of the orbits $\mathcal{O}rb(x,T) = \{T^n x \ ; n \geq 0\}$ of vectors x of X under the action of T. For instance when $\mathcal{O}rb(x,T)$ is dense in X, the vector x is said to be *hypercyclic*. The operator T itself is hypercyclic when there exists an $x \in X$ such that x is hypercyclic for T. It is not completely trivial to exhibit hypercyclic operators: the first example of such an operator was given by Rolewicz [**11**], who proved that if B denotes the standard backward weighted shift on $\ell_2(\mathbb{N})$, λB is hypercyclic for any complex number λ with $|\lambda| > 1$. Many more examples of hypercyclic operators on "classical" spaces can be found in the book [**4**]. It is a non-trivial result of Ansari [**1**] and Bernal-Gonzalez [**6**], relying on previous work of Salas [**12**] that any (real or complex) separable infinite-dimensional Banach space X supports a hypercyclic operator. Such a general operator has necessarily the form $T = I + N$, where N is a nuclear operator on X, so that a nuclear perturbation of the identity operator can indeed be hypercyclic. Obviously a finite rank perturbation of the identity operator can never be hypercyclic.

In [**14**] Shkarin investigated the following question: can a finite rank perturbation of a unitary operator on a complex separable infinite-dimensional Hilbert space be hypercyclic? This question came from the work of Salas [**13**] on supercyclicity of weighted shifts: T is said to be *supercyclic* (a weaker requirement than hypercyclicity) if there exists a vector

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$x \in X$ such that $\{\lambda T^n x\ ;\ n \geq 0,\ \lambda \in \mathbb{C}\}$ is dense in X. It is known ([**7**], see also [**9**] and [**10**]) that no hyponormal operator on a Hilbert space can be supercyclic. Salas thus proposed the following question: can a finite rank perturbation of a hyponormal operator on a Hilbert space be supercyclic? Shkarin answered in [**14**] this question in the affirmative, and proved: there exists a unitary operator V and an operator R of rank at most 2 acting on a Hilbert space H such that $V+R$ is hypercyclic on H. This yields an example of a contraction A and a rank 1 operator S on H such that $A+S$ is hypercyclic. But a natural question remained open in [**14**]:

Question 1.1. — *Does there exist a rank* 1 *perturbation of a unitary operator on a Hilbert space which is hypercyclic?*

Our aim in this paper is to answer Question 1.1 in the affirmative:

Theorem 1.2. — *There exists a unitary operator U and a rank* 1 *operator R on the complex Hilbert space $\ell_2(\mathbb{N})$ such that the operator $T = U + R$ is hypercyclic on $\ell_2(\mathbb{N})$.*

Our method of proof is rather different from the one employed in [**14**], the only common point being the criterion for hypercyclicity which we use: it is based on the properties of eigenvectors associated to eigenvalues of T which are of modulus 1, and was first introduced in [**2**]. We use here a recent refinement of this criterion which comes from [**8**], see Section 2 of this paper. The operators which we construct are intrinsically different from the ones of [**14**]: in [**14**] the operators live on the function space $L^2(\mathbb{T})$, and the operator $V + R$ (V unitary, R of rank 2) which is constructed is an operator induced by $V' + R'$ on an invariant subspace of $V' + R'$, where V' is the multiplication operator by z on $L^2(\mathbb{T})$ and R' is a rank one operator on $L^2(\mathbb{T})$. One of the key tools in the proof of [**14**] is a result of Belov [**5**] concerning the distribution of values of certain functions $\varphi : \mathbb{R} \to \mathbb{C}$ defined as lacunary trigonometric series.

Our approach here is much more elementary: our unitary operator U is a diagonal operator on $\ell_2(\mathbb{N})$ with unimodular diagonal coefficients, and these coefficients as well as the two vectors a and b in $\ell_2(\mathbb{N})$ which define $R = b \otimes a$ are constructed by induction in such a way that the eigenvectors associated to eigenvalues of modulus 1 of the operator $U + R$ can be explicitly written down. The main idea of the proof of Theorem 1.2 is presented in Section 2, and the inductive construction, which is more technical, is given in Section 3.

2. Main ingredients of the proof of Theorem 1.2

2.1. A criterion for hypercyclicity. — The criterion for hypercyclicity which we are going to use in the proof of Theorem 1.2 is stated in terms of eigenvectors associated to eigenvalues of modulus 1 of the operator. Roughly speaking, if T is a bounded linear operator on a complex separable Banach space X which has "plenty" of such eigenvectors, then T is hypercyclic. Here is the precise definition:

Definition 2.1. — We say that $T \in \mathcal{B}(X)$ has a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for any σ-measurable subset B of $\mathbb{T}$ with $\sigma(B) = 1$, we have $\overline{\text{sp}}[\ker(T - \lambda)\ ;\ \lambda \in B] = X$.

