TEN QUESTIONS IN LINEAR DYNAMICS

by

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Abstract. — Linear dynamical systems are systems of the form (X, T), where X is an infinite-dimensional separable Banach space and $T \in \mathcal{B}(X)$ is a bounded linear operator on X. We present and motivate ten questions concerning these systems, which bear on both topological and ergodic-theoretic aspects of the theory.

1. Introduction

The study of linear dynamics is the study of dynamical systems of the form (X, T), where X is a separable real or complex infinite-dimensional Banach space and $T \in \mathcal{B}(X)$ is a bounded linear operator on X. One investigates the behavior of the iterates T^n, $n \geq 0$, of T, and the properties of the orbits $\text{Orb}(x, T) = \{T^n x;\ n \geq 0\}$ of vectors x of X are of special interest. Roughly speaking, one may look at these dynamical systems from two different points of view:

— from the topological point of view: if U is a non-empty open subset of X, what can be said about the iterates $T^n(U)$ of this open set? A basic notion in this setting is that of topological transitivity: T is *topologically transitive* if, whenever U and V are two non-empty open subsets of X, there exists an integer $n \geq 0$ such that $T^n(U) \cap V$ is non-empty. Topological transitivity of T is equivalent to the fact that T is *hypercyclic*, i.e. that it admits a vector $x \in X$ whose orbit under the action of T is dense in X. Such vectors with dense orbits are called *hypercyclic vectors* for T, and the set of these vectors is usually denoted by $HC(T)$.

— from the measure-theoretic point of view: if $\mathcal{B}$ denotes the σ-algebra of Borel subsets of X and m is a Borel probability measure on X, one may consider T as a measurable transformation from $(X, \mathcal{B}, m)$ into itself. The game is then the following: given an operator $T \in \mathcal{B}(X)$, when is it possible to construct such a measure m which is invariant by T (i.e. such that $m(T^{-1}A) = m(A)$ for every $A \in \mathcal{B}$), and with respect to which T defines an ergodic transformation? Recall that T is said to be *ergodic* if whenever $A, B \in \mathcal{B}$ are

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two sets such that $m(A) > 0$ and $m(B) > 0$, there exists an integer $n \geq 0$ such that $m(T^{-n}A \cap B) > 0$.

Of course, topological and measurable dynamics are not two independent branches of dynamics, and there is a strong interplay between them. Here is a typical instance of such a phenomenon: suppose that $T \in \mathcal{B}(X)$ is such that it admits an invariant measure with respect to which it is ergodic, and that this measure m has full support in the sense that $m(U) > 0$ for every non-empty open subset U of X. Birkoff's ergodic theorem then implies that for every non-empty open subset U,

$$\frac{1}{N}\#\{1 \leq n \leq N \ ; \ T^n x \in U\} \to m(U) \quad \text{as } N \to +\infty \text{ for } m\text{-almost every } x \in X.$$

It follows immediately from this that T is hypercyclic, but also that it enjoys a stronger property: for m-almost every $x \in X$,

$$\underline{\mathrm{dens}}\{n \geq 0 \ ; \ T^n x \in U\} = \underline{\lim}_{N \to +\infty} \frac{1}{N}\#\{1 \leq n \leq N \ ; T^n x \in U\} > 0$$

for every non-empty open subset U of X. Vectors x enjoying this property are called *frequently hypercyclic* vectors, and when such vectors exist, T itself is called a *frequently hypercyclic* operator. A related notion is that of $\mathcal{U}$-frequent hypercyclicity: T is called a *$\mathcal{U}$-frequently hypercyclic* operator if it admits a vector $x \in X$ such that

$$\overline{\mathrm{dens}}\{n \geq 0 \ ; \ T^n x \in U\} = \overline{\lim}_{N \to +\infty} \frac{1}{N}\#\{1 \leq n \leq N \ ; T^n x \in U\} > 0$$

for every non-empty open subset U of X.

