NON-RECURRENCE SETS FOR WEAKLY MIXING LINEAR DYNAMICAL SYSTEMS

by

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Abstract. — We study non-recurrence sets for weakly mixing dynamical systems by using linear dynamical systems. These are systems consisting of a bounded linear operator acting on a separable complex Banach space X, which becomes a probability space when endowed with a non-degenerate Gaussian measure. We generalize some recent results of Bergelson, del Junco, Lemańczyk and Rosenblatt, and show in particular that sets $\{n_k\}$ such that $\frac{n_{k+1}}{n_k} \to +\infty$, or such that n_k divides n_{k+1} for each $k \geq 0$, are non-recurrence sets for weakly mixing linear dynamical systems. We also give examples, for each $r \geq 1$, of r-Bohr sets which are non-recurrence sets for some weakly mixing systems.

1. Introduction

The main topic of this paper is the study of recurrence and non-recurrence in the measure-theoretic framework, and for a particular class of dynamical systems, namely weakly mixing dynamical systems. Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of positive integers. The set $\{n_k \ ; \ k \geq 0\}$ is a *recurrence set*, or a *Poincaré set*, if for any dynamical system $(X, \mathcal{B}, m, T)$, where T is a measure-preserving transformation of a non-atomic probability space $(X, \mathcal{B}, m)$, the following is true: for any $A \in \mathcal{B}$ with $m(A) > 0$ there exists a $k \geq 0$ such that $m(T^{-n_k} A \cap A) > 0$.

Recurrence is a central topic in ergodic theory, and we refer the reader to one of the works [**14**], [**15**] or [**18**] and to the references therein for more information about it, as well as for applications to number theory and combinatorics. Standard examples of recurrence sets are the set $\mathbb{N}$ of all integers (this is the classical Poincaré recurrence theorem), the set of squares $\{k^2 \ ; \ k \geq 0\}$, and more generally any set $\{p(k); k \geq 0\}$ where p is a polynomial taking integer values on integers such that $p(0) = 0$. Also, any thick set, i.e. containing arbitrarily long blocks of integers, is a recurrence set, as well as any set of the form $(E - E) \cap \mathbb{N}$, where E is an infinite subset of $\mathbb{N}$. On the other hand, it is not difficult to exhibit non-recurrence sets: the set of odd integers is the easiest example, and rotations

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on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} \; ; \; |\lambda| = 1\}$ provide a wealth of non-recurrence sets: if $\{n_k\}$ is any set for which there exists a $\lambda \in \mathbb{T}$ and a $\delta > 0$ such that $|\lambda^{n_k} - 1| \geq \delta$ for all k, then $\{n_k\}$ is clearly not a recurrence set.

In the rest of the paper, we will say that $\{n_k\}$ is a *non-recurrence set for the dynamical system* $(X, \mathcal{B}, m, T)$, or simply that $\{n_k\}$ is *non-recurrent for the dynamical system* $(X, \mathcal{B}, m, T)$, if there exists a set $A \in \mathcal{B}$ with $m(A) > 0$ such that $m(T^{-n_k}A \cap A) = 0$ for each $k \geq 0$. Given a non-recurrence set $\{n_k\}$, it is interesting to try to construct $(X, \mathcal{B}, m, T)$ as above with additional properties: can we construct T ergodic? weakly mixing? The study of non-recurrence sets for weakly mixing dynamical systems was initiated by Bergelson, Del Junco, Lemańczyk and Rosenblatt in the recent paper [**7**], where they give several examples of weakly mixing non-recurrent systems and study the relationship between non-recurrence and rigidity. They prove here in particular that the generic transformation is both weakly mixing, rigid and non-recurrent, and that if the sequence $(n_k)_{k \geq 0}$ grows sufficiently fast, namely if it satisfies the condition

$$\sum_{k \geq 0} \frac{n_k}{n_{k+1}} < +\infty,$$

then $\{n_k\}$ is a non-recurrence set for some weakly mixing dynamical system.

Our aim here is to continue this study of non-recurrence sets for weakly mixing systems. We first provide new examples of such sets by using a rather particular class of dynamical systems, namely linear dynamical systems. A *linear dynamical system* consists of a bounded linear operator T on a separable complex infinite-dimensional Banach space X, and under certain conditions concerning, usually, the eigenvectors of T associated to eigenvalues of modulus 1, it is possible to construct a probability measure on X with respect to which T becomes a measure-preserving weakly mixing transformation. More details about this will be given in the next section. We will also need to use some results about *non-Jamison sequences*, which form a class of sequences connected to the study of partial power-boundedness of operators on separable spaces, and which appear naturally as well in the study of rigidity sequences (see [**11**]).

Here is our first result, which generalizes one of the results of [**7**] mentioned above:

Theorem 1.1. — *If $(n_k)_{k \geq 0}$ is a sequence of integers such that $\frac{n_{k+1}}{n_k}$ tends to infinity, the set $\{n_k\}$ is a non-recurrence set for some weakly mixing linear dynamical system. More generally, the same result holds if $(n_k)_{k \geq 0}$ is a non-Jamison sequence for which there exists a $\lambda_0 \in \mathbb{T}$ such that $\inf_{k \geq 0} |\lambda_0^{n_k} - 1| > 0$ (i.e. if $\{n_k\}$ is a non-recurrence set for some rotation on the unit circle).*

Our second result concerns sequences $(n_k)_{k \geq 0}$ such that n_k divides n_{k+1} for each $k \geq 1$.

Theorem 1.2. — *Let $(n_k)_{k \geq 0}$ be a sequence such that $n_k | n_{k+1}$ for each $k \geq 1$. Then for any $p \in \mathbb{Z}$, the set $\{n_k - p \; ; \; k \geq 0\} \cap \mathbb{N}$ is a non-recurrence set for some weakly mixing linear dynamical system. This applies in particular to the set $\{n_k\}$ itself.*

The proof of Theorem 1.2 uses the notion of rigidity, which was thoroughly explored in the context of weakly mixing systems in the paper [**7**], and also in [**11**]. Part of the proof of Theorem 1.2 relies on the fact that if $(n_k)_{k \geq 0}$ is a rigidity sequence in a certain strong sense, then $\{n_k - p\ ;\ k \geq 0\} \cap \mathbb{N}$ is a non-recurrence set for some weakly mixing dynamical system, whatever the choice of $p \in \mathbb{Z} \setminus \{0\}$. It seems to be an open question whether, if $(n_k)_{k\geq 0}$ is a rigidity sequence and $p \in \mathbb{Z} \setminus \{0\}$, the set $\{n_k - p\ ;\ k \geq 0\} \cap \mathbb{N}$ is always a non-recurrence set for some weakly mixing system.

A central open question in the paper [**7**] concerns lacunary sequences: if $(n_k)_{k\geq 0}$ is a lacunary sequence, i.e. if there exists an $a > 1$ such that $\frac{n_{k+1}}{n_k} > a$ for any k, then a result of Pollington [**23**] and de Mathan [**10**] is that there exists an element $\lambda = e^{2i\pi\theta} \in \mathbb{T}$ with θ irrational and a $\delta > 0$ such that $|\lambda^{n_k} - 1| > \delta$ for all k. So $\{n_k\}$ is a non-recurrence set for some ergodic dynamical system. Hence the following natural question:

Question 1.3. — [**7**] *If $(n_k)_{k\geq 0}$ is a lacunary sequence, does there always exist a weakly mixing dynamical system for which the set $\{n_k\}$ is not recurrent?*

Theorem 1.2 provides a positive answer to this question for sequences such that $n_k | n_{k+1}$ for each $k \geq 0$. Cutting and stacking constructions can also be used to exhibit some non-recurrent lacunary sets (see Theorem 6.1, which generalizes the result of [**7**] that the set $\{\frac{3^{k+1}-1}{2} - 1\ ;\ k \geq 0\}$ is non-recurrent for the Chacon transformation).

Our last result concerns sets which, for some fixed integer $r \geq 1$, are recurrent (in the topological sense) for all products of r rotations on the unit circle. In accordance with the terminology of [**22**], let us call such sets *r-Bohr sets*. It is an open question, equivalent to an old combinatorial problem of Veech [**26**] about syndetic sets and the Bohr topology on $\mathbb{Z}$, to know whether any set which is recurrent for all finite products of rotations (such a set is called a *Bohr set*) is necessarily topologically recurrent for all dynamical systems. See [**16**], [**18**], [**9**] or [**22**] for more about this question. It is shown in [**22**] that there exists for each $r \geq 1$ sets which are r-Bohr but not $(r+1)$-Bohr. Another construction of r-Bohr sets which are not Bohr is given in [**20**]. One of the interests of the sets constructed in [**20**] is that they have density zero, contrary to the sets of [**22**] which have positive density. Hence the sets of [**22**], which are non-recurrent for some product of $r + 1$ rotations on $\mathbb{T}$, are recurrent for all weakly mixing dynamical systems. We show here, as a consequence of Theorem 1.1, that the sets $\{n_k^{(r)}\}$ of [**20**] are r-Bohr, but are not recurrent for some weakly mixing dynamical systems:

Theorem 1.4. — *For any integer $r \geq 1$ there exist sets $\{n_k^{(r)}\}$ of integers which are r-Bohr, but which are non-recurrence sets for some weakly mixing linear dynamical systems.*

The paper is organized as follows: in Section 2 we recall some results about linear dynamical systems and Jamison sequences which will be needed for the proofs of the theorems. In Sections 3, 4 and 5 we present the proofs of Theorems 1.1, 1.2 and 1.4 respectively. Section 6 contains the proof of Theorem 6.1 concerning a construction of weakly mixing non-recurrent systems by cutting and stacking.

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2. Weakly mixing linear dynamical systems and Jamison sequences

In this section, which is expository, we briefly review some results concerning linear dynamical systems. They will be necessary for the proofs of the majority of the results in this paper. For a more detailed account, we refer the reader to the recent book [**5**].

