
KAZHDAN CONSTANTS, CONTINUOUS PROBABILITY MEASURES WITH LARGE FOURIER COEFFICIENTS AND RIGIDITY SEQUENCES

by

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To the memory of Jean-Pierre Kahane (1926-2017)

Abstract. — Exploiting a construction of rigidity sequences for weakly mixing dynamical systems by Fayad and Thouvenot, we show that for every integers p_1, \dots, p_r there exists a continuous probability measure μ on the unit circle \mathbb{T} such that

$$\inf_{k_1 \geq 0, \dots, k_r \geq 0} |\hat{\mu}(p_1^{k_1} \dots p_r^{k_r})| > 0.$$

This result applies in particular to the Furstenberg set $F = \{2^k 3^{k'}; k \geq 0, k' \geq 0\}$, and disproves a 1988 conjecture of Lyons inspired by Furstenberg's famous $\times 2 \times 3$ conjecture. We also estimate the modified Kazhdan constant of F and obtain general results on rigidity sequences which allow us to retrieve essentially all known examples of such sequences.

1. Introduction

Denote by \mathbb{T} the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$, by $\mathcal{M}(\mathbb{T})$ the set of (finite) complex Borel measures on \mathbb{T} and by $\mathcal{P}(\mathbb{T})$ the set of Borel probability measures on \mathbb{T} . The Fourier coefficients of $\mu \in \mathcal{M}(\mathbb{T})$ are defined here as

$$\hat{\mu}(n) = \int_{\mathbb{T}} \lambda^n d\mu(\lambda).$$

A measure $\mu \in \mathcal{P}(\mathbb{T})$ is said to be *continuous*, or *atomless*, if $\mu(\{\lambda\}) = 0$ for every $\lambda \in \mathbb{T}$. We denote the set of continuous probability measures on \mathbb{T} by $\mathcal{P}_c(\mathbb{T})$. According to a theorem of Wiener and the Koopman-von Neumann lemma, μ is continuous if and only if $\hat{\mu}(n)$ tends to zero as n tends to infinity along a sequence in \mathbb{N} of density one. For

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every $\mu \in \mathcal{P}(\mathbb{T})$, we define $\tilde{\mu}$ by setting $\tilde{\mu}(A) = \mu(A^c)$ for every Borel set $A \subseteq \mathbb{T}$, with $A^c = \{\bar{\lambda}; \lambda \in A\}$. Then $\nu := \mu * \tilde{\mu}$ has the property that $\hat{\nu}(n) = |\hat{\mu}(n)|^2 \geq 0$ for every $n \in \mathbb{Z}$, and ν belongs to $\mathcal{P}_c(\mathbb{T})$ as soon as μ does.

A conjecture of Russell Lyons. — Our aim in this paper is to study some non-lacunary sets of positive integers from a Fourier analysis point of view, and to construct some probability measures which have large Fourier coefficients on such sets. In particular, we disprove a 1988 conjecture of Lyons [32], called there Conjecture (C4), which reads as follows:

Lyons' Conjecture (C4): *If S is a non-lacunary semigroup of integers, and if $\mu \in \mathcal{P}_c(\mathbb{T})$, there exists an infinite sequence $(n_k)_{k \geq 1}$ of elements of S such that $\hat{\mu}(n_k) \rightarrow 0$ as $k \rightarrow +\infty$.*

This conjecture of Lyons is inspired by Furstenberg's famous conjecture concerning common invariant probability measures for two commuting automorphisms $T_p : \lambda \mapsto \lambda^p$ and $T_q : \lambda \mapsto \lambda^q$ of the unit circle \mathbb{T} when p and q are two multiplicatively independent integers (i.e. p and q are not both powers of the same integer). In this setting, Furstenberg's conjecture states that the only continuous probability measure on \mathbb{T} invariant by both T_p and T_q is the Lebesgue measure on \mathbb{T} . Furstenberg himself proved [22] that if S is any non-lacunary semigroup of integers (i.e. if S is not contained in any semigroup of the form $\{a^n; n \geq 0\}$, $a \geq 2$), the only infinite closed S -invariant subset of \mathbb{T} is \mathbb{T} itself. See [10] for an elementary proof of this result and the references mentioned in [15, Chapter 2] for several extensions. Since $S = \{p^k q^{k'}; k, k' \geq 0\}$ is a non-lacunary semigroup whenever p and q are multiplicatively independent, the only infinite closed subset of \mathbb{T} which is simultaneously T_p -invariant and T_q -invariant is \mathbb{T} . Starting with the work of Lyons in [32], Furstenberg's conjecture has given rise to an impressive amount of related questions and results, concerning in particular the dynamics of commuting group automorphisms. We refer the reader to the papers [36], [14] or [26] for example, as well as to the texts [30], [27] or [37] for surveys of results related to this conjecture, as well as for perspectives.

As written in [32], conjecture (C4) is a natural version of Furstenberg's conjecture about measures, but not involving invariance. If (C4) were true, it would imply an affirmative answer to the Furstenberg conjecture (if $\mu \in \mathcal{P}_c(\mathbb{T})$ is T_p - and T_q -invariant, applying (C4) to each of the measures $\mu_j := T_j(\mu)$, $j \in \mathbb{Z} \setminus \{0\}$, yields that $\hat{\mu}(j) = 0$ for every $j \in \mathbb{Z} \setminus \{0\}$).

Kazhdan sets and modified Kazhdan constants. — It turns out that Lyons' conjecture is related to an important property of subsets of \mathbb{Z} , namely that of being or not a *Kazhdan subset* of \mathbb{Z} . Kazhdan subsets Q of a second-countable topological group G are those for which there exists $\varepsilon > 0$ such that any strongly continuous representation π of G on a complex separable Hilbert space H admitting a vector $x \in H$ with $\|x\| = 1$ which is ε -invariant on Q (i.e. $\sup_{g \in Q} \|\pi(g)x - x\| < \varepsilon$) has a G -invariant vector. Such an ε is called a *Kazhdan constant for Q* , and the supremum of all ε 's with this property is the Kazhdan constant of Q . Groups with Property (T), also called Kazhdan groups, are those which admit a compact Kazhdan set. See the book [7] for more on Property (T) and its numerous important applications.

As suggested in [7, Sec. 7.12], it is of interest to study Kazhdan sets in groups which do not have Property (T), such as locally compact abelian groups, Heisenberg groups, etc. See [4] and also [17] for a study of such problems. In the case of the group \mathbb{Z} , the definition above is equivalent to the following one:

Definition 1.1. — (Kazhdan sets and constants) A subset $Q \subset \mathbb{Z}$ is said to be a *Kazhdan set* if there exists $\varepsilon > 0$ such that any unitary operator U acting on a complex separable Hilbert space H satisfies the following property: if there exists a vector $x \in H$ with $\|x\| = 1$ such that $\sup_{n \in Q} \|U^n x - x\| < \varepsilon$, then there exists a non-zero vector $y \in H$ such that $Uy = y$ (i.e. 1 is an eigenvalue of U). We will say in this case that (Q, ε) is a Kazhdan pair. We define the Kazhdan constant of Q as

$$\text{Kaz}(Q) = \inf_U \inf_{\|x\|=1} \sup_{q \in Q} \|U^q x - x\|,$$

where the first infimum is taken over all unitary operators U on H without fixed vectors.

It follows from [7, p. 30] that $0 \leq \text{Kaz}(Q) \leq \sqrt{2}$ for every $Q \subseteq \mathbb{Z}$.

Several characterizations of Kazhdan subsets of \mathbb{Z} were obtained in [4] as consequences of results applying to a much wider class of groups; self-contained proofs of these characterizations of Kazhdan subsets of \mathbb{Z} , involving only classical tools from harmonic analysis, were obtained in the paper [5]. One of the characterizations of generating Kazhdan sets obtained in [4, Th. 6.1] (see also [5, Th. 4.12]) runs as follows. Recall that Q is said to be *generating* in the group \mathbb{Z} if the smallest subgroup containing Q is \mathbb{Z} itself.

Theorem 1.2 ([4]). — *Let Q be a generating subset of \mathbb{Z} . Then Q is a Kazhdan subset of \mathbb{Z} if and only if there exists $\varepsilon' \in (0, \sqrt{2}]$ such that (Q, ε') is a modified Kazhdan pair, that is any unitary operator V acting on a complex separable Hilbert space H satisfies the following property: if there exists a vector $x \in H$ with $\|x\| = 1$ such that $\sup_{n \in Q} \|V^n x - x\| < \varepsilon'$, then V has at least one eigenvalue.*

We define now the modified Kazhdan constant of Q as

$$\widetilde{\text{Kaz}}(Q) = \inf_V \inf_{\|x\|=1} \sup_{q \in Q} \|V^q x - x\|,$$

where the first infimum is taken this time over unitary operators V on H without eigenvalues (that is, with continuous spectra). Therefore

$$0 \leq \text{Kaz}(Q) \leq \widetilde{\text{Kaz}}(Q) \leq \sqrt{2}$$

and for every $Q \subseteq \mathbb{Z}$, $\text{Kaz}(Q) = 0$ if and only if $\widetilde{\text{Kaz}}(Q) = 0$ if and only if Q is a non-Kazhdan set. The property of being or not a Kazhdan set can also be expressed in terms of Fourier coefficients of probability measures; see Section 5 for a discussion.

The characterization of Kazhdan subsets of \mathbb{Z} obtained by the authors in [4] (see also [5]) implies that the generating subsets Q of \mathbb{Z} which satisfy the property stated in (C4) (namely that there exists for every $\mu \in \mathcal{P}_c(\mathbb{T})$ an infinite sequence $(n_k)_{k \geq 1}$ of elements of Q such that $\widehat{\mu}(n_k) \rightarrow 0$ as $k \rightarrow +\infty$) are exactly the Kazhdan subsets of \mathbb{Z} with modified Kazhdan constant $\widetilde{\text{Kaz}}(Q) = \sqrt{2}$. Since $\sqrt{2}$ is the modified Kazhdan constant of \mathbb{Z} seen as a subset of itself, $\sqrt{2}$ is the maximal modified Kazhdan constant, and thus (C4) can be reformulated as: every generating non-lacunary semigroup S of integers is a Kazhdan subset of \mathbb{Z} with maximal modified Kazhdan constant $\sqrt{2}$. The relationship between Furstenberg $\times 2$ - $\times 3$ conjecture and modified Kazhdan constants can be also seen directly from Proposition 5.4 below.

