TOPOLOGICAL AND MEASURABLE UNIVERSALITY IN LINEAR DYNAMICS

by

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Abstract. — This is for the most part a survey paper, in which we present some results related to the construction of universal linear dynamical systems, i.e. of operators on Banach spaces which can represent, in various senses, certain classes of (nonlinear) dynamical systems. We first describe and generalize a result of Feldman proving the existence of universal linear systems in the topological sense. We then present the first example, due to Glasner and Weiss, of a universal linear system in the measurable framework. After this we report on a recent generalization of the Glasner-Weiss universality result, giving a simple and general criterion for an operator on a separable Banach space to be universal. The main ideas of the proof of this criterion are also presented.

1. Introduction

We present here some results concerning universality properties of linear dynamical systems, both from the topological and the measurable points of view. Precise definitions will be given below, but the intuitive definition of a universal system is that it "represents" all systems from a natural class. Topologically universal systems represent all homeomorphisms, or all self-maps, of compact spaces, while universal systems in the measurable framework represent all (invertible) ergodic systems on a standard probability space.

The first example of a topologically universal linear system was given by Feldman in [7]. After a very brief introduction to linear dynamical systems (Section 2), we present this result in Section 3 below, using a slightly different approach from the one of [7]. This allows us to obtain new classes of topologically universal linear systems (Theorem 3.3): these classes consist of all finite ℓ_p- or c_0-sums of operators on separable Banach spaces satisfying certain Properties (P) or (P') (see Definition 3.2 below), which stipulate the existence of a vector whose orbit has suitable summability properties. We also remark (Theorem 3.8) that universal operators in the sense of Rota and Caradus are, when suitably renormalized, universal for all self-maps of compact spaces.

We then move over to the measurable framework, and present in Section 4 a result of Glasner and Weiss, who obtained in [9] the first example of a universal linear system in this setting. Their example actually fits into a more general picture, since they constructed, for a large class of topological groups, universal representations for all free ergodic actions of

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the group. The approach of [9] is taken up in [11], where a simple criterion is obtained for an operator on a separable Banach space to induce a universal linear dynamical system. We present this criterion in Section 5, as Theorem 5.2: it states that operators satisfying a reinforced version of Property (P), called Property (Q) (see Definition 5.1 below), are universal for invertible ergodic systems on a standard probability space, while operators satisfying Property (Q'), which is a reinforced version of Property (P'), are universal for all ergodic systems. Thanks to this criterion, many classical hypercyclic operators can be shown to be universal. We also give in Section 5 some key ideas for the proof of the universality criterion of [11]. We propose along the way some challenging questions on universal systems, which concern in particular the possible links between topological and measurable universality.

2. Linear dynamical systems?

The study of linear dynamics is the study of dynamical systems of the form (Z, A), where Z is a real or complex infinite-dimensional separable Banach space and $A \in \mathcal{B}(Z)$ is a bounded linear operator on Z. One investigates the behavior of the iterates A^n, $n \geq 0$, of A, and the properties of the orbits $\text{Orb}(z, A) = \{A^n z;\ n \geq 0\}$ of vectors z of Z are of special interest. Roughly speaking, one may look at these dynamical systems from two different points of view:

— from the topological point of view: if U is a non-empty open subset of Z, what can be said about the iterates $A^n(U)$ of this open set? A basic notion in this setting is that of topological transitivity: A is *topologically transitive* if, whenever U and V are two non-empty open subsets of Z, there exists an integer $n \geq 0$ such that $A^n(U) \cap V$ is non-empty. Topological transitivity of A is equivalent to the fact that A is *hypercyclic*, i.e. that it admits a vector $z \in Z$ whose orbit under the action of A is dense in Z. Such vectors with dense orbits are called *hypercyclic vectors* for A, and the set of these vectors is usually denoted by $HC(A)$.

— from the measure theoretic point of view: if $\mathcal{B}$ denotes the σ-algebra of Borel subsets of Z and m is a Borel probability measure on Z, one may consider A as a measurable transformation from $(Z, \mathcal{B}, m)$ into itself. The game is then the following: given an operator $A \in \mathcal{B}(Z)$, when is it possible to construct such a measure m which is invariant by A (i.e. such that $m(A^{-1}B) = m(B)$ for every $B \in \mathcal{B}$), and with respect to which A defines an ergodic transformation? Recall that A is said to be *ergodic* if whenever $B, C \in \mathcal{B}$ are two sets such that $m(B) > 0$ and $m(C) > 0$, there exists an integer $n \geq 0$ such that $m(A^{-n}B \cap C) > 0$.

Of course, topological and measurable dynamics are not two independent branches of dynamics, and there is a strong interplay between them. Here is a typical instance of such a phenomenon: suppose that $A \in \mathcal{B}(Z)$ is such that it admits an invariant measure with respect to which it is ergodic, and that this measure m has full support in the sense that $m(U) > 0$ for every non-empty open subset U of Z. Birkhoff's ergodic theorem then implies that for every non-empty open subset U,

$$\frac{1}{N}\#\{1 \leq n \leq N\ ;\ \ A^n z \in U\} \to m(U) \quad \text{as } N \to +\infty \ \text{ for } m\text{-almost every } z \in Z.$$

It follows immediately from this that A is hypercyclic, but also that it enjoys a stronger property: for m-almost every $z \in Z$,

$$\underline{\text{dens}}\{n \geq 0\ ;\ A^n z \in U\} = \varliminf_{N \to +\infty} \frac{1}{N} \#\{1 \leq n \leq N\ ;\ A^n z \in U\} > 0$$

for every non-empty open subset U of Z. Vectors z enjoying this property are called *frequently hypercyclic* vectors, and when such vectors exist, A itself is called a *frequently hypercyclic* operator.