2.1. Linear dynamical systems. — A linear dynamical system consists of a pair (X, T), where X is an infinite-dimensional complex separable Banach space, and T is a bounded linear operator on X. Under some conditions on T and X, it is possible to construct a non-degenerate Gaussian measure m on X with respect to which T defines a measure-preserving weakly mixing transformation. Recall that if $\mathcal{B}$ denotes the σ-algebra of Borel subsets of X, a Borel probability measure m on X is said to be (centered) *Gaussian* if any element $x^* \in X^*$, considered as a complex-valued random variable on $(X, \mathcal{B}, m)$, has (centered) Gaussian law: for any Borel subset A of $\mathbb{C}$,

$$m(\{x \in X \ ; \ \langle x^*, x\rangle \in A\}) = \frac{1}{2\pi\sigma_{x^*}^2} \int_A e^{-\frac{(u^2+v^2)}{2\sigma_{x^*}^2}} dudv$$

for some $\sigma_{x^*} > 0$. The measure m is said to be *non-degenerate* if its topological support is the whole space X, i.e if $m(U) > 0$ for any non-empty open subset U of X. Whenever T is a measure-preserving transformation of $(X, \mathcal{B}, m)$, we will denote by U_T the associated Koopman operator on $L^2(X, \mathcal{B}, m)$: for $f \in L^2(X, \mathcal{B}, m)$, $U_T f = f \circ T$.

The study of operators on Banach spaces from the ergodic point of view was initiated by Flytzanis in [**12**], and developed later on in the two papers [**3**] and [**4**]. Among other things, a necessary and sufficient condition was obtained, under which a bounded operator on a complex infinite-dimensional separable Hilbert space H admits a non-degenerate invariant Gaussian measure with respect to which it defines an ergodic (or, equivalently here, weakly mixing) transformation of X. This condition involves eigenvectors of T associated to eigenvalues of modulus 1, which we call *unimodular eigenvectors*. It states roughly that if $T \in \mathcal{B}(H)$ has a huge supply of unimodular eigenvectors (which happens in very many concrete situations), T defines a weakly mixing measure-preserving transformation of H with respect to some non-degenerate Gaussian measure m. This condition is given in [**3**] and [**4**] in terms involving perfectly spanning sets of unimodular eigenvectors, but a simpler equivalent condition was recently obtained in the paper [**19**]:

Theorem 2.1. — [**3**], [**19**] *Let T be a bounded linear operator on a complex separable infinite-dimensional Hilbert space H. The following assertions are equivalent:*

(1) *for any countable subset Δ of $\mathbb{T}$, the linear span of the eigenspaces* $\ker(T - \lambda)$*, $\lambda \in \mathbb{T} \setminus \Delta$, is dense in H;*
(2) *there exists a non-degenerate Gaussian measure m on H such that T is a weakly mixing measure-preserving transformation of* $(H, \mathcal{B}, m)$.

We refer the reader to [**3**], [**4**], [**8**] or [**5**] for instance for many examples of such weakly mixing linear dynamical systems, living not only on Hilbert spaces but also on other Banach or Fréchet spaces. Extensions of Theorem 2.1 to the Banach space setting were obtained in [**4**] and [**5**], culminating in the recent paper [**6**] where it was shown that the implication (1) $\implies$ (2) remains true when X is an arbitrary complex separable Banach space.

Let us now present a particular class of operators, which give in a very handy way examples of Gaussian weakly mixing systems on a Hilbert space associated to prescribed continuous measures on $\mathbb{T}$.

2.2. Kalish-type operators. — These operators were introduced by Kalish in [**21**] in order to exhibit, for each closed subset F of $\mathbb{T}$, examples of operators on a Hilbert space whose spectrum and point spectrum coincide with F. The construction goes this way. On the space $L^2(\mathbb{T})$ of square-integrable functions on $\mathbb{T}$, consider the operator $T = M - J$, where M and J are defined as follows: M is the multiplication operator by the independent variable ζ on $L^2(\mathbb{T})$, and J is the integration operator: for any $f \in L^2(\mathbb{T})$, $Mf : \zeta \mapsto \zeta f(\zeta)$ and $Jf : \zeta \mapsto \int_{(1,\zeta)} f$, where, if $\zeta = e^{i\theta}$ with $0 \le \theta < 2\pi$, $(1,\zeta)$ denotes the arc $\{e^{i\alpha} \,;\, 0 \le \alpha \le \theta\}$, and $(\zeta, 1)$ denotes the arc $\{e^{i\alpha} \,;\, \theta \le \alpha \le 2\pi\}$. It is not difficult to check that for any $\lambda \in \mathbb{T}$ the characteristic function χ_λ of the arc $(\lambda, 1)$ satisfies $T\chi_\lambda = \lambda\chi_\lambda$, so that when χ_λ is non-zero, it is an eigenvector of T associated to the eigenvalue λ.

Let now σ be a continuous probability measure on $\mathbb{T}$, and let K denote its support. The closed linear span H_K in $L^2(\mathbb{T})$ of the eigenvectors χ_λ of T, $\lambda \in K$, is T-invariant, and so T induces an operator T_K on H_K. One sees easily that the eigenvector field (for T_K) $E : K \to H_K$ which maps λ to χ_λ is continuous, and that these eigenvectors $E(\lambda)$, $\lambda \in K$, span a dense subspace of H_K. Moreover the spectrum of T_K coincides with K, and T_K is invertible. It follows immediately from Theorem 2.1 that T_K admits a non-degenerate Gaussian measure with respect to which T_K becomes a weakly mixing measure-preserving transformation of H_K, but we actually know more, and there is a "canonical" way to associate a suitable Gaussian measure m to the eigenvector field E and the measure σ. One can construct a measure m such that for any $x, y \in H_K$,

$$\int_{H_K} \langle x, z\rangle \overline{\langle y, z\rangle} dm(z) = \int_{\mathbb{T}} \langle x, E(\lambda)\rangle \overline{\langle y, E(\lambda)\rangle} d\sigma(\lambda),$$

and the continuity of E on K combined with the fact that $\text{span}[E(\lambda) \,;\, \lambda \in K]$ is dense in H_K implies that m is non-degenerate. Also, it follows from the continuity of the measure σ (see [**3**] for details) that T_K is weakly mixing with respect to this measure m. Let us now define

$$\begin{aligned} V :\ & L^2(\mathbb{T}, \sigma) && \to L^2(\mathbb{T}, \sigma) \\ & f && \mapsto [\lambda \mapsto \lambda f(\lambda)] \end{aligned}$$

and

$$\begin{aligned} A :\ & L^2(\mathbb{T}, \sigma) && \to H_K \\ & f && \mapsto \int_{\mathbb{T}} f(\lambda) E(\lambda) d\sigma(\lambda). \end{aligned}$$

One sees easily that $TA = AV$, and so we have for all $x, y \in H_K$ and all $n \ge 0$

$$\int_{H_K} \langle x, T_K^n z\rangle \overline{\langle y, z\rangle} dm(z) = \int_{\mathbb{T}} \lambda^n \langle x, E(\lambda)\rangle \overline{\langle y, E(\lambda)\rangle} d\sigma(\lambda). \tag{1}$$

As explained in [**11**], these Kalish-type operators can be used to transfer properties of the measure σ into ergodic properties of the operator T_K. In particular, suppose that σ is such that $\hat{\sigma}(n_k) \to 1$ as $k \to +\infty$ for some strictly increasing sequence $(n_k)_{k\geq 0}$ of integers. It follows from (1) that

$$\int_{H_K} \langle x, (T_K^{n_k} - I)z\rangle \overline{\langle y, z\rangle} dm(z) \to 0 \text{ as } n_k \to +\infty \text{ for all } x, y \in H_K$$

and also (see [**11**] for details) that $U_{T_K}^{n_k} \to I$ in the Weak Operator Topology. Thus T_K is an example of a weakly mixing transformation which is *rigid* with respect to $(n_k)_{k\geq 0}$. The notion of rigid dynamical system was introduced by Furstenberg and Weiss in [**13**] in their study of mildly mixing dynamical systems (see also [**14**]).

Definition 2.2. — A measure-preserving transformation T of a non-atomic probability space $(X, \mathcal{B}, \mu)$ is said to be *rigid* if there exists a sequence $(n_k)_{k\geq 0}$ of integers such that for any $A \in \mathcal{B}$, $\mu(T^{-n_k}A \triangle A) \to 0$ as $n_k \to +\infty$.

Let us recall the following terminology of [**7**] and [**11**]: a sequence $(n_k)_{k\geq 0}$ is said to be a *rigidity sequence* if there exists a weakly mixing measure-preserving transformation which is rigid with respect to $(n_k)_{k\geq 0}$. Rigidity sequences can be characterized in terms of Fourier coefficients of measures:

Theorem 2.3. — [**7**], [**11**] *The following assertions are equivalent:*

(1) $(n_k)_{k\geq 0}$ *is a rigidity sequence;*
(2) *there exists a continuous probability measure* σ *on* $\mathbb{T}$ *such that* $\hat{\sigma}(n_k) \to 1$ *as* $k \to +\infty$.

If (2) in Theorem 2.3 above is satisfied, the Kalish-type operator T_K associated to σ is weakly mixing and rigid with respect to $(n_k)_{k\geq 0}$.

2.3. Partially power-bounded operators. — In the study of recurrence we are undertaking, an especially interesting class of linear dynamical systems is the class of partially power-bounded operators:

Definition 2.4. — If $(n_k)_{k\geq 0}$ is a strictly increasing sequence of integers, and T is a bounded linear operator on the Banach space X, T is said to be *partially power-bounded* with respect to $(n_k)_{k\geq 0}$ if $\sup_{k\geq 0} ||T^{n_k}||$ is finite.

In the rest of the paper we will denote by $\sigma_p(T) \cap \mathbb{T}$ the *unimodular point spectrum* of the operator T, i.e. the set of eigenvalues of T which are of modulus 1.