This is only a very brief introduction to some key concepts in linear dynamics, and for more information the reader is referred to one of the following references: the survey [**20**] presents a detailed picture of hypercyclicity and universality issues until the 90's. The two recent books [**7**] and [**21**] are references in the subject and contain a lot of material. The book [**21**] focuses on topological issues and contains a chapter on frequent hypercyclicity, where some results bearing on this subject are proved without having recourse to the ergodic-theoretic approach. The book [**7**] is more advanced, and the reader will find here in particular a presentation of linear dynamical systems from the measure-theoretic point of view.

My goal here is to present and motivate ten questions in linear dynamics. Some of them are definitely hard, while some others ought to be more accessible. This paper does not aim at making a list of all open questions in linear dynamics, and the questions I selected simply reflect my own interests (as of 2013, with an update in 2016). They are organized around three main topics: the Hypercyclicity Criterion, frequent hypercyclicity and ergodicity, and non-recurrence for weakly mixing systems.

2. Questions around the Hypercyclicity Criterion

One of the major open questions in hypercyclicity theory, which was answered in 2006 by De la Rosa and Read [**14**], was to know whether every hypercyclic operator T satisfied the so-called Hypercyclicity Criterion. The Hypercyclicity Criterion is a powerful tool for showing that an operator is hypercyclic, and its first version was given in Kitai's thesis in

1970. Many improvements and equivalent formulations were obtained afterwards, and we state here the most general version of the Hypercyclicity Criterion, which is due to Bès and Peris [**12**]:

Suppose that there exist two dense subsets D_1 and D_2 of X, a strictly increasing sequence $(n_k)_{k\geq 0}$ of integers and a sequence $(S_{n_k})_{k\geq 0}$ of maps from D_2 into X such that

(i) $T^{n_k}x \to 0$ *as* $k \to +\infty$ *for every* $x \in D_1$*;*

(ii) $S_{n_k}y \to 0$ *as* $k \to +\infty$ *for every* $y \in D_2$*;*

(iii) $T^{n_k}S_{n_k}y \to y$ *as* $k \to +\infty$ *for every* $y \in D_2$*.*

Then T is hypercyclic, and moreover the direct sum operator $T \oplus T$ is hypercyclic on the space $X \oplus X$ (that is, $X \times X$ endowed with the norm $||(x,y)|| = \max(||x||, ||y||)$ for instance, or with any equivalent norm).

It was shown by Bès and Peris in [**12**] that T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic. Until 2006, every known hypercyclic operator satisfied the Hypercyclicity Criterion, and the question was to know whether the fact that T was hypercyclic implied automatically that $T \oplus T$ was hypercyclic or not. De la Rosa and Read constructed in 2006 a counterexample [**14**], and shortly afterwards, Bayart and Matheron improved this counterexample and showed in [**7**] that many classical spaces, such as $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$ (in particular the Hilbert space) supported a hypercyclic operator which did not satisfy the Hypercyclicity Criterion. Then Shkarin obtained in 2010 in [**36**] an example of an operator T on a Banach space of the form $T = I + Q$, with Q a quasi-nilpotent operator, such that $T \oplus T$ was not hypercyclic. This enabled him to give an example of a hypercyclic C_0-semi-group $(T_t)_{t\geq 0}$ of operators on a Banach space such that the direct sum $(T_t \oplus T_t)_{t\geq 0}$ was not hypercyclic. Shkarin's construction is carried out on an ad-hoc Banach space, and the following question from [**36**] is quite natural:

Question 1. — [**36**] *Does there exist an operator T on a Hilbert space H such that T is hypercyclic, the spectrum $\sigma(T)$ of T is reduced to the singleton $\{1\}$, and $T \oplus T$ is not hypercyclic?*

The operators of [**36**] are quasi-nilpotent perturbations of the identity, and one might wonder whether it is possible to construct such counterexamples as compact, or even nuclear, perturbations of the identity. That such operators can be constructed does not seem clear at the moment, as being compact is a much stronger requirement than being quasi-nilpotent. An analogy (which has perhaps nothing to do with the matter at hand) is that, while there exist quasi-nilpotent operators on certain Banach spaces which have no non-trivial invariant closed subspace [**33**], it is known by a result of Lomonosov [**28**] that a compact operator always has a non-trivial closed invariant subspace. Let us summarize this as Question 2:

Question 2. — *Let T be a hypercyclic operator on a Banach space X, where T can be written as $T = I + K$ with K a compact operator. Does T necessarily satisfies the Hypercyclicity Criterion?*

If the answer to Question 2 were affirmative, the pathological spaces constructed recently by Argyros and Haydon [**1**] would give examples of spaces on which every hypercyclic operator automatically satisfies the Hypercyclicity Criterion (as every operator on one of these spaces has the form $\lambda I + K$, where λ is a scalar and K is a compact operator).

3. Questions around frequent hypercyclicity and ergodicity

Let us begin this section with a well-known heuristic statement: dynamics of linear operators, especially in their aspects connected to ergodic theory, are very much influenced by the properties of the eigenvectors of the operator. This was first discovered by Godefroy and Shapiro, who produced in [**19**] their well-known criterion that if the eigenvectors of an operator T associated to eigenvalues of modulus greater than 1 and smaller than 1, respectively, span a dense subspace of the space, then T is hypercyclic. This was developed by Bourdon and Shapiro in [**13**], and then in the works [**4**], [**5**], [**6**]. The notion of frequent hypercyclicity was introduced and investigated there, and the study of operators from the ergodic-theoretic point of view was also developed there, building on early work of Flytzanis [**17**]. The story went on afterwards (see for instance [**7**], [**22**], and [**8**], [**9**], [**30**], [**25**], [**29**], [**26**]), and I will only state for the time being one result which is obtained by putting together results of [**5**], [**6**], [**7**], [**22**], and [**8**]:

Let X be any complex separable infinite-dimensional Banach space, and $T \in \mathcal{B}(X)$ an operator whose unimodular eigenvectors (i.e. eigenvectors associated to eigenvalues of modulus 1*) satisfy the following assumption:*

$(\star)$ if D is any countable subset of the unit circle $\mathbb{T}$, the linear span of the eigenvectors of T associated to eigenvalues λ which belong to $\mathbb{T} \setminus D$ is dense in X.

Then there exists a (Gaussian) measure m on X with full support such that T defines an ergodic (actually, weakly mixing) transformation of $(X, \mathcal{B}, m)$. In particular, T is frequently hypercyclic. When X is a Hilbert space, the reverse of this implication holds true, and $T \in \mathcal{B}(H)$ is weakly mixing with respect to a Gaussian measure on H with full support if and only if it satisfies assumption $(\star)$.

Our first observation, which is implicit in Flytzanis' paper [**17**], is a rather intriguing one: a much weaker form of the statement above, which is easily obtained by combining results from [**4**] and [**22**], is that if assumption $(\star)$ is satisfied for an operator $T \in \mathcal{B}(X)$, then T is hypercyclic. Of course, any operator satisfying $(\star)$ has uncountably many eigenvalues of modulus 1, and these eigenvectors span a dense subspace of X. The surprising fact is that the answer to the following question was unknown until very recently: does there exist a hypercyclic operator T on a complex Banach space X whose unimodular eigenvectors span a dense subspace of X, but which has only countably many eigenvectors? One did not know either of any example of an operator which would be hypercyclic, have spanning unimodular eigenvectors, but which would not satisfy assumption $(\star)$ above. A related question of [**22**] involving frequent hypercyclicity asked whether a hypercyclic operator whose unimodular eigenvectors span a dense subspace of X was necessarily frequently hypercyclic. This question was interesting in particular for *chaotic operators* (and was stated first in this context in [**5**]), which are hypercyclic operators with a dense set of periodic points (it is not too difficult to see that an operator $T \in \mathcal{B}(X)$ is chaotic if and only if it is hypercyclic and its eigenvectors associated to eigenvalues which are N-th roots of 1 span a dense subspace of X). This important question remained open for several years, until Menet answered it in the negative in [**29**] by constructing chaotic operators on the spaces $\ell_p(\mathbb{N})$, $1 \le p < +\infty$, which are not frequently hypercyclic (and even not $\mathcal{U}$-frequently hypercyclic). The periodic points play a crucial role in the construction of [**29**], which leads to the following intriguing question:

Question 3. — *Let $T \in \mathcal{B}(X)$ be a hypercyclic operator on a complex separable Banach space X, admitting a family $(u_i)_{i\ge1}$ of unimodular eigenvectors associated to eigenvalues $(\lambda_i)_{i\ge1}$ which span the space X. Suppose additionally that the eigenvalues λ_i, $i \ge 1$, are rationally independent. Is T necessarily $\mathcal{U}$-frequently hypercyclic? frequently hypercyclic? ergodic?*

That the links between properties of the unimodular eigenvectors and frequent hypercyclicity (or ergodicity) are still not so well understood is attested anew by another question: it is known that operators admitting an ergodic measure with full support (we call such operators *ergodic* operators) need not have any unimodular eigenvalue. Two examples of such operators are given in [**6**] and they live on the spaces $c_0(\mathbb{N})$ and $C_0[0,1]$ (continuous functions on $[0,1]$ which vanish at the point 0) respectively. These two examples thus live on non-reflexive spaces. Examples of ergodic operators on a separable Hilbert space admitting only countably many unimodular eigenvectors were constructed very recently in [**26**], using some tools from [**25**], but the following question remains open:

Question 4. — *Let X be a reflexive complex separable Banach space, and let $T \in \mathcal{B}(X)$ be an ergodic operator (or a frequently hypercyclic operator, or even a $\mathcal{U}$-frequently hypercyclic operator) on X. Does T necessarily have a unimodular eigenvalue?*

A positive answer to Question 4 would shed much light on the next question, which concerns the links between frequent hypercyclicity and the Invariant Subset Problem :

Question 5. — *Let T be a frequently hypercyclic operator on a (reflexive) complex separable Banach space. Does T necessarily have a non-trivial invariant closed subset?*

The Read's type counterexamples of [**32**] or [**24**] have the property that all the non-zero vectors of the space are hypercyclic for the constructed operators - but none of these operators is frequently hypercyclic. It was proved in [**25**] that $\mathcal{U}$-frequently hypercyclic operators on reflexive spaces admit an invariant measure with full support. But an example of a frequently hypercyclic operator on a Hilbert space admitting no ergodic measure with full support was constructed in [**26**].

The differences between $\mathcal{U}$-frequent and frequent hypercyclicity deserve to be better understood too. Bayart and Rusza provided in [**9**] the first example of a $\mathcal{U}$-frequently hypercyclic operator which is not frequently hypercyclic (their counterexample is a bilateral weighted shift on $c_0(\mathbb{N})$). Such counterexamples were recently shown in [**26**] to exist on the Hilbert space, but the intricacy of the construction (relying on a development of the ideas of Menet in [**29**]) suggests the following question:

Question 6. — *Find "natural" examples of $\mathcal{U}$-frequently hypercyclic operators on a separable Hilbert space which are not frequently hypercyclic.*

It might be an idea to look for such counterexamples within the class of perturbations of unitary diagonal operators by weighted backward shifts, i. e. operators on $\ell_2(\mathbb{N})$ of the form $T = D_\lambda + B_\omega$, where D_λ is the diagonal operator on $\ell_2(\mathbb{N})$ with respect to the canonical basis $(e_n)_{n\geq 1}$ associated to a sequence $\lambda = (\lambda_n)_{n\geq 1}$ of unimodular complex numbers, and B_ω is the weighted unilateral backward shift with respect to $(e_n)_{n\geq 1}$ associated to a bounded sequence $\omega = (\omega_n)_{n\geq 1}$ of positive weights. Surprisingly, very little seems to be known concerning the dynamics of such operators (although they play a crucial role in the proof of some results in linear dynamics, like [**16**] or [**26**]):