This is an extremely brief and partial introduction to some key concepts in linear dynamics, and for more information the reader is referred to one of the following texts: the survey [12] presents a detailed picture of hypercyclicity and universality issues until the 90's. The two recent books [4] and [13] are references in the subject and contain a lot of material. The book [13] focuses on topological issues and contains a chapter on frequent hypercyclicity, where some results bearing on this subject are proved without having recourse to the measurable approach. The book [4] is more advanced, and the reader will find here in particular a presentation of linear dynamical systems from the measure-theoretic point of view.

The papers [5], [18], [10] (among many other interesting references) present some even more recent results on the ergodic theory of linear dynamical systems, which are not included in either of the two books [13], [4]. The paper [17] by Menet, which gives examples of chaotic operators which are not frequently hypercyclic, should definitely be mentioned too.

The reader wishing to know more about ergodic theory and dynamics in general can consult one of the books [23], [19], [15], or [8].

Unless stated otherwise, Banach spaces in this paper can be taken to be either real or complex.

3. Universal systems in the topological sense: the construction of Feldman

3.1. Topological universality. — We study in this section universality properties of operators in the topological sense. It makes sense to introduce two slightly different notions, depending on whether we wish the operator to represent all continuous self-maps of compact metrizable spaces, or simply all homeomorphisms of such spaces.

Definition 3.1. — Let A be a bounded operator on a separable Banach space Z.

- We say that A is *universal for homeomorphisms of compact spaces* if, for every compact metrizable space K and every homeomorphism T of K, there exists a compact A-invariant subset L of Z such that T is topologically conjugate to the map induced by A on L (i.e. there exists an homeomorphism $\phi : K \longrightarrow L$ such that $\phi \circ T = A \circ \phi$).
- We say that A is *universal for self-maps of compact spaces* if the same property holds true for all continuous self-maps $T : K \longrightarrow K$ of compact metrizable spaces K.

Feldman proved in [7] that there exists a universal operator for self-maps of compact spaces: if B denotes as usual the backward shift operator on the (real or complex) space $\ell^2(\mathbb{N})$, the infinite direct sum $2B_\infty = \bigoplus_{\ell_2} 2B$ of the operator $2B$ on the Hilbert space $H = \bigoplus_{\ell_2} \ell^2(\mathbb{N})$ is shown in [7] to be universal for self-maps of compact spaces. Several other universality results of this kind are proved in [7], among which we quote the following one: for any integer $r \geq 1$, denote by B_r the direct ℓ_2-sum of r copies of B on the real (resp. complex) Hilbert space H_r which is the direct ℓ_2-sum of r copies of the real (resp. complex) space $\ell^2(\mathbb{N})$. Then $2B_r$ is universal for all self-maps of compact subsets of $\mathbb{R}^r$ (resp. $\mathbb{C}^r$).

3.2. A generalization of Feldman's result. — Our aim is now to prove some more natural generalizations of the universality results mentioned above. These involve two properties which, for lack of a better terminology, we will call respectively Property (P) and Property (P'). These two properties will appear again in a reinforced form in our study of universality for ergodic systems (see Section 5 below).

Definition 3.2. — Let A be a bounded operator on a separable Banach space Z.

- We say that A *has Property (P)* if there exists a sequence $(z_n)_{n\in\mathbb{Z}}$ of vectors of Z such that $Az_n = z_{n+1}$ for every $n \in \mathbb{Z}$ (in which case we write $z_n = A^n z_0$ for every $n \in \mathbb{Z}$, even when A is not invertible) and
  (a) the series $\sum_{n\in\mathbb{Z}} A^{-n} z_0$ is unconditionally convergent in Z;
  (b) the vector z_0 does not belong to the closed linear span in Z of the vectors $A^{-n}z_0$, $n \in \mathbb{Z} \setminus \{0\}$.
- We say that A *has Property (P')* if it satisfies the requirements of Property (P) above and, moreover, the sequence $(z_n)_{n\in\mathbb{Z}}$ is such that $A^r z_0 = 0$ for some $r \in \mathbb{Z}$ (or, equivalently, such that $z_0 = 0$).

It is not difficult to exhibit examples of operators with Property (P) or (P'), for instance among bilateral (resp. unilateral) backward shifts on $\ell_p(\mathbb{Z})$, $p \geq 1$, or $c_0(\mathbb{Z})$ (resp. $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$). For instance the shift operator S defined on $\ell_2(\mathbb{Z})$ by $Se_n = 2e_{n-1}$ for every $n \geq 1$ and $Se_n = \frac{1}{2}e_{n-1}$ for every $n \leq 0$ has Property (P), while all multiples λB with $|\lambda| > 1$ of the backward shift on $\ell_2(\mathbb{N})$ have Property (P').

The main result of this section is Theorem 3.3 below. In its statement, infinite direct sums will be either ℓ_p-sums for some $p \geq 1$, or c_0-sums.

Theorem 3.3. — *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For each $k \geq 1$, let A_k be a bounded operator on a separable Banach space Z_k over $\mathbb{K}$.*

(1) *If all operators A_k have Property (P), the infinite direct sum $A = \bigoplus_{k\geq 1} A_k$ on the space $Z = \bigoplus_{k\geq 1} Z_k$ is universal for homeomorphisms of compact spaces.*
(2) *If all operators A_k have Property (P'), the infinite direct sum $A = \bigoplus_{k\geq 1} A_k$ is universal for self-maps of compact spaces.*
(3) *Let $r \geq 1$ be an integer. If the operators A_k, $1 \leq k \leq r$, have Property (P) (resp. Property (P')), the finite direct sum $\bigoplus_{1\leq k\leq r} A_k$ is universal for homeomorphisms (resp. self-maps) of compact subsets of $\mathbb{K}^r$.*

The proof uses essentially the same arguments as those of [7]; we present them in a slightly different way, so as to be coherent with our approach of universality questions in the measurable framework in Sections 4 and 5.