When T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, it is automatically hypercyclic. This is proved in [**2**]. The easiest way to check this spanning property of the eigenvectors is to exhibit a family $(K_i)_{i\geq 1}$ of compact perfect subsets of $\mathbb{T}$ and a family $(E_i)_{i\geq 1}$ of eigenvector fields $E_i : K_i \to X$ which are continuous on K_i and such that the vectors $E_i(\lambda)$, $i \geq 1, \lambda \in K_i$, span a dense subspace of X. This can be done rather easily by using the following theorem, which was proved in [**8**]:

Theorem 2.2. — *Let X be a complex separable infinite-dimensional Banach space, and let T be a bounded operator on X. Suppose that there exists a sequence $(u_i)_{i\geq 1}$ of vectors of X having the following properties:*

(i) *for each $i \geq 1$, u_i is an eigenvector of T associated to an eigenvalue μ_i of T, with $|\mu_i| = 1$ and the μ_i's all distinct;*
(ii) *$sp[u_i\ ;\ i \geq 1]$ is dense in X;*
(iii) *for any $i \geq 1$ and any $\varepsilon > 0$, there exists an $n \neq i$ such that $||u_n - u_i|| < \varepsilon$.*

Then there exists a family $(K_i)_{i\geq 1}$ of subsets of $\mathbb{T}$ which are homeomorphic to the Cantor set 2^ω and a family $(E_i)_{i\geq 1}$ of eigenvector fields $E_i : K_i \to X$ which are continuous on K_i for each i and which span X: $sp[E_i(\lambda)\ ;\ i \geq 1,\ \lambda \in K_i]$ is dense in X. So T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, and in particular T is hypercyclic.

Theorem 2.2 actually yields a stronger conclusion, as it is known that if T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, then it is frequently hypercyclic. See [**3**] for the definition of frequent hypercyclicity and for a proof of this statement in the Hilbert space setting, and [**8**] for a proof in the Banach space case. When the eigenvalues μ_i which appear in the assumption of Theorem 2.2 are N^{th} roots of 1, $N \geq 1$, then the operator is not only hypercyclic but chaotic (it is hypercyclic and has a dense set of periodic vectors). We will in the proof of Theorem 1.2 construct the operator T so that the assumptions of Theorem 2.2 are satisfied. It will become clear in the course of the proof that we can choose the μ_i's to be N^{th} roots of 1, and thus the operator of Theorem 1.2 can be made chaotic and frequently hypercyclic.

2.2. Eigenvectors of rank one perturbations of diagonal operators. — We are looking for a hypercyclic operator T on the space $\ell_2(\mathbb{N})$ endowed with the canonical basis $(e_n)_{n\geq 1}$ of the form $T = U + R$, where U is a unitary operator and R is an operator of rank 1. The unitary operator which we construct is a diagonal operator D defined by $De_n = \lambda_n e_n$, $n \geq 1$, where λ_n is for each $n \geq 1$ a complex number of modulus 1 with the λ_n's all distinct. The operator R has the form $R = b \otimes a$, where $a = \sum_{n\geq 1} a_n e_n$ and $b = \sum_{n\geq 1} b_n e_n$ are two elements of $\ell_2(\mathbb{N})$: $Rx = \langle x, b\rangle a$ for any $x \in \ell_2(\mathbb{N})$. Our aim is to define the coefficients λ_n and the numbers a_n and b_n in such a way that the operator $T = D + R$ satisfies the assumptions of Theorem 2.2.

Let $\lambda \in \mathbb{T}$ be a complex number of modulus 1. Then with the notation above, λ is an eigenvalue of the operator $T = D + R$ with associated eigenvector $u \in \ell_2(\mathbb{N}) \setminus \{0\}$ if and only if $(D+R)u=\lambda u$, i.e. $Du+\langle u,b\rangle a=\lambda u$, i.e. $(\lambda-D)u=\langle u,b\rangle a$. If $\lambda\not\in\{\lambda_n\ ;\ n\ge 1\}$, $\lambda-D$ is injective, and thus the equation above admits a non-zero solution u if and only if $a\in\mathrm{Ran}(\lambda-D)$, $a=(\lambda-D)a'$ where $a'\in\ell_2(\mathbb{N})$ is unique and $\langle a',b\rangle=1$. If $a=\sum_{n\ge1}a_ne_n$, then necessarily

$$a'=\sum_{n\ge1}\frac{a_n}{\lambda-\lambda_n}e_n,$$

and $\langle a',b\rangle=1$ means that

$$\sum_{n\ge1}\frac{a_n\bar{b}_n}{\lambda-\lambda_n}=1.$$

We can reformulate this observation as:

Lemma 2.3. — *If $\lambda\in\mathbb{T}\setminus\{\lambda_n\ ;\ n\ge1\}$, then λ is an eigenvalue of $D+R$ if and only if*

$$\sum_{n\ge1}\left|\frac{a_n}{\lambda-\lambda_n}\right|^2<+\infty \quad \textit{and} \quad \sum_{n\ge1}\frac{a_n\bar{b}_n}{\lambda-\lambda_n}=1.$$

In this case an associated eigenvector u is given by

$$u=\sum_{n\ge1}\frac{a_n}{\lambda-\lambda_n}e_n.$$

2.3. Strategy of the proof of Theorem 1.2. — Let $j:\{1,2,\ldots\}\longrightarrow\{1,2,\ldots\}$ be a function having the following properties:

- $j(1)=1$;
- $j(n)<n$ for every $n\ge2$;
- for any $k\ge1$ the set $\{n\ge2\ ;\ j(n)=k\}$ is infinite, i.e. j takes every value $k\ge1$ infinitely often.

