IP-DIRICHLET MEASURES AND IP-RIGID DYNAMICAL SYSTEMS: AN APPROACH VIA GENERALIZED RIESZ PRODUCTS

by

Sophie Grivaux

Abstract. — If $(n_k)_{k\geq 1}$ is a strictly increasing sequence of integers, a continuous probability measure σ on the unit circle $\mathbb{T}$ is said to be IP-Dirichlet with respect to $(n_k)_{k\geq 1}$ if $\hat{\sigma}(\sum_{k\in F} n_k) \to 1$ as F runs over all non-empty finite subsets F of $\mathbb{N}$ and the minimum of F tends to infinity. IP-Dirichlet measures and their connections with IP-rigid dynamical systems have been investigated recently by Aaronson, Hosseini and Lemańczyk. We simplify and generalize some of their results, using an approach involving generalized Riesz products.

1. Introduction

We will be interested in this paper in IP-Dirichlet probability measures on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} \ ; |\lambda| = 1\}$ with respect to a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers. Recall that a probability measure μ on $\mathbb{T}$ is said to be a *Dirichlet measure* when there exists a strictly increasing sequence $(p_k)_{k\geq 1}$ of integers such that the monomials z^{p_k} tend to 1 on $\mathbb{T}$ as k tends to infinity with respect to the norm of $L^p(\mu)$, where $1 \leq p < +\infty$. This is equivalent to requiring that the Fourier coefficients $\hat{\mu}(p_k)$ of the measure μ tend to 1 as k tends to infinity. If $(n_k)_{k\geq 1}$ is a (fixed) strictly increasing sequence of integers, we say that μ *is a Dirichlet measure with respect to the sequence* $(n_k)_{k\geq 1}$ if $\hat{\mu}(n_k) \to 1$ as $k \to +\infty$. Let $\mathcal{F}$ denote the set of all non-empty finite subsets of $\mathbb{N}$. The measure μ is said to be *IP-Dirichlet with respect to the sequence* $(n_k)_{k\geq 1}$ if

$$\hat{\mu}(\sum_{k\in F} n_k) \to 1 \qquad \text{as } \min(F) \to +\infty,\ F \in \mathcal{F}.$$

In other words: for all $\varepsilon > 0$ there exists a $k_0 \geq 0$ such that whenever F is a finite subset of $\{k_0, k_0+1, \ldots\}$,

$$|\hat{\mu}(\sum_{k\in F} n_k) - 1| \leq \varepsilon.$$

Our starting point for this paper is the work [**1**] by Aaronson, Hosseini and Lemańczyk, where IP-Dirichlet measures are studied in connection with rigidity phenomena for dynamical systems. Let $(X, \mathcal{B}, m)$ denote a standard non-atomic probability space and let

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T be a measure-preserving transformation of $(X, \mathcal{B}, m)$. Let again $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers.

Definition 1.1. — The transformation T is said to be *rigid with respect to* $(n_k)_{k\geq 1}$ if $m(T^{-n_k}A\triangle A) \to 0$ as $n_k \to +\infty$ for all sets $A \in \mathcal{B}$, or, equivalently, if for all functions $f \in L^2(X, \mathcal{B}, m)$, $||f \circ T^{n_k} - f||_{L^2(X,\mathcal{B},m)} \to 0$ as $k \to +\infty$.

Denote by σ_T the restricted spectral type of T, i.e. the spectral type of the Koopman operator U_T of T restricted to the space $L^2_0(X, \mathcal{B}, m)$ of functions of $L^2(X, \mathcal{B}, m)$ of mean zero (recall that $U_T f = f \circ T$ for every $f \in L^2(X, \mathcal{B}, m)$). Then it is not difficult to see that T is rigid with respect to $(n_k)_{k\geq 1}$ if and only if σ_T is a Dirichlet measure with respect to the sequence $(n_k)_{k\geq 1}$.

Rigidity phenomena for weakly mixing transformations have been investigated recently in the papers [**3**] and [**5**], where in particular the following question was considered: given a sequence $(n_k)_{k\geq 1}$ of integers, when is it true that there exists a weakly mixing transformation T of some probability space $(X, \mathcal{B}, m)$ which is rigid with respect to $(n_k)_{k\geq 1}$? When this is true, we say that $(n_k)_{k\geq 1}$ is a *rigidity sequence*. It was proved in [**3**] and [**5**] that $(n_k)_{k\geq 1}$ is a rigidity sequence if and only if there exists a continuous probability measure σ on $\mathbb{T}$ which is Dirichlet with respect to $(n_k)_{k\geq 1}$.

It is then natural to consider IP-rigidity for (weakly mixing) dynamical systems. This study was initiated in [**3**] and continued in [**1**].

Definition 1.2. — The system $(X, \mathcal{B}, m; T)$ is said to be *IP-rigid with respect to the sequence* $(n_k)_{k\geq 1}$ if for every $A \in \mathcal{B}$,

$$m(T^{\sum_{k\in F} n_k}A\triangle A) \to 0 \quad \text{ as } \min(F) \to +\infty,\ F \in \mathcal{F}.$$

Just as with the notion of rigidity, T is IP-rigid with respect to $(n_k)_{k\geq 1}$ if and only if σ_T is an IP-Dirichlet measure with respect to $(n_k)_{k\geq 1}$. Moreover, if we say that $(n_k)_{k\geq 1}$ is an *IP-rigidity sequence* when there exists a weakly mixing dynamical system $(X, \mathcal{B}, m; T)$ which is IP-rigid with respect to $(n_k)_{k\geq 1}$, then IP-rigidity sequences can be characterized in a similar fashion as rigidity sequences ([**1**, Prop. 1.2]): $(n_k)_{k\geq 1}$ is an IP-rigidity sequence if and only if there exists a continuous probability measure σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$.

IP-Dirichlet measures are studied in detail in the paper [**1**], and one of the important features which is highlighted there is the connection between the existence of a measure which is IP-Dirichlet with respect to a certain sequence $(n_k)_{k\geq 1}$ of integers, and the properties of the subgroups $G_p((n_k))$ of the unit circle associated to $(n_k)_{k\geq 1}$: for $1 \leq p < +\infty$,

$$G_p((n_k)) = \{\lambda \in \mathbb{T} \ ; \ \sum_{k\geq 1} |\lambda^{n_k} - 1|^p < +\infty\}$$

and for $p = +\infty$

$$G_\infty((n_k)) = \{\lambda \in \mathbb{T} \ ; \ |\lambda^{n_k} - 1| \to 0 \text{ as } k \to +\infty\}.$$

The main result of [**1**] runs as follows:

Theorem 1.3. — [**1**, Th. 2] *Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers. If μ is a probability measure on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$, then $\mu(G_2((n_k))) = 1$.*

The converse of Theorem 1.3 is false [**1**, Ex. 4.2], as one can construct a sequence $(n_k)_{k\geq 1}$ and a probability measure μ on $\mathbb{T}$ which is continuous, supported on $G_2((n_k))$ (which is uncountable), and not IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. On the other hand, if μ is a continuous probability measure such that $\mu(G_1((n_k))) = 1$, then μ is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$ [**1**, Prop. 1]. Again, this is not a necessary and sufficient condition for being IP-Dirichlet with respect to $(n_k)_{k\geq 1}$ [**1**]: if $(n_k)_{k\geq 1}$ is the sequence of integers defined by $n_1 = 1$ and $n_{k+1} = kn_k + 1$ for each $k \geq 1$, then there exists a continuous probability measure σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$, although $G_1((n_k)) = \{1\}$. Numerous examples of sequences $(n_k)_{k\geq 1}$ with respect to which there exist IP-Dirichlet continuous probability measures are given in [**1**] as well. For instance, such sequences are characterized among sequences $(n_k)_{k\geq 1}$ such that n_k divides n_{k+1} for each k, and among sequences which are denominators of the best rational approximants $\frac{p_k}{q_k}$ of an irrational number $\alpha \in (0, 1)$, obtained via the continued fraction expansion. It is also proved in [**1**] that sequences $(n_k)_{k\geq 1}$ such that the series $\sum_{k\geq 1}(n_k/n_{k+1})^2$ is convergent admit a continuous IP-Dirichlet probability measure.

Our aim in this paper is to simplify and generalize some of the results and examples of [**1**]. We first present an alternative proof of Theorem 1.3 above, which is completely elementary and much simpler than the proof of [**1**] which involves Mackey ranges over the dyadic adding machine. We then present a rather general way to construct IP-Dirichlet measures via generalized Riesz products. The argument which we use is inspired by results from [**10**] and [**8**, Section 4.2], where generalized Riesz products concentrated on some H_2-subgroups of the unit circle are constructed. Proposition 3.1 gives a bound from below on the Fourier coefficients of these Riesz products, and this enables us to obtain in Proposition 4.1 a sufficient condition on sets $\{n_k\}$ of the form

$$\{n_k\} = \bigcup_{k\geq 1} \{p_k, q_{1\,k}\, p_k, \ldots, q_{r_k,\,k}\, p_k\}, \tag{1}$$

where the $q_{j,k}$, $j = 1, \ldots r_k$, are positive integers and the sequence $(p_k)_{k\geq 1}$ is such that $p_{k+1} > q_{r_k,k}p_k$ for each $k \geq 1$, for the existence of an associated continuous generalized Riesz product which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. This condition is best possible (Proposition 4.2). As a consequence of Proposition 4.1, we retrieve and improve a result of [**1**] which runs as follows: if $(n_k)_{k\geq 1}$ is such that there exists an infinite subset S of $\mathbb{N}$ such that

$$\sum_{k\in S} \frac{n_k}{n_{k+1}} < +\infty \quad \text{and } n_k | n_{k+1} \text{ for each } k \notin S,$$

then there exists a continuous probability measure σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. This result is proved in [**1**] by constructing a rank-one weakly mixing system which is IP-rigid with respect to $(n_k)_{k\geq 1}$. Here we get a direct proof of this statement, where the condition $\sum_{k\in S}(n_k/n_{k+1}) < +\infty$ is replaced by the weaker condition $\sum_{k\in S}(n_k/n_{k+1})^2 < +\infty$.

