KAZHDAN SETS IN GROUPS AND EQUIDISTRIBUTION PROPERTIES

by

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Abstract. — Using functional and harmonic analysis methods, we study Kazhdan sets in topological groups which do not necessarily have Property (T). We provide a new criterion for a generating subset Q of a group G to be a Kazhdan set; it relies on the existence of a positive number ε such that every unitary representation of G with a (Q, ε)-invariant vector has a finite dimensional subrepresentation. Using this result, we give an equidistribution criterion for a generating subset of G to be a Kazhdan set. In the case where $G = \mathbb{Z}$, this shows that if $(n_k)_{k\geqslant 1}$ is a sequence of integers such that $(e^{2i\pi\theta n_k})_{k\geqslant 1}$ is uniformly distributed in the unit circle for all real numbers θ except at most countably many, $\{n_k\,;\,k \geqslant 1\}$ is a Kazhdan set in $\mathbb{Z}$ as soon as it generates $\mathbb{Z}$. This answers a question of Y. Shalom from [B. Bekka, P. de la Harpe, A. Valette, Kazhdan's property (T), Cambridge Univ. Press, 2008]. We also obtain characterizations of Kazhdan sets in second countable locally compact abelian groups, in the Heisenberg groups and in the group $\mathrm{Aff}_+(\mathbb{R})$. This answers in particular a question from [B. Bekka, P. de la Harpe, A. Valette, Kazhdan's property (T), op. cit.].

1. Introduction

A *unitary representation* of a topological group G on a Hilbert space H is a group morphism from G into the group $\mathscr{U}(H)$ of all unitary operators on H which is strongly continuous, i. e. such that the map $g \longmapsto \pi(g)x$ is continuous from G into H for all vectors $x \in H$. As all the representations we consider in this paper are unitary, we will often drop the word "unitary" and speak simply of representations of a group G on a Hilbert space H. In this paper the Hilbert spaces will always be supposed to be complex, and endowed with an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ which is linear in the first variable and antilinear in the second variable.

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Definition 1.1. — Let Q be a subset of a topological group G, ε a positive real number, and π a unitary representation of G on a Hilbert space H. A vector $x \in H$ is said to be (Q, ε)*-invariant for* π if

$$\sup_{g \in Q} ||\pi(g)x - x|| < \varepsilon ||x||.$$

A (Q, ε)-invariant vector for π is in particular non-zero. A G*-invariant vector* for π is a vector $x \in H$ such that $\pi(g)x = x$ for all $g \in G$.

The notions of Kazhdan sets and Kazhdan pairs will be fundamental in our work.

Definition 1.2. — A subset Q of a topological group G is a *Kazhdan set in* G if there exists $\varepsilon > 0$ such that the following property holds true: any unitary representation π of G on a complex Hilbert space H with a (Q, ε)-invariant vector has a non-zero G-invariant vector. In this case, the pair (Q, ε) is *Kazhdan pair*, and ε is a *Kazhdan constant for* Q. A group G has *Property* (T), or is a *Kazhdan group*, if it admits a compact Kazhdan set.

Property (T) is a rigidity property of topological groups which has been introduced by Kazhdan in [**26**] for locally compact groups, and which has spectacular applications to many fields. For instance, the groups $SL_n(\mathbb{R})$ and $SL_n(\mathbb{Z})$ have Property (T) if and only if $n \geqslant 3$. We refer the reader to the monograph [**6**] by Bekka, de la Harpe, and Valette for a comprehensive presentation of Kazhdan's Property (T) and its applications (see also [**19**]).

The aim of this paper is to identify and study Kazhdan sets in topological groups. For discrete groups with Property (T) the Kazhdan sets are known. Recall first the following definition.

Definition 1.3. — If Q is a subset of a group G, we denote by $\langle Q \rangle$ the smallest subgroup of G containing Q, i. e. the set of all elements of the form $g_1^{\pm 1} \dots g_n^{\pm 1}$, where $n \geqslant 1$ and $g_1, \dots, g_n$ belong to Q. We say that Q *generates* G, or *is generating in* G, if $\langle Q \rangle = G$.

Locally compact groups with Property (T) are compactly generated. In particular, discrete groups with Property (T) are finitely generated and it is known (see [**6**, Prop. 1.3.2]) that the Kazhdan subsets of a discrete group with Property (T) are exactly the generating subsets of the group. More generally [**6**, Prop. 1.3.2], a generating set of a locally compact group which has Property (T) is a Kazhdan set and, conversely, a Kazhdan set which has non-empty interior is necessarily a generating set.

For groups without Property (T) the results about Kazhdan sets and Kazhdan pairs are very sparse. It is known (see [**6**, Prop. 1.1.5]) that $(G, \sqrt{2})$ is a Kazhdan pair for every topological group G, so G is always a ("large") Kazhdan subset of itself. The main motivations for the present paper are two questions from [**6**, Sec. 7.12]. The first one is due to Y. Shalom:

Question 1.4. — [**6**, Sec. 7.12] "The question of knowing if a subset Q of $\mathbb{Z}$ is a Kazhdan set is possibly related to the equidistribution of the sequence $(e^{2i\pi n\theta})_{n \in Q}$ for θ irrational, in the sense of Weyl."

We refer the reader to the classical book [**28**] by Kuipers and Niederreiter for more information about equidistributed (sometimes called uniformly distributed) sequences. Recall that the Weyl Criterion ([**28**, Th. 2.1]) states that if $(x_k)_{k \geqslant 1}$ is a sequence of real numbers, $(e^{2i\pi x_k})_{k \geqslant 1}$ is equidistributed in $\mathbb{T}$ if and only if $\frac{1}{N} \sum_{k=1}^{N} e^{2i\pi h x_k}$ tends to 0 as N tends to infinity for every non-zero integer h. Hence if $(n_k)_{k \geqslant 1}$ is a sequence of elements of $\mathbb{Z}$, $(e^{2i\pi n_k\theta})_{k\geqslant 1}$ is equidistributed in $\mathbb{T}$ for every $\theta\in\mathbb{R}\backslash\mathbb{Q}$ if and only if $\frac{1}{N}\sum_{k=1}^{N}e^{2i\pi n_k\theta}$ tends to 0 as N tends to infinity for every $\theta\in\mathbb{R}\backslash\mathbb{Q}$. If χ_θ denotes, for every $\theta\in\mathbb{R}$, the character on $\mathbb{Z}$ associated to θ, this means that $\frac{1}{N}\sum_{k=1}^{N}\chi_\theta(n_k)$ tends to 0 as N tends to infinity for every $\theta\in\mathbb{R}\backslash\mathbb{Q}$.

The first remark about Question 1.4 is that it concerns Kazhdan *sets* and equidistributed *sequences*; notice that a rearrangement of the terms of a sequence can destroy its equidistribution properties. It is known [**28**, p. 135] that given a sequence of elements of the unit circle $\mathbb{T}$, there exists a certain rearrangement of the terms which is is equidistributed if and only if the original sequence is dense in $\mathbb{T}$. The second remark is that, as mentioned before, Kazhdan sets of $\mathbb{Z}$ are necessarily generating, while there are non-generating subsets Q of $\mathbb{Z}$, like $Q=p\mathbb{Z}$ with $p\geqslant 2$, for which the sequences $(e^{2i\pi pk\theta})_{k\in\mathbb{Z}}$ are equidistributed for all irrational θ's. So Question 1.4 may be rephrased as follows:

Question 1.5. — (a) Let Q be a Kazhdan subset of $\mathbb{Z}$. Does a certain rearrangement $(n_k)_{k\geqslant 1}$ of the elements of Q exist such that $(e^{2i\pi n_k\theta})_{k\geqslant 1}$ is equidistributed in $\mathbb{T}$ for every $\theta\in\mathbb{R}\backslash\mathbb{Q}$? Equivalently, is the sequence $(e^{2i\pi n\theta})_{n\in Q}$ dense in $\mathbb{T}$ for every $\theta\in\mathbb{R}\backslash\mathbb{Q}$?

(b) Let $Q=\{n_k\,;\,k\geqslant 1\}$ be a generating subset of $\mathbb{Z}$. Suppose that the sequence $(e^{2i\pi n_k\theta})_{k\geqslant 1}$ is equidistributed in $\mathbb{T}$ for every $\theta\in\mathbb{R}\backslash\mathbb{Q}$. Is Q a Kazhdan set in $\mathbb{Z}$?

We will prove in this paper that Question 1.5 (a) has a negative answer, a counterexample being provided by the set $Q=\{2^k+k\,;\,k\geqslant 1\}$ (see Example 6.4). On the other hand, one of the aims of this paper is to show that Question 1.5 (b) has a positive answer. Actually, we will consider Question 1.5 (b) in the more general framework of Moore groups, and answer it in the affirmative (Theorem 2.1).

The second question of [**6**, Sec. 7.12] runs as follows:

Question 1.6. — [**6**, Sec. 7.12] "More generally, what are the Kazhdan subsets of $\mathbb{Z}^k$, $\mathbb{R}^k$, the Heisenberg group, or other infinite amenable groups?"

We shall answer Question 1.6 in Section 6 by giving a complete description of Kazhdan sets in many classic groups which do not have Property (T), including the groups $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k\geqslant 1$, the Heisenberg groups of all dimensions, and the group $\mathrm{Aff}_+(\mathbb{R})$ of orientation-preserving affine homeomorphisms of $\mathbb{R}$.

2. Main results

Let us now describe our main results in more detail.

2.1. Equidistributed sets in Moore groups. — In order to state Question 1.5 (b) for more general groups, we first need to define equidistributed sequences. There are several possible ways of doing this. If $(g_k)_{k\geqslant 1}$ is a sequence of elements in a locally compact group G, uniform distribution of $(g_k)_{k\geqslant 1}$ in any of these senses requires a certain form of convergence, as N tends to infinity, of the means

$$\frac{1}{N}\sum_{k=1}^{N}\pi(g_k) \tag{2.1}$$

to the orthogonal projection P_π on the subspace of invariant vectors for π, for a certain class of unitary representations π of G. Veech [**37**], [**38**] calls $(g_k)_{k\geqslant 1}$ uniformly distributed in G if the convergence of the means (2.1) holds in the weak operator topology for all unitary representations of G (or, equivalently, for all irreducible unitary representations of G, provided G is supposed to be second countable). Unitary uniform distribution in the sense of Losert and Rindler [**29**], [**17**] requires the convergence in the strong operator topology of the means (2.1) for all irreducible unitary representations of the group, while Hartman uniform distribution only requires convergence in the strong operator topology for all finite dimensional unitary representations.

In this paper we deal with the following natural extension to general locally compact groups G of the equidistribution condition of Question 1.5 (b): if $(g_k)_{k\geqslant 1}$ is a sequence of elements of G, we require the sequence of means (2.1) to converge to 0 in the weak topology for all finite dimensional irreducible unitary representations of G except those belonging at most countably many equivalence classes of irreducible representations. In the case of the group $\mathbb{Z}$, sequences $(n_k)_{k\geqslant 1}$ of integers such that $(e^{2i\pi\theta n_k})_{k\geqslant 1}$ is uniformly distributed in $\mathbb{T}$ for all $\theta \in \mathbb{R}$ except countably many are said to be of first kind (see for instance [**21**]). The class of groups we will consider in relation to Question 1.5 (b) is the class of second countable Moore groups. Recall that G is said to be a *Moore group* if all irreducible representations of G are finite dimensional. Locally compact Moore groups are completely described in [**31**]: a Lie group is a Moore group if and only if it has a closed subgroup H such that H modulo its center is compact, and a locally compact group is a Moore group if and only if it is a projective limit of Lie groups which are Moore groups. See also the survey [**33**] for more information concerning the links between various properties of topological groups, among them the property of being a Moore group. Of course all locally compact abelian groups are Moore groups.

Here is the first main result of this paper.

Theorem 2.1. — *Let G be a second countable locally compact Moore group. Let $(g_k)_{k\geqslant 1}$ be a sequence of elements of G. Suppose that $(g_k)_{k\geqslant 1}$ satisfies the following equidistribution assumption:*

(2.2) *for all (finite dimensional) irreducible unitary representations π of G on a Hilbert space H, except those belonging to at most countably many equivalence classes,*

$$\frac{1}{N}\sum_{k=1}^{N}\langle \pi(g_k)x, y\rangle \xrightarrow[N\to+\infty]{} 0 \quad \textit{for every } x, y \in H.$$

– If $Q = \{g_k\,;\, k \geqslant 1\}$ generates G (in which case G has to be countable), then Q is a Kazhdan set in G.

– If Q is not assumed to generate G, Q becomes a Kazhdan set when one adds to it a suitable "small" perturbation. More precisely, if $(W_n)_{n\geqslant 1}$ is an increasing sequence of subsets of G such that $\bigcup_{n\geqslant 1} W_n = G$, there exists $n \geqslant 1$ such that $W_n \cup Q$ is a Kazhdan set in G.