2. Main results

The first main result of this paper is the following:

Theorem 2.1. — *Let p_1, \dots, p_r be positive distinct integers and set*

$$E = \{p_1^{k_1} \dots p_r^{k_r} ; k_1 \geq 0, \dots, k_r \geq 0\}.$$

There exists a continuous probability measure μ on \mathbb{T} such that

$$\inf_{k_1 \geq 0, \dots, k_r \geq 0} |\widehat{\mu}(p_1^{k_1} \dots p_r^{k_r})| > 0.$$

Equivalently,

$$\widetilde{\text{Kaz}}(E) < \sqrt{2}.$$

It should be noted that, as conjecture (C4) does not involve invariant measures, we do not assume in Theorem 2.1 that the integers p_j are multiplicatively independent. Notice also that the statement of Theorem 2.1 is well-known in the lacunary case: if $r = 1$ it suffices to consider the classical Riesz product associated to the sequence $(p^k)_{k \geq 0}$. In the non-lacunary case, Theorem 2.1 disproves Conjecture (C4), as well as the related conjectures (C5) and (C6) of [32] (which are both stronger than (C4)). It applies in particular to the Furstenberg set $F = \{2^k 3^{k'} ; k, k' \geq 0\}$ and shows the existence of a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

$$\inf_{k, k' \geq 0} \widehat{\mu}(2^k 3^{k'}) > 0.$$

In view of this result, one may naturally wonder for which values of $\delta \in (0, 1)$ there exists a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

$$\inf_{k, k' \geq 0} \widehat{\mu}(2^k 3^{k'}) \geq \delta,$$

or, equivalently, whether the Furstenberg set F is a Kazhdan set in \mathbb{Z} , and if yes, with which (modified) Kazhdan constant. In this direction, we prove the following result:

Theorem 2.2. — *Let $F = \{2^k 3^{k'} ; k, k' \geq 0\}$. Then $\widetilde{\text{Kaz}}(F) \leq 1$. More precisely, there exists for every $\delta \in (0, 1/2)$ a continuous probability measure μ on \mathbb{T} with nonnegative Fourier coefficients such that*

$$\inf_{k, k' \geq 0} \widehat{\mu}(2^k 3^{k'}) > \delta.$$

Rigidity sequences. — Our strategy for proving Theorem 2.1 is to construct measures $\mu \in \mathcal{P}_c(\mathbb{T})$ whose Fourier coefficients tend to 1 along a substantial part of the set $\{p_1^{k_1} \dots p_r^{k_r} ; k_1 \geq 0, \dots, k_r \geq 0\}$. In other words, we show that certain large subsets of this set form, when taken in a strictly increasing order, rigidity sequences in the sense of [8] or [18]. Recall that a dynamical system $(X, \mathcal{B}, m; T)$ on a Borel probability space is called *rigid* if there exists a strictly increasing sequence of integers $(n_k)_{k \geq 1}$ such that $\|U_T^{n_k} f - f\| \rightarrow 0$ as $k \rightarrow +\infty$ for every $f \in L^2(X, \mathcal{B}, m)$, where U_T denotes as usual the Koopman operator $f \mapsto f \circ T$ associated to T on $L^2(X, \mathcal{B}, m)$. Equivalently, $m(T^{-n_k} A \triangle A) \rightarrow 0$ as $k \rightarrow +\infty$ for every $A \in \mathcal{B}$. We say in this case that the system is rigid with respect to the sequence $(n_k)_{k \geq 1}$, or that $(n_k)_{k \geq 1}$ is a *rigidity sequence* for $(X, \mathcal{B}, m; T)$. The case where the system $(X, \mathcal{B}, m; T)$ is weakly mixing is particularly interesting, and is the object of the works [8] and [18]. A strictly increasing sequence $(n_k)_{k \geq 1}$ of integers is called a *rigidity sequence* if there exists a weakly mixing system which is rigid with respect to $(n_k)_{k \geq 1}$.

Using Gaussian dynamical systems, one can show that $(n_k)_{k \geq 1}$ is a rigidity sequence if and only if there exists a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\widehat{\mu}(n_k) \rightarrow 1$ as $k \rightarrow +\infty$. The study of rigidity sequences was initiated in [8] and [18]. Further works on this topic include the papers [1], [3], [2], [24], [21] [20] and [23] among others. The paper [21] by Fayad and Thouvenot is especially relevant here: the authors re-obtain a result of Adams [3], showing that whenever $(n_k)_{k \geq 1}$ is a rigidity sequence for an ergodic rotation on the circle, it is a rigidity sequence for a weakly mixing system. The proof of this result in [3] relies on an involved construction of a suitable weakly mixing system by cutting and stacking, while the authors of [21] proceed by a direct construction of suitable continuous probability measures: they show that if $\lambda^{n_k} \rightarrow 1$ for some $\lambda = e^{2i\pi\theta} \in \mathbb{T}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, there exists $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\widehat{\mu}(n_k) \rightarrow 1$.

The most important tool for proving Theorems 2.1 and 2.2 is the following theorem, which generalizes the result of Fayad and Thouvenot and provides some new examples of non-Kazhdan subsets of \mathbb{Z} :

Theorem 2.3. — *Let $(n_k)_{k \geq 0}$ be a strictly increasing sequence of integers. Suppose that the set*

$$C = \{ \lambda \in \mathbb{T} ; \lambda^{n_k} \rightarrow 1 \text{ as } k \rightarrow +\infty \}$$

is dense in \mathbb{T} . There exists for every $\varepsilon > 0$ a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

- (1) $\widehat{\mu}(n_k) \rightarrow 1$ as $k \rightarrow +\infty$;
- (2) $\sup_{k \geq 0} |\widehat{\mu}(n_k) - 1| < \varepsilon$.

In particular, $\{n_k ; k \geq 0\}$ is a non-Kazhdan subset of \mathbb{Z} .

Notice that C , like every subgroup of the circle group, is dense in \mathbb{T} as soon as it is infinite. We deduce from Theorem 2.3 the following two-dimensional statement, which is asymmetric and involves a uniformity assumption.

Theorem 2.4. — *Let $(m_k)_{k \geq 0}$ and $(n_{k'})_{k' \geq 0}$ be two strictly increasing sequences of integers. Let also $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be such that $\psi(k) \rightarrow +\infty$ as $k \rightarrow +\infty$, and set*

$$D_\psi = \{ (k, k') \in \mathbb{N}^2 ; 0 \leq k' \leq \psi(k) \}.$$

Suppose that the set

$$C'_\psi = \{ \lambda \in \mathbb{T} ; \lambda^{m_k n_{k'}} \rightarrow 1 \text{ as } k \rightarrow +\infty, (k, k') \in D_\psi \}$$

is dense in \mathbb{T} . There exists for every $\varepsilon > 0$ a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

- (1) $\widehat{\mu}(m_k n_{k'}) \rightarrow 1$ as $k \rightarrow +\infty$ with $(k, k') \in D_\psi$;
- (2) and

$$\sup_{k \geq 0, 0 \leq k' \leq \psi(k)} |\widehat{\mu}(m_k n_{k'}) - 1| < \varepsilon.$$

In particular, $\{m_k n_{k'} ; k \geq 0, 0 \leq k' \leq \psi(k)\}$ is a non-Kazhdan subset of \mathbb{Z} .

Given a doubly indexed sequence $(z_{k,k'})_{k,k' \geq 0}$ of complex numbers, saying that $z_{k,k'}$ converges to $z \in \mathbb{C}$ as $k \rightarrow +\infty$ with $(k, k') \in D_\psi$ means that there exists for every $\gamma > 0$ an integer k_0 such that $|z_{k,k'} - z| < \gamma$ for every $(k, k') \in \mathbb{N}^2$ with $k \geq k_0$ and $0 \leq k' \leq \psi(k)$.

Remark also that the assumption of Theorem 2.4 is in particular satisfied if the set

$$C' = \{ \lambda \in \mathbb{T} ; \lambda^{m_k n_{k'}} \rightarrow 1 \text{ as } k \rightarrow +\infty \text{ uniformly in } k' \}$$

is dense in \mathbb{T} .

Theorem 2.3 allows us to retrieve essentially all known examples of rigidity sequences (notable exceptions being the examples of [20] and [23]). We state separately as Corollaries 2.5 and 2.6 the parts of Theorems 2.3 and 2.4 dealing with rigidity sequences:

Corollary 2.5. — *Let $(n_k)_{k \geq 0}$ be a strictly increasing sequence of integers. Suppose that the set*

$$C = \{ \lambda \in \mathbb{T} ; \lambda^{n_k} \rightarrow 1 \text{ as } k \rightarrow +\infty \}$$

is dense in \mathbb{T} . Then $(n_k)_{k \geq 1}$ is a rigidity sequence.

Corollary 2.6. — *Let $(m_k)_{k \geq 0}$ and $(n_{k'})_{k' \geq 0}$ be two strictly increasing sequences of integers. Let also $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be such that $\psi(k) \rightarrow +\infty$ as $k \rightarrow +\infty$, and suppose that the set*

$$C'_\psi = \{ \lambda \in \mathbb{T} ; \lambda^{m_k n_{k'}} \rightarrow 1 \text{ as } k \rightarrow +\infty, (k, k') \in D_\psi \}$$

is dense in \mathbb{T} . Then there exists a continuous probability measure μ on \mathbb{T} such that $\widehat{\mu}(m_k n_{k'}) \rightarrow 1$ as $k \rightarrow +\infty$ with $(k, k') \in D_\psi$.

The proof of Theorem 2.3 builds on some ideas from [21]. While being an immediate consequence of Theorem 2.3, Corollary 2.5 admits a direct proof which is very much in the spirit of that of the main result of [21]. As Corollary 2.5 is of independent interest in the study of rigidity sequences, we will briefly sketch this direct proof in Section 4 of the paper.

Theorem 2.1 is obtained by first observing that the set $\{p_1^{k_1} \dots p_r^{k_r} ; p_1 \geq 0, \dots, p_r \geq 0\}$ can be split into r sets to which Theorem 2.4 (or Corollary 2.6) applies, and then considering a convex combination of the continuous measures obtained in this way.

Organization of the paper. — The paper is structured as follows. We give in Section 3 the proof of Theorems 2.3 and 2.4, and sketch in Section 4 a direct proof of Corollaries 2.5 and 2.6, essentially following the arguments of Fayad and Thouvenot in [21]. In Section 5, we recall a characterization of generating Kazhdan subsets of \mathbb{Z} from [4], and detail the links between several natural constants involved in this characterization. We explain in particular why the generating subsets of \mathbb{Z} which satisfy the property stated in (C4) are exactly the Kazhdan subsets of \mathbb{Z} with modified Kazhdan constant $\sqrt{2}$. Section 6 is devoted to applications: we prove Theorems 2.1 and 2.2, and show how to retrieve many examples of rigidity sequences, using Corollaries 2.5 and 2.6. We also provide an application of Theorem 2.2 to the study of the size of the exceptional set of values $\theta \in \mathbb{R}$ for which the sequence $(n_k \theta)_{k \geq 0}$ is not almost uniformly distributed modulo 1 with respect to a (finite) complex Borel measure $\nu \in \mathcal{M}(\mathbb{T})$, where $(n_k)_{k \geq 0}$ denotes the Furstenberg sequence: we show that this exceptional set is uncountable, thus providing a new example of a sublacunary sequence with uncountable exceptional set for (almost) uniform distribution.

3. Proof of Theorems 2.3 and 2.4

Given two integers $a < b$, we will when the context is clear denote by $[a, b]$ the set of integers k such that $a \leq k \leq b$.