Theorem 1.4. — *Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers for which there exists an infinite subset S of $\mathbb{N}$ such that*

$$\sum_{k\in S}\Big(\frac{n_k}{n_{k+1}}\Big)^2<+\infty \quad \text{and } n_k|n_{k+1} \text{ for each } k\notin S.$$

Then there exists a continuous generalized Riesz product σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$.

Using again sets of the form (1), we then show that the converse of Theorem 1.3 is false in the strongest possible sense, thus strengthening Example 4.2 of [**1**]:

Theorem 1.5. — *There exists a strictly increasing sequence $(n_k)_{k\geq 1}$ of integers such that $G_2((n_k))$ is uncountable, but no continuous probability measure is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$.*

The last section of the paper gathers some observations concerning the Erdös-Taylor sequence $(n_k)_{k\geq 1}$ defined by $n_1=1$ and $n_{k+1}=kn_k+1$, which is of interest in this context, as well as a generalization of Proposition 3.1 which shows that under the assumptions on the sequence $(n_k)_{k\geq 1}$ of either Corollary 3.2 or Theorem 1.4, there exist uncountably many dynamical systems which are weakly mixing and IP-rigid with respect to $(n_k)_{k\geq 1}$, and which have pairwise disjoint restricted maximal spectral types (Corollary 6.2).

Notation: In the whole paper, we will denote by $\{x\}$ the distance of the real number x to the nearest integer, by $\lfloor x\rceil$ the integer which is closest to x (if there are two such integers, we take the smallest one), and by $\langle x\rangle$ the quantity $x-\lfloor x\rceil$. Lastly, we denote by $[x]$ the integer part of x.

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2. An alternative proof of Theorem 1.3

Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers. Suppose that the measure μ on $\mathbb{T}$ is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. For every $\varepsilon>0$ there exists an integer k_0 such that for all sets $F\in\mathcal{F}$ with $\min(F)\geq k_0$, $\big|\widehat{\mu}\,\big(\sum_{k\in F}n_k\big)-1\big|\leq\varepsilon$. For every integer $N\geq k_0$, consider the quantities

$$\prod_{k=k_0}^{N}\frac{1}{2}\,(1+\lambda^{n_k})=2^{-(N-k_0+1)}\sum_{F\subseteq\{k_0,\dots,N\}}\lambda^{\sum_{k\in F}n_k}.$$

The notation on the righthand side of this display means that the sum is taken over all (possibly empty) finite subsets F of $\{k_0,\dots,N\}$. Integrating with respect to μ yields that

$$\int_{\mathbb{T}}\prod_{k=k_0}^{N}\frac{1}{2}\,(1+\lambda^{n_k})\,d\mu(\lambda)=2^{-(N-k_0+1)}\sum_{F\subseteq\{k_0,\dots,N\}}\widehat{\mu}\big(\sum_{k\in F}n_k\big),$$

so that

$$\Big|\int_{\mathbb{T}} \prod_{k=k_0}^{N} \frac{1}{2}\,(1+\lambda^{n_k})\,d\mu(\lambda)-1\Big| \le 2^{-(N-k_0+1)} \sum_{F\subseteq\{k_0,\dots,N\}} \Big|\widehat{\mu}\big(\sum_{k\in F} n_k\big)-1\Big| \le \varepsilon. \tag{2}$$

Let now

$$C=\Big\{\lambda\in\mathbb{T}\ \ ;\text{ the infinite product }\prod_{k=1}^{+\infty}\frac{1}{2}|1+\lambda^{n_k}|\text{ converges to a non-zero limit}\Big\}.$$

Observe that the set C does not depend on ε nor on k_0. For every $\lambda\in\mathbb{T}\setminus C$, the quantity $\prod_{k=k_0}^{N}\frac{1}{2}|1+\lambda^{n_k}|$ tends to 0 as $N\to+\infty$, and so by the dominated convergence theorem we get that

$$\int_{\mathbb{T}\setminus C} \prod_{k=k_0}^{N} \frac{1}{2}\,(1+\lambda^{n_k})\,d\mu(\lambda)\to 0 \qquad \text{as } N\to+\infty.$$

It then follows from (2) that

$$\limsup_{N\to+\infty}\Big|\int_{C} \prod_{k=k_0}^{N} \frac{1}{2}\,(1+\lambda^{n_k})\,d\mu(\lambda)-1\Big| \le \varepsilon$$

so that

$$\liminf_{N\to+\infty}\Big|\int_{C} \prod_{k=k_0}^{N} \frac{1}{2}\,(1+\lambda^{n_k})\,d\mu(\lambda)\Big| \ge 1-\varepsilon.$$

But

$$\Big|\int_{C} \prod_{k=k_0}^{N} \frac{1}{2}(1+\lambda^{n_k})\,d\mu(\lambda)\Big| \le \mu(C),$$

hence $\mu(C)\ge 1-\varepsilon$. This being true for any choice of ε in $(0,1)$, $\mu(C)=1$, and so the product $\prod_{k\ge1}\frac{1}{2}|1+\lambda^{n_k}|$ converges to a non-zero limit almost everywhere with respect to the measure μ. If we now write elements $\lambda\in C$ as $\lambda=e^{2i\pi\theta}$, $\theta\in[0,1)$, we have

$$\prod_{k\ge1}\frac{1}{2}|1+\lambda^{n_k}|=\prod_{k\ge1}|\cos(\pi\theta n_k)|.$$

Since $0<|\cos(\pi\theta n_k)|\le 1$ for all $k\ge1$, this means that the series $\sum_{k\ge1}1-|\cos(\pi\theta n_k)|$ is convergent. In particular $\{\theta n_k\}\to 0$ as $k\to+\infty$. As the quantities $1-|\cos(\pi\theta n_k)|$ and $\frac{\pi^2}{2}\{\theta n_k\}^2$ are equivalent as $k\to+\infty$, we obtain that the series $\sum_{k\ge1}\{\theta n_k\}^2$ is convergent. But

$$\big|1-\lambda^{n_k}\big|^2=\big|1-e^{2i\pi\theta n_k}\big|^2\le 4\pi^2\{\theta n_k\}^2,$$

and it follows from this that the series $\sum_{k\ge1}\big|1-\lambda^{n_k}\big|^2$ is convergent as soon as λ belongs to C. This proves our claim.

3. IP-Dirichlet generalized Riesz products

Our aim is now to give conditions on the sequence $(n_k)_{k\geq 1}$ which imply the existence of a generalized Riesz product which is continuous and IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. For information about classical and generalized Riesz products, we refer for instance the reader to the papers [**10**] and [**8**] and to the books [**7**] and [**12**].

Proposition 3.1. — *Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers. Suppose that there exists a sequence $(m_k)_{k\geq 1}$ of integers with $m_1 \geq 3$ such that*

$$n_{k+1} - 2\sum_{j=1}^{k} m_j n_j \geq 1 \quad \textit{for each } k \geq 1, \tag{3}$$

and

$$n_{k+1} - 2\sum_{j=1}^{k} m_j n_j \longrightarrow +\infty \quad \textit{as } k \longrightarrow +\infty. \tag{4}$$

For each $k \geq 1$, let $q_k \geq 1$ be an integer such that $q_k\pi\sqrt{2} \leq m_k + 2$. There exists a continuous generalized Riesz product σ on $\mathbb{T}$ such that for every finite subset $F \in \mathcal{F}$ and every integers j_k in $\{1, \ldots, q_k\}$, $k \in F$, one has

$$\hat{\sigma}\Big(\sum_{k\in F} j_k n_k\Big) \geq \prod_{k\in F}\Big(1 - 2\pi^2\Big(\frac{q_k}{m_k+2}\Big)^2\Big) \tag{5}$$

and

$$\hat{\sigma}\Big(\sum_{k\in F} n_k\Big) = \prod_{k\in F}\cos\Big(\frac{\pi}{m_k+2}\Big). \tag{6}$$

Proof. — For any integer $k \geq 1$, consider the polynomial P_k defined on $\mathbb{T}$ by

$$P_k(e^{2i\pi t}) = \frac{2}{m_k+2}\Big|\sum_{j=1}^{m_k+1}\sin\Big(\frac{j\pi}{m_k+2}\Big)e^{2i\pi jt}\Big|^2, \quad t \in [0,1].$$

Each P_k is a nonnegative trigonometric polynomial. Its spectrum is the set $\{-m_k, \ldots, m_k\}$ and a straightforward computation shows that $\hat{P}_k(0) = 1$. Condition (3), which is a dissociation condition, implies that the probability measures $\prod_{k=1}^{N} P_k\big(e^{2i\pi n_k t}\big)d\lambda(t)$ (where λ denotes here the normalized Lebesgue measure on $\mathbb{T}$) converge in the w^* topology as $N \to +\infty$ to a probability measure σ on $\mathbb{T}$, and that for each $F \in \mathcal{F}$ and each integers $j_k \in \{-m_k, \ldots, m_k\}$, $k \in F$,

$$\hat{\sigma}\Big(\sum_{k\in F} j_k n_k\Big) = \prod_{k\in F}\hat{P}_k(j_k),$$

while $\hat{\sigma}(n) = 0$ when n is not of this form. In particular

$$\hat{\sigma}\Big(\sum_{k\in F} n_k\Big) = \prod_{k\in F}\hat{P}_k(1).$$

Before getting into precise computation of these Fourier coefficients, let us prove that σ is a continuous measure: this follows from condition (4). If

$$\sum_{j=1}^{k} m_j n_j < n < n_{k+1} - \sum_{j=1}^{k} m_j n_j,$$

then $\hat{\sigma}(n) = 0$. So the Fourier transform of σ vanishes on successive intervals I_k of length $l_k = n_{k+1} - 2\sum_{j=1}^{k} m_j n_j - 1$. Since l_k tends to infinity with k by (4), it follows from the Wiener theorem that σ is continuous.