The proof of Theorem 1.2 will be carried out via an induction argument. As Step n, $n\ge1$, we define two unimodular numbers λ_n and μ_n, a complex number a_n and an n-tuple $b^{(n)}=(b_1^{(n)},\ldots,b_n^{(n)})$ of complex numbers such that the following properties hold true:

(1) if D_n denotes the diagonal operator on $\mathbb{C}^n$ with diagonal coefficients $\lambda_1,\ldots,\lambda_n$, with $\lambda_n\not\in\{\lambda_1,\ldots,\lambda_{n-1}\}$, and R_n denotes the rank 1 operator $b^{(n)}\otimes a^{(n)}$ on $\mathbb{C}^n$, i.e. $R_nx=\langle x,b^{(n)}\rangle a^{(n)}$ for any $x\in\mathbb{C}^n$, where $a^{(n)}=\sum_{j=1}^n a_je_j$ and $b^{(n)}=\sum_{j=1}^n b_j^{(n)}e_j$, then the operator $T_n=D_n+R_n$ acting on $\mathbb{C}^n$ has n distinct eigenvalues which are the unimodular numbers $\mu_1,\ldots,\mu_n$. Moreover μ_n does not belong to the set of distinct numbers $\{\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_{n-1}\}$, and the vector

$$u_i^{(n)}=\sum_{j=1}^n\frac{a_j}{\mu_i-\lambda_j}e_j$$

is an eigenvector of T_n associated to the eigenvalue μ_i. Additionally for any $n\ge1$, $\mathrm{sp}[u_i^{(n)}\ ;\ i=1,\ldots,n]=\mathrm{sp}[e_1,\ldots,e_n]$. Thus there exists a positive constant C_n such that for any $x \in \mathbb{C}^n$ with $x = \sum_{j=1}^n x_j e_j = \sum_{i=1}^n \alpha_i u_i^{(n)}$, we have

$$\sum_{i=1}^n |\alpha_i| \leq C_n \left(\sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}}$$

(2) $C_n > C_{n-1}$ and $C_n > 2$
(3) $0 < |a_n| < 2^{-n}$
(4) $|b_n^{(n)}| < 2^{-n}$
(5) we have

$$\left(\sum_{i=1}^{n-1} |b_i^{(n)} - b_i^{(n-1)}|^2 \right)^{\frac{1}{2}} < 2^{-n}$$

(6) for any $i = 1, \ldots, n-1$,

$$||u_i^{(n)} - u_i^{(n-1)}|| < \frac{2^{-n}}{C_{n-1}}$$

(7) $||u_{j(n)}^{(n)} - u_n^{(n)}|| < 2^{-n}$
(8) for any $k = 1, \ldots, n-1$ and any $i = 1, \ldots, k$, $||T_n u_i^{(k)} - \mu_i u_i^{(k)}|| < 3 \,.\, 2^{-(k-1)}$.

Suppose that the construction of the sequences $(\lambda_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$, $(a_n)_{n\geq 1}$ and $(b^{(n)})_{n\geq 1}$ has been carried out in such a way that properties (1)-(8) are satisfied. By (2) the vector $a = \sum_{n\geq 1} a_n e_n$ belongs to $\ell_2(\mathbb{N})$. By (4) and (5), we have

(9') $||b^{(n)} - b^{(n-1)}|| = || \sum_{i=1}^{n-1} (b_i^{(n)} - b_i^{(n-1)}) e_i + b_n^{(n)} e_n|| < 2.2^{-n} = 2^{-(n-1)}$

so that the sequence $(b^{(n)})_{n\geq 1}$ converges in $\ell_2(\mathbb{N})$ to a certain vector $b = \sum_{n\geq 1} b_n e_n$, with

(10') $||b^{(n)} - b|| \leq \sum_{j\geq n} ||b^{(j+1)} - b^{(j)}|| \leq \sum_{j\geq n} 2^{-j} < 2^{-(n-1)}$.

So it makes sense to define the rank one operator $R = b \otimes a$ on $\ell_2(\mathbb{N})$. Let D be the diagonal operator $D = \text{diag}(\lambda_n \ ; \ n \geq 1)$ on $\ell_2(\mathbb{N})$. We are going to show, using Theorem 2.2, that $D + R$ is then hypercyclic, which will prove Theorem 1.2.

Proof of Theorem 1.2 modulo the inductive construction. — For any $n \geq 1$, let P_n denote the canonical projection of $\ell_2(\mathbb{N})$ onto $\text{sp}[e_1, \ldots, e_n]$. For any $x = \sum_{j\geq 1} x_j e_j \in \ell_2(\mathbb{N})$, we have

$$T_n P_n x = T_n \left(\sum_{j=1}^n x_j e_j \right) = \sum_{j=1}^n \lambda_j x_j e_j + \langle x, b^{(n)} \rangle a^{(n)}.$$

Since $a^{(n)} \to a$, $b^{(n)} \to b$ and $\sup_{n\geq 1} ||b^{(n)}||$ is finite, $||T_n P_n x - (D+R)x||$ tends to zero as n tends to infinity. Applying this to $x = u_i^{(k)}$ yields that for any $k \geq 1$ and any $i = 1, \ldots, k$, $||T u_i^{(k)} - \mu_i u_i^{(k)}|| \leq 3 \,.\, 2^{-(k-1)}$ by (8), as $T_n P_n u_i^{(k)} = T_n u_i^{(k)}$ for any $n \geq k$. By (6) the sequence $(u_i^{(n)})_{n\geq i}$ converges as n tends to infinity to a certain vector $u_i \in \ell_2(\mathbb{N})$, which is nothing but

$$u_i = \sum_{j=1}^{+\infty} \frac{a_j}{\mu_i - \lambda_j} e_j.$$

It is a non zero vector, and making k tend to infinity in the inequalities above shows that $Tu_i = \mu_i u_i$, so that u_i is an eigenvector of T associated to the eigenvalue μ_i.