The equidistribution property (2.2) of the sequence $(g_k)_{k\geqslant 1}$ takes a more familiar form when the group G is supposed to be abelian: it is equivalent to requiring that condition (2.3) below holds true for all characters χ of the group except possibly countably many.

Theorem 2.2. — *Let G be a locally compact abelian group, and let $(g_k)_{k\geqslant 1}$ be a sequence of elements of G. Suppose that*

$$\frac{1}{N}\sum_{k=1}^{N}\chi(g_k)\xrightarrow[N\to+\infty]{}0 \tag{2.3}$$

for all characters χ on G, except at most countably many. If $Q=\{g_k\,;\,k\geqslant 1\}$ generates G, then Q is a Kazhdan set in G. If Q is not assumed to generate G, and if $(W_n)_{n\geqslant 1}$ is an increasing sequence of subsets of G such that $\bigcup_{n\geqslant 1}W_n=G$, there exists $n\geqslant 1$ such that $W_n\cup Q$ is a Kazhdan set in G.

Theorem 2.2 can thus be seen as a particular case of Theorem 2.1, except for the fact that there is no need to suppose that the group is second countable when it is known to be abelian. The case $G=\mathbb{Z}$ provides a positive answer to Question 1.5 (b) above.

2.2. Kazhdan sets and finite dimensional subrepresentations. — The proof of Theorem 2.2 relies on Theorem 2.3 below, which gives a new condition for a "small perturbation" of a subset Q of a group G to be a Kazhdan set in G. Theorem 2.3 constitutes the core of the paper, and has, besides the proofs of Theorems 2.1 and 2.2, several interesting applications which we will present in Sections 5 and 6.

Theorem 2.3. — *Let G be a topological group, and let $(W_n)_{n\geqslant 1}$ be an increasing sequence of subsets of G such that W_1 is a neighborhood of the unit element e of G and $\bigcup_{n\geqslant 1}W_n=G$. Let Q be a subset of G satisfying the following assumption:*

(*) *there exists a positive constant ε such that every unitary representation π of G on a Hilbert space H admitting a (Q,ε)-invariant vector has a finite dimensional subrepresentation.*

Then there exists an integer $n\geqslant 1$ such that $Q_n=W_n\cup Q$ is a Kazhdan set in G.

If the group G is locally compact, the same statement holds true for any increasing sequence $(W_n)_{n\geqslant 1}$ of subsets of G such that $\bigcup_{n\geqslant 1}W_n=G$.

The condition that W_1 be a neighborhood of e, which appears in the first part of the statement of Theorem 2.3, will be used in the proof in order to ensure the strong continuity of some infinite tensor product representations (see Proposition A.2). When G is locally compact, this assumption is no longer necessary (see Proposition A.1).

If the group G is locally compact and σ-compact, we can choose for $(W_n)_{n\geqslant 1}$ an increasing sequence of compact sets whose union is equal to G. If Q is a subset of G satisfying (*), there exists then by Theorem 2.3 a compact subset L of G such that $L\cup Q$ is a Kazhdan set in G. We thus retrieve a characterization of Property (T) for σ-compact locally compact groups due to Bekka and Valette [**5**], see also [**6**, Th. 2.12.9]. The original proof of this result relies on the Delorme-Guichardet theorem that such a group has Property (T) if and only if it has property (FH). See Section 5 for more details.

Theorem 2.3 admits a simpler formulation if we build the sequence $(W_n)_{n\geqslant 1}$ starting from a set which generates the group:

Corollary 2.4. — *Let G be a topological group. Let Q_0 be a subset of G which generates G and let Q be a subset of G. Suppose either that Q_0 has non-empty interior, or that G is a locally compact group. If Q satisfies assumption (*) of Theorem 2.3, then $Q_0\cup Q$ is a Kazhdan set in G.*

One of the main consequences of Corollary 2.4 is Theorem 2.5 below, which shows in particular that property (*) of Theorem 2.3 characterizes Kazhdan sets among generating sets (and which have non-empty interior – this assumption has to be added if the group is not supposed to be locally compact).

Theorem 2.5. — *Let G be a topological group and let Q be a subset of G which generates G. Suppose either that Q has non-empty interior or that G is locally compact. Then the following assertions are equivalent:*
(a) Q is a Kazhdan set in G;
(b) there exists a constant $\delta \in (0,1)$ such that every unitary representation π of G on a Hilbert space H admitting a vector $x \in H$ such that $\inf_{g \in Q} |\langle \pi(g)x, x\rangle| > \delta \|x\|^2$ has a finite dimensional subrepresentation;
(c) there exists a constant $\varepsilon > 0$ such that every unitary representation π of G on a Hilbert space H admitting a (Q,ε)-invariant vector has a finite dimensional subrepresentation.

The assumption that Q generates G cannot be dispensed with in Theorem 2.5: $Q = 2\mathbb{Z}$ is a subset of $\mathbb{Z}$ which satisfies property (c), but Q is clearly not a Kazhdan set in $\mathbb{Z}$. Condition (b) in Theorem 2.5 is easily seen to be equivalent to condition (c), which is nothing else than assumption (*) of Theorem 2.3. Its interest will become clearer in Section 6 below, where it will be used to obtain a characterization of Kazhdan sets in second countable locally compact abelian groups (Theorem 6.1). In the case of the group $\mathbb{Z}$, the characterization we obtain (Theorem 6.3) involves a classic class of sets in harmonic analysis, called *Kaufman sets*. We give in Section 6 several examples of "small" Kazhdan sets in $\mathbb{Z}$, describe Kazhdan sets in the Heisenberg groups H_n, $n \geqslant 1$ (Theorem 6.11), and also in the group $\mathrm{Aff}_+(\mathbb{R})$ (Theorem 6.14). These results provide an answer to Question 1.6.

The paper also contains an appendix which reviews some constructions of infinite tensor product representations on Hilbert spaces, used in the proof of Theorem 2.3.

3. Mixing properties for unitary representations and an abstract version of the Wiener theorem

3.1. Ergodic and mixing properties for unitary representations. — We first recall in this section some definitions and results concerning the structure of unitary representations of a topological group G. They can be found for instance in the book [**27**], the notes [**34**], and the paper [**8**] by Bergelson and Rosenblatt.

Recall that the class $\mathrm{WAP}(G)$ of *weakly almost periodic functions* on G is defined as follows: if $\ell^\infty(G)$ denotes the space of bounded functions on G, $f \in \ell^\infty(G)$ belongs to $\mathrm{WAP}(G)$ if the weak closure in $\ell^\infty(G)$ of the set $\{f(s^{-1}\cdot)\,;\, s \in G\}$ is weakly compact. For each $s \in G$, $f(s^{-1}\cdot)$ denotes the function $t \longmapsto f(s^{-1}t)$ on G. By comparison, recall that $f \in \ell^\infty(G)$ is an *almost periodic function* on G, written $f \in \mathrm{AP}(G)$, if the norm closure in $\ell^\infty(G)$ of $\{f(s^{-1}\cdot)\,;\, s \in G\}$ is compact. If π is a unitary representation of G on a Hilbert space H, the functions

$$\langle \pi(\cdot)x, y\rangle, \quad \big|\langle \pi(\cdot)x, y\rangle\big|, \quad \text{and} \quad \big|\langle \pi(\cdot)x, y\rangle\big|^2,$$

where x and y are any vectors of H, belong to $\mathrm{WAP}(G)$. For more on weakly almost periodic functions on a group, see for instance [**10**] or [**16**, Ch. 1, Sec. 9]. The interest of the class of weakly almost periodic functions on G in our context is that there exists on $\mathrm{WAP}(G)$ a unique G-invariant mean m. It satisfies

$$m(f(s^{-1}\,.\,)) = m(f(\,.\,s^{-1})) = m(f)$$

for every $f \in \mathrm{WAP}(G)$ and every $s \in G$. The abstract ergodic theorem then states that if π is a unitary representation of G on H, $m(\langle \pi(.)x, y\rangle) = \langle P_\pi x, y\rangle$ for every vectors $x, y \in H$, where P_π denotes the projection of H onto the space $E_\pi = \{x \in H\,;\, \pi(g)x = x \text{ for every } g \in G\}$ of G-invariant vectors for π. The representation π is *ergodic* (i. e. admits no non-zero G-invariant vector) if and only if $m(\langle \pi(.)x, y\rangle) = 0$ for every $x, y \in H$.

Following [**8**], let us now recall that the representation π is said to be *weakly mixing* if $m(|\langle \pi(.)x, x\rangle|) = 0$ for every $x \in H$, or, equivalently, $m(|\langle \pi(.)x, x\rangle|^2) = 0$ for every $x \in H$. Then $m(|\langle \pi(.)x, y\rangle|) = m(|\langle \pi(.)x, y\rangle|^2) = 0$ for every $x, y \in H$.

We will need the following characterization of weakly mixing representations.

Proposition 3.1. — *Let π be a unitary representation of G on a Hilbert space H. The following assertions are equivalent:*

(1) π is weakly mixing;
(2) π admits no finite dimensional subrepresentation;
(3) $\pi \otimes \overline{\pi}$ has no non-zero G-invariant vector.

Here $\overline{\pi}$ is the conjugate representation of π. The representation $\pi \otimes \overline{\pi}$ is equivalent to a representation of G on the space $HS(H)$ of Hilbert-Schmidt operators on H, which is often more convenient to work with. Recall that $HS(H)$ is a Hilbert space when endowed with the scalar product defined by the formula $\langle A, B\rangle = \mathrm{tr}(B^*A)$ for every $A, B \in HS(H)$. The space $H \otimes \overline{H}$, where $\overline{H}$ is the conjugate of H, is identified to $HS(H)$ by associating to each elementary tensor product $x \otimes \overline{y}$ of $H \otimes \overline{H}$ the rank-one operator $\langle\,.\,, y\rangle x$ on H. This map $\Theta : H \otimes \overline{H} \longrightarrow HS(H)$ extends into a unitary isomorphism, and we have for every $g \in G$ and every $T \in HS(H)$

$$\Theta\, \pi \otimes \overline{\pi}(g)\, \Theta^{-1}(T) = \pi(g)\, T\, \pi(g^{-1}).$$

We will, when needed, identify $\pi \otimes \overline{\pi}$ with this equivalent representation, and use it in particular in Section 3.3 to obtain a concrete description of the space $E_{\pi\otimes\overline{\pi}}$ of G-invariant vectors for $\pi \otimes \overline{\pi}$, which is identified to the subspace of $HS(H)$

$$\mathbf{E}_\pi = \{T \in HS(H)\,;\, \pi(g)\,T = T\,\pi(g) \text{ for every } g \in G\}\cdot$$

3.2. Compact unitary representations. — A companion to the property of weak mixing for unitary representation is that of compactness: given a unitary representation π of G on H a vector $x \in H$ is *compact* for π if the norm closure of the set $\{\pi(g)x\,;\, g \in G\}$ is compact in H. The representation π itself is said to be *compact* if every vector of H is compact for π. Compact representations decompose as direct sums of irreducible finite dimensional representations. The general structural result for unitary representations is given by the following result.

Proposition 3.2. — *A unitary representation π of G on a Hilbert space H decomposes as a direct sum of a weakly mixing representation and a compact representation:*

$$H = H_w \overset{\perp}{\oplus} H_c\,,$$

where H_w and H_c are both G-invariant closed subspaces of H, $\pi_w = \pi|_{H_w}$ is weakly mixing and $\pi_c = \pi|_{H_c}$ is compact. Hence π decomposes as a direct sum of a weakly mixing representation and finite dimensional irreducible subrepresentations.

See [**34**, Ch. 1], [**6**, Appendix M], [**8**] or [**12**] (in the amenable case) for detailed proofs of these results.

Now let π be a compact representation of G on a Hilbert space H, decomposed as a direct sum of irreducible finite dimensional representations of G. We sort out these representations by equivalence classes, and index the distinct equivalence classes by an index j belonging to a set J, which may be finite or infinite (and which is countable if H is separable). For every $j \in J$, we index by $i \in I_j$ all the representations appearing in the decomposition of π which are in the j-th equivalence class. More precisely, we can decompose H and π as

$$H = \bigoplus_{j \in J} \Big(\bigoplus_{i \in I_j} H_{i,j} \Big) \quad \text{and} \quad \pi = \bigoplus_{j \in J} \Big(\bigoplus_{i \in I_j} \pi_{i,j} \Big)$$

respectively, where the following holds true:

- for every $j \in J$, the spaces $H_{i,j}$, $i \in I_j$, are equal. We denote by K_j this common space, and by d_j its dimension (which is finite). We also write
$$\widetilde{H}_j = \bigoplus_{i \in I_j} H_{i,j}, \quad \text{so that} \quad H = \bigoplus_{j \in J} \widetilde{H}_j;$$
- for every $j \in J$, there exists an irreducible representation π_j of G on K_j such that $\pi_{i,j}$ is equivalent to π_j for every $i \in I_j$;
- if j, j' belong to J and $j \neq j'$, π_j and $\pi_{j'}$ are not equivalent.