When the sequence $(n_k)_{k\geq 0}$ is "too rich", such partially power-bounded operators with respect to $(n_k)_{k\geq 0}$ can have only countably many eigenvalues on the unit circle. These sequences $(n_k)_{k\geq 0}$ are called *Jamison sequences*, because of an old result of Jamison which states that the sequence (k) has this property.

Definition 2.5. — [**2**] A sequence $(n_k)_{k\geq 0}$ of integers is said to be a *Jamison sequence* when the following property holds true: if X is any complex separable Banach space and T is any bounded linear operator on X such that $\sup_{k\geq 0} ||T^{n_k}||$ is finite, then $\sigma_p(T) \cap \mathbb{T}$ is at most countable.

It can happen that for some sequences $(n_k)_{k\geq 0}$ (the non-Jamison sequences), there exists an operator on some complex separable Banach space X which is partially power-bounded with respect to $(n_k)_{k\geq 0}$ and which has uncountably many eigenvalues on the unit circle. In this case, there are hopes that T could define a weakly mixing transformation of $(X, \mathcal{B}, m)$ for some suitable Gaussian measure m on X.

The study of partially power-bounded operators started with a paper of Ransford [**24**], and the first example of a partially power-bounded operator with uncountable unimoduar point spectrum was given by Ransford and Roginskaya in [**25**]. Jamison and non-Jamison sequences were then further investigated in [**1**] and [**2**], and a complete characterization of Jamison sequences was obtained in [**2**]:

Theorem 2.6. — [**2**] *Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of integers with $n_0 = 1$. The following assertions are equivalent:*

(1) *$(n_k)_{k\geq 0}$ is a Jamison sequence, i.e. for any separable Banach space X and any bounded operator $T \in \mathcal{B}(X)$, $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$ implies that $\sigma_p(T) \cap \mathbb{T}$ is at most countable;*

(2) *there exists an $\varepsilon > 0$ such that for any $\lambda \in \mathbb{T} \setminus \{1\}$, $\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon$.*

Several examples of non-Jamison sequences were given in [**1**] and [**2**]. Among these are the sequences $(n_k)_{k\geq 0}$ such that $\frac{n_{k+1}}{n_k} \to +\infty$, the sequences $(n_k)_{k\geq 0}$ such that $n_k | n_{k+1}$ for each k and $\limsup_{k\to+\infty} \frac{n_{k+1}}{n_k} = +\infty$, etc. In all these cases (and in general, when $(n_k)_{k\geq 0}$ is not a Jamison sequence), there exists a separable X and $T \in \mathcal{B}(X)$ such that $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$ and $\sigma_p(T) \cap \mathbb{T}$ is uncountable. It was recently shown in [**11**] that one could always take X to be a Hilbert space, and, moreover, that the operator T could be constructed in such a way that it defines a weakly mixing measure-preserving transformation of $(H, \mathcal{B}, m)$ for some non-degenerate Gaussian measure m. This can be summarized as follows:

Theorem 2.7. — [**11**] *Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of integers such that $n_0 = 1$. The following assertions are equivalent:*

(1) *$(n_k)_{k\geq 0}$ is not a Hilbertian Jamison sequence, i.e. there exists a bounded linear operator T on a complex separable Hilbert space H such that $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$ and T defines a weakly mixing measure-preserving transformation of $(H, \mathcal{B}, m)$ for some non-degenerate Gaussian measure m (in particular $\sigma_p(T) \cap \mathbb{T}$ is uncountable);*

(2) *for any $\varepsilon > 0$ there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that $\sup_{k\geq 0} |\lambda^{n_k} - 1| < \varepsilon$.*

As can easily be guessed, the partial power-boundedness condition gives us a non-recurrence property, and its interest here is clear. We now have all our tools in hand for the proofs of Theorems 1.1, 1.2 and 1.4.

3. Proof of Theorem 1.1

We are going to prove that if $(n_k)_{k\geq 0}$ is a non-Jamison sequence with $n_0 = 1$ and if $\lambda_0 \in \mathbb{T}$ is such that $\inf_{k\geq 0} |\lambda_0^{n_k} - 1| > 0$, then there exists a bounded linear operator on a separable complex Hilbert space H which is weakly mixing and non-recurrent with respect to the set $\{n_k\}$. In spirit this statement follows from Theorem 2.7, but one needs to use some fine aspects of the construction in [**11**] in order to obtain it.

The proof of [**11**] yields for any fixed number $\delta > 0$ an operator $T \in \mathcal{B}(H)$ such that $\sup_{k\geq 0} ||T^{n_k}|| < 1 + \delta$ and T is a weakly mixing transformation of $(H, \mathcal{B}, m)$ for some non-degenerate Gaussian measure m. The operator is constructed as a sum $T = D + B$ of a diagonal operator D and a weighted backward shift B on the space $\ell_2(\mathbb{N})$ where:

• $D = \text{diag}(\lambda_n\ ;\ n \geq 1)$ is diagonal with respect to the canonical basis $(e_n)_{n\geq 1}$ of $\ell_2(\mathbb{N})$, and the λ_n's are distinct unimodular coefficients;

• B is defined as a weighted backward shift with $Be_1 = 0$ and $Be_n = \alpha_{n-1}e_{n-1}$ for $n \geq 2$, and $(\alpha_n)_{n\geq 1}$ is a very quickly decreasing sequence of positive numbers.

The construction uses an auxiliary function $j : \{2, +\infty\} \to \{1, +\infty\}$ which has the properties that $j(n) < n$ for each n, $j(2) = 1$, and j takes each value $r \geq 1$ infinitely often. The elements λ_n, $n \geq 1$, are constructed by induction on n, and the crucial tool for this construction is a distance $d_{(n_k)}$ on the unit circle which is associated to the sequence $(n_k)_{k\geq 0}$ in the following way: for $\lambda, \mu \in \mathbb{T}$,

$$d_{(n_k)}(\lambda, \mu) = \sup_{k\geq 0} |\lambda^{n_k} - \mu^{n_k}|.$$

Since $n_0 = 1$, $d_{(n_k)}$ is indeed a distance on $\mathbb{T}$. Now, the assumption that $(n_k)_{k\geq 0}$ is not a Jamison sequence implies that there exists an uncountable subset K of $\mathbb{T}$ containing the point 1 such that $(K, d_{(n_k)})$ is perfect. Then the construction of the diagonal coefficients λ_n goes as follows: one first chooses $\lambda_1 \in K$. Then if δ is a positive number, one proves that there exists a sequence (ε_n) of positive numbers going to zero very quickly such that if for each $n \geq 2$ we have $d_{(n_k)}(\lambda_n, \lambda_{j(n)}) < \varepsilon_n$, then for a suitable choice of the sequence of weights (α_n) the operator $T = D + B$ is such that $\sup_{k\geq 0} ||T^{n_k} - D^{n_k}|| < \delta$, and is weakly mixing. Observe also that T is invertible if the weights α_n are sufficiently small.

We now claim that we can ensure that $\sup_k ||T^{n_k} - I|| < 2\delta$. Indeed we have for each $k \geq 1$ that $||T^{n_k} - I|| \leq ||T^{n_k} - D^{n_k}|| + ||D^{n_k} - I||$, and $||D^{n_k} - I|| = \sup_{n\geq 1} |\lambda_n^{n_k} - 1|$. Hence $\sup_k ||D^{n_k} - I|| = \sup_k \sup_n |\lambda_n^{n_k} - 1| = \sup_n d_{(n_k)}(\lambda_n, 1)$. Now

$$d_{(n_k)}(\lambda_n, 1) \leq d_{(n_k)}(\lambda_n, \lambda_{j(n)}) + d_{(n_k)}(\lambda_{j(n)}, \lambda_{j\circ j(n)}) + \cdots + d_{(n_k)}(\lambda_{j^{[p-1]}(n)}, \lambda_{j^{[p]}(n)}),$$

where $j^{[k]}$ denotes the composition of j with itself k times and p is the unique integer such that $j^{[p-1]}(n) > 1$ and $j^{[p]}(n) = 1$ (recall that $j(2) = 1$). Hence

$$d_{(n_k)}(\lambda_n, 1) \leq \varepsilon_n + \varepsilon_{j(n)} + \ldots + \varepsilon_{j^{[p-1]}(n)} \leq \sum_{n\geq 2} \varepsilon_n.$$

If we take the ε_n's sufficiently small, we can ensure that $\sum_{n\geq 2} \varepsilon_n$ is less than any prescribed positive number, for instance less than δ. So we get that $\sup_k ||T^{n_k} - I|| < 2\delta$.

Going back to our initial assumption, we know that there exists $\lambda_0 \in \mathbb{T}$ such that $\inf_{k\geq 0} |\lambda_0^{n_k} - 1| = \delta > 0$. Let $T \in \mathcal{B}(H)$ be as above with $\sup_{k\geq 0} ||T^{n_k} - I|| < \frac{\delta}{2}$, and consider the operator $S = \lambda_0 T$. By construction, T is such that for any countable subset Δ of $\mathbb{T}$, the linear space spanned by the kernels $\ker(T - \lambda)$, $\lambda \in \mathbb{T} \setminus \Delta$, is dense in H. It is obvious that $\lambda_0 T$ satisfies the same property, and so by Theorem 2.1 S is weakly mixing with respect to some non-degenerate Gaussian measure m. For $\gamma \in (0, 1)$, let

$U_\gamma = B(e_1, \gamma)$ be the open ball of H centered at the vector e_1 and of radius γ. For $u \in U_\gamma$ we have for all $k \geq 1$

$$\begin{aligned} ||S^{n_k}u - u|| &= ||\lambda_0^{n_k}T^{n_k}u - u|| = ||\lambda_0^{n_k}(T^{n_k}u - u) + (\lambda_0^{n_k} - 1)u|| \\ &\geq |\lambda_0^{n_k} - 1|\,||u|| - \frac{\delta}{2}||u|| \\ &\geq \delta(1-\gamma) - \frac{\delta}{2}(1+\gamma) = \frac{\delta}{2} - \frac{3\delta}{2}\gamma > 2\gamma \end{aligned}$$

if γ is sufficiently small. Hence $S^{n_k}U_\gamma \cap U_\gamma = \varnothing$ if γ is sufficiently small, and so $\{n_k\}$ is a non-recurrence set for the operator S.