Let us now go back to the computation of the Fourier coefficients $\hat{\sigma}\Big(\sum_{k\in F} j_k n_k\Big)$. For each $q \in \{1, \dots, m_k\}$, we have

$$\hat{P}_k(q) = \frac{2}{m_k+2} \sum_{j=1}^{m_k+1-q} \sin\Big(\frac{(j+q)\pi}{m_k+2}\Big) \sin\Big(\frac{j\pi}{m_k+2}\Big). \tag{7}$$

Standard computations yield the following expression for $\hat{P}_k(q)$:

$$\begin{aligned} \hat{P}_k(q) &= \frac{1}{m_k+2}\Big((m_k+2-q)\cos\Big(\frac{q\pi}{m_k+2}\Big) + \sin\Big(\frac{q\pi}{m_k+2}\Big) \cdot \frac{\cos\big(\frac{\pi}{m_k+2}\big)}{\sin\big(\frac{\pi}{m_k+2}\big)}\Big) \\ &= \frac{1}{m_k+2}\Big((m_k+2-q)\cos\Big(\frac{q\pi}{m_k+2}\Big) + \cos\Big(\frac{(q-1)\pi}{m_k+2}\Big) \cdot \cos\Big(\frac{\pi}{m_k+2}\Big) \\ &\quad + \sin\Big(\frac{(q-1)\pi}{m_k+2}\Big) \cdot \frac{\cos^2\big(\frac{\pi}{m_k+2}\big)}{\sin\big(\frac{\pi}{m_k+2}\big)} \\ &= \cdots = \frac{1}{m_k+2}\Big((m_k+2-q)\cos\Big(\frac{q\pi}{m_k+2}\Big) \\ &\quad + \sum_{j=1}^{q} \cos\Big(\frac{(q-j)\pi}{m_k+2}\Big)\cos^j\Big(\frac{\pi}{m_k+2}\Big)\Big). \end{aligned} \tag{8}$$

Observe now that for every $x \in [0,1]$, $\cos x \geq 1 - x^2 \geq 0$. For each $k \geq 1$, $q_k \geq 1$ is an integer such that $q_k\pi\sqrt{2} \leq m_k+2$, and q belongs to the set $\{1, \dots, q_k\}$. So $(q-j)\pi \leq m_k+2$ for every $j \in \{0, \dots, q-1\}$. Thus

$$\cos\Big(\frac{q\pi}{m_k+2}\Big) \geq 1 - \pi^2\frac{q^2}{(m_k+2)^2} \quad \text{and} \quad \cos\Big(\frac{(q-j)\pi}{m_k+2}\Big) \geq 1 - \pi^2\frac{(q-j)^2}{(m_k+2)^2}.$$

Moreover, $\cos^j x \geq (1-x^2)^j \geq 1 - jx^2$ for every $x \in [0,1]$ and every $j \geq 1$, so that

$$\cos^j\Big(\frac{\pi}{m_k+2}\Big) \geq 1 - \pi^2\frac{j}{(m_k+2)^2}.$$

Putting things together, we obtain the estimate

$$\begin{aligned} \hat{P}_k(q) \geq \frac{1}{m_k+2}\Big((m_k+2-q)\Big(1-\pi^2\frac{q^2}{(m_k+2)^2}\Big) \\ + \sum_{j=1}^{q}\Big(1-\pi^2\frac{(q-j)^2}{(m_k+2)^2}\Big)\Big(1-\pi^2\frac{j}{(m_k+2)^2}\Big)\Big). \end{aligned}$$

Now, for every $j \in \{1, \dots, q-1\}$,

$$\Big(1 - \pi^2 \frac{(q-j)^2}{(m_k+2)^2}\Big)\Big(1 - \pi^2 \frac{j}{(m_k+2)^2}\Big) = 1 - \pi^2 \frac{(q-j)^2 + j}{(m_k+2)^2} + \pi^4 \frac{j(q-j)^2}{(m_k+2)^4}$$
$$\geq 1 - \pi^2 \frac{(q-j)^2 + j}{(m_k+2)^2} \geq 1 - 2\pi^2 \frac{q^2}{(m_k+2)^2}.$$

Summing over j and putting together terms, we eventually obtain that

$$\hat{P}_k(q) \geq \frac{1}{m_k+2}\Big((m_k+2-q)\Big(1 - \pi^2 \frac{q^2}{(m_k+2)^2}\Big) + q - 2\pi^2 \frac{q^3}{(m_k+2)^2}\Big)$$
$$\geq 1 - \frac{1}{m_k+2}(m_k+2-q)\pi^2\Big(\frac{q}{m_k+2}\Big)^2 - 2\pi^2\Big(\frac{q}{m_k+2}\Big)^3,$$

i.e. that

$$\hat{P}_k(q) \geq 1 - \pi^2\Big(\frac{q}{m_k+2}\Big)^2 - \pi^2\Big(\frac{q}{m_k+2}\Big)^3$$
$$\geq 1 - \pi^2\Big(\frac{q_k}{m_k+2}\Big)^2 - \pi^2\Big(\frac{q_k}{m_k+2}\Big)^3 \quad \text{for each } q \in \{1, \dots, q_k\}$$
$$\geq 1 - 2\pi^2\Big(\frac{q_k}{m_k+2}\Big)^2 \geq 0$$

since $q_k \pi \sqrt{2} \leq m_k + 2$. Assertion (5) follows directly from the fact that $\hat{\sigma}\big(\sum_{k \in F} j_k n_k\big) = \prod_{k \in F} \hat{P}_k(j_k)$. Assertion (6) is straightforward: the expression in the first line of the display (8) applied to $q = 1$ yields that $\hat{P}_k(1) = \cos(\pi/(m_k+2))$. This finishes the proof of Proposition 3.1. □

Proposition 3.1 may appear a bit technical at first sight, but it turns out to be quite easy to apply. As a first example, we use it to obtain another proof of a result of [**1**, Prop. 3.2]:

Corollary 3.2. — *Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers such that the series $\sum_{k\geq 1}(n_k/n_{k+1})^2$ is convergent. There exists a continuous generalized Riesz product σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$.*

Proof. — Without loss of generality we can assume that $\sum_{k\geq 1}(n_k/n_{k+1})^2 < 1/200$. Let $(\varepsilon_k)_{k\geq 1}$ be a sequence of real numbers with $0 < \varepsilon_k < 1/2$ for each $k \geq 2$, with $\varepsilon_1 = 0$, going to zero as k tends to infinity, and such that

$$\sum_{k\geq 1}\Big(\frac{1}{\varepsilon_{k+1}}\frac{n_k}{n_{k+1}}\Big)^2 < \frac{1}{50}.$$

Then $\varepsilon_{k+1} n_{k+1}/n_k > 7 > 6 + \varepsilon_k$, so that if we define $m_k = [(\varepsilon_{k+1} n_{k+1} - \varepsilon_k n_k)/2n_k]$ for each $k \geq 1$, each m_k is greater or equal to 3. Moreover

$$n_{k+1} - 2\sum_{j=1}^{k} m_j n_j \geq n_{k+1} - (\varepsilon_{k+1} n_{k+1} - \varepsilon_1 n_1) = (1 - \varepsilon_{k+1}) n_{k+1}$$

which tends to infinity as k tends to infinity, and is always greater than 1 because $\varepsilon_{k+1} < 1/2$ and $n_{k+1} \geq 2$ for each $k \geq 1$. Proposition 3.1 applies with this choice of the sequence

$(m_k)_{k\geq 1}$ and yields a continuous generalized Riesz product σ which satisfies

$$\hat{\sigma}\Big(\sum_{k\in F} n_k\Big) = \prod_{k\in F}\cos\Big(\frac{\pi}{m_k+2}\Big) \quad \text{for each } F \in \mathcal{F}.$$

Now m_k is equivalent as k tends to infinity to the quantity $\varepsilon_{k+1}n_{k+1}/2n_k$, so that the series $\sum_{k\geq 1} 1/(m_k+2)^2$ is convergent. Hence the infinite product $\prod_{k\geq 1}\cos\big(\pi/(m_k+2)\big)$ is convergent. For any $\varepsilon > 0$, let k_0 be such that $\prod_{k\geq k_0}\cos\big(\pi/(m_k+2)\big) \geq 1-\varepsilon$. If $F \in \mathcal{F}$ is such that $\min(F) \geq k_0$,

$$\hat{\sigma}\Big(\sum_{k\in F} n_k\Big) = \prod_{k\in F}\cos\Big(\frac{\pi}{m_k+2}\Big) \geq \prod_{k\geq k_0}\cos\Big(\frac{\pi}{m_k+2}\Big) \geq 1-\varepsilon,$$

and this proves that σ is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. □

4. An application to a special class of sets $\{n_k\}$

Proposition 3.1 applies especially well to a particular class of sequences $(n_k)_{k\geq 1}$, which we now proceed to investigate.

Proposition 4.1. — *Let $(p_l)_{l\geq 1}$ be a strictly increasing sequence of integers. For each $l \geq 1$, let $(q_{j,l})_{j=0,\dots,r_l}$ be a strictly increasing finite sequence of integers with $q_{0,l} = 1$, and set $q_l = q_{0,l} + q_{1,l} + \cdots + q_{r_l,l}$. Suppose that $p_{l+1} > q_{r_l,l}\, p_l$ for each $l \geq 1$, and that the series*

$$\sum_{l\geq 1}\left(\frac{q_l\, p_l}{p_{l+1}}\right)^2$$

is convergent. Let $(n_k)_{k\geq 1}$ be the strictly increasing sequence defined by

$$\{n_k\} = \bigcup_{l\geq 1}\{p_l, q_{1,l}\, p_l, \dots, q_{r_l,l}\, p_l\}$$

There exists a continuous generalized Riesz product σ on $\mathbb{T}$ which is IP-Dirichlet with respect to the sequence $(n_k)_{k\geq 1}$.