Without loss of generality, we will suppose that $\pi_{i,j} = \pi_j$ for every $i \in I_j$. However, we will keep the notation $H_{i,j}$ for the various orthogonal copies of the space K_j which appear in the decomposition of H, as discarding this notation may be misleading in some of the proofs presented below.

Let $A \in \mathscr{B}(H)$. We write A in block-matrix form with respect to the decompositions

$$H = \bigoplus_{j \in J} \Big(\bigoplus_{i \in I_j} H_{i,j} \Big) \quad \text{and} \quad H = \bigoplus_{j \in J} \widetilde{H}_j$$

as

$$A = \big(A_{u,v}\big)_{k,l \in J, u \in I_k, v \in I_l} \quad \text{and} \quad A = \big(\widetilde{A}_{k,l}\big)_{k,l \in J} \quad \text{respectively}\cdot$$

For every $j \in J$ and every $u, v \in I_j$, we denote by $i_{u,v}^{(j)}$ the identity operator from $H_{u,j}$ into $H_{v,j}$.

3.3. A formula for the projection $\mathbf{P}_\pi$ of $HS(H)$ on $\mathbf{E}_\pi$.— We now give an explicit formula for the projection $\mathbf{P}_\pi A$ of a Hilbert-Schmidt operator $A \in HS(H)$ on the following closed subspace of $HS(H)$:

$$\mathbf{E}_\pi = \{T \in HS(H)\,;\, \pi(g)\,T = T\,\pi(g) \text{ for every } g \in G\}.$$

We also compute the norm of $\mathbf{P}_\pi A$.

Proposition 3.3. — *Let π be a compact representation of G on H, written in the form $\pi = \bigoplus_{j \in J} \big(\bigoplus_{i \in I_j} \pi_j \big)$ as discussed in Section 3.2 above. For every operator $A \in HS(H)$, we* *have*

$$\mathbf{P}_\pi A = \sum_{j\in J} \frac{1}{d_j} \sum_{u,v\in I_j} \operatorname{tr}\big(A_{u,v}\big)\, i^{(j)}_{u,v} \qquad \textit{and} \qquad ||\mathbf{P}_\pi A||^2 = \sum_{j\in J} \frac{1}{d_j} \sum_{u,v\in I_j} \big|\operatorname{tr}\big(A_{u,v}\big)\big|^2.$$

The proof of Proposition 3.3 relies on the following straightforward lemma:

Lemma 3.4. — *The space $\mathbf{E}_\pi$ consists of the operators $T\in \mathrm{HS}(H)$ such that*

- *for every $k,l\in J$ with $k\neq l$, $\widetilde{T}_{k,l}=0$;*
- *for every $k\in J$ and every $u,v\in I_k$, there exists a complex number $\lambda_{u,v}$ such that $T_{u,v}=\lambda_{u,v}\, i^{(k)}_{u,v}$. Thus $\widetilde{T}_{k,k}=\big(\lambda_{u,v}\, i^{(k)}_{u,v}\big)_{u,v\in I_k}$.*

Proof of Lemma 3.4. — Let $T\in \mathbf{E}_\pi$. For every $k,l\in J$, $u\in I_k$ and $v\in I_l$, $\pi_k(g)\,T_{u,v} = T_{u,v}\,\pi_l(g)$ for every $g\in G$. Thus the operator $T_{u,v}$ intertwines the two representations π_k and π_l. If $T_{u,v}$ is non-zero, it follows from Schur's Lemma that $T_{u,v}$ is an isomorphism. The representations π_k and π_l are thus isomorphically (and hence unitarily) equivalent. Since π_k and π_l are not equivalent for $k\neq l$, it follows that $T_{u,v}=0$ in this case. If now $k=l$, Schur's Lemma again implies that $T_{u,v}=\lambda_{u,v}\, i^{(k)}_{u,v}$ for some scalar $\lambda_{u,v}$. Thus any operator $T\in\mathbf{E}_\pi$ satisfies the two conditions of the lemma. The converse is obvious. □

The proof of Proposition 3.3 is now easy.

Proof of Proposition 3.3. — Consider, for every $j\in J$ and $u,v\in I_j$, the one-dimensional subspace $\mathbf{E}^{(j)}_{u,v}$ of $HS(H)$ spanned by the operator $i^{(j)}_{u,v}$. These subspaces are pairwise orthogonal in $\mathrm{HS}(H)$, and by Lemma 3.4 we have

$$\mathbf{E}_\pi = \bigoplus_{j\in J}\Big(\bigoplus_{u,v\in I_j} \mathbf{E}^{(j)}_{u,v}\Big).$$

Hence, for every $A\in HS(H)$,

$$\mathbf{P}_\pi A = \sum_{j\in J}\sum_{u,v\in I_j} \Big\langle A, \frac{i^{(j)}_{u,v}}{||i^{(j)}_{u,v}||_{HS}}\Big\rangle \frac{i^{(j)}_{u,v}}{||i^{(j)}_{u,v}||_{HS}} = \sum_{j\in J}\frac{1}{d_j}\sum_{u,v\in I_j} \operatorname{tr}\big(A_{u,v}\big)\, i^{(j)}_{u,v},$$

which gives the two formulas we were looking for. □

Corollary 3.5. — *Let $\pi = \bigoplus_{j\in J}\big(\bigoplus_{i\in I_j}\pi_j\big)$ be a compact representation of G on H. Let $x=\bigoplus_{j\in J}\big(\bigoplus_{i\in I_j} x_{i,j}\big)$ and $y=\bigoplus_{j\in J}\big(\bigoplus_{i\in I_j} y_{i,j}\big)$ be two vectors of H, and let $A\in HS(H)$ be the rank-one operator $\langle\,\bullet\,,y\rangle x$. Then*

$$\mathbf{P}_\pi A = \sum_{j\in J}\frac{1}{d_j}\sum_{u,v\in I_j} \langle x_{u,j}, y_{v,j}\rangle\, i^{(j)}_{u,v}.$$

Proof. — For every $j\in J$ and $u,v\in I_j$, $A_{u,v}=\langle\,\bullet\,,y_{v,j}\rangle x_{u,j}$, so that $\operatorname{tr}\big(A_{u,v}\big) = \langle x_{u,j},y_{v,j}\rangle$. The result then follows from Proposition 3.3. □

3.4. An abstract version of the Wiener Theorem.— As recalled in Section 3.1, $\mathbf{E}_\pi$ is the space of G-invariant vectors for the representation $\pi \otimes \overline{\pi}$ on $HS(H)$, where for every $x, y \in H$, $x \otimes \overline{y}$ is identified with the rank-one operator $\langle \,.\,, y\rangle x$. For every pair (x, y) of vectors of H, denote by $\boldsymbol{b}_{x,y}$ the element of $K \otimes K$, with $K = \bigoplus_{j \in J} K_j$, defined by

$$\boldsymbol{b}_{x,y} = \sum_{j \in J} \frac{1}{\sqrt{d_j}} \Big(\sum_{i \in I_j} x_{i,j} \otimes \overline{y}_{i,j} \Big).$$

It should be pointed out that for a fixed index $j \in J$ the vectors $x_{i,j}$ and $y_{i,j}$ are understood in the formula above as belonging to the same space K_j (and not to the various orthogonal spaces $H_{i,j}$). So $\boldsymbol{b}_{x,y}$ is a vector of $K \otimes K$, not of $H \otimes H$. Thus

$$||\boldsymbol{b}_{x,y}||^2 = \sum_{j \in J} \frac{1}{d_j} \sum_{u, v \in I_j} \langle x_{u,j}, x_{v,j}\rangle \overline{\langle y_{u,j}, y_{v,j}\rangle}.$$

Combining Corollary 3.5 with the formula

$$m\big(|\langle \pi(\,.\,)x, y\rangle|^2\big) = \langle P_{\pi \otimes \overline{\pi}}\, x \otimes \overline{x}, y \otimes \overline{y}\,\rangle = \langle \mathbf{P}_\pi \langle \,.\,, x\rangle x, \langle \,.\,, y\rangle y\rangle$$

yields

Corollary 3.6. — *Let $\pi = \bigoplus_{j \in J} \big(\bigoplus_{i \in I_j} \pi_j \big)$ be a compact representation of G on H. For every vectors $x = \bigoplus_{j \in J} \big(\bigoplus_{i \in I_j} x_{i,j} \big)$ and $y = \bigoplus_{j \in J} \big(\bigoplus_{i \in I_j} y_{i,j} \big)$ of H, we have*

$$m\big(|\langle \pi(\,.\,)x, y\rangle|^2\big) = \sum_{j \in J} \frac{1}{d_j} \sum_{u, v \in I_j} \langle x_{u,j}, x_{v,j}\rangle \,.\, \overline{\langle y_{u,j}, y_{v,j}\rangle} = ||\boldsymbol{b}_{x,y}||^2. \tag{3.1}$$

We thus obtain the following abstract version of the Wiener Theorem for unitary representations of a group G:

Theorem 3.7. — *Let $\pi = \pi_w \oplus \pi_c$ be a unitary representation of G on a Hilbert space $H = H_w \oplus H_c$, where π_w is the weakly mixing part of π and π_c its compact part. Writing $\pi_c = \bigoplus_{j \in J} \big(\bigoplus_{i \in I_j} \pi_j \big)$ as above, we have for every vectors $x = x_w \oplus x_c$ and $y = y_w \oplus y_c$ of H*

$$m\big(|\langle \pi(\,.\,)x, y\rangle|^2\big) = ||\,\boldsymbol{b}_{x_c, y_c}||^2. \tag{3.2}$$

Proof. — As we have $m(|\langle \pi(\,.\,)x, y\rangle|^2) = m\big(|\langle \pi_w(\,.\,)x_w, y_w\rangle|^2\big) + m\big(|\langle \pi_c(\,.\,)x_c, y_c\rangle|^2\big)$ and $m(|\langle \pi_w(\,.\,)x_w, y_w\rangle|^2) = 0$, this follows from Corollary 3.6. □

We finally derive an inequality on the quantities $m\big(|\langle \pi(\,.\,)x, y\rangle|^2\big)$ for a compact representation π, which is a direct consequence of Corollary 3.6. This inequality will be a crucial tool for the proof of our main result, to be given in Section 4. Using the same notation as in the statement of Corollary 3.6, we denote by $x = \oplus_{j \in J}\, \widetilde{x}_j$ and $y = \oplus_{j \in J}\, \widetilde{y}_j$ the respective decompositions of the vectors x and y of H with respect to the decomposition $H = \oplus_{j \in J}\, \widetilde{H}_j$ of H. Applying the Cauchy-Schwarz inequality twice to (3.1) yields the following inequalities:

Corollary 3.8. — *Let π be a compact representation of G on H. For every vectors x and y of H, we have*

$$m\big(|\langle \pi(\,.\,)x, y\rangle|^2\big) \leqslant \sum_{j \in J} \frac{1}{d_j}\, ||\widetilde{x}_j||^2 .\, ||\widetilde{y}_j||^2 \leqslant \sum_{j \in J} ||\widetilde{x}_j||^2 .\, ||\widetilde{y}_j||^2.$$

3.5. Why is (3.2) an abstract version of the Wiener Theorem?— Theorem 3.7 admits a much simpler formulation in the case where G is an abelian group. If π is a compact representation of G, the formula (3.1) becomes

$$m\big(|\langle\pi(\,\boldsymbol{.}\,)x,y\rangle|^2\big)=\sum_{j\in J}\ \sum_{u,v\in I_j} x_{u,j}\,\overline{x}_{v,j}\,\overline{y}_{u,j}y_{v,j}$$

where $x_{i,j}$ and $y_{i,j}$, $i\in I_j$, $j\in J$, are simply scalars. Using the notation of Corollary 3.8, we have

$$m\big(|\langle\pi(\,\boldsymbol{.}\,)x,y\rangle|^2\big)=\sum_{j\in J}\Big|\sum_{u\in I_j} x_{u,j}\,\overline{y}_{u,j}\Big|^2=\sum_{j\in J}\big|\langle\widetilde{x}_j,\widetilde{y}_j\rangle\big|^2. \tag{3.3}$$

For every character $\chi\in\Gamma$ (where Γ denotes the dual group of G), we denote by E_χ the subspace of H

$$E_\chi=\{x\in H\,;\,\pi(g)x=\chi(g)x\text{ for every }g\in G\}$$

and by P_χ the orthogonal projection of H on E_χ. Each representation π_j, $j\in J$, being in fact a character χ_j on the group G, we can identify the space $\widetilde{H}_j$ with E_{χ_j}. Equation (3.3) then yields the following corollary:

Corollary 3.9. — *Let G be an abelian group, and let π be a representation of G on a Hilbert space H. Then we have for every $x,y\in H$*

$$m\big(|\langle\pi(\,\boldsymbol{.}\,)x,y\rangle|^2\big)=\sum_{j\in J}\big|\langle P_{E_{\chi_j}}x,P_{E_{\chi_j}}y\rangle\big|^2=\sum_{\chi\in\Gamma}\big|\langle P_{E_\chi}x,P_{E_\chi}y\rangle\big|^2.$$

In particular, if $x=y$,

$$m\big(|\langle\pi(\,\boldsymbol{.}\,)x,x\rangle|^2\big)=\sum_{\chi\in\Gamma}\big|\big|P_{E_\chi}x\big|\big|^4.$$

Specializing Corollary 3.9 to the case where $G=\mathbb{Z}$ yields that for any unitary operator U on H and any vectors $x,y\in H$,

$$\frac{1}{2N+1}\sum_{n=-N}^{N}\big|\langle U^nx,y\rangle\big|^2\xrightarrow[N\to+\infty]{}\sum_{\lambda\in\mathbb{T}}\big|\langle P_{\ker(U-\lambda\mathrm{Id}_H)}x,P_{\ker(U-\lambda\mathrm{Id}_H)}y\rangle\big|^2.$$

In particular, we have

$$\frac{1}{2N+1}\sum_{n=-N}^{N}\big|\langle U^nx,x\rangle\big|^2\xrightarrow[N\to+\infty]{}\sum_{\lambda\in\mathbb{T}}\big|\big|P_{\ker(U-\lambda\mathrm{Id}_H)}x\big|\big|^4. \tag{3.4}$$

If σ is a probability measure on the unit circle $\mathbb{T}$, the operator M_σ of multiplication by $e^{i\theta}$ on $L^2(\mathbb{T},\sigma)$ is unitary. Applying (3.4) to $U=M_\sigma$ and to $x=1$, the constant function equal to 1, we obtain Wiener's Theorem:

$$\frac{1}{2N+1}\sum_{n=-N}^{N}|\,\widehat{\sigma}(n)|^2\xrightarrow[N\to+\infty]{}\sum_{\lambda\in\mathbb{T}}\sigma(\{\lambda\})^2\ . \tag{3.5}$$

We refer the reader to [**1, 2, 3, 9, 14**] and the references therein for related aspects and generalizations of Wiener's theorem.

We now have all the necessary tools for the proof of Theorem 2.3, which we present in the next section.

4. Proof of Theorem 2.3

4.1. Notation. — Let $(W_n)_{n\geqslant 1}$ be an increasing sequence of subsets of G satisfying the assumptions of Theorem 2.3, and let Q be a subset of G. For each $n \geqslant 1$, we denote by Q_n the set $Q_n = W_n \cup Q$. Remark that G is the increasing union of the sets Q_n, $n \geqslant 1$. We also denote by ε_0 a positive constant such that assumption (*) holds true: any representation of G admitting a (Q, ε_0)-invariant vector has a finite dimensional subrepresentation.

In order to prove Theorem 2.3, we argue by contradiction, and suppose that Q_n is a non-Kazhdan set in G for every $n \geqslant 1$. We will then construct for every $\varepsilon > 0$ a representation π of G which admits a (Q, ε)-invariant vector, but is weakly mixing (which, by Proposition 3.1, is equivalent to the fact that π has no finite dimensional subrepresentation), and this will contradict (*).

4.2. Construction of a sequence $(\pi_n)_{n\geqslant 1}$ of finite dimensional representations of G. — The first step of the proof is to show that assumption (*) combined with the hypothesis that Q_n is a non-Kazhdan set for every $n \geqslant 1$ implies the existence of sequences of finite dimensional representations of G with certain properties.

Lemma 4.1. — *Let ϵ_0 be a positive constant such that assumption (*) holds true and suppose that Q_n is a non-Kazhdan set in G for every $n \geqslant 1$. For every sequence $(\varepsilon_n)_{n\geqslant 1}$ of positive real numbers decreasing to zero with $\varepsilon_1 \in (0, \varepsilon_0]$, there exist a sequence $(H_n)_{n\geqslant 1}$ of finite dimensional Hilbert spaces and a sequence $(\pi_n)_{n\geqslant 1}$ of unitary representations of G such that, for every $n \geqslant 1$, π_n is a representation of G on H_n and*

- *π_n has no non-zero G-invariant vector;*
- *π_n has a (Q_n, ε_n)-invariant unit vector $a_n \in H_n$: $||a_n|| = 1$ and*

$$\sup_{g\in Q_n} ||\,\pi_n(g)a_n - a_n|| < \varepsilon_n.$$

Proof of Lemma 4.1. — Let $n \geqslant 1$. Since Q_n is a not a Kazhdan set in G, there exists a representation ρ_n of G on a Hilbert space K_n which has no non-zero G-invariant vector, but is such that there exists a unit vector $x_n \in K_n$ with

$$\sup_{g\in Q_n} ||\,\rho_n(g)x_n - x_n|| < 2^{-n}.$$

Since $2^{-n} \leqslant \varepsilon_0$ for n large enough, assumption (*) implies that, for such integers n, ρ_n has a finite dimensional subrepresentation. By Proposition 3.1, ρ_n is not weakly mixing. This means that if we decompose K_n as $K_n = K_{n,w} \oplus K_{n,c}$ and ρ_n as $\rho_n = \rho_{n,w} \oplus \rho_{n,c}$, where $\rho_{n,w}$ and $\rho_{n,c}$ are respectively the weakly mixing and compact parts of π_n, $\rho_{n,c}$ is non-zero. Since ρ_n has no non-zero G-invariant vector, neither have $\rho_{n,w}$ nor $\rho_{n,c}$.

Decomposing x_n as $x_n = x_{n,w} \oplus x_{n,c}$, we have $1 = ||x_{n,w}||^2 + ||x_{n,c}||^2$. We claim that $\underline{\lim}_{n\to+\infty} ||\,x_{n,c}|| > 0$. Indeed, suppose that it is not the case. Then $\overline{\lim}_{n\to+\infty}||x_{n,w}|| = 1$. Since $||\rho_n(g)x_n - x_n||^2 = ||\rho_{n,w}(g)x_{n,w} - x_{n,w}||^2 + ||\rho_{n,c}(g)x_{n,c} - x_{n,c}||^2$ for every $g \in G$, we have

$$\sup_{g\in Q_n}\left|\left|\rho_{n,w}(g)\frac{x_{n,w}}{||x_{n,w}||} - \frac{x_{n,w}}{||x_{n,w}||}\right|\right| < \frac{2^{-n}}{||x_{n,w}||}$$

as soon as $x_{n,w}$ is non-zero. Since $\overline{\lim}_{n\to+\infty}||x_{n,w}|| = 1$, this implies that for any $\delta > 0$ there exists an integer n such that $\rho_{n,w}$ has a (Q_n, δ)-invariant vector of norm 1. Applying this to $\delta = \varepsilon_0$, there exists $n_0 \geqslant 1$ such that $\rho_{n_0,w}$ has a (Q_{n_0}, ε_0)-invariant vector, hence a

(Q, ε_0)-invariant vector. But $\rho_{n_0, w}$ is weakly mixing, so has no finite dimensional subrepresentation. This contradicts assumption (*). So we deduce that $\underline{\lim}_{n\to+\infty} ||\, x_{n,c}|| = \gamma > 0$. The same observation as above, applied to the representation $\rho_{n,c}$, shows that

$$\sup_{g \in Q_n} \Big|\Big| \rho_{n,c}(g)\frac{x_{n,c}}{||x_{n,c}||} - \frac{x_{n,c}}{||x_{n,c}||}\Big|\Big| < \frac{2^{-n}}{||x_{n,c}||}$$

for every n such that $x_{n,c}$ is non-zero, and thus that

$$\sup_{g \in Q_n} \Big|\Big| \rho_{n,c}(g)\frac{x_{n,c}}{||x_{n,c}||} - \frac{x_{n,c}}{||x_{n,c}||}\Big|\Big| < \frac{2^{-(n-1)}}{\gamma}$$

for infinitely many integers n. For these integers, $\rho_{n,c}$ is a compact representation for which $y_n = x_{n,c}/||x_{n,c}||$ is a $(Q_n, 2^{-(n-1)}/\gamma)$-invariant vector of norm 1. It has no non-zero G-invariant vector. Decomposing $\rho_{n,c}$ as a direct sum of finite dimensional representations, straightforward computations show that there exists for each such integer n a finite dimensional representation σ_n of G with a $(Q_n, 2^{-(n-2)}/\gamma)$-invariant vector but no non-zero G-invariant vector. Lemma 4.1 follows immediately by taking a suitable subsequence of $(\sigma_n)_{n\geqslant 1}$. □

4.3. Construction of weakly mixing representations of G with (Q, ε)-invariant vectors. — Let $\varepsilon > 0$ be an arbitrary positive number. Our aim is to show that there exists a weakly mixing representation of G with a (Q, ε)-invariant vector. We fix a sequence $(\varepsilon_n)_{n\geqslant 1}$ of positive numbers decreasing to zero so fast that the following properties hold:

(i) $0 < \varepsilon_n < \varepsilon_0$ for every $n \geqslant 1$, and $\sum_{n\geqslant 1} \varepsilon_n < \varepsilon^2/2$;

(ii) the sequence $(\frac{1}{(n+1)\varepsilon_n^2} \sum_{j=n}^{2n} \varepsilon_j^2)_{n\geqslant 1}$ tends to 0 as n tends to infinity.

We consider the representation $\boldsymbol{\pi} = \otimes_{n\geqslant 1} \pi_n$ of G on the infinite tensor product space $\boldsymbol{H} = \bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$, where the spaces H_n, the representations π_n and the vectors a_n are associated to ε_n for each $n \geqslant 1$ by Lemma 4.1. We refer to the appendix for undefined notation concerning infinite tensor products. We first prove the following fact:

Fact 4.2. — Under the assumptions above, $\boldsymbol{\pi}$ is a strongly continuous representation of G on $\boldsymbol{H}$ which has a (Q, ε)-invariant vector.

Proof of Fact 4.2. — In order to prove that $\boldsymbol{\pi}$ is well-defined and strongly continuous, it suffices to check that the assumptions of Proposition A.2 in the appendix hold true. For every $g \in G$ and $n \geqslant 1$, we have $|1 - \langle \pi_n(g)a_n, a_n\rangle| \leqslant ||\pi_n(g)a_n - a_n||$ so that

$$\sup_{g \in Q_n} |1 - \langle \pi_n(g)a_n, a_n\rangle| < \varepsilon_n.$$

By assumption (i), the series $\sum_{n\geqslant 1} \varepsilon_n$ is convergent. Since every element $g \in G$ belongs to all the sets Q_n except finitely many, the series $\sum_{n\geqslant 1} |1 - \langle \pi_n(g)a_n, a_n\rangle|$ is convergent for every $g \in G$. Moreover, it is uniformly convergent on Q_1, and hence on W_1. The function

$$g \longmapsto \sum_{n\geqslant 1} |1 - \langle \pi_n(g)a_n, a_n\rangle|$$

is thus continuous on W_1, which is a neighborhood of e. It follows then from Proposition A.2 that $\boldsymbol{\pi}$ is strongly continuous on $\boldsymbol{H}$. If G is locally compact, Proposition A.1 and the first part of the argument above suffice to show that $\boldsymbol{\pi}$ is strongly continuous, even when W_1 is not a neighborhood of e.