If $(n_k)_{k\geq 0}$ is a lacunary sequence, a result of de Mathan [**10**] and Pollington [**23**] (see also [**22**]) states that there exists indeed a $\lambda_0 \in \mathbb{T}$ such that $\inf_{k\geq 0} |\lambda_0^{n_k} - 1| > 0$. So any lacunary non-Jamison sequence is a non-recurrence set for some weakly mixing linear dynamical system. In particular, thanks to a result which we recalled in Section 2, this is the case if $\frac{n_{k+1}}{n_k}$ tends to infinity as k tends to infinity. Theorem 1.1 is proved.

Remark 3.1. — It is clearly not true that any non-Jamison sequence yields a non-recurrence set for some measure-preserving dynamical system. Indeed, if $(p_j)_{j\geq 1}$ is a sequence of integers growing very fast, it is not difficult to see (thanks to Theorem 2.3) that the sequence $(n_k)_{k\geq 0}$ defined by

$$\{n_k\} = \bigcup_{k\geq 1} \{p_j, 2p_j, \ldots, jp_j\}$$

is a non-Jamison sequence. But $\{n_k\}$ is a recurrence set, and of course there exists no element $\lambda_0 \in \mathbb{T}$ such that $\inf_k |\lambda_0^{n_k} - 1| > 0$.

Remark 3.2. — In view of the results of [**7**], one might also wonder whether it is true that whenever $(n_k)_{k\geq 0}$ is a non-Jamison sequence, the set $\{n_k - 1\}$ is a non-recurrence set for some weakly mixing linear dynamical system. This is not the case: consider, as in Remark 3.1 above, the set

$$\{n_k\} = \bigcup_{k\geq 1} \{p_j + 1, 2p_j + 1, \ldots, jp_j + 1\},$$

where $(p_j)_{j\geq 1}$ is a rapidly growing sequence of integers. Then (n_k) is a non-Jamison sequence, but $\{n_k - 1\}$ is a recurrence set. Observe that $\{n_k\}$ itself is a non-recurrence set for some weakly mixing linear dynamical system. One can show, using the same kind of arguments as in the proof of Theorem 1.1, that if $(n_k)_{k\geq 0}$ is a non-Jamison sequence and if there exists for each $\varepsilon > 0$ a $\lambda \in \mathbb{T}$ such that $\sup_k |\lambda^{n_k} - 1| < \varepsilon$ and $|\lambda^{n_k} - 1| \to 0$, then for every $p \in \mathbb{Z} \setminus \{0\}$ the set $\{n_k - p\} \cap \mathbb{N}$ is a non-recurrence set for some weakly mixing linear dynamical system.

4. Proof of Theorem 1.2

Let $(n_k)_{k\geq 0}$ be a sequence of integers such that $n_k | n_{k+1}$ for each $k \geq 0$. It is proved in [**7**] and [**11**] that $(n_k)_{k\geq 0}$ is a rigidity sequence for some weakly mixing dynamical system. As recalled in Section 2, this is equivalent to saying that there exists a continuous probability measure σ on $\mathbb{T}$ such that $\hat{\sigma}(n_k) \to 1$ as $n_k \to +\infty$. When $n_k | n_{k+1}$ for each $k \geq 0$ such measures do exist (they can be constructed as infinite convolutions of some suitable discrete measures), and one can even control the rate at which the Fourier coefficients $\hat{\sigma}(n_k)$ tend to 1. The following fact, due to J.-P. Kahane and contained in [**11**], is the crux of the proof of Theorem 1.2.

Lemma 4.1. — [**11**] *Let $(n_k)_{k\geq 0}$ be a sequence of integers such that $n_k|n_{k+1}$ for each $k \geq 0$. For any decreasing sequence $(a_k)_{k\geq 0}$ of positive numbers going to zero as k goes to infinity and such that $\sum_{k\geq 0} a_k = +\infty$, there exists a continuous probability measure σ on $\mathbb{T}$ such that*

$$|\hat{\sigma}(n_k) - 1| \leq a_k \quad \textit{for all } k \geq 0.$$

We will also need the following elementary fact:

Fact 4.2. — *There exist two sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ of positive real numbers and a partition of $\mathbb{N}$ in two infinite sets A and B such that the following properties hold true:*

(1) *the sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ decrease to zero, and $0 < a_k, b_k < 1$ for each $k \geq 0$;*
(2) *the series $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$ are divergent;*
(3) *the series $\sum_{k\in A} a_k^{\frac{1}{3}}$ and $\sum_{k\in B} b_k^{\frac{1}{3}}$ are convergent.*

Proof of Fact 4.2. — We construct by induction a very quickly increasing sequence of integers $(p_n)_{n\geq 0}$ and blocks $(a_k)_{k=p_{n-1}+1,\dots,p_n}$ and $(b_k)_{k=p_{n-1}+1,\dots,p_n}$ in such a way that

$$\sum_{k=p_{n-1}+1}^{p_n} a_k \geq \frac{1}{2} \quad \text{and} \quad \sum_{k=p_{n-1}+1}^{p_n} b_k^{\frac{1}{3}} \leq 2^{-n} \quad \text{for each odd integer } n$$

and

$$\sum_{k=p_{n-1}+1}^{p_n} a_k^{\frac{1}{3}} \leq 2^{-n} \quad \text{and} \quad \sum_{k=p_{n-1}+1}^{p_n} b_k \geq \frac{1}{2} \quad \text{for each even integer } n$$

and we set

$$A = \bigcup_{n \text{ even}} \{p_{n-1}+1, \dots, p_n\} \quad \text{and} \quad B = \bigcup_{n \text{ odd}} \{p_{n-1}+1, \dots, p_n\}.$$

We start with $p_0 = 0$, $p_1 = 1$, $a_1 = \frac{1}{2}$ and $b_1 = \frac{1}{8}$. Let now $r \geq 1$, and suppose that $p_1, \dots, p_{2r-1}$ have been constructed, as well as a_k and b_k for $k \leq p_{2r-1}$. We first choose p_{2r} so large that $(p_{2r} - p_{2r-1})b_{p_{2r-1}} \geq \frac{1}{2}$, then set $b_k = b_{p_{2r-1}}$ for $p_{2r-1}+1 \leq k \leq p_{2r}$. Then we choose, for $p_{2r-1}+1 \leq k \leq p_{2r}$, a_k positive and so small that $\sum_{k=p_{2r-1}+1}^{p_{2r}} a_k^{\frac{1}{3}} \leq 2^{-(2r)}$, with $a_{k+1} \leq a_k$ for each $k = p_{2r-1}+1, \dots, p_{2r}-1$. The construction of p_{2r+1} and of the numbers a_k and b_k for $p_{2r}+1 \leq k \leq p_{2r+1}$ is exactly the same, permuting the roles of a_k and b_k. $\square$

We will have to consider separately two cases in the proof of Theorem 1.2: the case where $p \in \mathbb{Z}$ is non-zero and the case where $p = 0$.

4.1. Proof of Theorem 1.2 when $p \in \mathbb{Z} \setminus \{0\}$. — Let $(a_k)_{k\geq 0}$, $(b_k)_{k\geq 0}$, A and B be given by Fact 4.2. Since $n_k | n_{k+1}$ for each $k \geq 0$ and the two series $\sum a_k$ and $\sum b_k$ are divergent, there exist by Lemma 4.1 two continuous probability measures σ and τ on $\mathbb{T}$ such that $|\hat{\sigma}(n_k) - 1| \leq a_k$ and $|\hat{\tau}(n_k) - 1| \leq b_k$ for all $k \geq 1$. Also, the two series $\sum_{k\in A} |\hat{\sigma}(n_k) - 1|^{\frac{1}{3}}$ and $\sum_{k\in B} |\hat{\tau}(n_k) - 1|^{\frac{1}{3}}$ are convergent.

Let T_K be the Kalish-type operator on H_K associated to the compact set $K = supp(\sigma)$, and T_L the operator on H_L associated to $L = supp(\tau)$ (where the symbol supp denotes the support of the measure). Let m_σ and m_τ be the two Gaussian measures, on H_K and H_L respectively, defined in Section 2.2: for any $x, y \in H_K$

$$\int_{H_K} \langle x, z\rangle \overline{\langle y, z\rangle} dm_\sigma(z) = \int_{\mathbb{T}} \langle x, E(\lambda)\rangle \overline{\langle y, E(\lambda)\rangle} d\sigma(\lambda)$$

and for any $u, v \in H_L$

$$\int_{H_L} \langle u, w\rangle \overline{\langle v, w\rangle} dm_\tau(w) = \int_{\mathbb{T}} \langle u, E(\lambda)\rangle \overline{\langle v, E(\lambda)\rangle} d\tau(\lambda),$$

where $E(\lambda) = \chi_\lambda$ for each $\lambda \in \mathbb{T}$. The measure m_σ (resp. m_τ) is non-degenerate, invariant with respect to T_K (resp. T_L), and T_K (resp. T_L) is weakly mixing on $(H_K, \mathcal{B}_K, m_\sigma)$ (resp. on $(H_L, \mathcal{B}_L, m_\tau)$). Our aim is now to show the following statement:

Proposition 4.3. — *For any $p \in \mathbb{Z}\setminus\{0\}$, the set $\{n_k - p\ ;\ k \geq 0\} \cap \mathbb{N}$ is a non-recurrence set for the weakly mixing system $T_K \times T_L$ on $(H_K \times H_L, \mathcal{B}_K \times \mathcal{B}_L, m_\sigma \times m_\tau)$.*