Proof. — We first give the proof of assertion (1) (the proof of assertion (2) is extremely similar, and we will not detail it). Let T be a homeomorphism of a compact metrizable space (K, d), and let $(x_k)_{k\geq 1}$ be a dense subset of K. For each $k \geq 1$, let $z_{0,k}$ be a vector of Z_k such that the sequence $(A_k^n z_{0,k})_{n\in\mathbb{Z}}$ satisfies assumptions (a) and (b) of Property (P).

Since the series $\sum_{n\in\mathbb{Z}} A^{-n}z_{0,k}$ is unconditionally convergent, there exists, for each $k\ge 1$, a positive constant M_k such that

$$\Big\| \sum_{n\in\mathbb{Z}} \theta_n\, A_k^{-n}z_{0,k} \Big\|_{Z_k} \le M_k \sup_{n\in\mathbb{Z}} |\theta_n| \tag{1}$$

for every bounded sequence $(\theta_n)_{n\in\mathbb{Z}}$ of scalars. Also, assumption (b) of Property (P) implies that there exists a functional z_k^* on Z_k such that $\langle z_k^*, z_{0,k}\rangle = 1$ and $\langle z_k^*, A_k^{-n}z_{0,k}\rangle = 0$ for every $n\in\mathbb{Z}\setminus\{0\}$. We now define a map $\phi:\ K \longrightarrow Z$ by setting

$$\phi(x) = \bigoplus\nolimits_{k\ge 1} \frac{2^{-k}}{M_k} \sum_{n\in\mathbb{Z}} d(T^n x, x_k) A_k^{-n} z_{0,k} \quad \text{for every } x\in K.$$

Since the direct sum is either an l_p- or a c_0-sum, (1) implies that $\phi(x)$ is well-defined for every $x\in K$ and that ϕ is a continuous map from K into Z. Let us now check that ϕ is injective: let x and y be two elements of K such that $\phi(x)=\phi(y)$. Then

$$\sum_{n\in\mathbb{Z}} d(T^n x, x_k) A_k^{-n} z_{0,k} = \sum_{n\in\mathbb{Z}} d(T^n y, x_k) A_k^{-n} z_{0,k}$$

for every $k\ge 1$. Applying the functional z_k^* on both sides, we obtain that $d(x,x_k) = d(y,x_k)$. This being true for every $k\ge 1$, and $(x_k)_{k\ge 1}$ being dense in K, this implies that $x=y$. Thus ϕ is injective, and an homeomorphism from K onto its image L, which is a compact subset of Z. It is straightforward to check that $\phi(Tx) = A\phi(x)$ for every $x\in K$, and it follows that T and $A:\ L\longrightarrow L$ are topologically conjugate.

Let us now prove assertion (3). Let $r\ge 1$, and suppose that the operators A_k, $1\le k\le r$, have Property (P). Let $(e_1,\dots e_k)$ denote the canonical basis of $\mathbb{K}^r$. Using the same notation as in the proof of assertion (1) above, we consider the map $\phi:\ K\longrightarrow Z$ defined by setting

$$\phi(x) = \bigoplus\nolimits_{k=1}^{r} \sum_{n\in\mathbb{Z}} \langle T^n x, e_k\rangle A_k^{-n} z_{0,k} \quad \text{for every } x\in K.$$

This map is well-defined and continuous on K, and injective: if x and y are two elements of K such that $\phi(x)=\phi(y)$, $\langle x, e_k\rangle = \langle y, e_k\rangle$ for every $1\le k\le r$, so that $x=y$. Hence T is topologically conjugate to the map induced by A on $L=\phi(K)$. □

As an application of assertion (3) of Theorem 3.3, we obtain for instance:

Corollary 3.4. — *Operators with Property* (P) *are universal for homeomorphisms of the Cantor space.*

In view of the proof of Theorem 3.3, the following observation is in order:

Remark 3.5. — Let A be a bounded operator on a separable Banach space Z over $\mathbb{K}$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Suppose that A satisfies Property (P), and let $z_0\in Z$ be an associated vector satisfying properties (a) and (b) of Definition 3.2. Let T be a homeomorphism of a compact metric space (K,d), and let $f:\ K\longrightarrow\mathbb{K}$ be a continuous scalar-valued function on K. The map $\phi_f:\ K\longrightarrow Z$ defined by setting

$$\phi_f(x) = \sum_{n\in\mathbb{Z}} f(T^n x) A^{-n} z_0 \quad \text{for every } x\in K$$

is well-defined and continuous on K. The image $\phi_f(K)$ of K is a compact subset L of Z, and if we denote by A_L the transformation induced by A on L, the map ϕ_f intertwines T

and A_L in the sense that $\phi_f \circ T = A_L \circ \phi_f$. So A_L is a factor of T, with ϕ_f a factor map (see for instance [23, p. 140] for the definition of topological factors). If the map f can be chosen in such a way that ϕ_f is injective (which is for instance the case in the situation considered in Corollary 3.4 above), then ϕ_f is a topological isomorphism between the two systems (K, T) and (L, A_L), which are hence topologically conjugate.

3.3. Universal operators in the sense of Rota and Caradus. — It is interesting to observe that, as a simple corollary of the existence of topologically universal operators, one can exhibit another class of universal operators, which originally appeared in quite another context. Rota in [21], and after him Caradus in [6], call a bounded operator A on a separable Banach space universal if it enjoys the following property: for any bounded operator $T \in \mathcal{B}(Z)$, there exists a non-zero scalar λ, a closed A-invariant subspace Z_0 of Z, and a linear isomorphism W from Z onto Z_0, such that $W(\lambda T) = AW$. In other words, some non-zero multiple of T is similar to a part of A. The first example of an operator with this universality property (the infinite direct sum $B_\infty = \oplus_{\ell_2} B$ of the backward shift B on $\ell^2(\mathbb{N})$) was given by Rota in [21]. Caradus then provided in [6] a simple condition for an operator on a Hilbert space to be universal in the sense above.