Proof. — As in the proof of Corollary 3.2, we can suppose that $\sum_{k\geq 1}(q_l p_l/p_{l+1})^2 < 1/400$, and consider a sequence $(\varepsilon_l)_{l\geq 1}$ going to zero as l tends to infinity with $\varepsilon_1 = 0$ and $0 < \varepsilon_l < 1/2$ for each $l \geq 2$, such that

$$\sum_{l\geq 1}\Big(\frac{1}{\varepsilon_{l+1}}\frac{q_l p_l}{p_{l+1}}\Big)^2 < \frac{1}{100}.$$

The same argument as in the proof of Corollary 3.2 shows that for $l \geq 1$ the integers $m_l = [(\varepsilon_{l+1}p_{l+1} - \varepsilon_l p_l)/(2p_l)]$ are greater or equal to 3, and that assumptions (3) and (4) of Proposition 3.1 are satisfied. As m_l is equivalent as l tends to infinity to $(\varepsilon_{l+1}p_{l+1})/(2p_l)$, we have that $q_l/(m_l+2)$ is equivalent to $(2q_l p_l)/(\varepsilon_{l+1}p_{l+1})$. Our assumption implies then that the series

$$\sum_{l\geq 1}\Big(\frac{q_l}{m_l+2}\Big)^2 \tag{9}$$

is convergent. Moreover, $q_l\pi\sqrt{2} < 5q_l < \frac{1}{2}\frac{\varepsilon_{l+1}p_{l+1}}{p_l}$. But $\frac{\varepsilon_{l+1}p_{l+1}}{2p_l} - \frac{\varepsilon_l}{2} \leq m_l + 1$, so that $\frac{\varepsilon_{l+1}p_{l+1}}{p_l} \leq 2(m_l+2)$. Hence $q_l\pi\sqrt{2} < m_l + 2$ for each $l \geq 2$. Applying Proposition 3.1 to the sequence $(p_l)_{l\geq 1}$, we get a continuous generalized Riesz product σ, and the estimates (5) yield that

$$\hat{\sigma}\Big(\sum_{l\in F}\Big(\sum_{j\in G_l} q_{j,l}\Big)p_l\Big) \geq \prod_{l\in F}\Big(1 - 2\pi^2\Big(\frac{q_l}{m_l+2}\Big)^2\Big)$$

for each set $F \in \mathcal{F}$ and each subsets G_l of $\{0,\ldots,r_l\}$, $l \in F$. In order to show that the measure σ is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$, it remains to observe that the product on the right-hand side is convergent by (9). We then conclude as in the proof of Corollary 3.2. □

The proof of Theorem 1.4 is now a straightforward corollary of Proposition 4.1. Recall that we wish to prove that if $(n_k)_{k\geq 1}$ is a sequence of integers for which there exists an infinite subset S of $\mathbb{N}$ such that

$$\sum_{k\in S}\Big(\frac{n_k}{n_{k+1}}\Big)^2 < +\infty \quad \text{and } n_k | n_{k+1} \text{ for each } k \notin S,$$

then there exists a continuous generalized Riesz product σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$.

Proof of Theorem 1.4. — Let $\Phi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $S = \{\Phi(l),\ l \geq 1\}$. Set $p_l = n_{\Phi(l)+1}$ for $l \geq 1$ and write for each $k \in \{\Phi(l)+1,\ldots,\Phi(l+1)\}$

$$n_k = s_{0,l}\, s_{1,l}\, \ldots s_{k-(\Phi(l)+1),l}\, p_l,$$

with $s_{0,l} = 1$ and $s_{j,l} \geq 2$ for each $j = 1,\ldots,\Phi(l+1) - (\Phi(l)+1)$. With the notation of Proposition 4.1 we have $r_l = \Phi(l+1) - (\Phi(l)+1)$ and

$$q_{k-(\Phi(l)+1),l} = s_{0,l}\, s_{1,l}\, \ldots s_{k-(\Phi(l)+1),l}$$

Hence $q_l = q_{0,l} + \cdots + q_{r_l,l} = s_{0,l} + s_{0,l}\, s_{1,l} + \cdots + s_{0,l}\, s_{1,l} \ldots s_{r_l,l}$. We have

$$\begin{aligned}\frac{q_l}{s_{0,l}\, s_{1,l}\ldots s_{r_l,l}} &= 1 + \frac{1}{s_{r_l,l}} + \frac{1}{s_{r_l-1,l}\, s_{r_l,l}} + \cdots + \frac{1}{s_{2,l}\ldots s_{r_l,l}} + \frac{1}{s_{1,l}\ldots s_{r_l,l}}\\ &\leq 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{r_l}} \quad \text{since } s_{j,l} \geq 2 \text{ for each } j = 1,\ldots,r_l\\ &\leq 2.\end{aligned}$$

This yields that $q_l \leq 2s_{0,l}s_{1,l}\ldots s_{r_l,l} = 2q_{r_l,l}$ for each $l \geq 1$. Our assumption that the series $\sum_{k\in S}(n_k/n_{k+1})^2$ is convergent means that the series $\sum_{l\geq 1}(q_{r_l,l}\, p_l/p_{l+1})^2$ is convergent. Hence the series $\sum_{l\geq 1}(q_l\, p_l/p_{l+1})^2$ is convergent and the conclusion follows from Proposition 4.1. □

Our next result shows the optimality of the assumption of Proposition 4.1 that the series $\sum_{l\geq 1}(q_l p_l/p_{l+1})^2$ is convergent.

Proposition 4.2. — *Let $(\gamma_l)_{l\geq 1}$ be any sequence of positive real numbers, going to zero as l goes to infinity, such that the series $\sum_{l\geq 1}\gamma_l^2$ is divergent, with $0 < \gamma_l < 1$ for each $l \geq 2$. Let $(r_l)_{l\geq 1}$ be a sequence of integers growing to infinity so slowly that the series $\sum_{l\geq 1}\gamma_l^2/r_l$ is divergent, with $r_l \geq 2$ for each $l \geq 1$. Define a sequence $(p_l)_{l\geq 1}$ of integers* *by setting* $p_1 = 1$ *and* $p_{l+1} = [r_l^2/\gamma_l]\, p_l + 1$. *For each* $l \geq 1$, *we have* $p_{l+1} > r_l\, p_l$. *Define a strictly increasing sequence* $(n_k)_{k\geq 1}$ *of integers by setting*

$$\{n_k\} = \bigcup_{l\geq 1} \{p_l, 2\,p_l, \ldots, r_l\, p_l\}.$$

Then no continuous measure σ *on the unit circle can be IP-Dirichlet with respect to the sequence* $(n_k)_{k\geq 1}$.

Proof. — We are going to show that $G_2((n_k)) = \{1\}$. It will then follow from Theorem 1.3 that no continuous probability measure on $\mathbb{T}$ can be IP-Dirichlet with respect to the sequence $(n_k)_{k\geq 1}$. Suppose that $\lambda \in \mathbb{T} \setminus \{1\}$ is such that

$$\sum_{k\geq 1} |\lambda^{n_k} - 1|^2 = \sum_{l\geq 1} \sum_{j=1}^{r_l} |\lambda^{jp_l} - 1|^2 < +\infty. \tag{10}$$

Let C be a positive constant such that for each $\theta \in \mathbb{R}$, $\frac{1}{C}\{\theta\} \geq |e^{2i\pi\theta} - 1| \geq C\{\theta\}$. Writing λ as $\lambda = e^{2i\pi\theta}$, $\theta \in [0, 1)$, we have that

$$|\lambda^{jp_l} - 1| \geq C\{jp_l\theta\} \quad \text{for each } l \geq 1 \text{ and } j = 1, \ldots, r_l. \tag{11}$$

Now $\{\theta p_l\} < 1/r_l$ for sufficiently large l. Else the set $\{\{j\theta p_l\}, j = 1, \ldots, r_l\}$ would form a $\{\theta p_l\}$-dense net of $[0, 1]$, and this would contradict the fact, implied by (10) and (11), that the quantity $\sum_{j=1}^{r_l} \{j\theta p_l\}^2$ tends to zero as l tends to infinity. Hence, for sufficiently large l, $\{j\theta p_l\} = j\{\theta p_l\}$ for every $j = 1, \ldots, r_l$, and thus the series $\sum_{l\geq 1} \sum_{j=1}^{r_l} j^2 |\lambda^{p_l} - 1|^2$ is convergent. As r_l tends to infinity with l, this means that the series

$$\sum_{l\geq 1} r_l^3\, |\lambda^{p_l} - 1|^2 \tag{12}$$

is convergent.