Let us now prove that the sequence $(u_i)_{i\geq 1}$ satisfies the assumptions of Theorem 2.2: assertion (i) is true by construction, as the μ_i's are all distinct. As for assertion (ii), let us consider a vector $x = \sum_{j=1}^{r} x_j e_j$ with finite support and $||x|| \leq 1$. Writing x as $x = \sum_{i=1}^{r} \alpha_i u_i^{(r)}$, we have by (1)

$$\begin{aligned} ||x - \sum_{i=1}^{r} \alpha_i u_i|| &\leq \left(\sum_{i=1}^{r} |\alpha_i|\right) \sup_{i=1,\dots,r} ||u_i - u_i^{(r)}|| \\ &\leq C_r\, ||x|| \sup_{i=1,\dots,r} \sum_{k\geq r+1} ||u_i^{(k)} - u_i^{(k-1)}|| \\ &\leq C_r \sum_{k\geq r+1} \frac{2^{-k}}{C_{k-1}} \leq 2^{-r} \end{aligned}$$

by (6). Hence for any $\varepsilon > 0$ there exists a vector $y \in \mathrm{sp}[u_j\ ;\ j \geq 1]$ such that $||x - y|| < \varepsilon$, and this proves assertion (ii). Assertion (iii) is a consequence of (7): for any $k \geq 1$ let A_k be the set $A_k = \{n \geq 2\ ;\ j(n) = k\}$. Observe that if $n \in A_k$, $n \geq k+1$. For any $n \in A_k$ we have $||u_k^{(n)} - u_n^{(n)}|| < 2^{-n}$ by (7). Let us estimate $||u_n - u_k||$:

$$\begin{aligned} ||u_n - u_k|| &\leq ||u_n - u_n^{(n)}|| + ||u_n^{(n)} - u_k^{(n)}|| + ||u_k^{(n)} - u_k|| \\ &\leq \sum_{m\geq n+1} ||u_n^{(m)} - u_n^{(m-1)}|| + 2^{-n} + \sum_{m\geq n+1} ||u_k^{(m)} - u_k^{(m-1)}|| \\ &\leq 2 \sum_{m\geq n+1} 2^{-m} + 2^{-n} = 5.2^{-n}. \end{aligned}$$

Thus if ε is any positive number, since A_k is infinite there exists an $n \in A_k$ such that $||u_n - u_k|| < \varepsilon$, and assertion (iii) of Theorem 2.2 is satisfied too. We have thus proved that T is hypercyclic, which proves Theorem 1.2 modulo the construction of λ_n, μ_n, a_n and $b^{(n)}$ for each $n \geq 1$. □

3. The induction step

In order to complete the proof of Theorem 1.2, we now have to carry out the induction step. Before starting, let us reformulate the first half of condition (1) in a more convenient way: saying that the operator $T_n = D_n + R_n$ acting on $\mathbb{C}^n$ has n distinct eigenvalues $\mu_1, \dots, \mu_n$ exactly means that we have

$$\text{(E)} \qquad \sum_{j=1}^{n} \frac{a_j \bar{b}_j^{(n)}}{\mu_i - \lambda_j} = 1 \qquad \text{for any } i = 1, \dots, n.$$

Let $M_n \in \mathcal{M}_n(\mathbb{C})$ be the matrix $M_n = (m_{ij})_{1\leq i,j\leq n}$ with $m_{ij} = \frac{1}{\mu_i - \lambda_j}$. These coefficients are well-defined, as we choose at each step k $\lambda_k \notin \{\mu_1, \dots, \mu_k\}$ and $\mu_k \notin \{\lambda_1, \dots, \lambda_k\}$.

Then equations (E) can be rewritten as the matrix equation

$$\begin{pmatrix} \dfrac{1}{\mu_1-\lambda_1} & \cdots & \dfrac{1}{\mu_1-\lambda_n} \\ \vdots & & \vdots \\ \dfrac{1}{\mu_n-\lambda_1} & \cdots & \dfrac{1}{\mu_n-\lambda_n} \end{pmatrix} \cdot \begin{pmatrix} a_1\overline{b}_1^{(n)} \\ a_2\overline{b}_2^{(n)} \\ \vdots \\ a_n\overline{b}_n^{(n)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \text{i.e.} \qquad M_n \begin{pmatrix} a_1\overline{b}_1^{(n)} \\ a_2\overline{b}_2^{(n)} \\ \vdots \\ a_n\overline{b}_n^{(n)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We are now ready to begin the construction.

• We start by taking $\lambda_1 = 1$ and $a_1 = 4^{-1}$ for instance. Then we take $\mu_1 \in \mathbb{T}$ with $\mu_1 \neq \lambda_1$ and $|\mu_1 - \lambda_1|$ so small (with $|\mu_1 - \lambda_1| < 1$ in particular) that if we set

$$\overline{b}_1^{(1)} = \frac{\mu_1 - \lambda_1}{a_1},$$

then $|b_1^{(1)}| < 2^{-1}$. Of course $Te_1 = \mu_1 e_1$.