Definition 3.6. — We say that a bounded operator A on a separable Hilbert space H *has Property* (C) if its kernel $\ker A$ is infinite dimensional and A is surjective.

If A is a bounded operator on a separable Hilbert space H which has Property (C), we denote by $\widetilde{A}$ the restriction of A to $(\ker A)^\perp$. It is an isomorphism from $(\ker A)^\perp$ onto H. Caradus' universality result runs as follows:

***Theorem 3.7* ([6]).** — *Let A be a bounded operator on a separable Hilbert space H which has Property (C). For any bounded operator T on H, and any non-zero scalar λ such that $|\lambda|.||T||.||\widetilde{A}^{-1}|| < 1$, λT is similar to a part of A.*

Thanks to this theorem, we can show that all operators with Property (C) are, after a suitable renormalization, universal for all self-maps of compact spaces:

Theorem 3.8. — *Let A be a bounded operator on a separable Hilbert space H which has Property (C). If $||\widetilde{A}^{-1}|| > 1$, A is universal for all self-maps of compact metric spaces.*

Proof. — Let λ be a scalar such that $1 < |\lambda| < ||\widetilde{A}^{-1}||$. By Theorem 3.3, the infinite direct sum operator $A_0 = \oplus_{\ell_2} \lambda B$ is universal for all self-maps of compact metric spaces. By Theorem 3.8, there exists a bounded operator W from H onto a closed subspace H_0 of H such that $WA_0 = AW$. It follows that the operator induced by A on H_0 is universal for all self-maps of compact spaces, and hence that A itself has the same property. □

3.4. Some open questions. — We conclude this section with some natural questions on topological universality. The first one concerns the possibility of characterizing topologically universal operators:

Question 3.9. — Is it possible to characterize the operators on a separable Hilbert space which are universal for homeomorphisms, or for self-maps, of compact spaces?

More concretely, it is a somewhat surprising fact that one has to take infinite direct sums of operators with Property (P) in Theorem 3.3 in order to obtain topologically universal systems for homeomorphisms or self-maps of compact spaces:

Question 3.10. — Are operators with Property (P) (resp. Property (P')) universal for homeomorphisms (resp. self-maps) of compact spaces?

The proof of Theorem 3.3 seems to point to a negative answer to Question 3.10, but I am not aware of any argument showing, for instance, that the operator $2B$ on $\ell^2(\mathbb{N})$ is not universal for self-maps of compact spaces.

4. Universal systems in the measurable sense: the construction of Glasner and Weiss

4.1. Universal representations of topological groups. — Glasner and Weiss introduced in [9] the notion of a universal representation of a topological group G:

Definition 4.1 **([9]).** — A representation $S = (S_g)_{g\in G}$ of a topological group G on a Banach space Z is called *universal* if for every ergodic probability-preserving free action $T = (T_g)_{g\in G}$ of G on a standard Lebesgue probability space $(X, \mathcal{B}, \mu)$, there exists a Borel probability measure ν on Z which is S-invariant, has full support, and is such that the actions of T and S of G on $(X, \mathcal{B}, \nu)$ and $(Z, \mathcal{B}_Z, \nu)$ respectively are isomorphic.

A universal representation of G thus models every possible free ergodic action of G on a probability space. Recall that $(T_g)_{g\in G}$ is said to be *free* if for every element $g \in G$ distinct from the identity, $\mu(\{x \in X;\, T_g x = x\}) = 0$, and *ergodic* if the following property holds true: if $B \in \mathcal{B}$ is such that $T_g^{-1}(B) = B$ for every $g \in G$, then $\mu(B) = 0$ or $\mu(B) = 1$. The existence of a universal representation is shown in [9] for a large class of groups:

Theorem 4.2 **([9]).** — *Let G be a topological group which belongs to one of the following classes:*

(1) countable discrete groups;
(2) locally compact, second countable, compactly generated groups;
(3) groups which can be written as the increasing union of a sequence of compact open subgroups.

Then G admits a universal representation $S = (S_g)_{g\in G}$ on a separable Hilbert space.

4.2. Universal operators. — When $G = \mathbb{Z}$, the main result of [9] states that there exists a bounded invertible operator on a separable Hilbert space which is *universal* in the sense of Definition 4.3 below. Recall (see for instance [23, Def. 2.4]) that two probability-preserving systems $(X_1, \mathcal{B}_1, m_1; T_1)$ and $(X_2, \mathcal{B}_2, m_2; T_2)$ are *isomorphic* if there exist sets $M_1 \in \mathcal{B}_1$, $M_2 \in \mathcal{B}_2$ with $m_1(M_1) = m_2(M_2) = 1$, $T_1(M_1) \subseteq M_1$, $T_2(M_2) \subseteq M_2$, and an invertible transformation $\Phi : M_1 \longrightarrow M_2$ such that $\Phi(T_1 x) = T_2\Phi(x)$ for every $x \in M_1$.

Definition 4.3. — Let A be a bounded operator on a separable Banach space Z.

- We say that A *universal for invertible ergodic systems* if it satisfies the following property: for every invertible ergodic dynamical system $(X, \mathcal{B}, \mu; T)$ on a standard Lebesgue probability space, there exists a probability measure ν on Z with full support which is A-invariant, and such that the systems $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$ are isomorphic.
- We say that A is *universal for ergodic systems* if the same property holds true for all ergodic dynamical systems $(X, \mathcal{B}, \mu; T)$ on a standard Lebesgue probability space.

Let us stress here that an important feature of universality in the sense of Definition 4.3 above is that the A-invariant measures ν on Z which induce all ergodic systems are required to have full support. We will come back to this at the end of Section 5.