Proof of Theorem 2.3. — Fix $\varepsilon \in (0, 1/2)$. The general strategy of the proof is the following: we construct a sequence $(\lambda_i)_{i \geq 1}$ of pairwise distinct elements of \mathbb{C} , as well as a strictly increasing sequence of integers $(N_p)_{p \geq 0}$ and, for every $p \geq 0$, a sequence $(a_i^{(p)})_{1 \leq i \leq 2^p}$ of positive weights with $\sum_{i=1}^{2^p} a_i^{(p)} = 1$, such that the probability measures

$$\mu_p = \sum_{i=1}^{2^p} a_i^{(p)} \delta_{\{\lambda_i\}}$$

satisfy certain properties stated below. At step p , we determine the elements λ_i for $2^{p-1} < i \leq 2^p$ as well as the integer N_p and the weights $a_i^{(p)}$, $1 \leq i \leq 2^p$, in such a way that $\lambda_1 = 1$ and $a_1^{(1)} = 1$, so that $\mu_0 = \delta_{\{1\}}$, $N_0 = 0$, and

- (1) for every $p \geq 1$, every $j \in [0, p-1]$ and every $k \in [N_j, N_{j+1}]$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) < 3\varepsilon(1 - \varepsilon)^j;$$

- (2) for every $p \geq 1$, every $q \in [0, p-1]$, $l \in [1, 2^{p-q})$, $r \in [1, 2^q]$,

$$|\lambda_{l2^{q+r}} - \lambda_r| < \eta_q$$

where $\eta_q = \frac{1}{4} \inf_{1 \leq i < j \leq 2^q} |\lambda_i - \lambda_j|$ for every $q \geq 1$, and $\eta_0 = 1$;

- (3) for every $p \geq 1$ and every $k \geq N_p$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) < \varepsilon(1 - \varepsilon)^{p+1};$$

- (4) for every $p \geq 1$, every $q \in [1, p]$ and every $r \in [1, 2^q]$,

$$\sum_{\{i \in [1, 2^p]; i \equiv r \pmod{2^q}\}} \mu_p(\{\lambda_i\}) \leq (1 - \varepsilon)^q.$$

Remark that property (2) implies that the sequence $(\lambda_i)_{i \geq 1}$ satisfies

- (5) for every $q \geq 0$, every $l \geq 0$, and every $r \in [1, 2^q]$,

$$|\lambda_{l2^{q+r}} - \lambda_r| < \eta_q.$$

Suppose that the sequences $(\lambda_i)_{i \geq 1}$, $(N_p)_{p \geq 0}$ and $(a_i^{(p)})_{1 \leq i \leq 2^p}$, $p \geq 0$, have been constructed so as to satisfy (1) to (4) above, and let μ be a w^* -limit point of the sequence $(\mu_p)_{p \geq 0}$ in $\mathcal{P}(\mathbb{T})$. Property (1) clearly implies that $\sup_{k \geq 0} |\widehat{\mu}(n_k) - 1| \leq 3\varepsilon$.

Claim 3.1. — We have $\widehat{\mu}(n_k) \rightarrow 1$ as $k \rightarrow +\infty$.

Proof. — For every $k \geq 0$, denote by $j_k \geq 0$ the unique integer j such that $k \in [N_j, N_{j+1})$. For every $p > j_k$, we have by (1)

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) < 3\varepsilon(1 - \varepsilon)^{j_k} \quad \text{so that} \quad \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu(\lambda) \leq 3\varepsilon(1 - \varepsilon)^{j_k}.$$

Since $j_k \rightarrow +\infty$ as $k \rightarrow +\infty$, $\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu(\lambda) \rightarrow 0$, i.e. $\widehat{\mu}(n_k) \rightarrow 1$. \square

Claim 3.2. — The probability measure μ is continuous.

Proof. — Fix $q \geq 1$, and consider for every $r \in [1, 2^q]$ the two arcs Γ_r and Δ_r of \mathbb{T} defined by

$$\Gamma_r = \{\lambda \in \mathbb{T}; |\lambda - \lambda_r| \leq \eta_q\} \quad \text{and} \quad \Delta_r = \{\lambda \in \mathbb{T}; |\lambda - \lambda_r| < \frac{3}{2}\eta_q\}.$$

The 2^q arcs Δ_r are pairwise disjoint. Indeed, for every $r, r' \in [1, 2^q]$ with $r \neq r'$, every $\lambda \in \Delta_r$ and every $\lambda' \in \Delta_{r'}$, we have by the definition of η_q that

$$|\lambda - \lambda'| \geq |\lambda_r - \lambda_{r'}| - 3\eta_q \geq 4\eta_q - 3\eta_q = \eta_q > 0.$$

So Δ_r and $\Delta_{r'}$ do not intersect.

Let us next estimate, for every $r \in [1, 2^q]$ and every $p \geq q$, the quantity $\mu_p(\Gamma_r)$. We have

$$\mu_p(\Gamma_r) = \sum_{\{i \in [1, 2^p]; \lambda_i \in \Gamma_r\}} \mu_p(\{\lambda_i\}).$$

Every $i \in [1, 2^p]$ can be written as $i = l2^q + s$ for some $l \geq 0$ and $s \in [1, 2^q]$. By (5), λ_i belongs to Γ_s . Since the arcs $\Delta_{r'}, r' \in [1, 2^q]$, are pairwise disjoint, it follows that

$$\mu_p(\Delta_r) = \mu_p(\Gamma_r) = \sum_{\{i \in [1, 2^p]; i \equiv r \pmod{2^q}\}} \mu_p(\{\lambda_i\}) \leq (1 - \varepsilon)^q$$

by (4). Also,

$$\mu_p\left(\bigcup_{r=1}^{2^q} \Gamma_r\right) = 1.$$

Since the arcs Γ_r are closed while the arcs Δ_r are open, going to the limit as p goes to infinity yields that $\mu(\Delta_r) \leq (1 - \varepsilon)^q$ for every $r \in [1, 2^q]$ and

$$\mu\left(\bigcup_{r=1}^{2^q} \Gamma_r\right) = 1.$$

If $\lambda \in \mathbb{T}$ is such that $\mu(\{\lambda\}) > 0$, there exists an $r \in [1, 2^q]$ such that $\lambda \in \Gamma_r \subset \Delta_r$. So $\mu(\{\lambda\}) \leq \mu(\Delta_r) \leq (1 - \varepsilon)^q$, a contradiction if q is sufficiently large. It follows that the measure μ is continuous. \square

By Claims 3.1 and 3.2, it suffices to construct $(\lambda_i)_{i \geq 0}$, $(N_p)_{p \geq 0}$ and $(a_i^{(p)})_{1 \leq i \leq 2^p}$, $p \geq 0$, satisfying (1) to (4) in order to prove Theorem 2.3. Recall that for $p = 0$, we set $\lambda_1 = 1$, $a_1^{(1)} = 1$ and $N_0 = 0$, so that $\mu_0 = \delta_{\{1\}}$.

For $p = 1$, we choose $\lambda_2 \in C$ distinct from λ_1 with $|\lambda_2 - \lambda_1| < 1$ and set $\mu_1 = (1 - \varepsilon)\delta_{\{1\}} + \varepsilon\delta_{\{\lambda_2\}}$. We have for every $k \geq 0$

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_1(\lambda) = \varepsilon |\lambda_2^{n_k} - 1| \leq 2\varepsilon < 3\varepsilon.$$

Hence property (1) is satisfied whatever the choice of N_1 . Since $\eta_0 = 1$ and $|\lambda_2 - \lambda_1| < 1$, property (2) is satisfied. We now have to choose N_1 in such a way that property (3) is satisfied. Since λ_2 belongs to C ,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_1(\lambda) = \varepsilon |\lambda_2^{n_k} - 1| \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty,$$

so we can choose N_1 so large that

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_1(\lambda) < \varepsilon(1 - \varepsilon)^2 \quad \text{for every } k \geq N_1.$$

Moreover, $\mu_1(\{1\}) = 1 - \varepsilon$ and $\mu_2(\{\lambda_2\}) = \varepsilon < 1 - \varepsilon$, so (4), which we only need to check for $q = p = 1$, is true. This terminates the construction for $p = 1$.

Suppose now that the construction has been carried out until step p , i.e. that the quantities λ_i , $i \in [1, 2^p]$, $(a_i^{(l)})_{1 \leq i \leq 2^l}$, and N_l , $l \in [0, p]$, have been constructed satisfying properties (1) to (4).

We construct by induction on $s \in [1, 2^p]$ elements λ_{2^p+s} of C , measures $\mu_{p,s} \in \mathcal{P}(\mathbb{T})$ of the form

$$\mu_{p,s} = \sum_{i=1}^{2^p+s} b_i^{(p,s)} \delta_{\{\lambda_i\}} \quad \text{with} \quad b_i^{(p,s)} > 0 \quad \text{and} \quad \sum_{i=1}^{2^p+s} b_i^{(p,s)} = 1,$$

and integers $N_{p,s}$ in such a way that the elements λ_i , $i \in [1, 2^{p+1}]$, are all distinct, $N_p < N_{p,1} < \dots < N_{p,2^p}$, and the following five properties are satisfied:

(a) for every $j \in [0, p-1]$ and every $k \in [N_j, N_{j+1}]$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 3\varepsilon(1 - \varepsilon)^j;$$

(b) for every $k \geq N_p$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 3\varepsilon(1 - \varepsilon)^p;$$

(c) for every $k \geq N_{p,s}$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 3\varepsilon(1 - \varepsilon)^{p+2};$$

(d) $\mu_{p,s}(\{\lambda_i\}) = \mu_p(\{\lambda_i\})$ for every $i \in (s, 2^p]$ and

$$\mu_{p,s}(\{\lambda_i\}) + \mu_{p,s}(\{\lambda_{2^p+i}\}) = \mu_p(\{\lambda_i\})$$

for every $i \in [1, s]$;

(e) $\mu_{p,s}(\{\lambda_i\}) \leq (1 - \varepsilon)^{p+1}$ for every $i \in [1, s] \cup [2^p + 1, 2^p + s]$.