• Suppose now that the construction has been carried out until Step $n-1$. We have to construct $\lambda_n \in \mathbb{T}$, $\mu_n \in \mathbb{T}$, $a_n \in \mathbb{C}$ and $b^{(n)} \in \mathbb{C}^n$ such that properties (1)-(8) hold true. First of all, let $\varepsilon > 0$ be a positive number which is so small that:

(a) $$0 < \varepsilon < 4^{-(n+1)}$$

(b) $$\prod_{j=1}^{n}(1+2^{-j}) \left(\sum_{j=1}^{n-1} \frac{1}{|\mu_{j(n)} - \lambda_j|^2} \right)^{\frac{1}{2}} \varepsilon < 4^{-(n+1)}$$

(c) $$\frac{1}{\min_{j=1,\ldots,n-1}|a_j|} \left(1 + \left(\sum_{j=1}^{n-1} |a_j|^2 \right)^{\frac{1}{2}} \right) \prod_{j=1}^{n}(1+2^{-j})\, \varepsilon < 2^{-n}.$$

We first construct the n^{th} diagonal coefficient λ_n of D_n: it is chosen very close to $\mu_{j(n)}$. More precisely: by the induction assumption $\mu_{j(n)}$ is an eigenvalue of the matrix M_{n-1}, so that

$$\sum_{j=1}^{n-1} \frac{a_j \overline{b}_j^{(n-1)}}{\mu_{j(n)} - \lambda_j} = 1.$$

It follows that there exists $\delta > 0$ such that for any $\lambda \in \mathbb{T}\setminus\{\lambda_1, \ldots, \lambda_{n-1}\}$ with $|\lambda - \mu_{j(n)}| < \delta$, we have

- $$\left| 1 - \sum_{j=1}^{n-1} \frac{a_j \overline{b}_j^{(n-1)}}{\lambda - \lambda_j} \right| < \varepsilon$$
- $$\prod_{j=1}^{n}(1+2^{-j}) \left(\sum_{j=1}^{n-1} \frac{1}{|\lambda - \lambda_j|^2} \right)^{\frac{1}{2}} \varepsilon < 4^{-(n+1)}$$
- $$\left(\sum_{j=1}^{n-1} |a_j|^2 \cdot \left| \frac{1}{\mu_{j(n)} - \lambda_j} - \frac{1}{\lambda - \lambda_j} \right|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

We choose $\lambda_n \in \mathbb{T} \setminus \{\lambda_1, \ldots, \lambda_{n-1}, \mu_1, \ldots, \mu_{n-1}\}$ such that $|\lambda_n - \mu_{j(n)}| < \delta$. We then have:

$$\left|1 - \sum_{j=1}^{n-1} \frac{a_j \bar{b}_j^{(n-1)}}{\lambda_n - \lambda_j}\right| < \varepsilon \tag{d}$$

$$\prod_{j=1}^{n} (1 + 2^{-j}) \left(\sum_{j=1}^{n-1} \frac{1}{|\lambda_n - \lambda_j|^2}\right)^{\frac{1}{2}} \varepsilon < 4^{-(n+1)} \tag{e}$$

$$\left(\sum_{j=1}^{n-1} |a_j|^2 \cdot \left|\frac{1}{\mu_{j(n)} - \lambda_j} - \frac{1}{\lambda_n - \lambda_j}\right|^2\right)^{\frac{1}{2}} < \varepsilon. \tag{f}$$

Once λ_n is chosen, the next step is to choose μ_n. We take $\mu_n \in \mathbb{T} \setminus \{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-1}\}$ with $|\mu_n - \lambda_n|$ so small that

$$\left|1 - \sum_{j=1}^{n-1} \frac{a_j \bar{b}_j^{(n-1)}}{\mu_n - \lambda_j}\right| < \varepsilon \tag{g}$$

$$\prod_{j=1}^{n} (1 + 2^{-j}) \left(\sum_{j=1}^{n-1} \frac{1}{|\mu_n - \lambda_j|^2}\right)^{\frac{1}{2}} \varepsilon < 4^{-(n+1)} \tag{h}$$

$$\left(\sum_{j=1}^{n-1} |a_j|^2 \cdot \left|\frac{1}{\mu_{j(n)} - \lambda_j} - \frac{1}{\mu_n - \lambda_j}\right|^2\right)^{\frac{1}{2}} < \varepsilon \tag{i}$$

and

$$\frac{|\mu_n - \lambda_n|}{|\mu_i - \lambda_n|} < \frac{2^{-n}}{C_{n-1}} \quad \text{for any } i = 1, \ldots, n-1 \tag{j}$$

$$||M_n^{-1}|| \leq (1 + 2^{-n})\, ||M_{n-1}^{-1}||. \tag{k}$$

It is easy to see that conditions (g), (h), (i) and (j) can be fullfilled if $|\mu_n - \lambda_n|$ is small enough. That condition (k) can be made to hold too is not so immediate, but not too hard either: first of all for any $\varepsilon' > 0$ there exists a $\delta' > 0$ such that if $|\mu_n - \lambda_n| < \delta'$, then

$$\left|\frac{\det M_{n-1}}{(\mu_n - \lambda_n) \det M_n} - 1\right| < \varepsilon'.$$

Indeed $(\mu_n - \lambda_n) \det M_n = \det \tilde{M}_n$, where $\tilde{M}_n$ is the matrix obtained from M_n by multiplying its last line by $(\mu_n - \lambda_n)$. If $|\mu_n - \lambda_n|$ is extremely small, the coefficients $(\tilde{M}_n)_{nj}$, $j = 1, \ldots, n-1$, are almost equal to zero, while $(\tilde{M}_n)_{nn} = 1$. Thus $\det \tilde{M}_n$ can be made as close as we wish to $\det M_{n-1}$, and it is possible to ensure that

$$\left|\frac{1}{(\mu_n - \lambda_n) \det M_n} - \frac{1}{\det M_{n-1}}\right| < \frac{\varepsilon'}{|\det M_{n-1}|},$$

from which it follows that

$$\left|\frac{\det M_{n-1}}{(\mu_n - \lambda_n) \det M_n} - 1\right| < \varepsilon'.$$