Proof. — Let us first observe that for any $x \in H_K$ and any $k \geq 1$,

$$\begin{aligned} \int_{H_K} |\langle x, (T_K^{n_k} - I)z\rangle|^2 dm_\sigma(z) &= \int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 |\langle x, E(\lambda)\rangle|^2 d\sigma(\lambda) \\ &\leq 2\pi\, ||x||^2 \int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 d\sigma(\lambda) \\ &\leq 4\pi\, ||x||^2\, \Re e(1 - \hat{\sigma}(n_k)) \leq 4\pi\, ||x||^2\, a_k, \end{aligned}$$

and in the same way

$$\begin{aligned} \int_{H_L} |\langle u, (T_L^{n_k} - I)w\rangle|^2 dm_\tau(w) &= \int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 |\langle u, E(\lambda)\rangle|^2 d\tau(\lambda) \\ &\leq 2\pi\, ||u||^2 \int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 d\tau(\lambda) \\ &\leq 4\pi\, ||u||^2\, \Re e(1 - \hat{\tau}(n_k)) \leq 4\pi\, ||u||^2\, b_k \end{aligned}$$

for all $u \in H_L$ and all $k \geq 1$. For $x \in H_K$, let us denote by f_x the function in $L^2(H_K, \mathcal{B}_K, m_\sigma)$ defined by $f_x(z) = \langle x, z\rangle$, and, for $u \in H_L$, by g_u the function in $L^2(H_L, \mathcal{B}_L, m_\tau)$ defined by $g_u(w) = \langle u, w\rangle$. We have by the inequalities above

$$(2) \qquad ||f_x \circ T_K^{n_k} - f_x||^2 \leq 4\pi\, ||x||^2\, a_k \quad \text{and} \quad ||g_u \circ T_L^{n_k} - g_u||^2 \leq 4\pi\, ||u||^2\, b_k,$$

and so we obtain that for all $x \in H_K$ and all $u \in H_L$,

$$\sum_{k\in A} ||f_x \circ T_K^{n_k} - f_x||^{\frac{2}{3}}_{L^2(H_K, m_\sigma)} < +\infty \quad \text{and} \quad \sum_{k\in B} ||g_u \circ T_L^{n_k} - g_u||^{\frac{2}{3}}_{L^2(H_L, m_\tau)} < +\infty.$$

We will now need a modification of an argument of [**7**]:

Lemma 4.4. — *Let x_0 be a non-zero vector of H_K, and let a, b, c and d be four real numbers with $a < b$ and $c < d$. Denote by $R_{a,b}^{c,d}$ the open rectangle of $\mathbb{C}$ defined by $R_{a,b}^{c,d} =]a, b[+i]c, d[$. Then the set*

$$U_{a,b}^{c,d} = \{x \in H_K \ ; \ \langle x_0, x\rangle \in R_{a,b}^{c,d}\}$$

is such that $m_\sigma(U_{a,b}^{c,d}) > 0$ and $\sum_{k\in A} m_\sigma(U_{a,b}^{c,d} \triangle T_K^{n_k} U_{a,b}^{c,d})$ is finite.

Proof of Lemma 4.4. — For simplicity, let us write $U_{a,b}^{c,d} = U$. Since U is a non-empty open subset of H_K and m_σ is non-degenerate, $m_\sigma(U) > 0$. Our aim is now to show that there exists a positive constant C such that for all k, $m_\sigma(U \setminus T_K^{-n_k} U) \leq C a_k^{\frac{1}{3}}$. Fix $\varepsilon > 0$, and suppose that x belongs to $U \setminus T_K^{-n_k} U$: we have $a < \Re e\langle x_0, x\rangle < b$ and $c < \Im m\langle x_0, x\rangle < d$, and either

$$\Re e\langle x_0, T_K^{n_k} x\rangle \leq a - \varepsilon \quad \text{or} \quad a - \varepsilon < \Re e\langle x_0, T_K^{n_k} x\rangle \leq a$$

or

$$b \leq \Re e\langle x_0, T_K^{n_k} x\rangle < b + \varepsilon \quad \text{or} \quad b + \varepsilon \leq \Re e\langle x_0, T_K^{n_k} x\rangle$$

or

$$\Im m\langle x_0, T_K^{n_k} x\rangle \leq c - \varepsilon \quad \text{or} \quad c - \varepsilon < \Im m\langle x_0, T_K^{n_k} x\rangle \leq c$$

or

$$d \leq \Im m\langle x_0, T_K^{n_k} x\rangle < d + \varepsilon \quad \text{or} \quad d + \varepsilon \leq \Im m\langle x_0, T_K^{n_k} x\rangle.$$

First observe that

$$\int_{\{x\in U \ ; \ \Re e\langle x_0, T_K^{n_k} x\rangle \leq a-\varepsilon\}} |\Re e\langle x_0, T_K^{n_k} x\rangle - \Re e\langle x_0, x\rangle|^2 dm_\sigma(x)$$

is bigger than

$$\varepsilon^2 \, m_\sigma(\{x \in U \ ; \ \Re e\langle x_0, T_K^{n_k} x\rangle \leq a - \varepsilon\}).$$

Thus we obtain from (2) that for all $k \geq 1$,

$$m_\sigma(\{x \in U \ ; \ \Re e\langle x_0, T_K^{n_k} x\rangle \leq a - \varepsilon\}) \leq \frac{4\pi \, ||x_0||^2 \, a_k}{\varepsilon^2}.$$

In the same way we get that the three quantities $m_\sigma(\{x \in U \ ; \ \Re e\langle x_0, T_K^{n_k} x\rangle \geq b - \varepsilon\})$, $m_\sigma(\{x \in U \ ; \ \Im m\langle x_0, T_K^{n_k} x\rangle \leq c - \varepsilon\})$ and $m_\sigma(\{x \in U \ ; \ \Im m\langle x_0, T_K^{n_k} x\rangle \geq d - \varepsilon\})$ are all smaller than $4\pi||x_0||^2 a_k \varepsilon^{-2}$. Now since f_{x_0} has complex Gaussian distribution, $\Re e f_{x_0}$ and $\Im m f_{x_0}$ are two real variables having the same Gaussian distribution, and so there exists a positive constant M such that for any $\varepsilon \in (0, 1)$ and any interval I of $\mathbb{R}$ of length ε,

$$m_\sigma(\{x \in H_K \ ; \ \Re e\langle x_0, x\rangle \in I\}) \leq M \, \varepsilon$$

and

$$m_\sigma(\{x \in H_K \ ; \ \Im m\langle x_0, x\rangle \in I\}) \leq M \, \varepsilon.$$

Since the measure σ is T_K-invariant,

$$m_\sigma(\{x \in H_K \ ; \ a-\varepsilon < \Re e\langle x_0, T_K^{n_k} x\rangle \leq a\}) = m_\sigma(\{x \in H_K \ ; \ a-\varepsilon < \Re e\langle x_0, x\rangle \leq a\}) \leq M\varepsilon$$

for all $k \geq 1$, and we have similar inequalities with the sets involving b, c and d. We deduce from all this that for all $k \geq 1$ and all $\varepsilon > 0$

$$m_\sigma(\{x \in U \; ; \Re e\langle x_0, T_K^{n_k} x\rangle \leq a\}) \leq \frac{4\pi ||x_0||^2 a_k}{\varepsilon^2} + M\varepsilon,$$

and so that

$$m_\sigma(U \setminus T_K^{n_k} U) \quad \leq \quad 4\left(\frac{4\pi ||x_0||^2 a_k}{\varepsilon^2} + M\varepsilon\right) \leq C\left(\frac{a_k}{\varepsilon^2} + \varepsilon\right)$$

for some numerical constant C. This being true for all $\varepsilon \in (0,1)$, we obtain by taking $\varepsilon = a_k^{\frac{1}{3}}$ that for all $k \geq 1$

$$m_\sigma(U \setminus T_K^{-n_k} U) \leq 2C\, a_k^{\frac{1}{3}}.$$

This implies that $m_\sigma(U \cap T_K^{-n_k} U) \geq m_\sigma(U) - 2C\, a_k^{\frac{1}{3}}$, and since $m_\sigma(T_K^{-n_k} U \triangle U) = 2m_\sigma(U) - 2m_\sigma(U \cap T_K^{-n_k} U)$, we eventually obtain that $m_\sigma(U \triangle T_K^{-n_k} U) = m_\sigma(U \triangle T_K^{n_k} U) \leq 4C\, a_k^{\frac{1}{3}}$. Since the series $\sum_{k\in A} a_k^{\frac{1}{3}}$ is convergent, this yields that

$$\sum_{k\in A} m_\sigma(T_K^{n_k} U \triangle U) < +\infty.$$

Lemma 4.4 is proved. □

Going back to the proof of Proposition 4.3, fix $p \in \mathbb{Z} \setminus \{0\}$ and let $U = U_{a,b}^{c,d}$ be one of the sets defined in Lemma 4.4. The argument runs now exactly as in [**7**].

We know that T_K is invertible, weakly mixing, and so that all its powers are ergodic. It follows that $T_K^p U$ is not contained in U up to a set of measure zero. Hence there exists a Borel subset C_0 of U, with $m_\sigma(C_0) > 0$, such that $m_\sigma(U \cap T_K^p C_0) = 0$. Let us set now, for a certain integer κ to be fixed later on,

$$C = T_K^p C_0 \setminus \bigcup_{k\geq\kappa,\, k\in A} T_K^{-(n_k - p)}(T_K^{n_k} U \triangle U).$$

We have

$$m_\sigma(C) \geq m_\sigma(C_0) - \sum_{k\geq\kappa, k\in A} m_\sigma(T_K^{n_k} U \triangle U).$$

Since the series on the right-hand side is convergent by Lemma 4.1, if we take κ sufficiently large we have that $m_\sigma(C) > 0$. Moreover, since $m_\sigma(U \cap T_K^p C_0) = 0$, C is contained in the complement of U, and $T_K^{n_k - p} C \subseteq T_K^{n_k} C_0 \setminus (T_K^{n_k} U \triangle U) \subseteq U$ for all $k \in A$ with $k \geq \kappa$ because $C_0 \subseteq U$. Hence we get that for all $k \geq \kappa$, $k \in A$, $m_\sigma(C \cap T_K^{n_k - p} C) = 0$. In order to obtain a set C' with positive measure which satisfies $m_\sigma(C' \cap T_K^{n_k - p} C') = 0$ for all $k \in A$, one has to use again the fact that all the powers of T_K are ergodic: if k_0 is the smallest integer such that $n_{k_0} - p \geq 1$, there exists a Borel subset C_{k_0} of C with $m_\sigma(C_{k_0}) > 0$ such that $m_\sigma(C_{k_0} \cap T_K^{n_{k_0} - p} C_{k_0}) = 0$, then a Borel subset C_{k_0+1} of C_{k_0} with $m_\sigma(C_{k_0+1}) > 0$ such that $m_\sigma(C_{k_0+1} \cap T_K^{n_{k_0+1} - p} C_{k_0+1}) = 0$, etc. until we get a Borel subset $C_{\kappa-1} = C'$ of C such that $m_\sigma(C') > 0$ and $m_\sigma(C' \cap T_K^{n_k - p} C') = 0$ for all $k \in A$, $k \geq k_0$.