Next, it is easy to check that the elementary vector $\boldsymbol{a} = \otimes_{n\geqslant 1} a_n$ of $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ satisfies $||\boldsymbol{a}|| = 1$ and $\sup_{g\in Q} ||\boldsymbol{\pi}(g)\boldsymbol{a} - \boldsymbol{a}|| < \varepsilon$. Indeed $||\boldsymbol{a}|| = \prod_{n\geqslant 1} ||a_n|| = 1$, and for every $g \in Q$ we have (using the fact that $Q \subseteq Q_n$ for every $n \geqslant 1$)

$$||\boldsymbol{\pi}(g)\boldsymbol{a} - \boldsymbol{a}||^2 = 2\,(1 - \mathrm{Re}\langle \boldsymbol{\pi}(g)\boldsymbol{a}, \boldsymbol{a}\rangle) \leqslant 2\left|1 - \prod_{n\geqslant 1} \langle \pi_n(g)a_n, a_n\rangle\right|$$
$$\leqslant 2 \sum_{n\geqslant 1} |1 - \langle \pi_n(g)a_n, a_n\rangle| < 2 \sum_{n\geqslant 1} \varepsilon_n.$$

Assumption (i) on the sequence $(\varepsilon_n)_{n\geq 1}$ implies that $\sup_{g\in Q} ||\boldsymbol{\pi}(g)\boldsymbol{a} - \boldsymbol{a}||^2 < \varepsilon^2$, and $\boldsymbol{a}$ is thus a (Q, ε)-invariant vector for $\boldsymbol{\pi}$. □

Using the notation of Section 3.2, we now decompose π_n and H_n as

$$\pi_n = \bigoplus_{j\in J_n} \Big(\bigoplus_{i\in I_{j,n}} \pi_{j,n}\Big) \quad \text{and} \quad H_n = \bigoplus_{j\in J_n} \Big(\bigoplus_{i\in I_{j,n}} H_{i,j,n}\Big)$$

respectively. Since H_n is finite dimensional, all the sets J_n and $I_{j,n}$, $j \in J_n$, are finite, and we assume that they are subsets of $\mathbb{N}$. For every $j \in J_n$, $H_{i,j,n} = K_{j,n}$. We also decompose $a_n \in H_n$ as $a_n = \bigoplus_{j\in J_n} \big(\bigoplus_{i\in I_{j,n}} a_{i,j,n}\big)$, and write $\widetilde{a}_{j,n} = \oplus_{i\in I_{j,n}} a_{i,j,n}$ for every $j \in J_n$. We have

$$(4.1) \qquad ||\widetilde{a}_{j,n}|| = \Bigg(\sum_{i\in I_{j,n}} ||\, a_{i,j,n}||^2\Bigg)^{\frac{1}{2}} \quad \text{and} \quad ||a_n|| = \Bigg(\sum_{j\in J_n} ||\widetilde{a}_{j,n}||^2\Bigg)^{\frac{1}{2}} = 1,$$

so that $||\widetilde{a}_{j,n}|| \leqslant 1$ for every $j \in J_n$. Also,

$$(4.2) \qquad ||\pi_n(g)a_n - a_n||^2 = \sum_{j\in J_n} \sum_{i\in I_{j,n}} ||\pi_{j,n}(g)a_{i,j,n} - a_{i,j,n}||^2 \quad \text{for every } g \in G,$$

so that

$$(4.3) \qquad \sup_{g\in Q_n} \Big(\sum_{i\in I_{j,n}} ||\pi_{j,n}(g)\, a_{i,j,n} - a_{i,j,n}||^2\Big)^{1/2} < \varepsilon_n \quad \text{for every } j \in J_n.$$

There are now two cases to consider.

- **Case 1.** We have $\underline{\lim}_{n\to+\infty} \max_{j\in J_n} ||\widetilde{a}_{j,n}|| = 0$.

Using Corollary 3.8 and the fact that $\sum_{j\in J_n} ||\,\widetilde{a}_{j,n}||^2 = ||a_n||^2 = 1$, we obtain that

$$m(\,|\langle \pi_n(\,\boldsymbol{.}\,)a_n, a_n\rangle|^2) \leqslant \sum_{j\in J_n} ||\,\widetilde{a}_{j,n}||^4 \leqslant \max_{j\in J_n} ||\,\widetilde{a}_{j,n}||^2 \,.\, \sum_{j\in J_n} ||\,\widetilde{a}_{j,n}||^2 \leqslant \max_{j\in J_n} ||\,\widetilde{a}_{j,n}||^2.$$

It follows from our assumption that $\underline{\lim}_{n\to+\infty} m(\,|\langle \pi_n(\,\boldsymbol{.}\,)a_n, a_n\rangle|^2) = 0$. So $\boldsymbol{\pi}$ is weakly mixing by Proposition A.3. We have thus proved in this case the existence of a weakly mixing representation of G with a (Q, ε)-invariant vector.

- **Case 2.** There exists $\delta > 0$ such that $\max_{j\in J_n} ||\widetilde{a}_{j,n}|| > \delta$ for every $n \geqslant 1$.

Let, for every $n \geqslant 1$, $j_n \in J_n$ be such that $||\widetilde{a}_{j_n,n}|| > \delta$. Set $I_n = I_{j_n,n}$, $\sigma_n = \pi_{j_n,n}$, $K_n = K_{j_n,n}$ and $b_{i,n} = a_{i,j_n,n}$ for every $i \in I_n$. Then σ_n is a non-trivial irreducible representation of G on the finite dimensional space K_n, and by (4.1) and (4.3) the finite family $(b_{i,n})_{i\in I_n}$ of vectors of K_n satisfies

$$(4.4)\qquad 1 \geqslant \Big(\sum_{i\in I_n} ||\, b_{i,n}||^2\Big)^{1/2} > \delta \quad\text{and}\quad \sup_{g\in Q_n}\Big(\sum_{i\in I_n} ||\sigma_n(g)\, b_{i,n} - b_{i,n}||^2\Big)^{1/2} < \varepsilon_n.$$

If we write

$$\widetilde{K}_n = \bigoplus_{i\in I_n} K_n, \quad \widetilde{b}_n = \bigoplus_{i\in I_n} b_{i,n}, \quad\text{and}\quad \widetilde{\sigma}_n = \bigoplus_{i\in I_n} \sigma_n,$$

this means that

$$(4.5)\qquad 1 \geqslant ||\widetilde{b}_n|| > \delta \quad\text{and}\quad \sup_{g\in Q_n} ||\widetilde{\sigma}_n(g)\widetilde{b}_n - \widetilde{b}_n|| < \varepsilon_n.$$

Now we again have to consider separately two cases.

– *Case 2.a.* There exists an infinite subset D of $\mathbb{N}$ such that whenever k and l are two distinct elements of D, σ_k and σ_l are not equivalent. Replacing the sequence $(\sigma_n)_{n\geqslant 1}$ by $(\sigma_n)_{n\in D}$, we can suppose without loss of generality that for every distinct integers m and n, with $m, n \geqslant 1$, σ_m and σ_n are not equivalent.

Consider for every $n \geqslant 1$ the representation

$$\rho_n = \widetilde{\sigma}_n \oplus \cdots \oplus \widetilde{\sigma}_{2n} \quad\text{of } G \text{ on}\quad \mathscr{H}_n = \widetilde{K}_n \oplus \cdots \oplus \widetilde{K}_{2n},$$

and the vector $b_n = \big(\sum_{k=n}^{2n} ||\widetilde{b}_k||^2\big)^{-\frac{1}{2}}\big(\widetilde{b}_n \oplus \cdots \oplus \widetilde{b}_{2n}\big)$ of $\mathscr{H}_n$, which satisfies $||b_n|| = 1$. For every $g \in Q_n$ we have, since Q_n is contained in Q_j for every $j \geqslant n$,

$$||\rho_n(g)b_n - b_n||^2 = \big(\sum_{k=n}^{2n} ||\widetilde{b}_k||^2\big)^{-1} \sum_{j=n}^{2n} ||\widetilde{\sigma}_j(g)\widetilde{b}_j - \widetilde{b}_j||^2 < \frac{1}{\delta^2(n+1)} \sum_{j=n}^{2n} \varepsilon_j^2$$

by (4.5). By assumption (ii) on the sequence $(\varepsilon_n)_{n\geqslant 1}$, we obtain that there exists an integer $n_0 \geqslant 1$ such that $\sup_{g\in Q_n} ||\rho_n(g)b_n - b_n|| < \varepsilon_n$ for every $n \geqslant n_0$. Let now $\boldsymbol{\rho} = \otimes_{n\geqslant n_0}\rho_n$ be the infinite tensor product of the representations ρ_n on the space $\boldsymbol{\mathscr{H}} = \bigotimes_{n\geqslant n_0}^{\boldsymbol{a}} \mathscr{H}_n$. An argument similar to the one given in Fact 4.2 shows that $\boldsymbol{\rho}$ is a strongly continuous representation of G on $\boldsymbol{\mathscr{H}}$ which has a (Q, ε)-invariant vector. It remains to prove that $\boldsymbol{\rho}$ is weakly mixing, and for this we will show that $m(|\langle\rho_n(\,.\,)b_n, b_n\rangle|^2)$ tends to zero as n tends to infinity. Recall that for every $n \geqslant 1$, the representations $\sigma_n, \ldots, \sigma_{2n}$ are mutually non-equivalent, so that, by Corollary 3.8, we have for every $n \geqslant 1$

$$m(|\langle\rho_n(\,.\,)b_n, b_n\rangle|^2) \leqslant \sum_{j=n}^{2n} \Big|\Big|\big(\sum_{k=n}^{2n} ||\widetilde{b}_k||^2\big)^{-\frac{1}{2}}\widetilde{b}_j\Big|\Big|^4 \leqslant \frac{1}{\delta^4(n+1)^2} \sum_{j=n}^{2n} ||\widetilde{b}_j||^4 \leqslant \frac{1}{\delta^4(n+1)}$$

by (4.5). So $m(|\langle\rho_n(\,.\,)b_n, b_n\rangle|^2)$ tends to zero as n tends to infinity. By Proposition A.3, $\boldsymbol{\rho}$ is weakly mixing. We have proved again in this case the existence of a weakly mixing representation of G with a (Q, ε)-invariant vector.

The other case we have to consider is when there exists an integer $n_1 \geqslant 1$ such that for every $n \geqslant n_1$, σ_n is equivalent to one of the representations $\sigma_1, \ldots, \sigma_{n_1}$. Indeed, if there is no such integer, we can construct a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of integers such that, for every $k \geqslant 1$, σ_{n_k} is not equivalent to one of the representations $\sigma_1, \ldots, \sigma_{n_{k-1}}$. The set $D = \{n_k\,;\, k \geqslant 1\}$ then has the property that whenever m and n are two distinct elements of D, σ_m and σ_n are not equivalent, and we are back to the setting of Case 2.*a*. Without loss of generality, we can suppose that σ_n is equal to σ_1 for every $n \geqslant 1$.

– *Case 2.b.* For every $n \geqslant 1$, σ_n is equal to σ_1. By (4.4), we have

$$1 \geqslant \Big(\sum_{i \in I_n} ||\, b_{i,\,n}||^2\Big)^{1/2} > \delta \quad \text{and} \quad \sup_{g \in Q_n} \Big(\sum_{i \in I_n} ||\sigma_1(g) b_{i,\,n} - b_{i,\,n}||^2\Big)^{1/2} < \varepsilon_n,$$

where all the vectors $b_{i,n}$, $i \in I_n$, belong to H_1. For each $n \geqslant 1$, set $c_n = \bigoplus_{i \in I_n} b_{i,\,n}$, seen as a vector of the infinite direct sum $H = \bigoplus_{j \geqslant 1} H_1$ by defining its j^{th} coordinate to be zero when j does not belong to I_n. Let also σ be the infinite direct sum $\sigma = \bigoplus_{j \geqslant 1} \sigma_1$ of σ_1 on H. Then we have, for every $n \geqslant 1$,

$$1 \geqslant ||c_n|| > \delta \quad \text{and} \quad \sup_{g \in Q_n} ||\sigma(g) c_n - c_n|| < \varepsilon_n.$$

Let now S be a finite subset of G. There exists an integer $n_S \geqslant 1$ such that $S \subseteq Q_n$ for every $n \geqslant n_S$, and hence

$$\sup_{g \in S} ||\sigma(g) c_n - c_n|| < \varepsilon_n \quad \text{for every } n \geqslant n_S.$$

It follows that σ has almost-invariant vectors for finite sets: for every $\delta > 0$ and every finite subset S of G, σ has an (S, δ)-invariant vector. This implies that σ_1 itself has almost-invariant vectors for finite sets (see [**34**, Lem. 1.5.4] or [**27**]). Since σ_1 is a finite dimensional representation, it follows that σ_1 has almost-invariant vectors. If $(v_n)_{n \geqslant 1}$ is a sequence of unit vectors of H_1 such that

$$\sup_{g \in G} ||\sigma(g) v_n - v_n|| < 2^{-n} \quad \text{for every } n \geqslant 1,$$

then any accumulation point of $(v_n)_{n \geqslant 1}$ is a non-zero G-invariant vector for σ_1. This contradicts our initial assumption on σ_1, and shows that the hypothesis of Case 2.b cannot be fulfilled.

Summing up our different cases, we have thus proved that there exists for every $\varepsilon > 0$ a representation of G with a (Q, ε)-invariant vector but no finite dimensional subrepresentation. This contradicts assumption (*) of Theorem 2.3, and concludes the proof.

5. Some consequences of Theorem 2.3

We begin this section by proving the two characterizations of Kazhdan sets obtained as consequences of Theorem 2.3.