Notice that
$$\left|\frac{1}{\det M_n}-\frac{\mu_n-\lambda_n}{\det M_{n-1}}\right|<\varepsilon'\frac{|\mu_n-\lambda_n|}{|\det M_{n-1}|}.$$
Then the formula $M_n^{-1}=\frac{1}{\det M_n}{}^t\mathrm{com}M_n$ yields that:
– the coefficients (n,j) and (i,n) of M_n^{-1}, $i,j=1,\ldots,n$, can be made arbitrarily small if $|\mu_n-\lambda_n|$ is small enough, as $({}^t\mathrm{com}M_n)_{nj}$ and $({}^t\mathrm{com}M_n)_{in}$ do not depend on $|\mu_n-\lambda_n|$, while $\det M_n$ can be made arbitrarily small with $|\mu_n-\lambda_n|$;
– the coefficients (i,j), $i,j=1,\ldots,n-1$ can be made very close to the coefficients $(M_{n-1}^{-1})_{ij}$. Indeed the dominant term in the computation of $({}^t\mathrm{com}M_n)_{ij}$ is the one involving $\frac{1}{\mu_n-\lambda_n}$, that is $\frac{1}{\mu_n-\lambda_n}({}^t\mathrm{com}M_{n-1})_{ij}$. So $(M_n^{-1})_{ij}$ can be made as close as we wish to
$$\frac{1}{(\mu_n-\lambda_n)\det M_n}({}^t\mathrm{com}M_{n-1})_{ij}=\frac{\det M_{n-1}}{(\mu_n-\lambda_n)\det M_n}(M_{n-1}^{-1})_{ij}.$$
Hence M_n^{-1} is very close to the matrix A_n for the operator norm on $\mathcal{M}_n(\mathbb{C})$, where $(A_n)_{ij}=(M_{n-1}^{-1})_{ij}$ for $i,j=1,\ldots,n-1$ and $(A_n)_{in}=(A_n)_{nj}=0$ for $i,j=1,\ldots,n$. Hence there exists $\gamma>0$ such that $||M_n^{-1}||\le(1+2^{-n})||M_{n-1}^{-1}||$ if $|\mu_n-\lambda_n|<\gamma$, and property (k) is satisfied if μ_n is sufficiently close to λ_n.
Now that λ_n and μ_n are constructed, it remains to fix a_n and $b^{(n)}$. We take first
$$a_n=2^{-(n+1)}|\mu_n-\lambda_n|.$$
There is now not much room for the choice of $b^{(n)}$: we must have
$$M_n\begin{pmatrix}a_1\overline{b}_1^{(n)}\\ \vdots\\ a_n\overline{b}_n^{(n)}\end{pmatrix}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\quad\text{i.e.}\quad\begin{pmatrix}a_1\overline{b}_1^{(n)}\\ \vdots\\ a_n\overline{b}_n^{(n)}\end{pmatrix}=M_n^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$
The numbers $\overline{b}_j^{(n)}$ are completely determined by these equations, and so we set
$$\overline{b}_i^{(n)}=\frac{1}{a_i}\sum_{j=1}^{n}(M_n^{-1})_{ij}.$$
It now remains to check that with this construction, properties (1)-(8) are satisfied:
- property (1) is true by construction, since
$$M_n\begin{pmatrix}a_1\overline{b}_1^{(n)}\\ \vdots\\ a_n\overline{b}_n^{(n)}\end{pmatrix}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$
- property (2) is trivially true if C_n is sufficiently large.
- as $a_n=2^{-(n+1)}|\mu_n-\lambda_n|$, $0<|a_n|<2^{-n}$, so (3) is true.
- let us now check property (5). We have
$$M_{n-1}\begin{pmatrix}a_1\overline{b}_1^{(n-1)}\\ \vdots\\ a_{n-1}\overline{b}_{n-1}^{(n-1)}\end{pmatrix}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

Hence

$$M_n \begin{pmatrix} a_1\overline{b}_1^{(n-1)} \\ \vdots \\ a_{n-1}\overline{b}_{n-1}^{(n-1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ c_n \end{pmatrix} \quad \text{where} \quad c_n = \sum_{j=1}^{n-1} \frac{a_j \overline{b}_j^{(n-1)}}{\mu_n - \lambda_j}.$$

By (g) we have $|1-c_n| < \varepsilon$, so that

$$\left|\left| M_n \begin{pmatrix} a_1(\overline{b}_1^{(n)} - \overline{b}_1^{(n-1)}) \\ \vdots \\ a_{n-1}(\overline{b}_{n-1}^{(n)} - \overline{b}_{n-1}^{(n-1)}) \\ a_n \overline{b}_n^{(n)} \end{pmatrix} \right|\right| = |1-c_n| < \varepsilon.$$

Hence

$$\left|\left| \begin{pmatrix} a_1(\overline{b}_1^{(n)} - \overline{b}_1^{(n-1)}) \\ \vdots \\ a_{n-1}(\overline{b}_{n-1}^{(n)} - \overline{b}_{n-1}^{(n-1)}) \\ a_n \overline{b}_n^{(n)} \end{pmatrix} \right|\right| < \varepsilon \, ||M_n^{-1}|| \leq \varepsilon \, (1+2^{-n}) \, ||M_{n-1}^{-1}|| \leq \ldots \leq \varepsilon \prod_{j=1}^{n} (1+2^{-j})$$

by (k) and the fact that $||M_1^{-1}|| = |\mu_1 - \lambda_1| < 1$, that is

$$\left(\sum_{j=1}^{n-1} |a_j|^2 \, |b_j^{(n)} - b_j^{(n-1)}|^2 + |a_n b_n^{(n)}|^2 \right)^{\frac{1}{2}} < \varepsilon \prod_{j=1}^{n} (1+2^{-j}).$$