Let us start with the construction of λ_{2^p+1} . By density of C , one can choose λ_{2^p+1} distinct from all the elements λ_i , $i \in [1, 2^p]$, with $|\lambda_{2^p+1} - \lambda_1|$ arbitrarily small. We define $\mu_{p,1}$ as

$$\begin{aligned} \mu_{p,1} &= \mu_p + \mu_p(\{1\}) \varepsilon (\delta_{\{\lambda_{2^p+1}\}} - \delta_{\{\lambda_1\}}) \\ &= \mu_p(\{1\}) (1 - \varepsilon) \delta_{\{\lambda_1\}} + \sum_{i=2}^{2^p} \mu_p(\{\lambda_i\}) \delta_{\{\lambda_i\}} + \mu_p(\{1\}) \varepsilon \delta_{\{\lambda_{2^p+1}\}}. \end{aligned}$$

In other words, we split the point mass $\delta_{\{\lambda_1\}}$ appearing in the expression of μ_p into $(1 - \varepsilon)\delta_{\{\lambda_1\}} + \varepsilon\delta_{\{\lambda_{2^p+1}\}}$. We have for every $k \geq 0$

$$\begin{aligned} (6) \quad \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,1}(\lambda) &\leq \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) + \mu_p(\{1\}) \varepsilon |\lambda_{2^p+1}^{n_k} - \lambda_1^{n_k}| \\ &\leq \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) + (1 - \varepsilon)^p \varepsilon |\lambda_{2^p+1}^{n_k} - \lambda_1^{n_k}| \end{aligned}$$

since $\mu_p(\{1\}) \leq (1 - \varepsilon)^p$ by (4) applied to $q = p$. If $|\lambda_{2^p+1} - \lambda_1|$ is sufficiently small, we have by (1) that

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,1}(\lambda) < 3\varepsilon(1 - \varepsilon)^j$$

for every $j \in [0, p-1]$ and every $k \in [N_j, N_{j+1}]$, i.e. that (a) holds true. Also (6) and (3) imply that for every $k \geq N_p$,

$$\int_{\mathbb{T}} |\lambda^{nk} - 1| d\mu_{p,1}(\lambda) < \varepsilon(1-\varepsilon)^{p+1} + 2\varepsilon(1-\varepsilon)^p < 3\varepsilon(1-\varepsilon)^p$$

so that (b) holds true. Since all the elements $\lambda_i, i \in [1, 2^p + 1]$, belong to C , there exists $N_{p,1} > N_p$ such that

$$\int_{\mathbb{T}} |\lambda^{nk} - 1| d\mu_{p,1}(\lambda) < 3\varepsilon(1-\varepsilon)^{p+2} \quad \text{for every } k \geq N_{p,1}.$$

Property (d) is clear from the expression of $\mu_{p,1}$, and property (e) is satisfied since $\mu_{p,1}(\{1\}) = \mu_p(\{1\})(1-\varepsilon) \leq (1-\varepsilon)^{p+1}$ and $\mu_{p,1}(\{\lambda_{2^p+1}\}) = \mu_p(\{1\})\varepsilon \leq \varepsilon(1-\varepsilon)^p \leq (1-\varepsilon)^{p+1}$ by (4) applied to $q = p$. Properties (a) to (e) are thus satisfied for $s = 1$.

Suppose now that $\lambda_{2^p+s'}, \mu_{2^p+s'}$, and $N_{2^p+s'}$ have been constructed for $s' < s$. Let $\lambda_{2^p+s} \in C \setminus \{\lambda_1, \dots, \lambda_{2^p+s-1}\}$ be very close to λ_s , and set

$$(7) \quad \mu_{p,s} = \mu_{p,s-1} + \mu_{p,s-1}(\{\lambda_s\})\varepsilon(\delta_{\{\lambda_{2^p+s}\}} - \delta_{\{\lambda_s\}})$$

This time, the mass point $\delta_{\{\lambda_s\}}$ appearing in μ_p is split as $(1-\varepsilon)\delta_{\{\lambda_s\}} + \varepsilon\delta_{\{\lambda_{2^p+s}\}}$. Since, for every $k \geq 0$,

$$(8) \quad \int_{\mathbb{T}} |\lambda^{nk} - 1| d\mu_{p,s}(\lambda) \leq \int_{\mathbb{T}} |\lambda^{nk} - 1| d\mu_{p,s-1}(\lambda) + (1-\varepsilon)^p \varepsilon |\lambda_{2^p+s}^{nk} - \lambda_s^{nk}|,$$

the induction assumption implies that (a) holds true provided $|\lambda_{2^p+s} - \lambda_s|$ is sufficiently small. As to (b), we have to consider separately the cases $N_p \leq k < N_{p,s-1}$ and $k \geq N_{p,s-1}$. If $|\lambda_{2^p+s} - \lambda_s|$ is sufficiently small, we have by (8) and (b) for $s-1$ that

$$\int_{\mathbb{T}} |\lambda^{nk} - 1| d\mu_{p,s}(\lambda) < 3\varepsilon(1-\varepsilon)^p \quad \text{for every } N_p \leq k < N_{p,s-1}.$$

By property (c) at step $s-1$ and (8),

$$\int_{\mathbb{T}} |\lambda^{nk} - 1| d\mu_{p,s}(\lambda) < \varepsilon(1-\varepsilon)^{p+2} + 2\varepsilon(1-\varepsilon)^p < 3\varepsilon(1-\varepsilon)^p$$

for every $k \geq N_{p,s-1}$. Hence (b) is satisfied at step s . Property (c) is satisfied if $N_{p,s}$ is chosen sufficiently large since all the elements $\lambda_i, i \in [1, 2^p + s]$, belong to C .

Property (d) follows from (7) and property (d) at step $s-1$. Indeed, $\mu_{p,s}(\{\lambda_i\}) = \mu_{p,s-1}(\{\lambda_i\})$ for every $i \notin \{s, 2^p + s\}$. Also, $\mu_{p,s-1}(\{\lambda_i\}) = \mu_p(\{\lambda_i\})$ for every $i \in [s, 2^p]$, so that $\mu_{p,s}(\{\lambda_i\}) = \mu_p(\{\lambda_i\})$ for every $i \in (s, 2^p]$. Observe next that $\mu_{p,s}(\{\lambda_i\}) + \mu_{p,s}(\{\lambda_{2^p+i}\}) = \mu_{p,s-1}(\{\lambda_i\}) + \mu_{p,s-1}(\{\lambda_{2^p+i}\}) = \mu_p(\{\lambda_i\})$ for every $i \in [1, s-1]$. Lastly, $\mu_{p,s}(\{\lambda_s\}) + \mu_{p,s}(\{\lambda_{2^p+s}\}) = \mu_{p,s-1}(\{\lambda_s\}) = \mu_p(\{\lambda_s\})$. So property (d) is true at step s .

As to property (e), we have $\mu_{p,s}(\{\lambda_i\}) = \mu_{p,s-1}(\{\lambda_i\})$ for every i distinct from λ_s and λ_{2^p+s} . So $\mu_{p,s}(\{\lambda_i\}) \leq (1-\varepsilon)^{p+1}$ for every $i \in [1, s] \cup [2^p + 1, 2^p + s]$. Also

$$\mu_{p,s}(\{\lambda_s\}) = \mu_{p,s-1}(\{\lambda_s\})(1-\varepsilon) = \mu_p(\{\lambda_s\})(1-\varepsilon) \leq (1-\varepsilon)^{p+1},$$

while $\mu_{p,s}(\{\lambda_{2^p+s}\}) = \mu_{p,s-1}(\{\lambda_s\})\varepsilon \leq (1-\varepsilon)^{p+1}$. So (e) holds true at step s . This terminates the construction of the measures $\mu_{p,s}$.

Let us now set $\mu_{p+1} = \mu_{p,2^p}$ and $N_{p+1} = N_{p,2^p}$. It remains to check that with these choices of $\lambda_i, i \in [1, 2^{p+1}]$, μ_{p+1} and N_{p+1} , properties (1) to (4) are satisfied.

By (a), property (1) is satisfied for every $j \in [0, p-1]$. The case where $j = p$ follows from (b). So (1) is true. Property (3) follows immediately from (c). Property (4) is a consequence of (d) and (e). Indeed, suppose first that $q \in [1, p]$. Then

$$\begin{aligned} \sum_{\{i \in [1, 2^{p+1}]; i \equiv r \pmod{2^q}\}} \mu_{p+1}(\{\lambda_i\}) &= \sum_{\{i \in [1, 2^p]; i \equiv r \pmod{2^q}\}} (\mu_{p+1}(\{\lambda_i\}) + \mu_{p+1}(\{\lambda_{2^p+i}\})) \\ &= \sum_{\{i \in [1, 2^p]; i \equiv r \pmod{2^q}\}} \mu_p(\{\lambda_i\}) \leq (1 - \varepsilon)^q. \end{aligned}$$

by (d) above and (4) at step p . If $q = p+1$, (4) follows immediately from (e). So it only remains to check (2).

Fix $q \in [0, p]$, $l \in [1, 2^{p+1-q}]$ and $r \in [1, 2^q]$. Consider first the case where $q = p$. In this case $l = 1$, and the quantities under consideration have the form $|\lambda_{2^{p+r}} - \lambda_r|$, with $r \in [1, 2^p]$. One can ensure in the construction that $|\lambda_{2^{p+r}} - \lambda_r| < \eta_p$ for every $r \in [1, 2^p]$ and then (2) holds true for $q = p$.

Suppose then that $q \in [0, p-1]$, and write l as $l = l' + \varepsilon 2^{p-q}$ with $\varepsilon \in \{0, 1\}$ and $l' \in [1, 2^{p-q}]$. Then $l 2^q + r = l' 2^q + r + \varepsilon 2^p$. Set $s = l' 2^q + r$. Then $1 \leq s \leq (2^{p-q} - 1) 2^q + 2^q = 2^p$, i.e. $s \in [1, 2^p]$. We have

$$|\lambda_{l 2^q + r} - \lambda_r| \leq |\lambda_{s + \varepsilon 2^p} - \lambda_s| + |\lambda_{l' 2^q + r} - \lambda_r|.$$

If $\varepsilon = 0$, the first term is zero; if $\varepsilon = 1$, it is equal to $|\lambda_{2^p+s} - \lambda_s|$, which can be assumed to be as small as we wish in the construction. As to the second term, it is less than η_q by property (2) at step p , since $l' \in [1, 2^{p-q}]$ and $r \in [1, 2^q]$ with $q \in [0, p-1]$. We can thus ensure that

$$|\lambda_{l 2^q + r} - \lambda_r| < \eta_q$$

for every $q \in [0, p]$, $l \in [1, 2^{p+1-q}]$, and $r \in [1, 2^q]$. So property (2) is satisfied at step $p+1$, and this concludes the proof of Theorem 2.3. \square

Theorem 2.4 is now a formal consequence of Theorem 2.3.

Proof of Theorem 2.4. — Recall that $D_\psi = \{(k, k') \in \mathbb{N}^2; 0 \leq k' \leq \psi(k)\}$ and

$$C'_\psi = \{\lambda \in \mathbb{T}; \lambda^{m_k n_{k'}} \rightarrow 1 \text{ as } k \rightarrow +\infty, (k, k') \in D_\psi\}.$$

Order the set $\{m_k n_{k'}; (k, k') \in D_\psi\}$ as a strictly increasing sequence $(p_l)_{l \geq 0}$ of integers. Since there exists for every integer $k_1 \geq 0$ an integer $l_1 \geq 0$ such that

$$\{p_l; l \geq l_1\} \subseteq \{m_k n_{k'}; (k, k') \in D_\psi, k \geq k_1\},$$

every element $\lambda \in C'_\psi$ has the property that $\lambda^{p_l} \rightarrow 1$ as $l \rightarrow +\infty$. By Theorem 2.3 applied to the sequence $(p_l)_{l \geq 1}$, there exists for every $\varepsilon > 0$ a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\hat{\mu}(p_l) \rightarrow 1$ as $l \rightarrow +\infty$ and $\sup_{l \geq 0} |\hat{\mu}(p_l) - 1| < \varepsilon$. Then

$$\sup_{k \geq 0, 0 \leq k' \leq \psi(k)} |\hat{\mu}(m_k n_{k'}) - 1| < \varepsilon.$$

Using this time the fact that there exists for every integer $l_2 \geq 0$ an integer $k_2 \geq 0$ such that

$$\{m_k n_{k'}; (k, k') \in D_\psi, k \geq k_2\} \subseteq \{p_l; l \geq l_2\},$$

we deduce that $\hat{\mu}(m_k n_{k'}) \rightarrow 1$ as $k \rightarrow +\infty$ with $(k, k') \in D_\psi$. Theorem 2.4 is proved. \square

4. A direct proof of Corollaries 2.5 and 2.6

We sketch in this section a direct proof of Corollary 2.5 (Corollary 2.6 is a formal consequence of it), following almost step by step the construction given in [21] and bypassing the additional technical difficulties of the proof of Theorem 2.3.