Let now $(\delta_l)_{l\geq 1}$ be a sequence of real numbers going to zero so slowly that the series $\sum_{l\geq 1} \frac{1}{r_l}\, \gamma_l^2\, \delta_l^2$ is divergent. Suppose that $|\lambda^{p_l} - 1| < \frac{\gamma_l}{r_l^2}\, \delta_l$ for infinitely many l. Then,

$$\left|\lambda^{\left[\frac{r_l^2}{\gamma_l}\right] p_l} - 1\right| < \delta_l \quad \text{for all these } l,$$

and by definition of p_{l+1}, $|\lambda^{p_{l+1}} - \lambda| < \delta_l$. Letting l tend to infinity along this set of integers, and remembering that $|\lambda^{p_{l+1}} - 1| \to 0$ as l tends to infinity, we get that $\lambda = 1$, which is contrary to our assumption. Hence $|\lambda^{p_l} - 1| \geq \frac{\gamma_l}{r_l^2}\, \delta_l$ for all integers l sufficiently large. Combining this with (12), this implies that the series

$$\sum_{l\geq 1} r_l^3\, \frac{\gamma_l^2}{r_l^4}\, \delta_l^2 = \sum_{l\geq 1} \frac{1}{r_l}\, \gamma_l^2\, \delta_l^2$$

is convergent, which is again a contradiction. So $G_2((n_k)) = \{1\}$ and we are done. □

Consider the sets $\{n_k\}$ given by Proposition 4.2. With the notation of Proposition 4.1, q_l is equivalent to $r_l^2/2$ as k tends to infinity, and the series $\sum_{l\geq 1} (q_l p_l / p_{l+1})^2$ is divergent because $(q_l p_l / p_{l+1})^2$ is equivalent to $\gamma_l^2/4$. This shows the optimality of the condition given in Proposition 4.1.

Looking at the construction of Proposition 4.2 from a different angle yields an example of a sequence $(n_k)_{k\geq 1}$ such that $G_2((n_k))$ is uncountable, but still no continuous probability measure on $\mathbb{T}$ can be IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. This is Theorem 1.5.

5. Proof of Theorem 1.5

Recall that we aim to construct a strictly increasing sequence $(n_k)_{k\geq 1}$ of integers such that $G_2((n_k))$ is uncountable, but no continuous probability measure on $\mathbb{T}$ is IP-Dirichlet with respect to the sequence $(n_k)_{k\geq 1}$. This sequence $(n_k)_{k\geq 1}$ will be of the kind considered in the previous section. Consider first the sequence $(p_l)_{l\geq 1}$ defined by $p_1 = 1$ and $p_{l+1} = \frac{l^2(l^2+1)}{2} p_l$ for all $l \geq 1$. We then define the sequence $(n_k)_{k\geq 1}$ by setting

$$\{n_k \;;\; k \geq 1\} = \bigcup_{l\geq 2} \{p_l, 2p_l, \ldots, l^2 p_l\}.$$

As $l^2 p_l < p_{l+1}$ for all $l \geq 2$, the sets $\{p_l, 2p_l, \ldots, l^2 p_l\}$ are consecutive sets of integers. Let $(M_l)_{l\geq 1}$ be the unique sequence of integers such that $\{n_{M_{l-1}+1}, \ldots, n_{M_l}\} = \{p_l, 2p_l, \ldots, l^2 p_l\}$ for each $l \geq 2$. We now know (see for instance [**2**] or [**5**] for a proof) that there exists a perfect uncountable subset K of $\mathbb{T}$ (which is actually a generalized Cantor set) such that

$$|\lambda^{p_l} - 1| \leq C \frac{p_l}{p_{l+1}} \qquad \text{for all } \lambda \in K \text{ and } l \geq 2,$$

where C is a positive universal constant. Hence for $\lambda \in K$, $l \geq 2$ and $j \in \{1, \ldots, l^2\}$ we have

$$|\lambda^{jp_l} - 1| \leq C\, j\, \frac{p_l}{p_{l+1}} \leq 2C\, l^2\, \frac{1}{l^4} = \frac{2C}{l^2}\cdot$$

Thus

$$\sum_{j=1}^{l^2} |\lambda^{jp_l} - 1|^2 \leq l^2\, \frac{4C^2}{l^4} = \frac{4C^2}{l^2}\cdot$$

Hence the series $\sum_{l\geq 2} \sum_{j=1}^{l^2} |\lambda^{jp_l} - 1|^2$ is convergent for all $\lambda \in K$, that is the series $\sum_{k\geq 1} |\lambda^{n_k} - 1|^2$ is convergent for all $\lambda \in K$. We have thus proved the first part of our statement, namely that $G_2((n_k))$ is uncountable.

Let now σ be a continuous probability measure on $\mathbb{T}$. The proof that σ cannot be IP-Dirichlet with respect to the sequence $(n_k)_{k\geq 1}$ relies on the following lemma:

Lemma 5.1. — *For all $l \geq 2$ and all $s \geq 1$, sp_l belongs to the set*

$$\Big\{\sum_{k\in F} n_k \;;\; F \in \mathcal{F},\ \min(F) \geq M_{l-1} + 1\Big\}.$$

Proof of Lemma 5.1. — It is clear that for all $n \geq 1$,

$$\{\sum_{j\in F} j \;;\; F \subseteq \{1, \ldots, n\},\ F \neq \varnothing\} = \{1, \ldots, \frac{n(n+1)}{2}\}.$$

Hence

$$\{\sum_{j\in F} jp_l \;;\; F \subseteq \{1, \ldots, l^2\},\ F \neq \varnothing\} = \{p_l, 2p_l \ldots, \frac{l^2(l^2+1)}{2} p_l\},$$

i.e.

$$\{\sum_{k\in F} n_k \ ; \ F\subseteq\{M_{l-1}+1,\dots,M_l\},\ F\neq\varnothing\}=\{p_l,2p_l\dots,p_{l+1}\}.$$

This proves Lemma 5.1 for $s\in\{1,\dots,\frac{l^2(l^2+1)}{2}\}$. Then since

$$\{\sum_{k\in F} n_k \ ; \ F\subseteq\{M_l+1,\dots,M_{l+1}\},\ F\neq\varnothing\}=\{p_{l+1},2p_{l+1}\dots,\frac{(l+1)^2((l+1)^2+1)}{2}p_{l+1}\},$$

we get that

$$\begin{aligned}\{\sum_{k\in F} n_k \quad &; \quad F\subseteq\{M_{l-1}+1,\dots,M_{l+1}\},\ F\neq\varnothing\}\\ &= \{p_l,2p_l,\dots,p_{l+1},p_{l+1}+p_l,p_{l+1}+2p_l,\dots,2p_{l+1},\dots\\ &\qquad \frac{(l+1)^2((l+1)^2+1)}{2}p_{l+1},\dots,\frac{(l+1)^2((l+1)^2+1)}{2}p_{l+1}+p_{l+1}\}\\ &= \{p_l,2p_l,\dots,\frac{l^2(l^2+1)}{2}\cdot\Big(\frac{(l+1)^2((l+1)^2+1)}{2}+1\Big)p_l\}.\end{aligned}$$

In particular $\{\sum_{k\in F} n_k \ ; \ F\subseteq\{M_{l-1}+1,\dots,M_{l+1}\},\ F\neq\varnothing\}$ contains the set

$$\{p_l,2p_l,\dots,\frac{l^2(l^2+1)}{2}\cdot\frac{(l+1)^2((l+1)^2+1)}{2}p_l\}.$$

Continuing in this fashion we obtain that for all $q\geq 1$,

$$\{\sum_{k\in F} n_k \quad ; \quad F\subseteq\{M_{l-1}+1,\dots,M_{l+q}\},\ F\neq\varnothing\}$$

contains the set

$$\{p_l,2p_l,\dots,\prod_{j=0}^{q}\frac{(l+j)^2((l+j)^2+1)}{2}p_l\}.$$

The conclusion of Lemma 5.1 follows from this. □

Suppose now that σ is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. Let $l_0\geq 2$ be such that for every $F\in\mathcal{F}$ with $\min(F)\geq M_{l_0-1}+1$, $|\hat\sigma(\sum_{k\in F}n_k)|\geq 1/2$. Then Lemma 5.1 implies that for all $s\geq 1$, $|\hat\sigma(sp_{l_0})|\geq 1/2$. This contradicts the continuity of the measure σ.

6. Additional results and comments

6.1. A remark about the Erdös-Taylor sequence. — Let $(n_k)_{k\geq 1}$ be the sequence of integers defined by $n_1=1$ and $n_{k+1}=kn_k+1$ for every $k\geq 1$. This sequence is interesting in our context because $G_1((n_k))=\{1\}$ while $G_2((n_k))$ is uncountable ([**6**], see also [**1**]): if $\lambda\in\mathbb{T}\setminus\{1\}$, there exists a positive constant ε such that $|\lambda^{n_k}-1|\geq\frac{\varepsilon}{k}$ for all $k\geq 1$. Indeed, if for some k we have $|\lambda^{n_k}-1|\leq\frac{\varepsilon}{k}$ with $\varepsilon=\frac{1}{2}|\lambda-1|$, then $|\lambda^{kn_k}-1|\leq\varepsilon$, so that $|\lambda^{n_{k+1}}-1|\geq|\lambda-1|-\varepsilon\geq\frac{1}{2}|\lambda-1|>0$. Hence if $\lambda\in\mathbb{T}\setminus\{1\}$ the series $\sum_{k\geq 1}|\lambda^{n_k}-1|$ is divergent. On the other hand, since the series $\sum_{k\geq 1}(n_k/n_{k+1})^2$ is convergent, $G_2((n_k))$ is uncountable. It is proved in [**1**] that there exists a continuous probability measure σ on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. This statement can also be seen as a consequence of Theorem 2.2 of [**9**]: it is shown there that there exists a continuous generalized Riesz product σ on $\mathbb{T}$ and a $\delta > 0$ such that

$$|\hat{\sigma}(\sum_{k\in F} n_k)| \geq \delta$$

for every $F \in \mathcal{F}$ such that $\min(F) > 4$. It is not difficult to see that this measure σ is in fact IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. We briefly give the argument below. It can be generalized to all sequences $(n_k)_{k\geq 1}$ such that the series $\sum_{k\geq 1}(n_k/n_{k+1})^2$ is convergent, thus yielding another proof of Corollary 3.2.