5.1. Proofs of Corollary 2.4 and Theorem 2.5. — Let us first prove Corollary 2.4.

Proof of Corollary 2.4. — Let Q_0 be a subset of G which has non-empty interior and which generates G. Denote for each $n \geqslant 1$ by $Q_0^{\pm n}$ the set $\{g_1^{\pm 1} \dots g_n^{\pm 1}\,;\, g_1, \dots, g_n \in Q_0\}$. Then $G = \bigcup_{n \geqslant 1} Q_0^{\pm n}$. Let g_0 be an element of the interior of Q_0. Then $g_0^{-1} Q_0$ is a neighborhood of e. There exists $n_0 \geqslant 1$ such that g_0^{-1} belongs to $Q_0^{\pm n_0}$, and thus $Q_0^{\pm (n_0+1)}$ is a neighborhood of e. If we set $W_n = Q_0^{\pm (n_0+n)}$ for $n \geqslant 1$, the sequence of sets $(W_n)_{n \geqslant 1}$ is increasing, W_1 is a neighborhood of e, and $(W_n)_{n \geqslant 1}$ satisfies the assumptions of Theorem 2.3. So if Q is a subset of G for which assumption (*) of Theorem 2.3 holds true, there exists $n \geqslant 1$ such that $Q_0^{\pm (n+n_0)} \cup Q$ is a Kazhdan set in G. Let $\varepsilon > 0$ be a Kazhdan constant for this set. Then $\varepsilon/(n + n_0)$ is a Kazhdan constant for $Q_0 \cup Q$, and $Q_0 \cup Q$

is a Kazhdan set in G. If G is locally compact, the same proof holds true without the assumption that Q_0 has non-empty interior. □

Proof of Theorem 2.5. — Let us first show that (a) implies (b). Suppose that Q is a Kazhdan set, and let $0 < \varepsilon < \sqrt{2}$ be a Kazhdan constant for Q. Let $\delta = \sqrt{1-\varepsilon^2/2}$ and consider a representation π of G on a Hilbert space H for which there is a vector $x \in H$ with $||x|| = 1$ such that $\inf_{g\in Q} |\langle \pi(g)x, x\rangle| > \delta$. Then the representation $\pi \otimes \overline{\pi}$ of G on $H \otimes \overline{H}$ verifies

$$2\mathrm{Re}\langle \pi \otimes \overline{\pi}(g)x \otimes \overline{x}, x \otimes \overline{x}\rangle = 2|\langle \pi(g)x, x\rangle|^2 > 2 - \varepsilon^2$$

for every $g \in Q$. Hence $\|\pi \otimes \overline{\pi}(g)x \otimes \overline{x} - x \otimes \overline{x}\| < \varepsilon$ for every $g \in Q$ and $\pi \otimes \overline{\pi}$ has a non-zero G-invariant vector. It follows from Proposition 3.1 that π has a finite dimensional subrepresentation. Thus (*b*) is true. That (b) implies (c) is straightforward, and that (c) implies (a) is a consequence of Corollary 2.4. □

5.2. Property (T) in σ-compact locally compact groups. — As a consequence of Theorem 2.3, we retrieve a characterization of Property (T) due to Bekka and Valette [**5**], [**6**, Th. 2.12.9], valid for σ-compact locally compact groups, which states the following:

Theorem 5.1 ([**5**]). — *Let G be a σ-compact locally compact group. Then G has Property (T) if and only if every unitary representation of G with almost-invariant vectors has a non-trivial finite dimensional subrepresentation.*

The proof of [**5**] relies on the equivalence between Property (T) and Property (FH) for such groups [**6**, Th. 2.12.4]. As a direct consequence of Theorem 2.3, we will derive a new proof of Theorem 5.1 which does not involve property (FH).

If Q is a subset of a topological group G, and if π is a unitary representation of G on a Hilbert space H, we say that π has *Q-almost-invariant vectors* if it has (Q, ε)-invariant vectors for every $\varepsilon > 0$. The same argument as in [**6**, Prop. 1.2.1] shows that Q is a Kazhdan set in G if and only if every representation of G with Q-almost-invariant vectors has a non-zero G-invariant vector. As a direct corollary of Theorem 2.5, we obtain the following characterization of Kazhdan sets which generate the group:

Corollary 5.2. — *Let Q be a subset of a locally compact group G which generates G. Then Q is a Kazhdan set in G if and only if every representation π of G with Q-almost-invariant vectors has a non-trivial finite dimensional subrepresentation.*

Proof of Corollary 5.2. — The only thing to prove is that if every representation π of G with Q-almost-invariant vectors has a non-trivial finite dimensional representation, Q is a Kazhdan set. For this it suffices to show the existence of an $\varepsilon > 0$ such that assumption (*) of Theorem 2.3 holds true. The argument is exactly the same as the one given in [**6**, Prop. 1.2.1]: suppose that there is no such ε, and let, for every $\varepsilon > 0$, π_ε be a representation of G with a (Q, ε)-invariant vector but no finite dimensional subrepresentation. Then $\pi = \bigoplus_{\varepsilon>0} \pi_\varepsilon$ has Q-almost-invariant vectors but no finite dimensional subrepresentation (this follows immediately from [**6**, Prop. A.1.8]), contradicting our initial assumption. □

Proof of Theorem 5.1. — It is clear that Property (T) implies that every representation of G with almost-invariant vectors has a non-trivial finite dimensional subrepresentation. Conversely, suppose that every representation of G with almost-invariant vectors has a non-trivial finite dimensional subrepresentation. Using the same argument as in the proof of Corollary 5.2, we see that there exists a compact subset Q of G such that assumption

(*) of Theorem 2.3 holds true. Choosing for $(W_n)_{n\geqslant 1}$ an increasing sequence of compact subsets of G such that $\bigcup_{n\geqslant 1} W_n = G$, Theorem 2.3 implies that there exists an $n \geqslant 1$ such that $W_n \cup Q$ is a Kazhdan set in G. Since $W_n \cup Q$ is compact, G has Property (T). □

5.3. Equidistribution assumptions: proofs of Theorems 2.2 and 2.1. — Let G be a second countable locally compact group, and let π be a unitary representation of G on a separable Hilbert space H. Such a representation can be decomposed as a direct integral of irreducible unitary representations over a Borel space (see for instance [**6**, Sec. F.5] or [**15**]). More precisely, there exists a finite positive measure μ on a standard Borel space Z, a measurable field $z \mapsto H_z$ of Hilbert spaces over Z, and a measurable field of irreducible representations $z \mapsto \pi_z$, where each π_z is a representation of G on H_z, such that π is unitarily equivalent to the direct integral $\pi_\mu = \int_Z^\oplus \pi_z \, d\mu(z)$ on $\mathscr{H} = \int_Z^\oplus H_z \, d\mu(z)$. The Hilbert space $\mathscr{H}$ is the set of equivalence classes of square integrable vector fields $z \mapsto x_z$, with $x_z \in H_z$, with respect to the measure μ; π_μ is the representation of G on $\mathscr{H}$ defined by $\pi_\mu(g)x = [z \mapsto \pi_z(g)x_z]$ for every $g \in G$ and $x \in \mathscr{H}$.

Proof of Theorem 2.1. — Our aim is to show that, under the hypothesis of Theorem 2.1, assumption (*) of Theorem 2.3 is satisfied. Let π be a representation of G on a Hilbert space H. Since G is second countable, we can suppose that H is separable. Suppose that π admits a $(Q, 1/2)$-invariant vector $x \in H$ and, using the notation and the result recalled above, write

$$\pi = \int_Z^\oplus \pi_z \, d\mu(z), \quad x = [\, z \longmapsto x_z \,], \quad \text{and} \quad H = \int_Z^\oplus H_z \, d\mu(z).$$

We have for every $k \geqslant 1$

$$\operatorname{Re}\langle \pi(g_k)x, x\rangle = \operatorname{Re}\int_Z \langle \pi_z(g_k)x_z, x_z\rangle \, d\mu(z) = 1 - \frac{1}{2}\,||\pi(g_k)x - x||^2 > \frac{7}{8}$$

so that

$$\operatorname{Re}\int_Z \frac{1}{N}\sum_{k=1}^{N} \langle \pi_z(g_k)x_z, x_z\rangle \, d\mu(z) > \frac{7}{8} \quad \text{for every } N \geqslant 1. \tag{5.1}$$

Now, assumption (2.2) of Theorem 2.1 states that there exists a countable set $\mathcal{C}_0$ of equivalence classes of irreducible representations such that

$$\frac{1}{N}\sum_{k=1}^{N} \langle \pi(g_k)x, x\rangle \longrightarrow 0 \quad \text{as } N \longrightarrow +\infty \tag{5.2}$$

for every irreducible representation π whose equivalence class $[\pi]$ does not belong to $\mathcal{C}_0$ and every vector x in the underlying Hilbert space. It follows from (5.2) that the set $Z_0 = \{z \in Z \,;\, [\pi_z] \in \mathcal{C}_0\}$ satisfies $\mu(Z_0) > 0$, and there exists $[\pi_0] \in \mathcal{C}_0$ such that $\mu(\{z \in Z \,;\, \pi_z \text{ and } \pi_0 \text{ are equivalent}\}) > 0$. Hence π_0 is a subrepresentation of π. Since all irreducible representations of G are supposed to be finite dimensional, π has a finite dimensional subrepresentation. So assumption (*) of Theorem 2.3 is satisfied. As Q generates G, it now follows from Theorem 2.5 that Q is a Kazhdan set in G. □

Proof of Theorem 2.2. — The proof of Theorem 2.2 is exactly the same as that of Theorem 2.1, using the fact that if G is a locally compact abelian group (not necessarily second countable), any unitary representation of G is equivalent to a direct integral of irreducible representations (see for instance [**15**, Th. 7.36]). □

6. Examples and applications

We present in this section some examples of Kazhdan sets in different kinds of groups, some statements being obtained as consequences of Theorems 2.3 or 2.5. We do not try to be exhaustive, and our aim here is rather to highlight some interesting phenomena which appear when looking for Kazhdan sets, as well as the connections of these phenomena with some remarkable properties of the group. We begin with the simplest case, that of locally compact abelian (LCA) groups.

6.1. Kazhdan sets in locally compact abelian groups. — Let G be a second countable LCA group, the dual group of which we denote by Γ. If σ is a finite Borel measure on Γ, recall that its Fourier-Stieljes transform is defined by

$$\widehat{\sigma}(g) = \int_{\Gamma} \gamma(g)\, d\sigma(\gamma) \quad \text{for every } g \in G.$$

It is an easy consequence of the spectral theorem for unitary representations that if Q is a subset of a second countable LCA group G, Q is a Kazhdan set in G if and only if there exists $\varepsilon > 0$ such that any probability measure σ on Γ with $\sup_{g\in Q} |\widehat{\sigma}(g) - 1| < \varepsilon$ satisfies $\sigma(\{1\}) > 0$, where 1 denotes the trivial character on G. Using Theorem 2.5 combined with the spectral theorem for unitary representations again, we obtain the following stronger characterization of Kazhdan sets which generate the group in any second countable LCA group.

Theorem 6.1. — *Let G be a second countable LCA group, and let Q a subset of G which generates G. The following assertions are equivalent:*

(1) Q is a Kazhdan set in G;
(2) there exists $\delta \in (0, 1)$ such that any probability measure σ on Γ with $\inf_{g\in Q} |\widehat{\sigma}(g)| > \delta$ has a discrete part;
(3) there exists $\varepsilon > 0$ such that any probability measure σ on Γ with $\sup_{g\in Q} |\widehat{\sigma}(g) - 1| < \varepsilon$ has a discrete part.

Theorem 6.1 becomes particularly meaningful in the case of the group $\mathbb{Z}$, as it yields a characterization of Kazhdan subsets of $\mathbb{Z}$ involving some classic sets in harmonic analysis, introduced by Kaufman in [**24**]. They are called *w-sets* by Kaufman [**25**], and *Kaufman sets* (**Ka** sets) by other authors, such as Hartman [**20**], [**21**].

Definition 6.2. — Let Q be a subset of $\mathbb{Z}$, and let $\delta \in (0, 1)$.

• We say that Q belongs to the class **Ka** if there exists a finite complex-valued continuous Borel measure μ on $\mathbb{T}$ such that $\inf_{n\in Q} |\hat{\mu}(n)| > 0$, and to the class δ-**Ka** if there exists a finite complex-valued continuous Borel measure μ on $\mathbb{T}$ with $\mu(\mathbb{T}) = 1$ such that $\inf_{n\in Q} |\hat{\mu}(n)| > \delta$.

• We say that Q belongs to the class $\mathbf{Ka}^+$ if there exists a continuous probability measure σ on $\mathbb{T}$ such that $\inf_{n\in Q} |\widehat{\sigma}(n)| > 0$, and to the class δ-$\mathbf{Ka}^+$ if there exists a continuous probability measure σ on $\mathbb{T}$ such that $\inf_{n\in Q} |\widehat{\sigma}(n)| > \delta$.