In particular

$$\left(\sum_{j=1}^{n-1} |a_j|^2 \, |b_j^{(n)} - b_j^{(n-1)}|^2 \right)^{\frac{1}{2}} < \varepsilon \prod_{j=1}^{n} (1+2^{-j}) \tag{l}$$

so that

$$\min_{j=1,\ldots,n-1} |a_j| \left(\sum_{j=1}^{n-1} |b_j^{(n)} - b_j^{(n-1)}|^2 \right)^{\frac{1}{2}} < \varepsilon \prod_{j=1}^{n} (1+2^{-j}).$$

By (c) we get that

$$\left(\sum_{j=1}^{n-1} |b_j^{(n)} - b_j^{(n-1)}|^2 \right)^{\frac{1}{2}} < 2^{-n},$$

which is property (5).

• property (4) is a consequence of the equations

$$\sum_{j=1}^{n} \frac{a_j \overline{b}_j^{(n)}}{\mu_n - \lambda_j} = 1, \quad \text{i.e.} \quad \sum_{j=1}^{n-1} \frac{a_j \overline{b}_j^{(n)}}{\mu_n - \lambda_j} + \frac{a_n \overline{b}_n^{(n)}}{\mu_n - \lambda_n} = 1$$

and

$$\sum_{j=1}^{n-1} \frac{a_j \overline{b}_j^{(n-1)}}{\mu_{j(n)} - \lambda_j} = 1.$$

We have

$$\begin{aligned}
a_n \overline{b}_n^{(n)} &= (\mu_n - \lambda_n) \left(1 - \sum_{j=1}^{n-1} \frac{a_j \overline{b}_j^{(n)}}{\mu_n - \lambda_j} \right) = (\mu_n - \lambda_n) \left(\sum_{j=1}^{n-1} \left(\frac{a_j \overline{b}_j^{(n-1)}}{\mu_{j(n)} - \lambda_j} - \frac{a_j \overline{b}_j^{(n)}}{\mu_n - \lambda_j} \right) \right) \\
&= (\mu_n - \lambda_n) \left(\sum_{j=1}^{n-1} a_j \left(\frac{1}{\mu_{j(n)} - \lambda_j} - \frac{1}{\mu_n - \lambda_j} \right) \overline{b}_j^{(n-1)} \right. \\
&+ \left. \sum_{j=1}^{n-1} \frac{a_j}{\mu_n - \lambda_j} (\overline{b}_j^{(n-1)} - \overline{b}_j^{(n)}) \right).
\end{aligned}$$

Thus

$$\begin{aligned}
|a_n \overline{b}_n^{(n)}| &\leq |\mu_n - \lambda_n| \left(\sum_{j=1}^{n-1} |a_j|^2 \left| \frac{1}{\mu_{j(n)} - \lambda_j} - \frac{1}{\mu_n - \lambda_j} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n-1} |b_j^{(n-1)}|^2 \right)^{\frac{1}{2}} \\
&+ |\mu_n - \lambda_n| \left(\sum_{j=1}^{n-1} |a_j|^2 |b_j^{(n-1)} - b_j^{(n)}|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n-1} \frac{1}{|\mu_n - \lambda_j|^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

Now by (i) and (l), we have

$$|a_n \overline{b}_n^{(n)}| \leq |\mu_n - \lambda_n| \left(\varepsilon \, ||b^{(n-1)}|| + \varepsilon \prod_{j=1}^{n} (1 + 2^{-j}) \left(\sum_{j=1}^{n-1} \frac{1}{|\mu_n - \lambda_j|^2} \right)^{\frac{1}{2}} \right).$$

We have seen in Section 2.3 that properties (4) and (5) at Step $j \leq n-1$ imply that $||b^{(j)} - b^{(j-1)}|| \leq 2^{-(j-1)}$ (this is assertion (9')), so that $||b^{(n-1)}|| \leq \sum_{j=2}^{n-1} 2^{-(j-1)} \leq 1$. Combining this with property (h), we obtain that

$$|a_n \overline{b}_n^{(n)}| < |\mu_n - \lambda_n| (\varepsilon + 4^{-(n+1)}).$$

Since $\varepsilon < 4^{-(n+1)}$ by (a),

$$|b_n^{(n)}| < 2 \, . \, 4^{-(n+1)} \frac{|\mu_n - \lambda_n|}{|a_n|}.$$

As $a_n = 2^{-(n+1)} |\mu_n - \lambda_n|$, this yields that $|b_n^{(n)}| < 2^{-n}$, and (4) holds true.

- property (6) is easy: for $i = 1, \ldots, n-1$,

$$||u_i^{(n-1)} - u_i^{(n)}|| = \frac{|a_n|}{|\mu_i - \lambda_n|} = 2^{-(n+1)} \frac{|\mu_n - \lambda_n|}{|\mu_i - \lambda_n|} < 2^{-n}$$

by (j). So (6) is true.