The measure σ of [**9**] is constructed in the following way: let Δ be the function defined for $t \in \mathbb{R}$ by $\Delta(t) = \max(1 - 6|t|, 0)$. If K is the function $\mathbb{R}$ given by the expression

$$K(t) = \frac{1}{2\pi}\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2, \quad t \in \mathbb{R}$$

and K_α is defined for each $\alpha > 0$ by $K_\alpha(t) = \alpha K(\alpha t)$, $t \in \mathbb{R}$, then $\Delta(x) = \hat{K}_{\frac{1}{6}}(x)$ for every $x \in \mathbb{R}$. The function $\Delta * \Delta$ is a $\mathcal{C}^2$ function on $\mathbb{R}$ which is supported on $[-\frac{1}{3}, \frac{1}{3}]$, takes positive values on $]-\frac{1}{3}, \frac{1}{3}[$, and attains its maximum at the point 0. Hence its derivative vanishes at the point 0. Let $a > 0$ be such that the function $\varphi = a\Delta * \Delta$ satisfies $\varphi(0) = 1$. We have also $\varphi'(0) = 0$, and so there exists a constant $c \geq 0$ and a $\gamma \in (0, \frac{1}{3})$ such that for all x with $|x| < \gamma$, $\varphi(x) \geq 1 - cx^2$. Lastly, recall that $\varphi(x) = \widehat{aK^2_{\frac{1}{6}}}(x)$ for all $x \in \mathbb{R}$. Consider now the sequence $(P_j)_{j\geq 1}$ of trigonometric polynomials defined on $\mathbb{T}$ in the following way: for $j \geq 1$ and $t \in \mathbb{R}$,

$$P_j(e^{it}) = \sum_{s\in\mathbb{Z}} \varphi(\frac{s}{j})e^{ist}.$$

This is indeed a polynomial of degree at most $\lfloor\frac{j}{3}\rfloor$, since $\varphi(\frac{s}{j}) = 0$ as soon as $\frac{s}{j} \geq \frac{1}{3}$. We now claim that P_j takes only nonnegative values on $\mathbb{T}$: indeed, consider for each $j \geq 1$ and $t \in \mathbb{R}$ the function $\Phi_{j,t}$ defined by $\Phi_{j,t}(x) = jK^2_{\frac{1}{6}}(j(x+t))$, $x \in \mathbb{R}$. Its Fourier transform is then given by $\hat{\Phi}_{j,t}(\xi) = e^{i\xi t}\widehat{K^2_{\frac{1}{6}}}(\frac{\xi}{j}) = e^{i\xi t}\Delta * \Delta(\frac{\xi}{j})$. Thus $P_j(e^{it}) = a\sum_{s\in\mathbb{Z}} \hat{\Phi}_{j,t}(s)$. Applying the Poisson formula to the function $\Phi_{j,t}$, we get that $P_j(e^{it}) = 2\pi a\sum_{s\in\mathbb{Z}} \Phi_{j,t}(2\pi s) = 2\pi a\sum_{s\in\mathbb{Z}} jK^2_{\frac{1}{6}}(j(2\pi s + t)) \geq 0$. Hence $P_j(e^{it})$ is nonnegative for all $t \in \mathbb{R}$, $\hat{P}_j(0) = 1$ and $\hat{P}_j(1) = \varphi(\frac{1}{j}) \geq 1 - \frac{c}{j^2}$ as soon as $j \geq j_0$, where $j_0 = \lfloor\frac{1}{\gamma}\rfloor + 1$. Consider then for $m \geq j_0$ the nonnegative polynomials Q_m defined by

$$Q_m(e^{it}) = \prod_{j=j_0}^{m} P_j(e^{in_jt}), \quad t \in \mathbb{R}.$$

Since the degree of P_j is less than $\lfloor\frac{j}{3}\rfloor$ and $n_{j+1} > \frac{jn_j}{3}$, $\hat{Q}_m(0) = 1$ for each $m \geq 1$ and the polynomials Q_m converge in the w^*-topology to a generalized Riesz product σ on $\mathbb{T}$ which is continuous and such that for every set $F \in \mathcal{F}$ with $\min(F) \geq j_0$,

$$\hat{\sigma}(\sum_{k\in F} n_k) \geq \prod_{k\in F}(1 - \frac{c}{k^2}).$$

It follows that σ is an IP-Dirichlet measure with respect to the sequence $(n_k)_{k\geq 1}$.

6.2. A sequence $(n_k)_{k\ge 1}$ with respect to which there exists a continuous Dirichlet measure, but such that $G_\infty((n_k)) = \{1\}$. — The examples of sequences $(n_k)_{k\ge 1}$ given in [**3**] and [**5**] for which there exists a continuous probability measure σ on $\mathbb{T}$ such that $\hat{\sigma}(n_k) \to 1$ as $k \to +\infty$ all share the property that $|\lambda^{n_k} - 1| \to 0$ for some $\lambda \in \mathbb{T} \setminus \{1\}$. One may thus wonder whether there exists a sequence $(n_k)_{k\ge 1}$ with respect to which there exists a continuous Dirichlet probability measure σ, and such that $G_\infty((n_k)) = \{\lambda \in \mathbb{T} ; |\lambda^{n_k} - 1| \to 0\} = \{1\}$. The answer is yes, and an ad hoc sequence $(n_k)_{k\ge 1}$ can be constructed from the Erdös-Taylor sequence above. Changing notations, let us denote by $(p_k)_{k\ge 1}$ this sequence defined by $p_1 = 1$ and $p_{k+1} = kp_k + 1$ for each $k \ge 1$. For each integer $q \ge 1$, consider the finite set

$$\mathcal{P}_q = \{\sum_{k\in F} p_k \; ; \; F \neq \emptyset, \; F \subseteq \{2^q + 1, \ldots, 2^{q+1}\}\}.$$

The set $\bigcup_{q\ge 1} \mathcal{P}_q$ can be written as $\{n_k \; ; \; k \ge 1\}$, where $(n_k)_{k\ge 1}$ is a strictly increasing sequence of integers. Let now σ be a continuous probability measure which is IP-Dirichlet with respect to the Erdös-Taylor sequence $(p_k)_{k\ge 1}$:

$$\hat{\sigma}(\sum_{k\in F} p_k) \to 1 \quad \text{as} \quad \min(F) \to +\infty, \; F \in \mathcal{F}.$$

This implies that $\hat{\sigma}(n_k) \to 1$ as $k \to +\infty$. Indeed, let $\varepsilon > 0$ and k_0 be such that $|\hat{\sigma}(\sum_{k\in F} p_k) - 1| < \varepsilon$ for all $F \in \mathcal{F}$ with $\min(F) \ge k_0$. Let q_0 be such that $2^{q_0} + 1 \ge k_0$. Then $|\hat{\sigma}(n_k) - 1| < \varepsilon$ for all k such that n_k belongs to the union $\bigcup_{q\ge q_0} \mathcal{P}_q$. Since all the sets $\mathcal{P}_q$ are finite, $|\hat{\sigma}(n_k) - 1| < \varepsilon$ for all but finitely many k.

It remains to prove that $G_\infty((n_k)) = \{1\}$, and the argument for this is very close to one employed in [**1**]. Let $\varepsilon \in (0, 1/16)$ for instance, and suppose that $\lambda \in \mathbb{T}$ is such that $|\lambda^{n_k} - 1| < \varepsilon$ for all k larger than some k_0. We claim then that if q_0 is such that $2^{q_0} + 1 \ge k_0$, then we have for all q larger than q_0

$$\sum_{k=2^q+1}^{2^{q+1}} |\lambda^{p_k} - 1| < 2C^2\varepsilon, \tag{13}$$

where $C > 0$ is a constant such that $\{t\}/C \le |e^{2i\pi t} - 1| \le C\{t\}$ for all $t \in \mathbb{R}$. Indeed, our assumption that $|\lambda^{n_k} - 1| < \varepsilon$ for all $k \ge k_0$ implies that for all $q \ge q_0$ and all disjoint finite subsets F and G of the set $\mathcal{P}_q$,

$$\{\sum_{k\in F} p_k\theta\} < C\varepsilon, \quad \{\sum_{k\in G} p_k\theta\} < C\varepsilon \quad \text{and} \quad \{\sum_{k\in F\sqcup G} p_k\theta\} < C\varepsilon$$

where $\lambda = e^{2i\pi\theta}$ with $\theta \in [0, 1)$ and $F \sqcup G$ denotes the disjoint union of F and G. Now the same argument as in [**1**, Prop. 1.1] yields that

$$\langle \sum_{k\in F\sqcup G} p_k\theta\rangle = \langle\sum_{k\in F} p_k\theta\rangle + \langle\sum_{k\in G} p_k\theta\rangle.$$

Setting

$$A_{q,+} = \{k \in \{2^q + 1, \ldots 2^{q+1}\} \; ; \; \langle p_k\theta\rangle \ge 0\}$$

and

$$A_{q,-} = \{k \in \{2^q + 1, \ldots 2^{q+1}\} \; ; \; \langle p_k\theta\rangle < 0\},$$

this implies that

$$\sum_{k\in A_{q,+}} \{p_k\theta\} < C\varepsilon \quad \text{and} \quad \sum_{k\in A_{q,-}} \{p_k\theta\} < C\varepsilon.$$

Hence

$$\sum_{k=2^q+1}^{2^{q+1}} \{p_k\theta\} < 2C\varepsilon \quad \text{so that} \quad \sum_{k=2^q+1}^{2^{q+1}} |\lambda^{p_k}-1| < 2C^2\varepsilon \quad \text{for all } q\geq q_0.$$

Suppose now that $\lambda\neq 1$, and set $\varepsilon=|\lambda-1|/(4C^2)$. Then (13) above implies that there exists an infinite subset E of $\mathbb{N}$ such that $|\lambda^{p_k}-1|\leq (2C^2\varepsilon)/k$ for all $k\in E$. If it were not the case, we would have $|\lambda^{p_k}-1|>(2C^2\varepsilon)/k$ for all k large enough, so that

$$\sum_{k=2^q+1}^{2^{q+1}} |\lambda^{p_k}-1| > 2C^2\varepsilon \sum_{k=2^q+1}^{2^{q+1}} \frac{1}{k} \geq 2C^2\varepsilon\,\frac{2^{q+1}-2^q}{2^q} \geq 2C^2\varepsilon \tag{14}$$

for all q large enough, which is a contradiction with (13). This proves the existence of the set E. Now for all $k\in E$

$$|\lambda^{p_{k+1}}-1| \geq |\lambda-1| - |\lambda^{kp_k}-1| \geq |\lambda-1| - k|\lambda^{p_k}-1| \geq 4C^2\varepsilon - 2C^2\varepsilon = 2C^2\varepsilon.$$

But this stands again in contradiction with (13), and we infer from this that λ is necessarily equal to 1. Thus $G_\infty((n_k))=\{1\}$, and we are done.