We apply now exactly the same procedure to the operator T_L: if $u_0 \in H_L$ is a non-zero vector, $a', b', c', d' \in \mathbb{R}$ with $a' < b'$ and $c' < d'$, applying Lemma 4.4 to the set

$$V_{a',b'}^{c',d'} = \{u \in H_L \ ; \ \langle u_0, u\rangle \in R_{a',b'}^{c',d'}\}$$

with T_L in place of T_K and m_τ in place of m_σ we obtain a Borel subset D' of H_L such that $m_\tau(D') > 0$ and $m_\tau(D' \cap T_L^{n_k - p} D') = 0$ for all $k \in B$ with $k \geq k_0$.

It is now not difficult to see that $C' \times D'$ is a set of positive measure which is non-recurrent with respect to the set $\{n_k - p \ ; \ k \geq k_0\} \cap \mathbb{N}$ for the weakly mixing operator $T_K \times T_L$ on $H_K \times H_L$ (indeed, any union of finitely many non-recurrence sets for weakly mixing transformations is in its turn a non-recurrence set for a weakly mixing system). This finishes the proof of Proposition 4.3. □

Theorem 1.2 is proved in the case where p is non-zero.

4.2. Proof of Theorem 1.2 in the case where $p = 0$**.** — The case where p is equal to zero is a bit different, and uses the fact that, since n_k divides n_{k+1} for each k, $(n_k)_{k\geq 0}$ is in particular lacunary. Hence there exists a $\lambda_0 \in \mathbb{T}$ such that for all $k \geq 0$, $|\lambda_0^{n_k} - 1| \geq \delta$. Consider now, as in the proof of Theorem 1.1, the operators $S_K = \lambda_0 T_K$ on H_K and $S_L = \lambda_0 T_L$ on H_L, where T_K and T_L are the operators defined above. The same argument as in the proof of Theorem 1.1 shows that S_K and S_L are weakly mixing with respect to some Gaussian measures on H_K and H_L respectively, but we need to be a bit more precise here: we need to know that the measure m_σ itself is S_K-invariant, and that S_K defines a weakly mixing transformation of $(H_K, \mathcal{B}_K, m_\sigma)$ (and the same thing for the operator S_L on H_L). The fact that m_σ is S_K-invariant is an immediate consequence of the fact that Gaussian measures are rotation-invariant. In order to check that S_K is weakly mixing with respect to m_σ, it suffices to show (see [**3**] or [**5**] for details) that for all $x, y \in H_K$,

$$\frac{1}{N} \sum_{n=1}^{N} \left| \int_{H_K} \langle x, S_K^n z\rangle \overline{\langle y, z\rangle} dm_\sigma(z) \right|^2 \to 0 \quad \text{as } N \to +\infty.$$

But

$$\begin{aligned} \left| \int_{H_K} \langle x, S_K^n z\rangle \overline{\langle y, z\rangle} dm_\sigma(z) \right| &= \left| \int_{H_K} \lambda_0^n \langle x, T_K^n z\rangle \overline{\langle y, z\rangle} dm_\sigma(z) \right| \\ &= \left| \int_{H_K} \langle x, T_K^n z\rangle \overline{\langle y, z\rangle} dm_\sigma(z) \right| \end{aligned}$$

and since T_K is weakly mixing the conclusion follows.

We are now going to prove the following proposition:

Proposition 4.5. — *The set $\{n_k\}$ is a non-recurrence set for the weakly mixing system $S_K \times S_L$ on $(H_K \times H_L, \mathcal{B}_K \times \mathcal{B}_L, m_\sigma \times m_\tau)$.*

Proof. — We have proved in Lemma 4.4 that given a non-zero vector $x_0 \in H_K$ and $a < b$, $c < d$, the set

$$U_{a,b}^{c,d} = \{x \in H_K \ ; \ \Re e\langle x_0, x\rangle \in]a, b[+ i]c, d[\}$$

satisfies

$$\sum_{k\in A} m_\sigma(U^{c,d}_{a,b}\triangle T_K^{n_k}U^{c,d}_{a,b}) < +\infty.$$

Now take $0 < a < b$ and $0 < c < d$ such that for each $k \geq 1$, the subsets $R^{c,d}_{a,b} =]a, b[+i]c, d[$ and $\lambda_0^{n_k} R^{c,d}_{a,b}$ do not intersect. This is possible since $|\lambda_0^{n_k} - 1| \geq \delta$ for each $k \geq 1$. For such a choice of a, b, c and d, the series

$$\sum_{k\in A} m_\sigma(U^{c,d}_{a,b} \cap S_K^{-n_k}U^{c,d}_{a,b})$$

is convergent. Indeed, suppose that x belongs to $U^{c,d}_{a,b} \cap S_K^{-n_k}U^{c,d}_{a,b}$. Then $\langle x_0, \lambda_0^{n_k}T_K^{n_k}x\rangle \in R^{c,d}_{a,b}$, i.e. $\lambda_0^{n_k}\langle x_0, T_K^{n_k}x\rangle \in R^{c,d}_{a,b}$, i.e. $\langle x_0, T_K^{n_k}x\rangle \in \lambda_0^{-n_k} R^{c,d}_{a,b}$. Since $R^{c,d}_{a,b}$ and $\lambda_0^{n_k}R^{c,d}_{a,b}$ do not intersect, it follows that $\langle x_0, T_K^{n_k}x\rangle$ does not belong to $R^{c,d}_{a,b}$, i.e. that x belongs to $U^{c,d}_{a,b} \setminus T_K^{-n_k}U^{c,d}_{a,b}$. As $\sum_{k\in A} m_\sigma(U^{c,d}_{a,b}\triangle T_K^{-n_k}U^{c,d}_{a,b})$ is finite, we obtain that $\sum_{k\in A} m_\sigma(U^{c,d}_{a,b} \cap S_K^{n_k}U^{c,d}_{a,b})$ is finite as well. Setting

$$C = U^{c,d}_{a,b} \setminus \bigcup_{k\geq\kappa, k\in A} U^{c,d}_{a,b} \cap S_K^{n_k}U^{c,d}_{a,b},$$

we obtain if κ is large enough that $m_\sigma(C) > 0$. It is clear that $C \cap S_K^{n_k}C = \varnothing$ for all $k \in A$, $k \geq \kappa$. The same argument as in Proposition 4.3 above shows that there exists a Borel subset C' of C such that $m_\sigma(C') > 0$ and $m_\sigma(C' \cap S_K^{n_k}C') = 0$ for all $k \in A$. In a similar fashion we obtain a Borel subset D' of H_L with $m_\tau(D') > 0$ such that $m_\tau(D' \cap S_L^{n_k}D') = 0$ for all $k \in B$, and we deduce from this that the set $C' \times D'$ is non-recurrent with respect to the set $\{n_k\}$ for the weakly mixing transformation $T_K \times T_L$ of $(H_K \times H_L, \mathcal{B}_K \times \mathcal{B}_L, m_\sigma \times m_\tau)$. This finishes the proof of Proposition 4.5. □

Theorem 1.2 is proved. We thus obtain a positive answer to Question 1.3 in the case where the set $\{n_k\}$ is such that $n_k | n_{k+1}$ for each $k \geq 0$.

4.3. Some remarks. — The divisibility assumption on the n_k's is used in the proof of Theorem 1.1 in two places: first, we need it in order to construct the two measures σ and τ with $|\hat{\sigma}(n_k) - 1| \leq a_k$ and $|\hat{\tau}(n_k) - 1| \leq b_k$ for each $k \geq 1$, and, second, we deduce from it that the sequence $(n_k)_{k\geq 0}$ is lacunary. It is not difficult to see that the proof yields in fact the following more general result:

Theorem 4.6. — *Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence having the following property:*

> *for each sequence $(a_k)_{k\geq 0}$ of positive numbers decreasing to zero, there exists a continuous measure σ on the unit circle such that $|\hat{\sigma}(n_k) - 1| \leq a_k$ for all k.*

Then for each $p \in \mathbb{Z} \setminus \{0\}$, the set $\{n_k - p \,;\, k \geq 0\} \cap \mathbb{N}$ is a non-recurrence set for some weakly mixing linear dynamical system. If moreover $\{n_k\}$ is non-recurrent with respect to some rotation on $\mathbb{T}$ (in particular if it is lacunary), then the set $\{n_k\}$ itself is a non-recurrence set for some weakly mixing linear dynamical system.

Of course, any sequence $(n_k)_{k\geq 0}$ satisfying the assumption of Theorem 4.6 is a rigidity sequence. It is natural to wonder whether it is true that any rigidity sequence satisfies this assumption. This is not the case, as shown below.