The universal operators constructed in [9] are defined as shift operators on certain weighted two-sided ℓ_p-spaces of sequences, for $1 < p < +\infty$. Equivalently, they are weighted shift operators on $\ell_p(\mathbb{Z})$. The proof of [9] involves an ergodic theorem for random walks on groups due to Jones, Rosenblatt, and Tempelman [14]. This result states that if η is a symmetric strictly aperiodic probability measure on $\mathbb{Z}$, the following property holds true: for any probability-preserving dynamical system $(X, \mathcal{B}, \mu; T)$ and for any function $f \in L^p(X, \mathcal{B}, \mu)$, with $1 < p < +\infty$, the powers $A_\eta^n f$ of the random walk operator on $\mathbb{Z}$ defined by

$$A_\eta f(x) = \sum_{k\in\mathbb{Z}} f(T^k x)\, \eta(k)$$

converge, for almost every $x \in X$, to the projection of f onto the subspace of $L^p(X, \mathcal{B}, \mu)$ consisting of T-invariant functions. This ergodic theorem can be applied for instance to the measure $\eta = (\delta_{-1} + \delta_0 + \delta_1)/3$ on $\mathbb{Z}$. The shift operator constructed in [9] is defined using the weights w_k, $k \in \mathbb{Z}$, defined by $w_k = \sum_{n\geq 1} p_n\, \eta^{*n}(k)$ for every $k \in \mathbb{Z}$, where $(p_n)_{n\geq 1}$ is a sequence of positive real numbers such that $\sum_{n\geq 1} p_n = 1$ and $\sup\,(p_n/p_{n+1}) < +\infty$. The shift operator S is defined on

$$\ell_p(\mathbb{Z}, w) := \{\xi = (\xi_k)_{k\in\mathbb{Z}};\ \sum_{k\in\mathbb{Z}} |\xi_k|^p\, w_k < +\infty\}$$

by the formula $S\xi = (\xi_{k+1})_{k\in\mathbb{Z}}$ for every $\xi \in \ell_p(\mathbb{Z}, w)$. It is shown in [9] that S is bounded, and the ergodic theorem of [14] implies then that for any function $f \in L^{2p}(X, \mathcal{B}, \mu)$ the sequence $(f(T^k x))_{k\in\mathbb{Z}}$ belongs to $\ell_p(\mathbb{Z}, w)$ for μ-almost every $x \in X$. Setting

$$\begin{aligned} \Phi_f : (X, \mathcal{B}, \mu) &\longrightarrow (\ell_p(\mathbb{Z}, w), \mathcal{B}_{\ell_p(\mathbb{Z},w)}, \nu_f) \\ x &\longmapsto (f(T^k x))_{k\in\mathbb{Z}} \end{aligned}$$

where ν_f is the image measure of μ by the map Φ_f on $\ell_p(\mathbb{Z}, w)$ (i.e. $\nu_f(B) = \mu(\Phi_f^{-1}(B))$ for any Borel subset B of $\ell_p(\mathbb{Z}, w)$), one easily checks that Φ_f intertwines the actions of T on $(X, \mathcal{B}, \mu)$ and of S on $\ell_p(\mathbb{Z}, w)$. The map Φ_f is thus a factor map, whatever the choice of the function f. The last, and most difficult, steps of the proof of [9] are then to ensure that Φ_f is a conjugacy of dynamical systems, and that the measure ν_f on $\ell_p(\mathbb{Z}, w)$ has full support (if the systems $(X, \mathcal{B}, \mu\,; T)$ and $(\ell_p(\mathbb{Z}, w), \mathcal{B}_{\ell_p(\mathbb{Z},w)}, \nu_f\,; S)$ are conjugate, they are isomorphic – see the beginning of Chapter 2 in [23] for more details on conjugacy of measure-preserving systems). The map Φ_f needs thus to be constructed in such a way that, for every $C \in \mathcal{B}$, there exists $B \in \mathcal{B}_{\ell_p(\mathbb{Z},w)}$ such that $\mu(\Phi_f^{-1}(B) \triangle C) = 0$, and $\mu(\{x \in X\,;\, \Phi_f(x) \in U\}) > 0$ for every non-empty open subset U of $\ell_p(\mathbb{Z}, w)$.

5. Some more universal operators in the measurable framework

5.1. A general criterion for universality. — Our aim in the paper [11] was to present an alternative construction of universal operators, which is elementary in the sense that it avoids the use of an ergodic theorem such as the one of [14]. The construction given there is also more flexible than that of [9], yields some rather simple criteria for universality, and allows us to show the existence of universal operators on a large class of Banach spaces. Moreover, this construction makes it possible to exhibit operators which are universal for *all* ergodic dynamical systems, not only for invertible ones. In order to emphasize the parallel between the topological and the measurable contexts in universality questions, we introduce the following reinforcements of Properties (P) and (P'):

Definition 5.1. — Let A be a bounded operator on a separable Banach space Z.

- We say that A *has Property (Q)* if there exists a sequence $(z_n)_{n\in\mathbb{Z}}$ of vectors of Z such that $Az_n = z_{n+1}$ for every $n \in \mathbb{Z}$ (in which case we write $z_n = A^n z_0$ for every $n \in \mathbb{Z}$) and
  (a) the series $\sum_{n\in\mathbb{Z}} A^{-n} z_0$ is unconditionally convergent in $\mathbb{Z}$;
  (b) z_0 does not belong to the closed linear span of the vectors $A^{-n}z_0$, $n \in \mathbb{Z} \setminus \{0\}$;
  (c) the linear span of the vectors $A^{-n}z_0$, $n \in \mathbb{Z}$, is dense in Z.
- We say that A *has Property (Q')* if it satisfies the requirements of Property (Q) above and, moreover, the sequence $(z_n)_{n\in\mathbb{Z}}$ is such that $A^r z_0 = 0$ for some $r \in \mathbb{Z}$ (or, equivalently, such that $z_0 = 0$).