Question 7. — *Is it possible to characterize the pairs of parameters (λ, ω) such that $D_\lambda + B_\omega$ is hypercyclic? ($\mathcal{U}$-)frequently hypercyclic? ergodic?*

We finish this section with an open question about the spectrum of frequently hypercyclic operators:

Question 8. — *Which compact subsets K of $\mathbb{C}$ can be realized as the spectrum of a frequently hypercyclic operator?*

A compact subset K of $\mathbb{C}$ can be realized as the spectrum of a hypercyclic operator on a complex Hilbert space if and only if every connected component of K intersects the unit circle [**37**]. On the contrary, it is shown by Shkarin in the same paper [**37**] that no operator T on a Banach space X whose spectrum contains an isolated point can be frequently hypercyclic. This is the case for instance of any operator T such that $\sigma(T) = \{1\}$. A strengthening of this result was obtained by Beise, who showed in [**10**] that under certain additional conditions, the spectrum of a frequently hypercyclic operator cannot be contained in the closed unit disk and intersect the unit circle in only finitely many points. But it is not known if the spectrum of a frequently hypercyclic operator can coincide with the set $[0, 2]$, for instance.

4. Questions around non-recurrence for weakly mixing systems

Before stating our questions on this topic, a few reminders about weakly mixing systems are in order. Recall that Birkoff's ergodic theorem implies that T is ergodic if and only if

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} m(T^{-n}A \cap B) = m(A)\, m(B) \quad \text{for every sets } A, B \in \mathcal{B}.$$

A (much) stronger notion than ergodicity is that of *strong mixing*: here one requires that $m(T^{-n}A \cap B) \to m(A)\, m(B)$ as $n \to +\infty$. Intuitively, the two events $T^{-n}A$ and B become asymptotically independent as n tends to infinity: $m(T^{-n}A \cap B)$ becomes closer and closer to $m(T^{-n}A)\, m(B) = m(A)\, m(B)$. Between these two notions of ergodicity and strong mixing stands the notions of weak-mixing: T is *weakly mixing* if

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} |m(T^{-n}A \cap B) - m(A)\, m(B)| = 0 \quad \text{for every } A, B \in \mathcal{B},$$

or equivalently if for each sets $A, B \in \mathcal{B}$, $m(T^{-n}A \cap B) \to m(A)\, m(B)$ as n tends to infinity along a certain subset D of $\mathbb{N}$ of density 1. Also, T is weakly mixing if and only if $T \times T$ is ergodic on $(X \times X, \mathcal{B} \otimes \mathcal{B}, m \otimes m)$. For more information about such notions and examples of weakly mixing systems, the reader can consult one of the books [**38**], [**18**], or [**27**]. In the setting of linear dynamical systems, an operator T on a complex Hilbert space is weakly mixing with respect to a Gaussian measure with full support if and only if its unimodular eigenvectors satisfy assumption $(\star)$ [**5**].

The Poincaré recurrence theorem states that whenever T is a measure-preserving transformation of $(X, \mathcal{B}, m)$ and $A \in \mathcal{B}$ is such that $m(A) > 0$, there exists an integer $n \geq 1$ such that $m(T^{-n}A \cap A) > 0$; in other words, the set $\mathbb{N}$ is a recurrence set for any measure-preserving transformation. If D is an infinite subset of $\mathbb{N}$, D is called a *recurrence set* if for any measure-preserving system $(X, \mathcal{B}, m; T)$ and any $A \in \mathcal{B}$ with $m(A) > 0$, there exists an integer $n \in D$ such that $m(T^{-n}A \cap A) > 0$. If now $(n_k)_{k \geq 0}$ is a strictly increasing sequence of integers, we say that $\{n_k \,;\, k \geq 0\}$ is a *non-recurrence set for the system* $(X, \mathcal{B}, m; T)$ if there exists a set $A \in \mathcal{B}$ with $m(A) > 0$ such that $m(T^{-n_k}A \cap A) = 0$ for every $k \geq 0$.