6.3. IP-Dirichlet systems with disjoint spectral measures. — We gave in Proposition 3.1 a condition on a sequence $(n_k)_{k\geq 1}$ implying the existence of a generalized Riesz product on $\mathbb{T}$ which is IP-Dirichlet with respect to $(n_k)_{k\geq 1}$. Actually, the flexibility of the construction allows us to show that there are uncountably many disjoint such Riesz products. Recall that two probability measures σ and σ' on $\mathbb{T}$ are said to be *disjoint* if there exist two disjoint Borel subsets A and B of $\mathbb{T}$ such that $\sigma(A)=\sigma'(B)=1$ and $\sigma(B)=\sigma'(A)=0$. When this is the case, we write $\sigma\perp\sigma'$.

Proposition 6.1. — *Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of integers. Suppose that there exists a sequence $(m_k)_{k\geq 1}$ of integers with $m_1\geq 3$ such that*

$$n_{k+1} - 4\sum_{j=1}^{k} m_j n_j \geq 1 \quad \textit{for each } k\geq 1, \tag{15}$$

and

$$n_{k+1} - 4\sum_{j=1}^{k} m_j n_j \longrightarrow +\infty \quad \textit{as } k\longrightarrow +\infty. \tag{16}$$

Let Θ be the set of all sequences $(\theta_k)_{k\geq 1}$ of real numbers such that $\theta_k\in\{1,\sqrt{\pi}\}$ for each $k\geq 1$.

For each $k\geq 1$, let $q_k\geq 1$ be an integer such that $q_k\pi\sqrt{2}\leq m_k+2$. For each sequence $\theta\in\Theta$, the continuous generalized Riesz product

$$\sigma_\theta = w^* - \lim_{N\to+\infty} \prod_{k=1}^{N} \frac{2}{[\theta_k m_k]+2}\,\Big|\sum_{j=1}^{[\theta_k m_k]+1} \sin\Big(\frac{j\pi}{[\theta_k m_k]+2}\Big) e^{2i\pi j n_k t}\Big|^2 d\lambda(t)$$

is such that for every finite subset $F \in \mathcal{F}$ and every integers j_k in $\{1, \ldots, q_k\}$, $k \in F$, one has

$$\hat{\sigma}_\theta\Big(\sum_{k\in F} j_k n_k\Big) \geq \prod_{k\in F}\Big(1 - 2\pi^2\Big(\frac{q_k}{[\theta_k m_k]+2}\Big)^2\Big) \tag{17}$$

and

$$\hat{\sigma}_\theta\Big(\sum_{k\in F} n_k\Big) = \prod_{k\in F}\cos\Big(\frac{\pi}{[\theta_k m_k]+2}\Big). \tag{18}$$

Moreover, if θ and θ' are two elements of Θ such that $\theta_k \neq \theta'_k$ for infinitely many integers $k \geq 1$, then for all integers $n, p \geq 1$ the two measures σ_θ^{*n} and $\sigma_{\theta'}^{*p}$ are disjoint.

As a consequence of Proposition 6.1, we obtain:

Corollary 6.2. — *If the sequence $(n_k)_{k\geq 1}$ satisfies the assumptions of either Corollary 3.2, Proposition 4.1 or Theorem 1.4, there exist uncountably many dynamical systems which are weakly mixing and IP-rigid with respect to $(n_k)_{k\geq 1}$, and which have reduced maximal spectral types which are pairwise disjoint.*

Proof. — Let σ_θ, $\theta \in \Theta$, be one of the measures associated to the sequence $(n_k)_{k\geq 1}$ obtained in the proof of Proposition 6.1. Observe that σ_θ is a continuous symmetric measure. Following the proof of [**1**, Prop. 1.2], let $(X_\theta, \mathcal{B}_\theta, m_\theta, T_\theta)$ be the Gauss dynamical system with spectral measure σ_θ. This system is weakly mixing and IP-rigid with respect to $(n_k)_{k\geq 1}$. It is well-known (see for instance [**4**, Ch. 14, Sec. 3, Th. 1]) that the reduced maximal spectral type of this system (i.e. the maximal spectral type of the Koopman operator U_{T_θ} acting on the set $L^2_0(X_\theta, \mathcal{B}_\theta, m_\theta)$ of functions of $L^2(X_\theta, \mathcal{B}_\theta, m_\theta)$ of mean 0) is equal to

$$\tau_\theta = \frac{1}{e-1}\sum_{n\geq 1}\frac{\sigma_\theta^{*n}}{n!}.$$

We claim that if θ and θ' are two elements of Θ with infinitely many distinct coordinates, then the two measures τ_θ and $\tau_{\theta'}$ are disjoint.

For each $n, p \geq 1$, there exist by Proposition 6.1 two disjoint Borel subsets $A_{\theta,n,p}$ and $A_{\theta',n,p}$ of $\mathbb{T}$ such that $\sigma_\theta^{*n}(A_{\theta,n,p}) = 1$, $\sigma_{\theta'}^{*p}(A_{\theta,n,p}) = 0$, $\sigma_{\theta'}^{*p}(A_{\theta',n,p}) = 1$ and $\sigma_\theta^{*n}(A_{\theta',n,p}) = 0$. For each $n \geq 1$, let $B_{\theta,n} = \cap_{s\geq 1} A_{\theta,n,s}$ and $B_{\theta',p} = \cap_{r\geq 1} A_{\theta',r,p}$. For each $n, p \geq 1$, the sets $B_{\theta,n}$ and $B_{\theta',p}$ are disjoint since $A_{\theta,n,p} \cap A_{\theta',n,p} = \varnothing$. Also $\sigma_{\theta'}^{*p}(B_{\theta,n}) = \sigma_\theta^{*n}(B_{\theta',p}) = 0$ while $\sigma_\theta^{*n}(B_{\theta,n}) = \sigma_{\theta'}^{*p}(B_{\theta',p}) = 1$. Set $E_\theta = \bigcup_{n\geq 1} B_{\theta,n}$ and $E_{\theta'} = \bigcup_{p\geq 1} B_{\theta',p}$. The two sets E_θ and $E_{\theta'}$ are disjoint. Also

$$\tau_\theta(E_\theta) = \frac{1}{e-1}\sum_{n\geq 1}\frac{\sigma_\theta^{*n}(E_\theta)}{n!} \geq \frac{1}{e-1}\sum_{n\geq 1}\frac{\sigma_\theta^{*n}(B_{\theta,n})}{n!} = \frac{1}{e-1}\sum_{n\geq 1}\frac{1}{n!} = 1.$$

Hence $\tau_\theta(E_\theta) = 1$. Moreover,

$$\tau_{\theta'}(E_\theta) = \frac{1}{e-1}\sum_{p\geq 1}\frac{\sigma_{\theta'}^{*p}(E_\theta)}{p!} = 0 \quad \text{since } \sigma_{\theta'}^{*p}(B_{\theta,n}) = 0 \ \text{ for each } n \geq 1.$$

In the same way we prove that $\tau_{\theta'}(E_{\theta'}) = 1$ while $\tau_\theta(E_{\theta'}) = 0$. We have thus proved that τ_θ and $\tau_{\theta'}$ are disjoint measures, and this yields Corollary 6.2. □

Proof of Proposition 6.1. — Let $\theta \in \Theta$. Since $[\theta_j m_j] \le \theta_j m_j + 1 \le \sqrt{\pi} m_j + 1 \le (\sqrt{\pi} + 1/3) m_j < 4m_j$ for every $j \ge 1$, conditions (15) and (16) of Proposition 6.1 imply that conditions (3) and (4) of Proposition 3.1 are true for the sequence $([\theta_k m_k])_{k\ge 1}$. So all of Proposition 6.1 but the last statement follows from Proposition 3.1. Denote for each $\theta \in \Theta$ by $P_{\theta,k}$ the polynomial on $\mathbb{T}$ defined by

$$P_{\theta,k}(e^{2i\pi t}) = \frac{2}{[\theta_k m_k]+2} \Big| \sum_{j=1}^{[\theta_k m_k]+1} \sin\Big(\frac{j\pi}{[\theta_k m_k]+2}\Big) e^{2i\pi jt}\Big|^2.$$

Let θ and θ' be two elements of Θ which have infinitely many distinct coordinates. Without loss of generality we can suppose that there is an infinite subset I of the integers such that $\theta_k = \sqrt{\pi}$ and $\theta'_k = 1$ for each $k \in I$. Let $n, p \ge 1$ be two integers. The following lemma, whose proof essentially follows from that of Th. 1.2 in the paper [**11**] of Peyrière (see also [**7**]), gives a criterion for the two measures σ_θ^{*n} and $\sigma_{\theta'}^{*p}$ to be disjoint:

Lemma 6.3. — *Let $\theta, \theta' \in \Theta$ and $n, p \ge 1$. Suppose that there exists a sequence $(j_k)_{k\ge 1}$ of integers with $|j_k| \le m_k$ for each $k \ge 1$ such that*

$$\sum_{k\ge 1} \Big| \hat{P}_{\theta,k}(j_k)^n - \hat{P}_{\theta',k}(j_k)^p \Big|^2 = +\infty. \tag{19}$$

Then the measures σ_θ^{*n} and $\sigma_{\theta'}^{*p}$ are disjoint.