It is not difficult to see that operators satisfying either Property (Q) or Property (Q') satisfy the so-called Frequent Hypercyclicity Criterion (see [4] or [13]), and are thus frequently hypercyclic. Many of the classical operators satisfying the Frequent Hypercyclicity Criterion have Property (Q) or Property (Q'). The main result of [11] can be stated as follows:

Theorem 5.2 **([11]).** — *Let A be a bounded operator on a real or complex separable Banach space Z.*

- *If A has Property (Q), A is universal for invertible ergodic systems.*
- *If A has Property (Q'), A is universal for all ergodic systems.*

Example 5.3. — The operators λB, $|\lambda| > 1$, acting on $\ell_p(\mathbb{N})$, $1 \le p < +\infty$, or $c_0(\mathbb{N})$, are universal for all ergodic systems.

Theorem 5.2 allows us to characterize in [11] universal operators among unilateral or bilateral weighted shifts on ℓ_p or c_0, to show the existence of universal operators on Banach spaces containing a complemented subspace with a symmetric basis, and to give a criterion for universality of operators on complex Banach spaces in terms of unimodular eigenvectors. We quote here one of the most useful results along these lines, thanks to which it is possible for example to study the universality of adjoints of multipliers on $H^2(\mathbb{D})$, or that of a classic Kalish-type operator on $L^2(\mathbb{T})$ (see [11] or [4] for unexplained terms).

Theorem 5.4 **([11]).** — *Let A be a bounded operator on a separable complex Banach space Z.*

- *Suppose that A admits an eigenvector field E which is analytic in a neighborhood of the unit circle $\mathbb{T}$, and that $\overline{\text{span}}\big[E(\lambda);\ \lambda \in \mathbb{T}\big] = Z$. Then A is universal for invertible ergodic systems.*
- *If the eigenvector field E is analytic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ and $\overline{\text{span}}\big[E(\lambda);\ \lambda \in \mathbb{T}\big] = Z$, then A is universal for ergodic systems.*

Our aim in the rest of this section is to help the reader of [9] or [11] to develop an intuition of the proofs of such universality results as those of Theorem 4.2 or Theorem 5.2. In the rest of this section, all probability-preserving systems will be assumed to be invertible.

5.2. Symbolic dynamics. — The basic intuition comes from symbolic dynamics, and the coding of a probability-preserving system thanks to partitions of the space X. Let $(X, \mathcal{B}, \mu\,; T)$ be a probability-preserving system, and let $\mathcal{P} = \{P_i\,;\, i \in I\}$ be a partition of X indexed by a certain set I. Denote by Y the space $I^{\mathbb{Z}}$ of two-sided sequences of elements of I, endowed with the product topology associated to the discrete topology on I. Let also $\sigma : Y \longrightarrow Y$ be the shift homeomorphism defined by $\sigma((y_n)_{n\in\mathbb{Z}}) = (y_{n+1})_{n\in\mathbb{Z}}$ for every $y = (y_n)_{n\in\mathbb{Z}} \in Y$. Define a map $\phi_{\mathcal{P}} : X \longrightarrow Y$ by letting, for each $x \in X$ and each $n \in \mathbb{Z}$, the n-th coordinate $(\phi_{\mathcal{P}}(x))_n$ of $\phi_{\mathcal{P}}(x)$ be the unique index $i \in I$ such that $T^n x$ belongs to P_i. It is straightforward to check that $\phi_{\mathcal{P}} : (X, \mathcal{B}, \mu\,; T) \longrightarrow (Y, \mathcal{B}_Y, \phi_{\mathcal{P}}\mu\,; \sigma)$ is a factor map, where $\phi_{\mathcal{P}}\mu$ is the image of the measure μ by the map $\phi_{\mathcal{P}}$. This map $\phi_{\mathcal{P}}$ is not necessarily a conjugacy of dynamical systems, but it is so when the partition $\mathcal{P}$ is a *generator* for T, i.e. when the infinite join $\bigvee_{n\in\mathbb{Z}} T^n\mathcal{P}$ of the partitions $T^n\mathcal{P}$, $n \in \mathbb{Z}$, is equal to $\mathcal{B}$ up to sets of measure 0 (i.e. for every $B \in \mathcal{B}$ there exists $C \in \bigvee_{n\in\mathbb{Z}} T^n\mathcal{P}$ such that $\mu(B \triangle C) = 0$):

Proposition 5.5. — *Let $(X, \mathcal{B}, \mu\,; T)$ be a probability-preserving system, and let $\mathcal{P}$ be a partition of X. The following assertions are equivalent:*

(a) *$\phi_{\mathcal{P}}$ is a conjugacy between the two systems $(X, \mathcal{B}, \mu\,; T)$ and $(Y, \mathcal{B}_Y, \phi_{\mathcal{P}}\mu\,; \sigma)$;*
(b) *there exists a subset $M \in \mathcal{B}$ of X with $\mu(M) = 1$ such that $\phi_{\mathcal{P}}$ is injective on M;*
(c) *$\mathcal{P}$ is a generator for T.*

We refer the reader to [23, Ch. 4] for a presentation of basic facts concerning partitions. Proposition 5.5 is proved for instance on pages 97-98 of [23].