It is not difficult to see that a non-recurrence set for a weakly mixing system must have density 0. Non-recurrence sets for weakly mixing systems have been investigated in the two papers [**11**] and [**23**], and it was shown for instance in [**23**], using linear dynamical systems, that sets $\{n_k \,;\, k \geq 0\}$ such that n_k divides n_{k+1} for each $k \geq 0$ are non-recurrence sets for weakly mixing systems. A question of [**11**] is the following: if $\{n_k \,;\, k \geq 0\}$ is a lacunary set, is it necessarily a non-recurrence set for some weakly mixing dynamical system? Recall that $\{n_k \,;\, k \geq 0\}$ is called *lacunary* if there exists an $a > 1$ such that $n_{k+1}/n_k \geq a$ for each $k \geq 0$. Under this assumption, there exist uncountably many elements λ of the unit circle $\mathbb{T}$ such that $\inf_{k \geq 0} |\lambda^{n_k} - 1| > 0$ ([**15**], [**31**]), and so $\{n_k \,;\, k \geq 0\}$ is clearly a non-recurrence set for some ergodic rotation of the unit circle. It would be very exciting to be able to generalize the methods of [**23**] to obtain an answer to the following question:

Question 9. — *Let $\{n_k \,;\, k \geq 0\}$ be a lacunary set. Does there exist a linear dynamical system which is weakly mixing and non-recurrent with respect to the set $\{n_k \,;\, k \geq 0\}$?*

This does not seem to be easy. A way to approach this question would be to understand better operators which are partially power-bounded in the measure-theoretic sense. More precisely, recall that an operator T on a Banach space X is said to be *partially power-bounded* with respect to a sequence $(n_k)_{k \geq 0}$ if $\sup_{k \geq 0} ||T^{n_k}||$ is finite. Sequences $(n_k)_{k \geq 0}$ with $n_0 = 1$ for which there exists a partially power-bounded operator T on a separable Banach space X with uncountable unimodular point spectrum are studied in [**34**], [**35**], [**2**], [**3**], and [**16**]. The following statement is proved in [**16**]:

There exists a weakly mixing transformation T of a complex separable Hilbert space H with $\sup_{k \geq 0} ||T^{n_k}|| < +\infty$ *if and only if the sequence $(n_k)_{k \geq 0}$ has the following property: for every $\varepsilon > 0$, there exists a $\lambda \in \mathbb{T}$ such that* $\sup_{k \geq 0} |\lambda^{n_k} - 1| < \varepsilon$. *Such sequences are called* non-Jamison sequences.

The philosophy of the non-recurrence results proved in [**22**] is the following: using the assumption on $(n_k)_{k \geq 0}$, one constructs a weakly mixing T such that the iterates

T^{n_k}, $k \geq 0$ are controlled, and preferably tending in some sense to the identity operator (for instance if $(n_k)_{k\geq 0}$ is a non-Jamison sequence, one can construct a T with $\sup_{k\geq 0} ||T^{n_k}|| < +\infty$, and with a little more work, one can manage to ensure, for a fixed $\varepsilon > 0$, that $\sup_{k\geq 0} ||T^{n_k}|| < 1 + \varepsilon$). Then, since $(n_k)_{k\geq 0}$ is a lacunary sequence, one can find $\delta > 0$ and $\lambda_0 \in \mathbb{T}$ such that $|\lambda_0^{n_k} - 1| > \delta$ for all $k \geq 0$. The operator $S = \lambda_0 T$ is now weakly mixing, and if we have been careful enough, it will be non-recurrent with respect to the set $\{n_k \, ; \, k \geq 0\}$. In order to be able to extend this method to larger classes of sequences, one needs to be able to find classes of sequences $(n_k)_{k\geq 0}$ for which the answer to the following question is positive:

Question 10. — *For which sequences $(n_k)_{k\geq 0}$ is it true that there exists a bounded operator T on a complex Banach (or Hilbert) space X which is weakly mixing with respect to some Gaussian measure m with full support, and such that $\sup_{k\geq 0} ||T^{n_k}x||$ is finite for m-almost every $x \in X$?*

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