Example 4.7. — Let $(q_k)_{k\geq 0}$ be a sequence of integers tending to infinity. Consider the sequence $(n_k)_{k\geq 0}$ defined by $n_0 = 1$ and $n_{k+1} = q_k n_k + 1$ for all $k \geq 0$. Since $\frac{n_{k+1}}{n_k}$ tends to infinity, $(n_k)_{k\geq 0}$ is a rigidity sequence. But if σ is a continuous measure on $\mathbb{T}$ such that $\hat{\sigma}(n_k)$ tends to 1, then $|\hat{\sigma}(n_k) - 1| > \frac{1}{q_k^4}$ for infinitely many integers k. Indeed, suppose that $|\hat{\sigma}(n_k) - 1| \leq \frac{1}{q_k^4}$ for all k's except finitely many. Then we have for all k sufficiently large

$$\begin{aligned}
\int_{\mathbb{T}} |\lambda - 1| d\sigma(\lambda) &= \int_{\mathbb{T}} |\lambda^{n_{k+1}} - \lambda^{q_k n_k}| d\sigma(\lambda) \\
&\leq \left(\int_{\mathbb{T}} |\lambda^{n_{k+1}} - 1|^2 d\sigma(\lambda)\right)^{\frac{1}{2}} + q_k \left(\int_{\mathbb{T}} |\lambda^{n_k} - 1|^2 d\sigma(\lambda)\right)^{\frac{1}{2}} \\
&\leq (2\Re e(1 - \hat{\sigma}(n_{k+1})))^{\frac{1}{2}} + q_k\ (2\Re e(1 - \hat{\sigma}(n_k)))^{\frac{1}{2}} \\
&\leq 2\sqrt{2}\,(\frac{1}{q_{k+1}^2} + \frac{1}{q_k})\cdot
\end{aligned}$$

Letting k go to infinity, we obtain that $\hat{\sigma}(1) = 1$, which is impossible since σ is supposed to be continuous. Hence $|\hat{\sigma}(n_k) - 1| > \frac{1}{q_k^4}$ for infinitely many k's. If $(q_k)_{k\geq 0}$ goes to infinity extremely slowly, we thus see that any "rigidity measure" associated to the sequence $(n_k)_{k\geq 0}$ has Fourier coefficients going to 1 along some sub-sequence of the sequence $(n_k)_{k\geq 0}$ slower than any prescribed rate.

5. Proof of Theorem 1.4

Let us begin by recalling briefly here some of the results of [**20**]. If $r \geq 1$ is an integer, sets $\{n_k^{(r)}\}$ are constructed which have the property of being recurrent in the topological sense for all products of r rotations on $\mathbb{T}$. This means for all $(\lambda_1, \ldots, \lambda_r) \in \mathbb{T}^r$ and all $\varepsilon > 0$, there exists an integer $k \geq 0$ such that

$$\max_{i=1,\ldots,r} |\lambda_i^{n_k^{(r)}} - 1| < \varepsilon.$$

But these sets are non-recurrent for some dynamical system on some compact space, namely for some suitable product of $2^{r-1} + 1$ rotations on $\mathbb{T}$: there exist $\mu_0, \ldots, \mu_{2^{r-1}} \in \mathbb{T}$ and $\delta > 0$ such that for all $k \geq 0$,

$$\min_{j=0,\ldots,2^{r-1}} |\mu_j^{n_k^{(r)}} - 1| > \delta.$$

The sets $\{n_k^{(r)}\}$ have the following form: $\{n_k^{(r)}\} = \{n_{k,0}^{(r)}\} \cup \bigcup_{A\subseteq\{1,\ldots,r-1\}} \{n_{k,A}^{(r)}\}$, with

$$\{n_{k,0}^{(r)}\} = \bigcup_{N\geq 1} \{H_N q + 1\ ;\ 1 \leq q \leq Q_N^{(r)}\} := \bigcup_{N\geq 1} B_{N,0}^{(r)}$$

and

$$\{n_{k,A}^{(r)}\} = \bigcup_{N\geq 1} \{H_N \Delta_{N,A}^{(r)}(L_N j + 1)\ ;\ 1 \leq j \leq \Theta_N^{(r)}\} := \bigcup_{N\geq 1} B_{N,A}^{(r)}$$

for $A \subseteq \{1, \ldots, r-1\}$, $A \neq \varnothing$, and

$$\{n_{k,\varnothing}^{(r)}\} = \bigcup_{N\geq 1} \{H_N \Delta_{N,\varnothing}^{(r)}\} := \bigcup_{N\geq 1} B_{N,\varnothing}^{(r)}.$$

Here $(L_N)_{N\geq 1}$ is any rapidly increasing sequence of integers, the sequences $(\Delta_{N,A}^{(r)})_{N\geq 1}$, $A \subseteq \{1, \ldots, r-1\}$, $(\Theta_N^{(r)})_{N\geq 1}$ and $(Q_N^{(r)})_{N\geq 1}$ depend from the sequence $(L_N)_{N\geq 1}$, and the sequence $(H_N)_{N\geq 1}$ in an extremely rapidly increasing sequence of integers independent from all the other parameters: none of the sequences $(L_N)_{N\geq 1}$, $(\Delta_{N,A}^{(r)})_{N\geq 1}$, $(\Theta_N^{(r)})_{N\geq 1}$ or $(Q_N^{(r)})_{N\geq 1}$ depend from $(H_N)_{N\geq 1}$. As explained in [**20**], for a fixed $N \geq 1$ all the blocks $B_{N,0}^{(r)}$ and $B_{N,A}^{(r)}$ are necessarily intertwined, but for different N's they are very far away one from another: if at step $N+1$ the integer H_{N+1} is chosen extremely large with respect to all the quantities appearing at step N, any block $B_{N+1,0}^{(r)}$ or $B_{N+1,A}^{(r)}$ is very far away from any block $B_{N,0}^{(r)}$ or $B_{N,A}^{(r)}$.

Our aim is now to show that if the parameters L_N and H_N are suitably chosen at each step N, the set $\{n_k^{(r)}\}$ (which is recurrent for all products of r rotations) is non-recurrent for some weakly mixing linear dynamical system on a Hilbert space. As we have seen in Subsections 4.1 and 4.2, any finite union of non-recurrence sets for weakly mixing linear dynamical systems is again a non-recurrence set for some weakly mixing linear dynamical system. So it suffices to construct operators T_0 and T_A, $A \subseteq \{1, \ldots, r-1\}$ on Hilbert spaces such that $\{n_{k,0}^{(r)}\}$ (resp. $\{n_{k,A}^{(r)}\}$) is a non-recurrence set for T_0 (resp T_A). But this follows easily from Theorem 1.1. The first ingredient is the following easy lemma:

Lemma 5.1. — *Whatever the choice of L_N at step N and the values of $\Delta_{N,A}^{(r)}$ and $\Theta_N^{(r)}$, provided that the sequence $(H_N)_{N\geq 1}$ grows sufficiently fast the sets*

$$\bigcup_{N\geq 1} \{H_N q\ ;\ 1 \leq q \leq Q_N^{(r)}\}$$

and

$$\bigcup_{N\geq 1} \{H_N \Delta_{N,A}^{(r)} L_N j\ ;\ 1 \leq j \leq \Theta_N^{(r)}\}, \quad A \subseteq \{1, \ldots, r-1\},\ A \neq \varnothing$$

and

$$\bigcup_{N\geq 1} \{H_N \Delta_{N,\varnothing}^{(r)}\}$$

generate non-Jamison sequences.

Proof of Lemma 5.1. — By Lemma 2.4 of [**20**], there exists a perfect subset K of $\mathbb{T}$, with $1 \in K$, such that for each $\lambda \in K$ and each $N \geq 1$, $|\lambda^{H_N} - 1| \leq M\, \frac{H_N}{H_{N+1}}$, where M is a numerical constant. Hence

$$|\lambda^{H_N q} - 1| \leq M\, \frac{H_N Q_N^{(r)}}{H_{N+1}} < 2^{-N} \quad \text{for all } q \text{ with } 1 \leq q \leq Q_N^{(r)}$$

if H_{N+1} is sufficiently large. This holds true for all $\lambda \in K$ and $N \geq 1$. Let now $\varepsilon > 0$. If N_0 is such that $2^{-N_0} < \varepsilon$, then $|\lambda^{H_N q} - 1| < \varepsilon$ for all $\lambda \in K$, $N \geq N_0$ and $1 \leq q \leq Q_N^{(r)}$. Since the set K is perfect and contains the point 1, one can find $\lambda \in K$ as close to 1 as we wish, but not equal to 1. Hence there exists a $\lambda \in K$, $\lambda \neq 1$, such that $|\lambda^{H_N q} - 1| < \varepsilon$ for all $N \geq 1$ and $1 \leq q \leq Q_N^{(r)}$. This shows that the set $\bigcup_{N \geq 1} \{H_N q \ ; \ 1 \leq q \leq Q_N^{(r)}\}$ generates a Jamison sequence. The proof is exactly the same for the other sets, using that H_{N+1} can be chosen much larger than $H_N \Delta_{N,A}^{(r)} L_N \Theta_N^{(r)}$ for each subset A of $\{1, \ldots, r-1\}$. □

The second fact we need is that if the sequences $(L_N)_{N \geq 1}$ and $(H_N)_{N \geq 1}$ grows sufficiently fast, there exist elements λ_0 and λ_A of the unit circle, $A \subseteq \{1, \ldots, r-1\}$, such that

$$|\lambda_0^{n_{k,0}^{(r)}} - 1| > \frac{1}{2} \quad \text{and} \quad |\lambda_A^{n_{k,A}^{(r)}} - 1| > \frac{1}{2}$$

for each k. See Proposition 4.5 of [**20**] for the proof, which is very similar to the proof of Lemma 5.1 above. Hence we can apply Theorem 1.1, and we obtain that the sets $\{n_{k,0}^{(r)}\}$ and $\{n_{k,A}^{(r)}\}$ are non-recurrence sets for some weakly mixing linear dynamical systems T_0 and T_A, respectively, acting on a Hilbert space. This finishes the proof of Theorem 1.4.

6. Some further non-recurrent lacunary sets

In this subsection we expand on a result of [**7**], where the authors show that the Chacon transformation is non-recurrent with respect to a certain lacunary set. Our Theorem 6.1 is a mild generalization of this, and is inspired by the proof of Proposition 3.10 in [**7**].