Our characterization of Kazhdan subsets of $\mathbb{Z}$ is given by Theorem 6.3 below:

Theorem 6.3. — *Let Q a subset of $\mathbb{Z}$ which generates $\mathbb{Z}$. Then Q is a Kazhdan set in $\mathbb{Z}$ if and only if there exists a $\delta \in (0,1)$ such that Q does not belong to δ-**Ka**$^+$.*

It is interesting to remark [**21**] that a set Q belongs to **Ka** if and only if it belongs to δ-**Ka** for every $\delta \in (0,1)$. There is no similar statement for the class **Ka**$^+$: any sufficiently lacunary subset of $\mathbb{Z}$, such as $Q = \{3^k + k\ ;\ k \geqslant 1\}$, is easily seen to belong to **Ka**$^+$ (it suffices to consider an associated Riesz product – see for instance [**22**] for details); but the same reasoning as in Example 6.4 below shows that this set Q is a Kazhdan subset of $\mathbb{Z}$. Thus there exists by Theorem 6.3 a $\delta \in (0,1)$ such that Q does not belong to δ-**Ka**$^+$.

We present now some typical examples of Kazhdan sets in $\mathbb{Z}$ or $\mathbb{R}$ obtained using the above characterizations. The first one provides a negative answer to Question 1.5 (a).

Example 6.4. — The set $Q = \{2^k + k\,;\, k \geqslant 0\}$ is a Kazhdan set in $\mathbb{Z}$ and there are irrational numbers θ such that $(e^{2i\pi n\theta})_{n\in Q}$ is not dense in $\mathbb{T}$. In particular, no rearrangement $(m_k)_{k\geqslant 1}$ of the elements of Q exists such that $(e^{2i\pi m_k\theta})_{k\geqslant 1}$ is equidistributed in $\mathbb{T}$ for every irrational number θ.

Proof. — The sequence $(n_k)_{k\geqslant 0}$ defined by $n_k = 2^k + k$ for every $k \geqslant 0$ satisfies the relation $2n_k = n_{k+1} + k - 1$ for every $k \geqslant 0$. Let σ be a probability measure on $\mathbb{T}$ such that $\sup_{k\geqslant 0} |\widehat{\sigma}(n_k) - 1| < 1/18$. Since, by the Cauchy-Schwarz inequality,

$$|\widehat{\sigma}(k) - 1| \leqslant \int_{\mathbb{T}} |\lambda^k - 1| d\sigma(\lambda) \leqslant \sqrt{2}\, |\widehat{\sigma}(k) - 1|^{1/2} \quad \text{for every } k \in \mathbb{Z},$$

we have

$$\begin{aligned} |\widehat{\sigma}(k-1) - 1| &\leqslant 2 \int_{\mathbb{T}} |\lambda^{n_k} - 1| d\sigma(\lambda) + \int_{\mathbb{T}} |\lambda^{n_{k+1}} - 1| d\sigma(\lambda) \\ &\leqslant 2\sqrt{2}\, |\widehat{\sigma}(n_k) - 1|^{1/2} + \sqrt{2}\, |\widehat{\sigma}(n_{k+1}) - 1|^{1/2} \end{aligned}$$

for all $k \geqslant 1$, so that $\sup_{k\geqslant 0} |\widehat{\sigma}(k) - 1| < 1$. Since

$$\frac{1}{N} \sum_{k=1}^{N} \widehat{\sigma}(k) = \int_{\mathbb{T}} \Big(\frac{1}{N} \sum_{k=1}^{N} \lambda^k \Big) d\sigma(\lambda) \longrightarrow \sigma(\{1\}) \quad \text{as } N \longrightarrow +\infty,$$

we have $\sigma(\{1\}) > 0$. So $Q = \{n_k\,;\, k \geqslant 0\}$ is a Kazhdan set in $\mathbb{Z}$. But $(n_k)_{k\geqslant 0}$ being lacunary, it follows from a result due independently to Pollington [**35**] and De Mathan [**30**] that there exists a subset A of $[0,1]$ of Hausdorff measure 1 such that for every θ in A, the set $Q\theta = \{n_k\theta\ ;\ k \geqslant 0\}$ is not dense modulo 1. One of these numbers θ is irrational, and the conclusion follows. □

Example 6.5. — The set $Q' = \{2^k\,;\, k \geqslant 0\}$ is not a Kazhdan set in $\mathbb{Z}$.

Proof. — The fact that Q' is not a Kazhdan set in $\mathbb{Z}$ relies on the observation that 2^k divides 2^{k+1} for every $k \geqslant 0$. Using the same construction as the one of [**13**, Prop. 3.9], we consider for any fixed $\varepsilon > 0$ a decreasing sequence $(a_j)_{j\geqslant 1}$ of positive real numbers with $a_1 < \varepsilon/(2\pi)$ such that the series $\sum_{j\geqslant 1} a_j$ is divergent. Then the infinite convolution of two-points Dirac measures

$$\sigma = \mathop{\ast}_{j\geqslant 1} \big((1 - a_j)\delta_{\{1\}} + a_j \delta_{\{e^{i\pi 2^{-j+1}}\}} \big)$$

is a well-defined probability measure on $\mathbb{T}$, which is continuous by the assumption that the series $\sum_{j\geqslant 1} a_j$ diverges. For every $k \geqslant 0$,

$$\widehat{\sigma}(2^k) = \prod_{j\geqslant 1}\big(1 - a_j + a_j e^{i\pi 2^{k-j+1}}\big) = \prod_{j\geqslant k+1}\big(1 - a_j(1 - e^{i\pi 2^{k-j+1}})\big).$$

As $|1 - a_j(1 - e^{i\pi 2^{k-j+1}})| \leqslant 1$, it follows that

$$|\widehat{\sigma}(2^k) - 1| \leqslant \sum_{j\geqslant k+1} a_j |1 - e^{i\pi 2^{k-j+1}}| \leqslant \pi\, a_{k+1}\, 2^{k+1} \sum_{j\geqslant k+1} 2^{-j} = 2\pi a_{k+1} < \varepsilon$$

for every $k \geqslant 0$. This proves that Q' is not a Kazhdan set in $\mathbb{Z}$. □

Example 6.6. — If p is a non-constant polynomial with integer coefficients such that $p(\mathbb{Z})$ is included in $a\mathbb{Z}$ for no integer a with $|a| \geqslant 2$, then $Q = \{p(k)\,;\, k \geqslant 0\}$ is a Kazhdan set in $\mathbb{Z}$.

Proof. — Our assumption that $p(\mathbb{Z})$ is included in $a\mathbb{Z}$ for no integer a with $|a| \geqslant 2$ implies that Q generates $\mathbb{Z}$. Since the sequence $(\lambda^{p(k)})_{k\geqslant 0}$ is uniformly distributed in $\mathbb{T}$ for every $\lambda = e^{2i\pi\theta}$ with θ irrational (see for instance [**28**, Th. 3.2]), Theorem 2.1 implies that Q is a Kazhdan set in $\mathbb{Z}$. □

Example 6.7. — Let p be a non-constant real polynomial, and let $Q = \{p(k)\,;\, k \geqslant 0\}$. Then $(-\delta, \delta) \cup Q$ is a Kazhdan subset of $\mathbb{R}$ for any $\delta > 0$.

Proof. — Write p as $p(x) = \sum_{j=0}^{d} a_j x^j$, $d \geqslant 1$, and let $r \in \{1, \ldots, d\}$ be such that $a_r \neq 0$. It is well-known (see for instance [**28**, Th. 3.2]) that the sequence $(e^{2i\pi t p(k)})_{k\in\mathbb{Z}}$ is uniformly distributed in $\mathbb{T}$ as soon as ta_r is irrational. This condition excludes only countably many values of t. Set now $W_n = (-n, n)$ for every integer $n \geqslant 1$. Thanks to Theorem 2.3, we obtain that there exists $n \geqslant 1$ such that $(-n, n) \cup Q$ is a Kazhdan set in $\mathbb{R}$. Let $\varepsilon > 0$ be a Kazhdan constant for this set. Fix $\delta > 0$. In order to prove that $(-\delta, \delta) \cup Q$ is a Kazhdan set in $\mathbb{R}$, we consider a positive number γ, which will be fixed later on, and let σ be a probability measure on $\mathbb{R}$ such that $\sup_{t\in(-\delta,\delta)\cup Q} \big|\widehat{\sigma}(t) - 1\big| < \gamma$. For any $a \in \mathbb{N}$ and any $t \in (-\delta, \delta)$,

$$2(1 - \operatorname{Re}\widehat{\sigma}(at)) = \int_{\mathbb{R}} \big|e^{iatx} - 1\big|^2 d\sigma(x) \leqslant a^2 \int_{\mathbb{R}} \big|e^{itx} - 1\big|^2 d\sigma(x) \leqslant 2a^2 \operatorname{Re}(1 - \widehat{\sigma}(t))$$

so that $\sup_{t\in(\delta,\delta)} (1 - \operatorname{Re}\widehat{\sigma}(at)) < a^2\gamma$. If we choose $a > n/\delta$ and $\gamma < \min(\varepsilon, \varepsilon^2/(2a^2))$, we obtain that $\sup_{t\in(-n,n)\cup Q} \big|1 - \widehat{\sigma}(t)\big| < \varepsilon$, and since ε is a Kazhdan constant for $(-n, n) \cup Q$, $\sigma(\{0\}) > 0$. Hence γ is a Kazhdan constant for $(-\delta, \delta) \cup Q$. □

Remark 6.8. — It is necessary to add a small interval to the set Q in order to turn it into a Kazhdan subset of $\mathbb{R}$, even when Q generates a dense subgroup of $\mathbb{R}$. Indeed, consider the polynomial $p(x) = x + \sqrt{2}$. The set $Q = \{k + \sqrt{2}\,;\, k \geqslant 0\}$ is not a Kazhdan set in $\mathbb{R}$: for any $\varepsilon > 0$, let $b \in \mathbb{N}$ be such that $|e^{2i\pi b\sqrt{2}} - 1| < \varepsilon$. The measure σ defined as the Dirac mass at the point $2\pi b$ satisfies $\sup_{k\geqslant 0} |\widehat{\sigma}(k + \sqrt{2}) - 1| < \varepsilon$, so that Q is not a Kazhdan set in $\mathbb{R}$.

We finish this section by exhibiting a link between Kazhdan subsets of $\mathbb{Z}^d$ and Kazhdan subsets of $\mathbb{R}^d$, $d \geqslant 1$. Let Q be a subset of $\mathbb{Z}^d$. Seen as a subset of $\mathbb{R}^d$, Q is never a Kazhdan set. But as a consequence of Theorem 2.3, we see that Q becomes a Kazhdan set in $\mathbb{R}^d$ if we add a small perturbation to it.

Proposition 6.9. — *Fix an integer $d \geqslant 1$, and let $(W_n)_{n\geqslant 1}$ be an increasing sequence of subsets of $\mathbb{R}^d$ such that $\bigcup_{n\geqslant 1} W_n = \mathbb{R}^d$. Let Q be a Kazhdan subset of $\mathbb{Z}^d$. There exists an $n \geqslant 1$ such that $W_n \cup Q$ is a Kazhdan set in $\mathbb{R}^d$. Also, $B(0,\delta) \cup Q$ is a Kazhdan subset of $\mathbb{R}^d$ for any $\delta > 0$, where $B(0,\delta)$ denotes the open unit ball of radius δ for the Euclidean norm on $\mathbb{R}^d$.*

Proof. — Let $\varepsilon > 0$ be a Kazhdan constant for Q, seen as a subset of $\mathbb{Z}^d$. Let π be a representation of $\mathbb{R}^d$ on a separable Hilbert space H which admits a $(Q, \varepsilon^2/2)$-invariant vector $x \in H$. Without loss of generality we can suppose that π is a direct integral on a Borel space Z, with respect to a finite measure μ on Z, of a family $(\pi_z)_{z\in\mathbb{Z}}$ of irreducible representations of $\mathbb{R}^d$. So π is a representation of $\mathbb{R}^d$ on $L^2(Z,\mu)$. We write elements f of $L^2(Z,\mu)$ as $f = (f_z)_{z\in Z}$. We suppose that $||x|| = 1$; our hypothesis implies that

$$\sup_{\boldsymbol{t}\in Q} \left|1 - \langle \pi(\boldsymbol{t})x, x\rangle\right| < \frac{\varepsilon^2}{2}\cdot$$

Each representation π_z acts on vectors $\boldsymbol{t} = (t_1, \ldots, t_d)$ of $\mathbb{R}^d$ as $\pi_z(\boldsymbol{t}) = \exp(2i\pi\langle \boldsymbol{t}, \boldsymbol{\theta}_z\rangle)$ for some vector $\boldsymbol{\theta}_z = (\theta_{1,z}, \ldots, \theta_{d,z})$ of $\mathbb{R}^d$. Hence

$$\sup_{\boldsymbol{t}\in Q} \left|1 - \int_Z e^{2i\pi\langle \boldsymbol{t}, \boldsymbol{\theta}_z\rangle} |x_z|^2 d\mu(z)\right| < \frac{\varepsilon^2}{2}\cdot$$