• in order to prove property (7), we have to estimate

$$\begin{aligned}
||u^{(n)}_{j(n)}-u^{(n)}_n|| &= \left|\left|\sum_{j=1}^{n}\frac{a_j}{\mu_{j(n)}-\lambda_j}e_j-\sum_{j=1}^{n}\frac{a_j}{\mu_n-\lambda_j}e_j\right|\right|\\
&= \left(\sum_{j=1}^{n-1}|a_j|^2\left|\frac{1}{\mu_{j(n)}-\lambda_j}-\frac{1}{\mu_n-\lambda_j}\right|^2\right)^{\frac{1}{2}}+|a_n|\left|\frac{1}{\mu_{j(n)}-\lambda_n}-\frac{1}{\mu_n-\lambda_n}\right|\\
&< \varepsilon+|a_n|\,\frac{|\mu_n-\mu_{j(n)}|}{|\mu_{j(n)}-\lambda_n|\,.\,|\mu_n-\lambda_n|}.
\end{aligned}$$

by (i). Now as $a_n=2^{-(n+1)}|\mu_n-\lambda_n|$,

$$\begin{aligned}
|a_n|\frac{|\mu_n-\mu_{j(n)}|}{|\mu_{j(n)}-\lambda_n|\,.\,|\mu_n-\lambda_n|} &\leq 2^{-(n+1)}\,\frac{|\mu_n-\lambda_n|+|\lambda_n-\mu_{j(n)}|}{|\mu_{j(n)}-\lambda_n|}\\
&\leq 2^{-(n+1)}\left(1+\frac{|\mu_n-\lambda_n|}{|\mu_{j(n)}-\lambda_n|}\right)\\
&< 2^{-(n+1)}\,(1+2^{-n})
\end{aligned}$$

by (j). It follows then from (a) that

$$||u^{(n)}_{j(n)}-u^{(n)}_n||\leq\varepsilon+2^{-(n+1)}\,(1+2^{-n})<2^{-n},$$

so (7) is true.

• lastly, we have to estimate the quantities $||T_nu_i^{(k)}-\mu_iu_i^{(k)}||$ for $k=1,\dots,n-1$ and $i=1,\dots,k$: since $T_ku_i^{(k)}=\mu_iu_i^{(k)}$, we have

$$||T_nu_i^{(k)}-\mu_iu_i^{(k)}||=||\sum_{p=k+1}^{n}(T_p-T_{p-1})u_i^{(k)}||\leq\sum_{p=k+1}^{n}||(T_p-T_{p-1})u_i^{(k)}||.$$

Since $u_i^{(k)}$ belongs to $\mathrm{sp}[e_1,\dots,e_k]$, we have $T_pu_i^{(k)}=D_ku_i^{(k)}+R_pu_i^{(k)}$ for $p\geq k$, so that

$$(T_p-T_{p-1})u_i^{(k)} = (R_p-R_{p-1})u_i^{(k)}=\langle u_i^{(k)},b^{(p)}-b^{(p-1)}\rangle a^{(p)}$$

for $p\geq k+1$. Thus

$$||(T_p-T_{p-1})u_i^{(k)}|| \leq ||b^{(p)}-b^{(p-1)}||\,.\,||u_i^{(k)}||\,.\,||a^{(p)}||.$$

By the induction assumption and (5) which we have already proved for $p=n$, we know that (9') holds true for any $p\leq n$: $||b^{(p)}-b^{(p-1)}||\leq2^{-(p-1)}$ for $k+1\leq p\leq n$. Moreover for $k+1\leq p\leq n$, $||a^{(p)}||\leq1$ by (3) which is true until step n, and so it remains to prove that $||u_i^{(k)}||\leq3$ for any $k=1,\dots,n-1$. By the induction assumption and (6), we have $||u_i^{(j)}-u_i^{(j-1)}||\leq2^{-j}$ for $i+1\leq j\leq n-1$. Hence

$$||u_i^{(k)}-u_i^{(i)}||\leq\sum_{j=i+1}^{k}||u_i^{(j)}-u_i^{(j-1)}||\leq\sum_{j=i+1}^{k}2^{-j}\leq2^{-i}$$

for any $1 \leq k \leq n-1$. So $||u_i^{(k)}|| \leq 2^{-i} + ||u_i^{(i)}||$. Now for any $i \leq n-1$, we have by (7) of the induction assumption that $||u_{j(i)}^{(i)} - u_i^{(i)}|| \leq 2^{-i}$ so that $||u_i^{(k)}|| \leq 2.2^{-i} + ||u_{j(i)}^{(i)}||$. Then since $i \leq n-1$ and $j(i) < i$ we can again estimate

$$||u_{j(i)}^{(i)}|| \leq ||u_{j(i)}^{(j(i))}|| + 2^{-j(i)} < 2.2^{-j(i)} + ||u_{j(j(i))}^{(j(i))}||.$$

Since $j(m) < m$ for every $m \geq 2$, there exists for each $i \leq n-1$ an integer s_i such that $j^{[s_i-1]}(i) > j^{[s_i]}(i)$ and $j^{[s_i]}(i) = 1$, where $j^{[s]}(i)$ denotes for each $s \geq 1$ the s^{th} iterate of the function j. Thus

$$||u_i^{(k)}|| \leq 2(2^{-i} + 2^{-j(i)} + 2^{-j(j(i))} + \ldots + 2^{-j^{[s_i-1]}(i)} + 2^{-1}) + ||u_1^{(1)}|| \leq 3.$$

So $||(T_p - T_{p-1})u_i^{(k)}|| \leq 3\,.\,2^{-(p-1)}$ for any $k+1 \leq p \leq n$. This yields that

$$||T_n u_i^{(k)} - \mu_i u_i^{(k)}|| < 3 \sum_{p=k+1}^{n} 2^{-(p-1)} \leq 3\,.\,2^{-(k-1)}$$

and this estimate proves (8).

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