Theorem 6.1. — *For each strictly increasing sequence $(n_k)_{k \geq 0}$ of integers, write n_{k+1} as $n_{k+1} = p_k n_k + r_k$, with p_k and r_k non-negative integers. If $p_k \geq 3$ for each k and if the series $\sum_{k \geq 1} \frac{r_k}{p_k n_k}$ is convergent, then the set $\{n_k - 1 \ ; \ k \geq 1\}$ is a non-recurrence set for some weakly mixing transformation.*

We can retrieve from this the examples of [**7**]: we can always write n_{k+1} as $n_{k+1} = p_k n_k + r_k$ with $0 \leq r_k < n_k$. If the series $\sum_{k \geq 1} \frac{n_k}{n_{k+1}}$ is convergent, then $\sum_{k \geq 1} \frac{r_k}{p_k n_k}$ is obviously convergent as well, and so the set $\{n_k - 1\}$ is a non-recurrence set for some weakly mixing transformation. And if (n_k) is the sequence defined by $n_0 = 1$ and $n_{k+1} = 3n_k + 1$ for each $k \geq 0$, then $n_k = 1 + 3 + \ldots + 3^k$ for $k \geq 1$. Since the series $\sum_{k \geq 1} \frac{r_k}{p_k n_k}$ is convergent, the set $\{n_k - 1\} = \{\frac{3^{k+1} - 1}{2} - 1\}$ is a non-recurrence set for a certain weakly mixing rank-one transformation (and the proof of Theorem 6.1 shows that this is the Chacon transformation).

Proof. — Observe first that if there exists a k_0 such that $r_k = 0$ for all $k \geq k_0$, the conclusion of Theorem 6.1 follows from Theorem 1.2, since in this case n_k divides n_{k+1} for each $k \geq k_0$. Hence we can suppose without loss of generality that $r_k \geq 1$ for infinitely many k's. The proof uses the standard cutting and stacking method. Let us denote by $\mathcal{I}_k$ the tower of height h_k which we have at step k of the construction. At step $k+1$, we cut this tower into p_k sub-towers $\mathcal{I}_{k,1}, \ldots, \mathcal{I}_{k,p_k}$ having a basis which is an interval of length $\frac{1}{p_k}$ times the length of the basis of $\mathcal{I}_k$. If $r_k \geq 1$, set $a_k = \lfloor \frac{p_k}{3} \rfloor$, and stack on top of each other, and in this order, the towers $\mathcal{I}_{k,1}, \ldots, \mathcal{I}_{k,a_k}$, one spacer, $\mathcal{I}_{k,a_k+1}, \ldots, \mathcal{I}_{k,p_k}$, and then

$r_k - 1$ spacers. If $r_k = 0$, we simply stack the p_k towers $\mathcal{I}_{k,1}, \dots, \mathcal{I}_{k,p_k}$ on top of each other. So if we start with a tower $\mathcal{I}_1$ of height $n_1 \geq 3$ with a basis which is an interval of length l_1, the k^{th} tower $\mathcal{I}_k$ has height n_k and a basis which is an interval of length $l_k = \frac{l_{k-1}}{p_{k-1}}$. Since the series $\sum_{k\geq 1} \frac{r_k}{p_k n_k}$ is convergent, with a suitable choice of l_1 we can ensure that this defines a measure-preserving transformation T of the interval $[0,1]$ (with respect to the Lebesgue measure). It is clear that T is ergodic, and the usual argument shows that T is weakly mixing: suppose that $f \in L^2([0,1])$ is an eigenfunction of the Koopman operator U_T, associated to an eigenvalue $\lambda \in \mathbb{T}$: $f(Tx) = \lambda f(x)$ a.e. on $[0,1]$, hence for almost every $x \in [0,1]$ we have for each $n \geq 1$ $f(T^n x) = \lambda^n f(x)$. As T is ergodic, if f is non-zero we can suppose without loss of generality that $|f| = 1$ a.e.. Fix $\varepsilon > 0$. Since $r_k \geq 1$ for infinitely many k's, we can find an integer k such that $r_k \geq 1$ and a function $g \in L^2([0,1])$ with the following two properties: $||f - g||_2 < \varepsilon$, and g is constant on each level of the tower $\mathcal{I}_k$. If τ denotes one of the levels of the towers $\mathcal{I}_{k,1}, \dots, \mathcal{I}_{k,a_k-1}$ (appearing at the bottom of $\mathcal{I}_{k+1}$), one sees easily that g has the same value on the level τ and on the level $T^{a_k n_k+1}\tau$. So if E denotes the set which is the union of all these levels τ, we have

$$\left(\int_E |f(T^{a_k n_k+1}x) - f(x)|^2 dx\right)^{\frac{1}{2}} \leq 2\varepsilon.$$

Hence

$$|\lambda^{a_k n_k+1} - 1| \left(\int_E |f(x)|^2\right)^{\frac{1}{2}} \leq 2\varepsilon.$$

Since $|f| = 1$ a.e., we get that $|\lambda^{a_k n_k+1} - 1|\sqrt{m(E)} \leq 2\varepsilon$. Now, as the measure of E is bigger than $\frac{1}{4}$, this yields that $|\lambda^{a_k n_k+1} - 1| \leq 8\varepsilon$. If τ' denotes now one of the levels of the towers $\mathcal{I}_{k,a_k+1}, \dots, \mathcal{I}_{k,2a_k}$ (appearing just after the first added spacer in the tower $\mathcal{I}_{k+1}$), the same argument shows that g has the same value on the level τ' and on the level $T^{a_k n_k}\tau'$, and so if E' denote the union of all these levels τ' we get that $|\lambda^{a_k n_k} - 1|\sqrt{m(E')} \leq 2\varepsilon$, from which it follows that $|\lambda^{a_k n_k} - 1| \leq 8\varepsilon$. Hence $|\lambda - 1| \leq 16\varepsilon$ for each $\varepsilon > 0$, so $\lambda = 1$ and T is weakly mixing.

Let us now prove that $\{n_k - 1\}$ is a non-recurrence set for T. Let A denote the first added spacer (in the construction of $\mathcal{I}_2$). It is contained in any of the towers $\mathcal{I}_k$, $k \geq 3$. To visualize the action of T^{n_k-1} on A, suppose that this spacer A is painted red. Let τ be a red level of the tower $\mathcal{I}_{k+1}$: if τ is a level of one of the sub-towers $\mathcal{I}_{k,j}$, $j \in \{1, \dots, a_k - 1, a_k + 1, \dots, p_k - 1\}$, then $T^{n_k}\tau$ is also a red level, and so $T^{n_k-1}\tau$ cannot be red. If τ is a red level of $\mathcal{I}_{k,a_k}$, T^{n_k+1} maps τ on a red level, and the two levels below are not red, so $T^{n_k-1}\tau$ cannot be red either. Lastly, we have to consider the case where τ is a red level of $\mathcal{I}_{k,p_k}$: here in general T^{n_k-1} does not map τ onto a level of $\mathcal{I}_{k+1}$, and one has to go over to the towers $\mathcal{I}_{k+2}$, $\mathcal{I}_{k+3}$, etc. in order to see the action of T^{n_k-1} on τ. Here is may happen that T^{n_k-1} maps some piece of τ on some red level of a tower $\mathcal{I}_{k+2}$, $\mathcal{I}_{k+3}$, etc. What we eventually get is that for each $k \geq 3$, $T^{n_k-1}(A \setminus \mathcal{I}_{k,p_k}) \cap A = \varnothing$. Set

$$C = A \setminus \bigcup_{k\geq\kappa} \mathcal{I}_{k,p_k},$$

where κ is a sufficiently large integer. For each $k \geq 3$, the measure of $\mathcal{I}_{k,p_k}$ is less than $\frac{1}{p_k n_k}$ and, since the series $\sum_k \frac{1}{p_k n_k}$ is convergent, the set C has positive measure if κ is large enough. Moreover, $T^{n_k-1}C$ is contained in $T^{n_k-1}(A \setminus \mathcal{I}_{k,p_k})$ for each $k \geq \kappa$, and the set $T^{n_k-1}(A \setminus \mathcal{I}_{k,p_k})$ does not intersect A. Hence it does not intersect C either, and $T^{n_k-1}C \cap C = \varnothing$ for all $k \geq \kappa$. Then the same argument as in the proof of Proposition 4.3 shows that there exists a Borel subset C' of C of positive measure with the property that $m(T^{n_k-1}C' \cap C') = 0$ for each $k \geq 1$. This proves that $\{n_k - 1\}$ is a non-recurrence set for T. □

As a corollary to Theorem 6.1 we obtain:

Corollary 6.2. — *Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of integers such that, if we write n_{k+1} as $n_{k+1} = p_k n_k + r_k$, $p_k \geq 3$ and $0 \leq r_k < n_k$, the series $\sum_{k\geq 1} \frac{r_k}{p_k n_k}$ is convergent. Then for each $p \geq 1$ the set $\{n_k - p\} \cap \mathbb{N}$ is a non-recurrence set for some weakly mixing dynamical system.*

Proof. — For each $k \geq 1$ we have $n_{k+1} - p + 1 = p_k(n_k - p + 1) + r_k + p_k(p-1)$, with $r_k + p_k(p-1) \geq 0$. Now, since $n_{k+1} \geq 3n_k$ for each k, the series $\sum \frac{1}{n_k}$ is convergent, and it follows that the series

$$\sum_{k\geq 0} \frac{r_k + p_k(p-1)}{p_k(n_k - p + 1)}$$

is convergent. Applying Theorem 6.1, we can construct a weakly mixing system for which the set $\{n_k - p\} \cap \mathbb{N}$ is non-recurrent. □

It would be interesting to know whether Theorem 6.1 and Corollary 6.2 remain true when r_k is not supposed to be non-negative, but when one supposes only that the series $\sum_{k\geq 0} \frac{|r_k|}{p_k n_k}$ is convergent.

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