Proof. — Using the notation of the proof of Theorem 2.3, we construct a sequence $(\lambda_i)_{i \geq 1}$ of pairwise distinct elements of C , as well as a strictly increasing sequence of integers $(N_p)_{p \geq 0}$, such that the measures

$$\mu_p = 2^{-p} \sum_{i=1}^{2^p} \delta_{\{\lambda_i\}}, \quad p \geq 0$$

satisfy

(1') for every $p \geq 1$, every $j \in [0, p-1]$ and every $k \in [N_j, N_{j+1}]$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) < 2^{-(j-1)};$$

(2') for every $p \geq 1$, every $q \in [0, p-1]$, $l \in [1, 2^{p-q}]$, $r \in [1, 2^q]$,

$$|\lambda_{l2^{q+r}} - \lambda_r| < \eta_q$$

where $\eta_q = \frac{1}{4} \inf_{1 \leq i < j \leq 2^q} |\lambda_i - \lambda_j|$ for every $q \geq 1$, and $\eta_0 = 1$;

(3') for every $p \geq 1$ and every $k \geq N_p$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) < 2^{-(p+1)}.$$

Again, property (2') implies that

(4') for every $q \geq 0$, every $l \geq 0$, and every $r \in [1, 2^q]$,

$$|\lambda_{l2^{q+r}} - \lambda_r| < \eta_q.$$

Then any w^* -limit point μ of $(\mu_p)_{p \geq 0}$ will be a continuous measure which satisfies $\widehat{\mu}(n_k) \rightarrow 1$ as $k \rightarrow +\infty$.

For $p = 0$, we set $\lambda_1 = 1$, $N_0 = 0$, and $\mu_0 = \delta_{\{1\}}$. For $p = 1$, we choose $\lambda_2 \in C \setminus \{1\}$ with $|\lambda_2 - 1| < 1$ and set $\mu_1 = \frac{1}{2}(\delta_{\{1\}} + \delta_{\{\lambda_2\}})$. We have

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_1(\lambda) = \frac{1}{2} |\lambda_2^{n_k} - 1| \leq 1 < 2 \quad \text{for every } k \geq 0.$$

Hence property (1') is satisfied whatever the choice of N_1 . Since $|\lambda_2 - 1| < 1$, (2') is true. If N_1 is chosen sufficiently large, μ_1 satisfies (3').

Suppose now that the construction has been carried out until step p . We can then construct by induction on $s \in [1, 2^p]$ measures $\mu_{p,s}$ which satisfy

(a') for every $j \in [0, p-1]$ and every $k \in [N_j, N_{j+1}]$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 2^{-(j-1)};$$

(b') for every $k \geq N_p$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 2^{-(p-1)};$$

(c') for every $k \geq N_{p,s}$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 2^{-(p+2)}.$$

We define $\mu_{p,1}$ as

$$\mu_{p,1} = \mu_p + 2^{-(p+1)}(\delta_{\{\lambda_{2^{p+1}}\}} - \delta_{\{\lambda_1\}})$$

where $\lambda_{2^{p+1}} \in C \setminus \{\lambda_1, \dots, \lambda_{2^p}\}$ is such that $|\lambda_{2^{p+1}} - \lambda_1|$ is very small. Then for every $k \geq 0$,

$$(9) \quad \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,1}(\lambda) \leq \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_p(\lambda) + 2^{-(p+1)} |\lambda_{2^{p+1}}^{n_k} - \lambda_1^{n_k}|.$$

It follows that (a') holds true for $\mu_{p,1}$, provided that $|\lambda_{2^{p+1}} - \lambda_1|$ is sufficiently small. Also, we have by (9) and (3') that for every $k \geq N_p$,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,1}(\lambda) < 2^{-(p+1)} + 2^{-p} < 2^{-(p-1)}$$

which is (b'). If $N_{p,1}$ is sufficiently large, (c') is true.

Supposing now that $s \geq 2$ and that the construction has been carried out for every $s' < s$, we choose $\lambda_{2^{p+s}} \in C \setminus \{\lambda_1, \dots, \lambda_{2^{p+s-1}}\}$ very close to λ_s , and set

$$\mu_{p,s} = \mu_{p,s-1} + 2^{-(p+1)}(\delta_{\{\lambda_{2^{p+s}}\}} - \delta_{\{\lambda_s\}})$$

Since, for every $k \geq 0$,

$$(10) \quad \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) \leq \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s-1}(\lambda) + 2^{-(p+1)} |\lambda_{2^{p+s}}^{n_k} - \lambda_s^{n_k}|,$$

the induction assumption implies that (a') holds true provided $|\lambda_{2^{p+s}} - \lambda_s|$ is sufficiently small. As to (b'), we consider separately the cases $N_p \leq k < N_{p,s-1}$ and $k \geq N_{p,s-1}$. If $|\lambda_{2^{p+s}} - \lambda_s|$ is sufficiently small,

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 2^{-(p-1)} \quad \text{for every } N_p \leq k < N_{p,s-1}.$$

By property (c') at step $s-1$ and (10),

$$\int_{\mathbb{T}} |\lambda^{n_k} - 1| d\mu_{p,s}(\lambda) < 2^{-(p+2)} + 2^{-p} < 2^{-(p-1)}$$

for every $k \geq N_{p,s-1}$. Hence (b') is satisfied at step s . Property (c') is satisfied if $N_{p,s}$ is chosen sufficiently large. This terminates the construction of the measures $\mu_{p,s}$.

We then set $\mu_{p+1} = \mu_{p,2^p}$ and $N_{p+1} = N_{p,2^p}$ and check as in the proof of Theorem 2.3 that properties (1'), (2'), and (3') are satisfied. \square

Remark 4.1. — Suppose that the set

$$C' = \{\lambda \in \mathbb{T} ; \lambda^{n_k n_{k'}} \rightarrow 1 \text{ as } k \rightarrow +\infty \text{ uniformly in } k'\}$$

is dense in \mathbb{T} . It is natural to wonder whether there exists a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\widehat{\mu}(m_k n_{k'}) \rightarrow 1$ as $k \rightarrow 1$ uniformly in k' . The following example shows that it is not the case: set $m_k = 2^k$ and $n_{k'} = k'$ for every $k, k' \geq 0$. The set

$$C' = \{\lambda \in \mathbb{T}; \lambda^{m_k n_{k'}} \rightarrow 1 \text{ as } k \rightarrow +\infty \text{ uniformly in } k'\}$$

contains all 2^k -th roots of 1, and so is dense in \mathbb{T} . Suppose that $\mu \in \mathcal{P}(\mathbb{T})$ is such that $\widehat{\mu}(2^k k') \rightarrow 1$ as $k \rightarrow +\infty$ uniformly in k' . Then there exists an integer $k_0 \geq 1$ such that $|\widehat{\mu}(2^{k_0} k')| \geq 1/2$ for every $k' \geq 0$. Consider the measure $\nu = T_{2^{k_0}}(\mu)$. Since $\widehat{\nu}(n) = \widehat{\mu}(2^{k_0} n)$ for every $n \in \mathbb{Z}$, ν cannot be continuous. Also, $\nu(\{\lambda_0\}) = \mu(\{\lambda \in \mathbb{T}; \lambda^{2^{k_0}} = \lambda_0\})$ for every $\lambda_0 \in \mathbb{T}$, and so the measure μ itself cannot be continuous.

So the conclusion of Corollary 2.6 seems to be essentially optimal.

5. From Conjecture (C4) to the study of some non-Kazhdan subsets of \mathbb{Z}

5.1. Kazhdan constants and Fourier coefficients of probability measures. — We begin this section by recalling a characterization of generating Kazhdan subsets of \mathbb{Z} , obtained in [4, Th. 6.1] (see also [5, Th. 4.12]) and presenting some facts concerning the (modified) Kazhdan constants of such sets. We state it here in a slightly modified way (condition (ii) is not exactly the same as in [5, Th. 4.12]), and include a discussion concerning the links between the various constants appearing in the equivalent conditions.

Theorem 5.1. — *Let Q be a generating subset of \mathbb{Z} . Then Q is a Kazhdan subset of \mathbb{Z} if and only if one of the following equivalent assertions holds true:*

- (i) *there exists $\varepsilon \in (0, \sqrt{2})$ such that (Q, ε) is a modified Kazhdan pair. Equivalently, $\widetilde{\text{Kaz}}(Q) \geq \varepsilon$;*
- (ii) *there exists $\gamma \in (0, 1)$ such that any measure $\mu \in \mathcal{P}(\mathbb{T})$ with $\sup_{n \in Q} (1 - \Re e \widehat{\mu}(n)) < \gamma$ has a discrete part;*
- (iii) *there exists $\delta \in (0, 1)$ such that any measure $\mu \in \mathcal{P}(\mathbb{T})$ with $\inf_{n \in Q} |\widehat{\mu}(n)| > \delta$ has a discrete part.*

Moreover:

- (i) is satisfied for $\varepsilon \in (0, \sqrt{2})$ if and only if (ii) is satisfied for $\gamma = \varepsilon^2/2$;
- if (ii) is satisfied for $\gamma \in (0, 1)$, (iii) is satisfied for $\delta = \sqrt{1 - \gamma}$, while if (iii) is satisfied for $\delta \in (0, 1)$, (ii) is satisfied for $\gamma = 1 - \delta$;
- hence if (i) is satisfied for $\varepsilon \in (0, \sqrt{2})$, (iii) is satisfied for $\delta = \sqrt{1 - \varepsilon^2/2}$, while if (iii) is satisfied for $\delta \in (0, 1)$, (i) holds true for $\varepsilon = \sqrt{2(1 - \delta)}$.

We prove briefly here the statement concerning the relations between the constants ε , γ , and δ appearing in (i), (ii), and (iii) respectively, following [4] and [5].

Proof. — Suppose that (i) is satisfied for $\varepsilon \in (0, \sqrt{2})$, and let $\mu \in \mathcal{P}(\mathbb{T})$. Consider the unitary operator $U = M_\lambda$ of multiplication by λ on $L^2(\mathbb{T}, \mu)$. Let f be the function constantly equal to 1. Then $\|U^n f - f\|^2 = 2(1 - \Re e \widehat{\mu}(n))$. If $\sup_{n \in Q} (1 - \Re e \widehat{\mu}(n)) < \varepsilon^2/2$, U has an eigenvalue since $\widetilde{\text{Kaz}}(Q) \geq \varepsilon$, and so μ has a discrete part.

Conversely, suppose that (ii) is satisfied for $\gamma \in (0, 1)$. Let U be a unitary operator on a separable Hilbert space H , and let $x \in H$ with $\|x\| = 1$ be such that

$$\sup_{n \in Q} \|U^n x - x\| < \sqrt{2\gamma}.$$

The proof of [5, Th. 4.6] shows then that there exists $\mu \in \mathcal{P}(\mathbb{T})$ such that

$$2 \sup_{n \in Q} (1 - \Re e \widehat{\mu}(n)) = \sup_{n \in Q} \|U^n x - x\|^2 < 2\gamma.$$

So $\sup_{n \in Q} (1 - \Re e \widehat{\mu}(n)) < \gamma$. By (ii), μ has a discrete part, and so U has an eigenvalue. Hence $\widetilde{\text{Kaz}}(Q) \geq \sqrt{2\gamma}$.