The coding of a system via a partition which is a generator for the system thus appears as a most natural tool to exhibit universal models (of course, outside the linear setting). We quote here a few important results along these lines, and refer the reader to the fundamental paper [24] by Weiss, and its sequel [22] by Shilon and Weiss, for more on this topic. The statement of some of these results involve the entropy of the initial system. We will not define entropy here (see for instance [23, Ch. 4]), and the reader not familiar with entropy can understand it simply as a measure of the complexity of the system. A first result concerning the construction of universal models goes back to Rokhlin [20]:

***Theorem 5.6* ([20]).** — *Let $(X, \mathcal{B}, \mu\,; T)$ be an ergodic system on a standard probability space. If its entropy $h(T)$ is finite, there exists a countable generator $\mathcal{P}$ for T. It follows that the shift σ on $\mathbb{N}^{\mathbb{Z}}$ is a universal model for all ergodic systems of finite entropy.*

This result was strengthened by Krieger [16], who showed that systems with finite entropy admit finite generators:

***Theorem 5.7* ([16]).** — *Let $(X, \mathcal{B}, \mu\,; T)$ be an ergodic system on a standard probability space. If its entropy $h(T)$ is finite, T admits a finite generator $\mathcal{P}$. The number p of elements of $\mathcal{P}$ can be chosen in such a way that*

$$e^{h(T)} \le p \le e^{h(T)+1}.$$

It follows that for every positive integer a, the shift σ on $\{0, 1, \ldots, a-1\}^{\mathbb{Z}}$ is a universal model for ergodic systems of entropy strictly less that $\ln a$. In particular, the shift on $\{0, 1\}^{\mathbb{Z}}$ is a universal model for ergodic systems of zero entropy.

5.3. An attempt at a proof of Theorem 5.2. — With these results at hand, let us try to see how we could represent ergodic systems of finite entropy as linear dynamical systems. Let A be a bounded operator on a Banach space Z over $\mathbb{K}$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, satisfying Property (Q), and let $z_0 \in Z$ be a vector for which assumptions (a) to (c) of Definition 3.1 hold true. For simplicity's sake, we will suppose here that A is invertible.

Let $(X,\mathcal{B},\mu\,;T)$ be a system of finite entropy. By Theorem 5.7 above, there exists a finite generator $\mathcal{P}=\{P_1,\dots,P_p\}$ of T. Let $a_1,\dots,a_p$ be distinct scalars, and let $f:\ X \longrightarrow \mathbb{K}$ be the function defined by $f(x)=a_i$ for every $x\in P_i$, $i=1,\dots,p$. The map Φ_f associated to f by setting

$$\Phi_f(x)=\sum_{k\in\mathbb{Z}} f(T^kx)A^{-k}z_0 \qquad \text{for every } x\in X$$

is well-defined. It is a factor map between the systems $(X,\mathcal{B},\mu\,;T)$ and $(Z,\mathcal{B}_Z,\nu_f\,;A)$, where ν_f is the probability measure on Z defined as the image measure of μ by the map Φ_f. Let us now show that Φ_f is a conjugacy between these two systems: since the partition $\mathcal{P}$ is a generator for T, it suffices to prove that for any integer r and any indices $i_{-r},\dots,i_r$ belonging to $\{1,\dots,p\}$, there exists $B\in\mathcal{B}_Z$ such that, up to a set of μ-measure zero,

$$\Phi_f^{-1}(B)=\{x\in X\,;\,\forall\,k\in\{-r,\dots,r\},\ T^kx\in P_{i_k}\}.$$

It is here that assumption (b) on the sequence $(A^{-k}z_0)_{k\in\mathbb{Z}}$ comes into play: since A is invertible, it implies that there exists for every $k\in\mathbb{Z}$ a functional $z_k^*\in Z^*$ such that $\langle z_k^*,A^{-k}z_0\rangle=1$ and $\langle z_k^*,A^{-j}z_0\rangle=0$ for every $j\in\mathbb{Z}$, $j\neq k$. Hence $\langle z_k^*,\Phi_f(x)\rangle=f(T^kx)$ for every $k\in\mathbb{Z}$ and every $x\in X$. Consider now the Borel subset B of Z defined by

$$B=\{z\in Z\,;\,\forall\,k\in\{-r,\dots,r\},\ \langle z_k^*,z\rangle=a_{i_k}\}.$$

Then $\Phi_f^{-1}(B)=\{x\in X\,;\forall\,k\in\{-r,\dots,r\},\ f(T^kx)=a_{i_k}\}$, and since the scalars $a_1,\dots,a_p$ are distinct, the definition of f implies that

$$\Phi_f^{-1}(B)=\{x\in X\,;\,\forall\,k\in\{-r,\dots,r\},\ T^kx\in P_{i_k}\}$$

up to a set of μ-measure zero. It thus follows that the two systems $(X,\mathcal{B},\mu\,;T)$ and $(Z,\mathcal{B}_Z,\nu_f\,;A)$ are conjugate via the map Φ_f, and are hence isomorphic.

So all ergodic systems with finite entropy are represented, in a certain sense, by the operator A. But the measures ν_f constructed above never have full support (the map Φ_f is essentially bounded on X), and it is not easy to get an intuition of how the supports of these measures ν_f might look. In order to force the measures ν_f to have full support, one needs to exploit the bicyclicity assumption (c) on the vector z_0. Let U be a non-empty open subset of Z. By assumption (c), there exists a vector $v_0\in Z$ of the form

$$v_0=\sum_{|k|\le r}\alpha_kA^{-k}z_0,$$

where the scalars α_k are all distinct, such that the open ball $B(v_0,2\varepsilon)$ is contained in U for some $\varepsilon>0$. Let $\alpha=\max_{|k|\le r}|\alpha_k|>0$. There exists $r_0\ge 0$ such that

$$\Big\|\sum_{|k|>r_0}\beta_kA^{-k}z_0\Big\|\le\frac{\varepsilon}{\alpha}\sup_{|k|>r_0}|\beta_k| \tag{2}$$

for every bounded sequence $(\beta_k)_{k\in\mathbb{Z}}$ of scalars. Adding if necessary some terms $\alpha_kA^{-k}z_0$ of extremely small norm to the vector v_0, we can suppose without loss of generality that $r\ge r_0$.

Suppose now that the partition $\mathcal{P}$ considered above has been constructed in such a way that it contains a Rokhlin tower of height $2r+1$, i.e. that there exists a subset E of $\mathcal{B}$ with $\mu(E) > 0$ such that the sets $E, T(E), \ldots, T^{2r}(E)$ are pairwise disjoint and belong to $\mathcal{P}$. For instance we number the elements of the partition $\mathcal{P}$ in such a way that $P_i = T^{-r+i-1}E$ for every $1 \le i \le 2r+1$ (and $p \ge 2r+1$). Let now $(a_i)_{1\le i\le p}$ be a sequence of distinct scalars with the following properties:

- $a_i = \alpha_{-r+i-1}$ for every $1 \le i \le 2r+1$;
- $|a_i| \le \alpha$ for every $2r+1 < i \le p$.