We postpone the proof for the moment, and show that the assumption of Lemma 6.3 is satisfied.

Let $(j_k)_{k\ge 1}$ be a sequence of integers such that $j_k = o(m_k)$ as k tends to infinity. Then

$$\hat{P}_{\theta,k}(j_k) = 1 - \frac{\pi^2}{2}\frac{j_k^2}{\theta_k^2 m_k^2} + O(\frac{j_k^3}{m_k^3}) \quad \text{as } k \to +\infty.$$

Indeed we have from (8) that

$$\begin{aligned}
\hat{P}_{\theta,k}(j_k) \quad &= \quad \Big(1 - \frac{j_k}{[\theta_k m_k]+2}\Big)\cos\Big(\frac{j_k \pi}{[\theta_k m_k]+2}\Big) \\
&+ \frac{1}{[\theta_k m_k]+2} \sum_{j=1}^{j_k} \cos\Big(\frac{(j_k - j)\pi}{[\theta_k m_k]+2}\Big)\cos^j\Big(\frac{\pi}{[\theta_k m_k]+2}\Big) \\
&= \Big(1 - \frac{j_k}{[\theta_k m_k]+2}\Big)\Big(1 - \frac{\pi^2}{2}\frac{j_k^2}{([\theta_k m_k]+2)^2} + o(\frac{j_k^2}{m_k^2})\Big) \\
&+ \frac{1}{[\theta_k m_k]+2} \sum_{j=1}^{j_k} \Big(1 - \frac{\pi^2}{2}\frac{(j_k - j)^2}{([\theta_k m_k]+2)^2} + o(\frac{j_k^2}{m_k^2})\Big)\Big(1 - \frac{\pi^2}{2}\frac{j}{([\theta_k m_k]+2)^2} + o(\frac{j_k^2}{m_k^2})\Big) \\
&= 1 - \frac{j_k}{[\theta_k m_k]+2} - \frac{\pi^2}{2}\frac{j_k^2}{([\theta_k m_k]+2)^2} + O(\frac{j_k^3}{m_k^3}) \\
&+ \frac{j_k}{[\theta_k m_k]+2} - \frac{\pi^2}{2}\frac{1}{([\theta_k m_k]+2)^3}\sum_{j=1}^{j_k}((j_k - j)^2 + j) + O(\frac{j_k^3}{m_k^3}).
\end{aligned}$$

Now $\sum_{j=1}^{j_k} j = \frac{1}{2} j_k(j_k+1)$ while $\sum_{j=1}^{j_k}(j_k-j)^2 = \frac{1}{6}(j_k-1)j_k(2j_k-1)$. It follows that

$$\hat{P}_{\theta,k}(j_k) = 1 - \frac{\pi^2}{2}\frac{j_k^2}{([\theta_k m_k]+2)^2} + O(\frac{j_k^3}{m_k^3}) = 1 - \frac{\pi^2}{2}\frac{j_k^2}{\theta_k^2 m_k^2} + O(\frac{j_k^3}{m_k^3}).$$

In the same way

$$\hat{P}_{\theta',k}(j_k) = 1 - \frac{\pi^2}{2}\frac{j_k^2}{\theta_k'^2 m_k^2} + O(\frac{j_k^3}{m_k^3}) \quad \text{as } k \to +\infty.$$

It follows that

$$\begin{aligned} \Big|\hat{P}_{\theta,k}(j_k)^n - \hat{P}_{\theta',k}(j_k)^p\Big| &= \Big|\frac{n\pi^2}{2}\frac{j_k^2}{\theta_k^2 m_k^2} - \frac{p\pi^2}{2}\frac{j_k^2}{\theta_k'^2 m_k^2}\Big| + O(\frac{j_k^3}{m_k^3}) \\ &= \frac{\pi^2}{2}\frac{j_k^2}{m_k^2}\Big|\frac{n}{\theta_k^2} - \frac{p}{\theta_k'^2}\Big| + O(\frac{j_k^3}{m_k^3}). \end{aligned}$$

Remember now that for each $k \in I$, $\theta_k = \sqrt{\pi}$ and $\theta'_k = 1$, and that I is an infinite set. Hence for every $k \in I$,

$$\Big|\frac{n}{\theta_k^2} - \frac{p}{\theta_k'^2}\Big| = \Big|\frac{n}{\pi} - p\Big| > 0.$$

So

$$\Big|\hat{P}_{\theta,k}(j_k)^n - \hat{P}_{\theta',k}(j_k)^p\Big|^2 \sim \frac{\pi^4}{4}\Big|\frac{n}{\pi} - p\Big|^2\Big(\frac{j_k}{m_k}\Big)^4 \quad \text{as } k \to +\infty,\ k \in I.$$

If the sequence $(j_k)_{k\ge1}$ is chosen in such a way that $j_k = o(m_k)$ as k tends to infinity and $\sum_{k\in I}(\frac{j_k}{m_k})^4 = +\infty$, condition (19) is satisfied for all integers $n, p \ge 1$. The conclusion then follows from Lemma 6.3. □

Proof of Lemma 6.3. — As mentioned already above, this proof is extremely close to that of [**11**, Th. 1.2], but we include it for completeness's sake. Denote by μ_θ the measure σ_θ^{*n}, and by $\mu_{\theta'}$ the measure $\sigma_{\theta'}^{*p}$. For every $k \ne l$ we have $\hat{\mu}_\theta(j_k n_k) = \hat{P}_{\theta,k}(j_k)^n$, $\hat{\mu}_\theta(j_l n_l) = \hat{P}_{\theta,l}(j_l)^n$ and

$$\hat{\mu}_\theta(j_k n_k - j_l n_l) = \hat{P}_{\theta,k}(j_k)^n \hat{P}_{\theta,l}(j_l)^n = \hat{\mu}_\theta(j_k n_k)\hat{\mu}_\theta(j_l n_l).$$

Also $\hat{\mu}_{\theta'}(j_k n_k) = \hat{P}_{\theta',k}(j_k)^p$, $\hat{\mu}_{\theta'}(j_l n_l) = \hat{P}_{\theta',l}(j_l)^p$ and

$$\hat{\mu}_{\theta'}(j_k n_k - j_l n_l) = \hat{P}_{\theta',k}(j_k)^p \hat{P}_{\theta',l}(j_l)^p = \hat{\mu}_{\theta'}(j_k n_k)\hat{\mu}_{\theta'}(j_l n_l).$$

All the Fourier coefficients of the measures μ_θ and $\mu_{\theta'}$ are real. Consider the functions $f_{\theta,k}$ and $f_{\theta',k}$ defined on $\mathbb{T}$ by $f_{\theta,k}(e^{2i\pi t}) = e^{2i\pi j_k n_k t} - \hat{\mu}_\theta(j_k n_k)$ and $f_{\theta',k}(e^{2i\pi t}) = e^{2i\pi j_k n_k t} - \hat{\mu}_{\theta'}(j_k n_k)$, $t \in [0,1)$. Then the functions $(f_{\theta,k})_{k\ge1}$ form an orthogonal family in $L^2(\mu_\theta)$, and $||f_{\theta,k}||^2_{L^2(\mu_\theta)} = 1 - |\hat{\mu}_\theta(j_k n_k)|^2 \le 1$. It follows that if $(b_k)_{k\ge1}$ is any square-summable sequence of complex numbers, the series $\sum_{k\ge1} b_k f_{\theta,k}$ converges in $L^2(\mu_\theta)$. In the same way, the series $\sum_{k\ge1} b_k f_{\theta',k}$ converges in $L^2(\mu_{\theta'})$. Suppose that μ_θ and $\mu_{\theta'}$ are not disjoint. Then we can write $\mu_\theta = \mu_{\theta,a} + \mu_{\theta,s}$, where $\mu_{\theta,a}$ is absolutely continuous with respect to $\mu_{\theta'}$ and $\mu_{\theta,s}$ and $\mu_{\theta'}$ are disjoint. Write $d\mu_{\theta,a} = \varphi d\mu_{\theta'}$, where $\varphi \in L^1(\mu_{\theta'})$. Let $\varepsilon > 0$ and let A be a Borel subset of $\mathbb{T}$ such that $\mu_{\theta'}(A) > 0$ and $\varphi > \varepsilon$ on A. Consider the measure ν

on $\mathbb{T}$ defined by $d\nu = \varepsilon \mathbf{1}_A d\mu_{\theta'}$. Then $\nu \le \mu_{\theta'}$ and $\nu \le \mu_\theta$, and the two series $\sum_{k\ge1} b_k f_{\theta,k}$ and $\sum_{k\ge1} b_k f_{\theta',k}$ converge in $L^2(\nu)$. Hence the series

$$\sum_{k\ge1} b_k(f_{\theta,k} - f_{\theta',k}) = \sum_{k\ge1} b_k(\hat{\mu}_\theta(j_k n_k) - \hat{\mu}_{\theta'}(j_k n_k)) = \sum_{k\ge1} b_k(\hat{P}_{\theta,k}(j_k)^n - \hat{P}_{\theta',k}(j_k)^p)$$

is convergent. This being true for any square-summable sequence $(b_k)_{k\ge1}$, it follows that the series

$$\sum_{k\ge1} \left| \hat{P}_{\theta,k}(j_k)^n - \hat{P}_{\theta',k}(j_k)^p \right|^2$$

is convergent, which contradicts our assumption (19). The two measures μ_θ and $\mu_{\theta'}$ are hence disjoint. □

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Sophie Grivaux, CNRS, Laboratoire Paul Painlevé, UMR 8524, Université Lille 1, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France • *E-mail :* `grivaux@math.univ-lille1.fr`