Suppose next that property (ii) is satisfied for $\gamma \in (0, 1)$. Let $\mu \in \mathcal{P}(\mathbb{T})$ be such that $\inf_{n \in Q} |\widehat{\mu}(n)| > \sqrt{1 - \gamma}$. Set $\nu = \mu * \widetilde{\mu}$. Then $\inf_{n \in Q} \widehat{\nu}(n) > 1 - \gamma$. It follows that $\sup_{n \in Q} (1 - \widehat{\nu}(n)) < \gamma$, and ν has a discrete part. So μ itself has a discrete part.

Lastly, suppose that (iii) is satisfied for $\delta \in (0, 1)$. Let $\mu \in \mathcal{P}(\mathbb{T})$ be a measure satisfying $\sup_{n \in Q} (1 - \Re e \widehat{\mu}(n)) < 1 - \delta$. Then $\inf_{n \in Q} |\widehat{\mu}(n)| \geq \inf_{n \in Q} \Re e \widehat{\mu}(n) > \delta$, so μ has a discrete part. \square

Remark 5.2. — Given a subset Q of \mathbb{Z} , one can prove, using the spectral theorem for unitary operators, that the following assertions are equivalent (see [5, Th. 4.6]):

- (i') Q is a Kazhdan subset of \mathbb{Z} , i.e. there exists $\varepsilon \in (0, \sqrt{2})$ such that (Q, ε) is a Kazhdan pair;
- (ii') there exists $\gamma \in (0, 1)$ such that any measure $\mu \in \mathcal{P}(\mathbb{T})$ with $\sup_{n \in Q} (1 - \Re e \widehat{\mu}(n)) < \gamma$ is such that $\mu(\{1\}) > 0$.

Moreover (i') holds true for a certain constant $\varepsilon \in (0, \sqrt{2})$ (i.e. $\text{Kaz}(Q) \geq \varepsilon$) if and only if (ii') holds true for $\gamma = \varepsilon^2/2$.

It is interesting to note that these two conditions (i') and (ii') are not equivalent to the natural version (iii') of (iii) (namely, that there exists $\delta \in (0, 1)$ such that any measure $\mu \in \mathcal{P}(\mathbb{T})$ with $\inf_{n \in Q} |\widehat{\mu}(n)| > \delta$ satisfies $\mu(\{1\}) > 0$). Indeed, (iii') is satisfied for any Dirac mass $\delta_{\{\lambda\}}$, $\lambda \in \mathbb{T}$. The proof that (ii) implies (iii) in Theorem 5.1 above uses in a crucial way the fact that if $\mu \in \mathcal{P}(\mathbb{T})$ is such that $\mu * \widetilde{\mu}$ has a discrete part, μ itself has a discrete part. But $\mu * \widetilde{\mu}$ may very well satisfy $\mu * \widetilde{\mu}(\{1\}) > 0$ while $\mu(\{1\}) = 0$, and so (ii') does not imply (iii').

Theorem 5.1 is related to Conjecture (C4) in the following way:

Corollary 5.3. — *Let Q be a generating subset of \mathbb{Z} . The following assertions are equivalent:*

- (α) Q is a Kazhdan subset of \mathbb{Z} with $\widetilde{\text{Kaz}}(Q) = \sqrt{2}$;
- (β) any measure $\mu \in \mathcal{P}_c(\mathbb{T})$ satisfies $\inf_{n \in Q} |\widehat{\mu}(n)| = 0$;
- (γ) any measure $\mu \in \mathcal{P}_c(\mathbb{T})$ satisfies $\liminf_{\substack{|n| \rightarrow +\infty \\ n \in Q}} |\widehat{\mu}(n)| = 0$.

Proof. — The equivalence between (α) and (β) follows immediately from Theorem 5.1. So only the implication (β) \implies (γ) requires a proof. Suppose that any $\mu \in \mathcal{P}_c(\mathbb{T})$ satisfies $\inf_{n \in Q} |\widehat{\mu}(n)| = 0$. We want to show that the conclusion can be reinforced into $\liminf_{\substack{|n| \rightarrow +\infty \\ n \in Q}} |\widehat{\mu}(n)| = 0$. Let $\rho \in \mathcal{P}_c(\mathbb{T})$ be a Rajchman measure with positive coefficients, that is such that $\lim_{|n| \rightarrow +\infty} \widehat{\rho}(n) = 0$ and $\widehat{\rho}(n) > 0$ for every $n \in \mathbb{Z}$. Consider the measure $\nu = (\mu * \widetilde{\mu} + \rho)/2$. It is continuous and satisfies $\nu(n) > 0$ for every $n \in \mathbb{Z}$. Since $\inf_{n \in Q} \widehat{\nu}(n) = 0$ and $\nu(n) > 0$ for every $n \in \mathbb{Z}$, $\liminf_{\substack{|n| \rightarrow +\infty \\ n \in Q}} \widehat{\nu}(n) = 0$. Hence $\liminf_{\substack{|n| \rightarrow +\infty \\ n \in Q}} |\widehat{\mu}(n)|^2 = 0$, and the conclusion follows. \square

So Conjecture (C4) is equivalent to the statement that any non-lacunary semigroup of integers has modified Kazhdan constant $\sqrt{2}$. We can also estimate the Fourier coefficients of a continuous probability measure on \mathbb{T} which is T_2 - and T_3 -invariant in terms of the modified Kazhdan constant of the Furstenberg set. Notice that Proposition 5.4 is meaningful only if $\tilde{\kappa} > 0$.

Proposition 5.4. — *Let $F = \{2^k 3^{k'} ; k, k' \geq 0\}$ and set $\tilde{\kappa} = \widetilde{\text{Kaz}}(F)$. Let μ be a continuous probability measure on \mathbb{T} which is T_2 - and T_3 -invariant. Then*

$$|\hat{\mu}(j)| \leq 1 - \frac{\tilde{\kappa}^2}{2} \quad \text{for every } j \in \mathbb{Z} \setminus \{0\}.$$

Proof of Proposition 5.4. — Set, for every $j \in \mathbb{Z} \setminus \{0\}$, $\mu_j = T_j \mu$. Then μ_j is a continuous measure which satisfies $\hat{\mu}_j(2^k 3^{k'}) = \hat{\mu}(j)$ for every $k, k' \geq 0$. It follows that if $\delta \in (0, 1)$ is such that (iii) of Theorem 5.1 is satisfied, $\delta \geq |\hat{\mu}(j)|$. Hence, by Theorem 5.1 again, $\tilde{\kappa} \leq \sqrt{2(1 - |\hat{\mu}(j)|)}$. \square

Remark 5.5. — Although a generating subset Q of \mathbb{Z} is a Kazhdan set if and only if $\widetilde{\text{Kaz}}(Q) > 0$, there is no link between the Kazhdan constant and the modified Kazhdan constant of Q . Indeed, there exist Kazhdan subsets Q of \mathbb{Z} with maximal modified constant $\widetilde{\text{Kaz}}(Q) = \sqrt{2}$ and arbitrarily small Kazhdan constant $\text{Kaz}(Q)$. This relies on the following observation, which can be extracted from the proof of [5, Th 7.1] and results from Proposition 6.10 below.

Proposition 5.6. — *Let $(n_k)_{k \geq 0}$ be a strictly increasing sequence of integers with $n_0 = 1$ such that $(n_k \theta)_{k \geq 0}$ is uniformly distributed modulo 1 for every $\theta \in \mathbb{R} \setminus D$, where D is countable subset of \mathbb{R} . Then the set $Q = \{n_k ; k \geq 0\}$ is a Kazhdan subset of \mathbb{Z} which satisfies $\widetilde{\text{Kaz}}(Q) = \sqrt{2}$.*

Consider, for every integer $p \geq 2$, the set $Q_p = p\mathbb{N} + 1$. By Proposition 5.6, Q_p is a Kazhdan subset of \mathbb{Z} with $\widetilde{\text{Kaz}}(Q_p) = \sqrt{2}$. But the measure $\mu = \delta_{\{e^{2i\pi/p}\}}$ satisfies

$$\sup_{n \in Q_p} (1 - \Re e \hat{\mu}(n)) = 1 - \cos(2\pi/p).$$

Hence $\text{Kaz}(Q_p) \leq \sqrt{2(1 - \cos(2\pi/p))}$, which can be arbitrarily small if p is sufficiently large.

6. Applications

6.1. Proof of Theorem 2.1. — Our first and main application of Theorem 2.4 (or Corollary 2.6) is Theorem 2.1, which solves in particular Conjecture (C4) and shows that the invariance assumption on the measure is indeed essential in the statement of Furstenberg's $\times 2$ - $\times 3$ conjecture.

Proof of Theorem 2.1. — If $r = 1$, Theorem 2.1 claims the existence, for every integer $p \geq 2$, of a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\inf_{k \geq 0} |\hat{\mu}(p^k)| > 0$. As mentioned in Section 2, this statement is well-known: it suffices to consider the classical Riesz product associated to the sequence $(p^k)_{k \geq 0}$. One can also show, either as in [8] or [18], or as an application of Corollary 2.5, that $(p^k)_{k \geq 0}$ is a rigidity sequence, so that there exists $\mu \in \mathcal{P}_c(\mathbb{T})$ with $\hat{\mu}(p^k) \rightarrow 1$ as $k \rightarrow +\infty$.

Suppose now that $r \geq 2$, and consider, for every fixed index $1 \leq j \leq r$, the set

$$C'_j = \{e^{2i\pi n p_j^{-l}}; n, l \geq 0\}$$

of roots of all powers of p_j . It is dense in \mathbb{T} , and has the following property: there exists for every $\lambda \in C'_j$ an integer l_j such that $\lambda^{p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}} = 1$ for every $k_j \geq l_j$ and $k_i \geq 0$, $1 \leq i \leq r$ with $i \neq j$. Hence

$$\sup_{\substack{k_i \geq 0 \\ 1 \leq i \leq r, i \neq j}} |\lambda^{p_1^{k_1} \dots p_r^{k_r}} - 1| \rightarrow 0 \quad \text{as } k_j \rightarrow +\infty.$$

Consider the two sequences $(m_k)_{k \geq 0}$ and $(n_{k'})_{k' \geq 0}$ obtained by setting $m_k = p_j^k$, $k \geq 0$, and ordering the set

$$\{p_1^{k_1} \dots p_{j-1}^{k_{j-1}} p_{j+1}^{k_{j+1}} \dots p_r^{k_r}; k_i \geq 0, 1 \leq i \leq r \text{ with } i \neq j\}$$

as a strictly increasing sequence $(n_{k'})_{k' \geq 0}$, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function such that

$$\{p_1^{k_1} \dots p_{j-1}^{k_{j-1}} p_{j+1}^{k_{j+1}} \dots p_r^{k_r}; 0 \leq k_i \leq k, 1 \leq i \leq r \text{ with } i \neq j\}$$

is contained in the set $\{n_{k'}; 0 \leq k' \leq \psi(k)\}$ for every $k \geq 0$. By Corollary 2.6, there exists a measure $\mu_j \in \mathcal{P}_c(\mathbb{T})$ such that $\widehat{\mu}_j(p_1^{k_1} \dots p_r^{k_r}) \rightarrow 1$ as $k_j \rightarrow +\infty$ with $0 \leq k_i \leq k_j$, $1 \leq i \leq r$ with $i \neq j$. Replacing, for every $1 \leq j \leq r$, μ_j by $\mu_j * \widetilde{\mu}_j$, we can suppose without loss of generality that $\widehat{\mu}_j(n) \geq 0$ for every $n \in \mathbb{Z}$.