Just as before, let $f : X \longrightarrow \mathbb{K}$ be the function defined by $f(x) = a_i$ for every $x \in P_i$, $1 \le i \le p$. For every $x \in E$ we have

$$\Phi_f(x) = \sum_{|k|\le r} \alpha_k A^{-k} z_0 + \sum_{|k|>r} f(T^k x) A^{-k} z_0$$

so that

$$||\Phi_f(x) - v_0|| = ||\sum_{|k|>r} f(T^k x) A^{-k} z_0||.$$

Since f is essentially bounded by α on E and $r > r_0$, (2) implies that $||\Phi_f(x) - v_0|| \le \varepsilon$ for μ-almost every $x \in E$. It follows that $\nu_f(U) > 0$.

The measure ν_f constructed via this argument thus gives positive measure to a fixed non-empty open subset of Z. Since we want ν_f to have full support, we need to ensure that $\nu_f(U_n) > 0$ for every $n \ge 1$, where $(U_n)_{n\ge 1}$ is a countable basis of open subsets of Z. The natural idea is to construct by induction a sequence $(f_n)_{n\ge1}$ of finite-valued functions from X into $\mathbb{K}$ such that, for every $n \ge 1$, $\nu_{f_n}(U_n) > 0$, and $\nu_{f_n}(U_k)$ is essentially equal to $\nu_{f_k}(U_k)$ for every $1 \le k \le n-1$. If each function f_n is a suitably small perturbation of the previous function f_{n-1} (i.e. if for instance the norm $||f_n - f_{n-1}||$ in $L^2(X, \mathcal{B}, \mu)$ is extremely small), the functions Φ_{f_n} will converge, in $L^2(X, \mathcal{B}, \mu\,; Z)$, to a certain function $\Phi \in L^2(X, \mathcal{B}, \mu\,; Z)$ as n tends to infinity. This map Φ will easily be seen to be a factor map between the systems $(X, \mathcal{B}, \mu\,; T)$ and $(Z, \mathcal{B}_Z, \nu\,; A)$, where ν is the image measure of μ by Φ. Observe that Φ will never be of the form Φ_f for some essentially bounded function f on $(X, \mathcal{B}, \mu)$. If a proper control is kept of the quantities $\nu_{f_n}(U_p)$, $n \ge p$, the measure ν will have full support. The main difficulty will be to construct the successive functions f_n (i.e. the successive partitions $\mathcal{P}_n$ of the space X) in such a way that Φ is a conjugacy between the two systems $(X, \mathcal{B}, \mu\,; T)$ and $(Z, \mathcal{B}_Z, \nu\,; A)$. This relies on a careful construction of Rokhlin towers at each step, combined with technical modifications and precise estimates. We also have to take into account the fact that, when proving Theorem 5.2, we cannot in general suppose that $(X, \mathcal{B}, \mu\,; T)$ has finite entropy.

5.4. Weiss' universal model. — Some constructions of this type are carried out in [24] and [22], where universal models for various classes of systems are constructed using this approach. We quote in particular the following remarkable result of Weiss, which shows that minimal universal models for ergodic systems (whatever their entropy) exist. The universal model of Weiss is a subshift of $\overline{\mathbb{N}}^{\mathbb{Z}}$, where $\overline{\mathbb{N}} = \{1, 2, \ldots, \infty\}$ is endowed with the usual topology (the point ∞ is the limit of the integers n, $n \in \mathbb{N}$, as n goes to infinity).

Theorem 5.8. — *There exists a closed subset Y of $\overline{\mathbb{N}}^{\mathbb{Z}}$ which is invariant under the action of the shift $\sigma : \overline{\mathbb{N}}^{\mathbb{Z}} \longrightarrow \overline{\mathbb{N}}^{\mathbb{Z}}$, such that $(Y\sigma)$ is a minimal system, and, for every ergodic system $(X, \mathcal{B}, \mu\,; T)$ on a standard probability space, there exists a countable generator $\mathcal{P}$*

for T such that $\phi_{\mathcal{P}}(x)$ belongs to Y for μ-almost every $x \in X$. The shift σ on Y is thus a universal model for all ergodic systems.

5.5. Some open questions. — In view of the approach presented above, and the results of [9], it seems rather likely that Theorem 5.2 above can be extended to obtain a general criterion for representations of certain topological groups G to be universal in the sense of Definition 4.1:

Question 5.9. — For which topological groups G does there exist a general criterion, in the flavor of Theorem 5.2, for a representation of G to be universal for all free ergodic actions of G on a standard probability space?

We next recall the following open question from [9]:

***Question 5.10* ([9]).** — Does every locally compact second countable group admit a universal representation?

In another direction, it would be very interesting to obtain other examples of universal operators exhibiting different features from the ones presented here. Several questions in this direction are proposed in [11]; we single out the following one:

Question 5.11. — Does there exist a universal operator for (invertible) ergodic systems which has no unimodular eigenvalue?

The last problem we would like to propose here concerns the possible links between topological and measurable universality. As we have seen in the previous sections, infinite direct sums of operators with Property (P) are universal in the topological sense for homeomorphisms of compact spaces, while operators with Property (Q) are universal in the measurable sense for the class of invertible ergodic systems. Now, Property (P) and (Q) are very close to each other, and the proofs of Theorem 3.3 and Theorem 5.2 present some obvious similarities. It is thus natural to wonder whether this is more than a mere coincidence:

Question 5.12. — Are there links between universality in the topological framework and in the measurable framework?

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