Let now $\rho \in \mathcal{P}_c(\mathbb{T})$ be such that $\widehat{\rho}(n) > 0$ for every $n \in \mathbb{Z}$, and set

$$\mu = \frac{1}{r+1} \left(\sum_{j=1}^r \mu_j + \rho \right).$$

Then μ is a continuous probability measure on \mathbb{T} with $\widehat{\mu}(n) > 0$ for every $n \in \mathbb{Z}$. Moreover, we have

$$(11) \quad \liminf \widehat{\mu}(p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}) \geq \frac{1}{r+1} \quad \text{as } \max(k_1, \dots, k_r) \rightarrow +\infty.$$

Indeed, if $(k_1^{(l)}, \dots, k_r^{(l)})_{l \geq 1}$ is an infinite sequence of elements of \mathbb{N}^r , one can extract from it a sequence (still denoted by $(k_1^{(l)}, \dots, k_r^{(l)})_{l \geq 1}$) with the following property: there exists $1 \leq j \leq r$ such that $k_i^{(l)} \leq k_j^{(l)}$ for every $1 \leq i \leq r$. Then

$$\liminf_{l \rightarrow +\infty} \widehat{\mu}(p_1^{k_1^{(l)}} \dots p_r^{k_r^{(l)}}) \geq \frac{1}{r+1} \liminf_{l \rightarrow +\infty} \widehat{\mu}_j(p_1^{k_1^{(l)}} \dots p_r^{k_r^{(l)}}) = \frac{1}{r+1}.$$

This yields (11). Since $\widehat{\mu}(n) > 0$ for every $n \geq 0$, it follows that

$$\inf_{\substack{k_i \geq 0 \\ 1 \leq i \leq r}} \widehat{\mu}(p_1^{k_1} \dots p_r^{k_r}) > 0,$$

and Theorem 2.1 is proved. □

6.2. The case of the Furstenberg set. — Theorem 2.1 applies to the Furstenberg set $F = \{2^k 3^{k'} ; k, k' \geq 0\}$ and shows the existence of a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

$$\inf_{k, k' \geq 0} \widehat{\mu}(2^k 3^{k'}) > 0$$

(the fact that the measure μ can be supposed to have nonnegative Fourier coefficients can be extracted from the proof of Theorem 2.1, or deduced formally from Theorem 2.1 by considering the measure $\mu * \widetilde{\mu}$). By Corollary 5.3, this means that $\widehat{\text{Kaz}}(F) < \sqrt{2}$.

As mentioned in the introduction, it is natural to look for the optimal constant $\delta \in (0, 1)$ for which there exists a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

$$(12) \quad \inf_{k, k' \geq 0} \widehat{\mu}(2^k 3^{k'}) \geq \delta$$

This is equivalent to asking whether F is a Kazhdan set in \mathbb{Z} , and if yes, with which (modified) Kazhdan constant. The best result which can be obtained via the methods presented here is that there exists a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ satisfying (12) for every $\delta \in (0, 1/2)$: this is the content of Theorem 2.2, which we now prove.

Proof of Theorem 2.2. — The proof goes along the same lines as that of Theorem 2.1, but it requires the full force of Theorem 2.4 rather than the weaker statement of Corollary 2.6.

Fix $\delta \in (0, 1/2)$. There exist by Theorem 2.4 two measures $\mu_1, \mu_2 \in \mathcal{P}_c(\mathbb{T})$ such that

$$|\widehat{\mu}_1(2^k 3^{k'})| \geq \sqrt{2\delta} \quad \text{for every } k \geq 0 \text{ and every } 0 \leq k' \leq k$$

and

$$|\widehat{\mu}_2(2^k 3^{k'})| \geq \sqrt{2\delta} \quad \text{for every } k' \geq 0 \text{ and every } 0 \leq k \leq k'.$$

The measure $\mu = \frac{1}{2}(\mu_1 * \widetilde{\mu}_1 + \mu_2 * \widetilde{\mu}_2)$ has nonnegative Fourier coefficients and satisfies $\widehat{\mu}(2^k 3^{k'}) \geq \delta$ for every $k, k' \geq 0$.

It then follows from Theorem 5.1 that if $\{2^k 3^{k'} ; k, k' \geq 0\}$ is a Kazhdan subset of \mathbb{Z} , its modified Kazhdan constant must be less than $\sqrt{2(1-\delta)}$ for every $\delta \in (0, 1/2)$, so must be at most 1. \square

That the bound $1/2$ can be further improved does not seem clear at all, and we do not know whether there exists for every $\delta \in [1/2, 1)$ a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that

$$\inf_{k, k' \geq 0} \widehat{\mu}(2^k 3^{k'}) \geq \delta.$$

Question 6.1. — Is the Furstenberg set $\{2^k 3^{k'} ; k, k' \geq 0\}$ a Kazhdan set in \mathbb{Z} ?

Note that a lacunary semigroup $\{a^n ; n \geq 0\}$, $a \geq 2$, cannot be a Kazhdan set (see [5, Ex. 5.2]). We also observe that Theorem 2.4 immediately yields

Corollary 6.2. — For any function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ with $\psi(k) \rightarrow +\infty$ as $k \rightarrow +\infty$, the sets

$$\{2^k 3^{k'} ; k \geq 0, 0 \leq k' \leq \psi(k)\} \quad \text{and} \quad \{2^k 3^{k'} ; k' \geq 0, 0 \leq k \leq \psi(k')\}$$

are non-Kazhdan sets in \mathbb{Z} .

Along the same lines, one can also ask for which values of $\delta \in (0, 1]$ there exists a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\liminf \widehat{\mu}(2^k 3^{k'}) \geq \delta$ as $\max(k, k') \rightarrow +\infty$. The proof of Theorem 2.1 allows us to exhibit a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ with nonnegative Fourier coefficients (namely $\mu = (\mu_1 + \mu_2)/2$) such that $\liminf \widehat{\mu}(2^k 3^{k'}) \geq 1/2$ as $\max(k, k') \rightarrow +\infty$. Again, we do not know whether the constant $1/2$ can be improved. The strongest statement which could be expected in this direction is the existence of a measure $\mu \in \mathcal{P}_c(\mathbb{T})$ such that $\widehat{\mu}(2^k 3^{k'}) \rightarrow 1$ as $\max(k, k') \rightarrow +\infty$. This would show that the Furstenberg sequence is a rigidity sequence for weakly mixing dynamical systems. This natural question is raised in Remark 3.12 (b) of [8] and we record it anew here:

Question 6.3. — Is the Furstenberg sequence a rigidity sequence for weakly mixing dynamical systems?

6.3. Examples of rigidity sequences. — Corollaries 2.5 and 2.6 allow us to retrieve directly all known examples of rigidity sequences from [8], [18], [2], [1] and [21]. The only examples of rigidity sequences not covered by our results are those of [20] and [23]. Indeed, Fayad and Kanigowski construct in [20] examples of rigidity sequences $(n_k)_{k \geq 0}$ such that $\{\lambda^{n_k}; k \geq 0\}$ is dense in \mathbb{T} for every $\lambda = e^{2i\pi\theta} \in \mathbb{T}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and there exist for every integer $p \geq 2$ infinitely many integers k such that p does not divide n_k . So such sequences never satisfy the assumption of Corollary 2.5. Griesmer strengthens this result in [23] by showing the existence of rigidity sequences $(n_k)_{k \geq 0}$ such that $\{n_k; k \geq 0\}$ is dense in \mathbb{Z} in the Bohr topology.

We briefly list here some of the examples of rigidity sequences which can be obtained from Corollaries 2.5 and 2.6. Our first example is that of Fayad and Thouvenot in [21].

Example 6.4. — [21] If the sequence $(n_k)_{k \geq 0}$ is such that there exists $\lambda = e^{2i\pi\theta} \in \mathbb{T}$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, such that $\lambda^{n_k} \rightarrow 1$ as $k \rightarrow +\infty$, $(n_k)_{k \geq 0}$ is a rigidity sequence.

This result of [21] follows directly from Corollary 2.5. Indeed, if $\lambda^{n_k} \rightarrow 1$ with $\lambda = e^{2i\pi\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda^{pn_k} \rightarrow 1$ for every $p \in \mathbb{Z}$. Since θ is irrational, the set $\{\lambda^p; p \in \mathbb{Z}\}$ is dense in \mathbb{T} , and Corollary 2.5 applies.

Example 6.5. — [8], [18] If $(n_k)_{k \geq 0}$ is a strictly increasing sequence of integers such that $n_k | n_{k+1}$ for every $k \geq 0$, $(n_k)_{k \geq 0}$ is a rigidity sequence.

Indeed, under the assumption of Example 6.5, the set $C = \{\lambda \in \mathbb{T}; \lambda^{n_k} \rightarrow 1\}$ contains all n_k -th roots of 1, $k \geq 0$, and is hence dense in \mathbb{T} .

Corollary 2.6 shows that Example 6.5 can be improved into

Example 6.6. — Let $(m_k)_{k \geq 0}$ be a strictly increasing sequence of integers such that $m_k | m_{k+1}$ for every $k \geq 0$. Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function. Order the set $\{k' m_k; k \geq 0 \text{ and } 1 \leq k' \leq \psi(k)\}$ as a strictly increasing sequence $(n_k)_{k \geq 0}$. Then $(n_k)_{k \geq 0}$ is a rigidity sequence.

Indeed, the set $C' = \{\lambda \in \mathbb{T}; \lambda^{k' m_k} \rightarrow 1 \text{ as } k \rightarrow +\infty \text{ uniformly in } k'\}$ contains all m_k -th roots of 1, and is dense in \mathbb{T} . So Corollary 2.6 applies.

For instance, if $(r_k)_{k \geq 0}$ is any sequence of positive integers, the sequence $(n_k)_{k \geq 0}$ obtained by ordering the set $\{k' 2^k; k \geq 0, 1 \leq k' \leq r_k\}$ in a strictly increasing sequence is a rigidity sequence. This provides new examples of rigidity sequences $(n_k)_{k \geq 0}$ such that $\frac{n_{k+1}}{n_k} \rightarrow 1$ as $k \rightarrow +\infty$.

Example 6.7. — Let $(r_k)_{k \geq 0}$ be any sequence of positive integers with $r_k \rightarrow +\infty$ as $k \rightarrow +\infty$. The sequence $(n_l)_{l \geq 0}$ obtained by ordering in a strictly increasing fashion the set $\{j2^k; k \geq 0, 1 \leq j \leq r_k\}$ is a rigidity sequence which satisfies $\frac{n_{l+1}}{n_l} \rightarrow 1$ as $l \rightarrow +\infty$.

Proof. — It suffices to show that for every $\varepsilon > 0$ and every l sufficiently large there exists $l' > l$ such that $\frac{n_{l'}}{n_l} < 1 + \varepsilon$.

– Suppose first that $n_l = j2^k$ for some $k \geq 0$ and some $1/\varepsilon < j < r_k$. Then taking $n_{l'} = (j+1)2^k$, we have $\frac{n_{l'}}{n_l} = \frac{j+1}{j} < 1 + \varepsilon$.

– Suppose next that $n_l = j2^k$ for some $k \geq 0$ and some $1 \leq j \leq 1/\varepsilon$. Fix an integer p such that $2^{-p} < \varepsilon$. If l is sufficiently large, we have $r_{k-p} > 2^p/\varepsilon$. Set $j' = j2^p$. Since $j' \leq 2^p/\varepsilon < r_{k-p}$, the integer $n_{l'} = (j'+1)2^{k-p}$ appears in the sequence $(n_l)_{l \geq 0}$. Also, since $n_{l'} = (j'+1)2^{k-p} > j2^k = n_l$, we have $l' > l$, and

$$\frac{n_{l'}}{n_l} = \frac{(j'+1)2^{k-p}}{j2^k} = \frac{(j'+1)}{j} 2^{-p} \leq \frac{j+2^{-p}}{j} < 1 + 2^{-p} < 1 + \varepsilon.$$

– The last case we have to deal with is when $n_l = r_k 2^k$ for some $k \geq 0$. Let $j' \geq 1$ be such that $j' \leq r_k/2 < j'+1$. Then $j' < r_{k+1}$, and if we set $n_{l'} = (j'+1)2^{k+1}$, the integer $n_{l'}$ appears in the sequence $(n_l)_{l \geq 0}$. We have

$$\frac{n_{l'}}{n_l} = \frac{(j'+1)2^{k+1}}{r_k 2^k} = \frac{2(j'+1)}{r_k} \leq 1 + \frac{2}{r_k} < 1 + \varepsilon$$

if k is sufficiently large, and this terminates the proof. \square

Example 6.8. — [1] (a) Let $(d_k)_{k \geq 0}$ be a strictly increasing sequence of positive integers of density zero. There exists a strictly increasing sequence of integers $(n_k)_{k \geq 0}$ which is a rigidity sequence and satisfies $n_k \leq d_k$ for every $k \geq 0$.

(b) Let $(d_k)_{k \geq 0}$ be a sequence of real numbers with $d_k \geq k$ for every $k \geq 0$ and $\lim_{k \rightarrow +\infty} d_k/k = +\infty$. There exists a strictly increasing sequence of integers $(n_k)_{k \geq 0}$ which is a rigidity sequence and satisfies $n_k \leq d_k$ for every $k \geq 0$.

This has been proved by Aaronson in [1, Th. 4]; a simpler construction with the weaker conclusion that $n_k \leq d_k$ for infinitely many k was given in [8, Prop. 3.18]. The proof given below uses Corollary 2.5 and a result of Bugeaud [16].

Proof. — As the statement (a) is a simple consequence of (b), we only give the proof of (b). Set $g_0 = 1$ and $g_k = d_k/k$ for every $k \geq 1$. Then $(g_k)_{k \geq 0}$ is a sequence of reals with $g_k \geq 1$ for every $k \geq 0$ which tends to infinity (notice that for (a) this holds since $(d_k)_{k \geq 0}$ is a sequence of density zero). Using (a particular case of) [16, Th. 1], we obtain that there exists for every fixed irrational number θ an increasing sequence $(n_k)_{k \geq 0}$ of positive integers such that $n_k \leq kg_k = d_k$ for every $k \geq 1$ and $\exp(2i\pi\theta)^{n_k} \rightarrow 1$. It follows from Example 6.4 that $(n_k)_{k \geq 0}$ is a rigidity sequence. \square

Example 6.9. — Let $(m_k)_{k \geq 0}$ be a strictly increasing sequence of positive integers with $m_{k+1} - m_k \rightarrow +\infty$. There exists a strictly increasing sequence of integers $(n_k)_{k \geq 0}$ which is a rigidity sequence and satisfies $m_k \leq n_k < m_{k+1}$ for every $k \geq 0$.

Proof. — The proof is exactly the same as the preceding one, replacing the result from [16] by [9, Obs. 1.36]. \square

6.4. Exceptional sets for (almost) uniform distribution. — Let $(n_k)_{k \geq 0}$ be a strictly increasing sequence of integers, and let $\nu \in \mathcal{M}(\mathbb{T})$ be a (finite) complex Borel measure on \mathbb{T} . We stress that ν is not necessarily a probability measure. Given $\theta \in \mathbb{R}$, the sequence $(n_k \theta)_{k \geq 0}$ is said ([31], [29, p. 53]) to be *almost uniformly distributed with respect to ν* if there exists a strictly increasing sequence $(N_j)_{j \geq 1}$ of positive integers such that for every arc $I \subset \mathbb{T}$ whose endpoints are not atoms (mass-points) for ν one has

$$\lim_{j \rightarrow +\infty} \frac{1}{N_j} \# \{n \leq N_j : \exp(2i\pi n_k \theta) \in I\} = \nu(I).$$

The analog of Weyl's criterion states that $(n_k \theta)_{k \geq 0}$ is almost uniformly distributed with respect to ν if and only if there exists a strictly increasing sequence $(N_j)_{j \geq 1}$ of positive integers such that

$$\lim_{j \rightarrow \infty} \frac{1}{N_j} \sum_{k=1}^{N_j} \exp(m 2i\pi n_k \theta) \text{ exists for every } m \in \mathbb{Z}.$$

In this case, the limit is $\hat{\nu}(m)$. It can also be proved that $(n_k \theta)_{k \geq 0}$ is almost uniformly distributed with respect to ν if and only if there exists a strictly increasing sequence $(N_j)_{j \geq 1}$ of positive integers such that

$$\frac{1}{N_j} \sum_{k=1}^{N_j} f(e^{2i\pi n_k \theta}) \rightarrow \int_{\mathbb{T}} f d\mu \quad \text{as } j \rightarrow +\infty \quad \text{for every } f \in \mathcal{C}(\mathbb{T}).$$

We now denote by $W((n_k)_{k \geq 0}, \nu)$, the *exceptional set of almost uniform distribution* of (n_k) with respect to ν . This is the set of all $\theta \in \mathbb{R}$ such that $(n_k \theta)_{k \geq 0}$ is not almost uniformly distributed with respect to ν . We will write $U((n_k)_{k \geq 0}, \nu)$ for the exceptional set of (classical) uniform distribution of (n_k) with respect to ν , which corresponds to the case where $N_j = j$ for every $j \geq 1$.

The size of the exceptional set $U((n_k)_{k \geq 0}, \nu)$ has been studied in many works, in particular in the case where ν is the normalized Lebesgue measure on \mathbb{T} . In this case, we write it as $U((n_k)_{k \geq 0})$. If the sequence $(n_k)_{k \geq 0}$ is lacunary, $U((n_k)_{k \geq 0})$ is uncountable, and even of Hausdorff dimension 1 ([19], see also [25]). See also [35] and [33] for a stronger result. On the other hand, it is known (see [11], [13]) that among various natural classes of random sequences of integers, almost all sequences $(n_k)_{k \geq 0}$ satisfy $U((n_k)_{k \geq 0}) = \mathbb{Q}$. These typical random sequences $(n_k)_{k \geq 0}$ are sublacunary, i.e. satisfy $n_{k+1}/n_k \rightarrow 1$ as $k \rightarrow +\infty$. Nonetheless, examples of sublacunary sequences $(n_k)_{k \geq 0}$ with $U((n_k)_{k \geq 0})$ uncountable were constructed in [19] (see also [6]). Concerning the size of $W((n_k)_{k \geq 0}, \nu)$ we refer for instance to [34], [25] and [28]. See also [15] for other references.

Our results about the size of $W((n_k)_{k \geq 0}, \nu)$ rely on the following generalization of Proposition 5.6, which provides a link between the size of the exceptional set $W((n_k)_{k \geq 0}, \nu)$ and the modified Kazhdan constant of the set $\{n_k ; k \geq 0\}$.

Proposition 6.10. — *Let $(n_k)_{k \geq 0}$ be a strictly increasing sequence of positive integers with $n_0 = 1$, and let $\nu \in \mathcal{M}(\mathbb{T})$ with $\nu \neq \delta_{\{1\}}$. If $W((n_k)_{k \geq 0}, \nu)$ is finite or countable infinite, $Q = \{n_k ; k \geq 0\}$ is a Kazhdan subset of \mathbb{Z} , and*

$$\widetilde{\text{Kaz}}(Q) \geq \sqrt{2(1 - \Re \hat{\nu}(1))}.$$

Proof. — Fix $\gamma \in (0, 1 - \Re \widehat{\nu}(1))$, and let μ be a probability measure on \mathbb{T} such that $\sup_{k \geq 0} (1 - \Re \widehat{\mu}(n_k)) < \gamma$. Then

$$1 - \Re \int_{\mathbb{T}} \left(\frac{1}{N} \sum_{k=1}^N \lambda^{n_k} \right) d\mu(\lambda) < \gamma \quad \text{for every } N \geq 1.$$

Suppose that the measure μ is continuous. Since there exists a strictly increasing sequence $(N_j)_{j \geq 1}$ of integers such that

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \lambda^{n_k} \rightarrow \widehat{\nu}(1) \quad \text{as } j \rightarrow +\infty \quad \text{for every } \lambda \in \mathbb{T} \setminus C,$$

where C is a finite or countable infinite subset of \mathbb{T} , we have $1 - \Re \widehat{\nu}(1) \leq \gamma$, which contradicts our initial assumption. So μ has a discrete part. It then follows from Theorem 5.1 that the modified Kazhdan constant of Q is at least $\sqrt{2(1 - \Re \widehat{\nu}(1))}$. \square

The following result provides an example of a nonlacunary semigroup $(n_k)_{k \geq 0}$ whose associated exceptional sets $W((n_k)_{k \geq 0}, \nu)$ with respect to ν are uncountable for a large class of measures $\nu \in \mathcal{M}(\mathbb{T})$.

Theorem 6.11. — Denote by $(n_k)_{k \geq 0}$ the sequence obtained by ordering the Furstenberg set $F = \{2^k 3^{k'}; k, k' \geq 0\}$ in a strictly increasing fashion. For every measure $\nu \in \mathcal{M}(\mathbb{T})$ such that $\Re \widehat{\nu}(1) < 1/2$, the set $W((n_k)_{k \geq 0}, \nu)$ is uncountable.

Proof of Theorem 6.11. — Fix $\nu \in \mathcal{M}(\mathbb{T})$, and suppose that $U((n_k)_{k \geq 0}, \nu)$ is at most countable. Since $\widetilde{\text{Kaz}}(F) \leq 1$ by Theorem 2.2, it follows from Proposition 6.10 that $\sqrt{2(1 - \Re \widehat{\nu}(1))} \leq 1$, i.e. that $\Re \widehat{\nu}(1) \geq 1/2$. This proves Theorem 6.11. \square

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