Consider now the representation ρ of $\mathbb{Z}^d$ on $L^2(Z,\mu)$ defined by $\rho(\boldsymbol{n})\, f : z \longmapsto e^{2i\pi\langle \boldsymbol{n}, \boldsymbol{\theta}_z\rangle} f_z$ for every $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and every $f \in L^2(Z,\mu)$. We have

$$\sup_{\boldsymbol{n}\in Q} ||\rho(\boldsymbol{n})x - x||^2 \leqslant 2 \sup_{\boldsymbol{n}\in Q} \left|1 - \langle \rho(\boldsymbol{n})x, x\rangle\right| < \varepsilon^2,$$

and since ε is a Kazhdan constant for Q as a subset of $\mathbb{Z}^d$, ρ has a non-zero $\mathbb{Z}^d$-invariant vector. There exists hence $f \in L^2(Z,\mu)$ with $||f|| = 1$ such that $\rho(\boldsymbol{n})f = f$ for every $\boldsymbol{n} \in \mathbb{Z}^d$. Fix a representative of $f \in L^2(Z,\mu)$, and set $Z_0 = \{z \in Z\,;\, f_z \neq 0\}$. Then $\mu(Z_0) > 0$. For every $z \in Z_0$ we have $e^{2i\pi\langle \boldsymbol{n}, \boldsymbol{\theta}_z\rangle} = 1$ for every $\boldsymbol{n} \in \mathbb{Z}^d$, which implies that $\boldsymbol{\theta}_z \in \mathbb{Z}^d$. For each $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, let $Z_{\boldsymbol{n}} = \{z \in Z_0\,;\, \theta_{i,z} = n_i \text{ for each } i \in \{1, \ldots, n\}\}$ and $H_{\boldsymbol{n}} = \{f \in L^2(Z,\mu)\,;\, f = 0\ \mu\text{-a.e. on } Z\backslash Z_{\boldsymbol{n}}\}$. We have $\bigcup_{\boldsymbol{n}\in\mathbb{Z}^d} Z_{\boldsymbol{n}} = Z_0$, so there exists $\boldsymbol{n}_0 \in \mathbb{Z}^d$ such that $\mu(Z_{\boldsymbol{n}_0}) > 0$. Each subspace $H_{\boldsymbol{n}}$ is easily seen to be invariant for π, and the representation $\pi_{\boldsymbol{n}}$ induced by π on $H_{\boldsymbol{n}}$ is given by $\pi_{\boldsymbol{n}}(\boldsymbol{t})\, f : z \longmapsto e^{2i\pi\langle \boldsymbol{t}, \boldsymbol{n}\rangle} f_z$ for every $\boldsymbol{t} \in \mathbb{R}^d$ and every $f \in H_{\boldsymbol{n}}$. So π admits a subrepresentation of dimension 1 as soon as $H_{\boldsymbol{n}}$ is non-zero, i.e. as soon as $\mu(Z_{\boldsymbol{n}}) > 0$. Since $\mu(Z_{\boldsymbol{n}_0}) > 0$, π admit a subrepresentation of dimension 1. An application of Theorem 2.5 now shows that $W_n \cup Q$ is a Kazhdan set in $\mathbb{R}^d$ for some $n \geqslant 1$. If we choose $W_n = B(0,n)$ for every $n \geqslant 1$, and proceed as in the proof of Example 6.7, we obtain that $B(0,\delta) \cup Q$ is a Kazhdan set in $\mathbb{R}^d$ for every $\delta > 0$. □

We now move out of the commutative setting, and present a characterization of Kazhdan sets in the Heisenberg groups H_n.

6.2. Kazhdan sets in the Heisenberg groups H_n. — The Heisenberg group of dimension $n \geqslant 1$, denoted by H_n, is formed of triples $(t, \boldsymbol{q}, \boldsymbol{p})$ of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n+1}$. The group operation is given by

$$(t_1, \boldsymbol{q}_1, \boldsymbol{p}_1) \cdot (t_2, \boldsymbol{q}_2, \boldsymbol{p}_2) = (t_1 + t_2 + \frac{1}{2}(\boldsymbol{p}_1 \cdot \boldsymbol{q}_2 - \boldsymbol{p}_2 \cdot \boldsymbol{q}_1), \boldsymbol{q}_1 + \boldsymbol{q}_2, \boldsymbol{p}_1 + \boldsymbol{p}_2),$$

where $\boldsymbol{p}\cdot\boldsymbol{q}$ denotes the scalar product of two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ of $\mathbb{R}^n$. Irreducible unitary representations of H_n are completely classified (see for instance [**36**, Ch. 2], or [**15**, Cor. 6.51]): there are two distinct families of such representations, which we denote respectively by $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$:

– the representations belonging to the family $(\mathcal{F}_1)$ are representations of H_n on $L^2(\mathbb{R}^n)$. They are parametrized by an element of $\mathbb{R}$, which we write as $\pm\lambda$ with $\lambda > 0$. Then $\pi_{\pm\lambda}(t, \boldsymbol{q}, \boldsymbol{p})$, $(t, \boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2n+1}$, acts on $L^2(\mathbb{R}^n)$ as

$$\pi_{\pm\lambda}(t, \boldsymbol{q}, \boldsymbol{p})\, u \;:\; \boldsymbol{x} \longmapsto e^{i(\pm\lambda t \pm\sqrt{\lambda}\boldsymbol{q}\cdot\boldsymbol{x} + \frac{\lambda}{2}\boldsymbol{q}\cdot\boldsymbol{p})} u(\boldsymbol{x} + \sqrt{\lambda}\,\boldsymbol{p})$$

where u belongs to $L^2(\mathbb{R}^n)$. These representations have the following important property, which will appear again in the next subsection:

Fact 6.10. — For every $\pm\lambda \in \mathbb{R}$ and every $u \in L^2(\mathbb{R}^n)$,

$$\langle \pi_{\pm\lambda}(t, \boldsymbol{q}, \boldsymbol{p})\, u, u\rangle \longrightarrow 0 \quad \text{as} \quad |\boldsymbol{p}| \longrightarrow +\infty.$$

Proof. — This follows directly from the dominated convergence theorem if u has compact support in $\mathbb{R}^n$. It then suffices to approximate $u \in L^2(\mathbb{R}^n)$ by functions with compact support to get the result. □

– the representations belonging to the family $(\mathcal{F}_2)$ are one-dimensional. They are parametrized by elements $(\boldsymbol{y}, \boldsymbol{\eta})$ of $\mathbb{R}^{2n}$: for every $(t, \boldsymbol{q}, \boldsymbol{p}) \in H_n$,

$$\pi_{\boldsymbol{y},\boldsymbol{\eta}}(t, \boldsymbol{p}, \boldsymbol{q}) = e^{i(\boldsymbol{y}\cdot\boldsymbol{q} + \boldsymbol{\eta}\cdot\boldsymbol{p})}.$$

We denote by π_n the projection $(t, \boldsymbol{q}, \boldsymbol{p}) \longmapsto (\boldsymbol{q}, \boldsymbol{p})$ of H_n onto $\mathbb{R}^{2n}$. Our main result concerning Kazhdan sets in H_n is the following:

Theorem 6.11. — *Let Q be a subset of the Heisenberg group H_n, $n \geqslant 1$. The following assertions are equivalent:*

(1) Q is a Kazhdan set in H_n;
(2) $\pi_n(Q)$ is a Kazhdan set in $\mathbb{R}^{2n}$.

Proof. — We set $Q_0 = \pi_n(Q)$. The proof of Theorem 6.11 relies on the same kind of ideas as those employed in the proof of Proposition 6.9. We start with the easy implication, which is that (1) implies (2). Suppose that Q is a Kazhdan set in H_n, and let $\varepsilon > 0$ be a Kazhdan constant for Q. Let σ be a probability measure on $\mathbb{R}^{2n}$ such that

$$\sup_{(\boldsymbol{q},\boldsymbol{p})\in Q_0} \left|\widehat{\sigma}(\boldsymbol{q}, \boldsymbol{p}) - 1\right| = \sup_{(\boldsymbol{q},\boldsymbol{p})\in Q_0} \left|\int_{\mathbb{R}^{2n}} e^{i(\boldsymbol{y}\cdot\boldsymbol{q} + \boldsymbol{\eta}\cdot\boldsymbol{p})} d\sigma(\boldsymbol{y}, \boldsymbol{\eta}) - 1\right| < \frac{\varepsilon^2}{2} \tag{6.1}$$

and consider the representation ρ of H_n on $L^2(\mathbb{R}^{2n}, \sigma)$ defined by

$$\rho(t, \boldsymbol{q}, \boldsymbol{p}) f \;:\; (\boldsymbol{y}, \boldsymbol{\eta}) \longmapsto e^{i(\boldsymbol{y}\cdot\boldsymbol{q} + \boldsymbol{\eta}\cdot\boldsymbol{p})} f(\boldsymbol{y}, \boldsymbol{\eta})$$

for every $(t, \boldsymbol{q}, \boldsymbol{p}) \in H_n$ and every $f \in L^2(\mathbb{R}^{2n}, \sigma)$. Then (6.1) implies that the constant function $\mathbb{1}$ is a (Q, ε)-invariant vector for ρ. Since (Q, ε) is a Kazhdan pair in H_n, it follows that ρ admits a non-zero H_n-invariant function $f \in L^2(\mathbb{R}^{2n}, \sigma)$. Fix a representative of f, and consider the subset A of $\mathbb{R}^{2n}$ consisting of pairs $(\boldsymbol{y}, \boldsymbol{\eta})$ such that $f(\boldsymbol{y}, \boldsymbol{\eta}) \neq 0$. Then $\sigma(A) > 0$, and for every $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2n}$, σ-almost every element $(\boldsymbol{y}, \boldsymbol{\eta})$ of A satisfies $\boldsymbol{y}\cdot\boldsymbol{q} + \boldsymbol{\eta}\cdot\boldsymbol{p} \in 2\pi\mathbb{Z}$. Hence σ-almost every element $(\boldsymbol{y}, \boldsymbol{\eta})$ of A has the property that $\boldsymbol{y}\cdot\boldsymbol{q} + \boldsymbol{\eta}\cdot\boldsymbol{p} \in 2\pi\mathbb{Z}$ for every $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{Q}^{2n}$. By continuity, σ-almost every element $(\boldsymbol{y}, \boldsymbol{\eta})$ of A has the property that $\boldsymbol{y}\cdot\boldsymbol{q} + \boldsymbol{\eta}\cdot\boldsymbol{p}$ belongs to $2\pi\mathbb{Z}$ for every $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2n}$, so that

$(\boldsymbol{y},\boldsymbol{\eta}) = (\mathbf{0},\mathbf{0})$. We have thus proved that $\sigma(\{\mathbf{0},\mathbf{0}\}) > 0$, and it follows that Q_0 is a Kazhdan set in $\mathbb{R}^{2n}$.

Let us now prove the converse implication. Suppose that Q_0 is a Kazhdan set in $\mathbb{R}^{2n}$, and let $0 < \varepsilon < 3$ be a Kazhdan constant for Q_0. Let π be a unitary representation of H_n on a separable Hilbert space H, which admits a $(Q, \frac{\varepsilon}{8})$-invariant vector $x \in H$ of norm 1. We write as usual π as a direct integral $\pi = \int_Z^{\oplus} \pi_z \, d\mu(z)$, where μ is a finite Borel measure on a standard Borel space Z, and x as $(x_z)_{z\in Z}$, with $\int_Z ||x_z||^2 d\mu(z) = 1$. We have

$$\sup_{(t,\boldsymbol{q},\boldsymbol{p})\in Q} \Big|1 - \int_Z \langle \pi_z(t,\boldsymbol{q},\boldsymbol{p})x_z, x_z\rangle d\mu(z)\Big| < \frac{\varepsilon}{8}. \tag{6.2}$$

For every $z \in Z$, the irreducible representation π_z belongs to one of the two families $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$. If π_z belongs to $(\mathcal{F}_1)$, we write it as $\pi_{\pm\lambda_z}$ for some $\pm\lambda_z \in \mathbb{R}$, and if π belongs to $(\mathcal{F}_2)$, as $\pi_{\boldsymbol{y}_z,\boldsymbol{\eta}_z}$ for some $(\boldsymbol{y}_z,\boldsymbol{\eta}_z) \in \mathbb{R}^{2n}$. Let, for $i = 1, 2$, Z_i be the subset of Z consisting of the elements $z \in Z$ such that π_z belongs to $(\mathcal{F}_i)$. We have $x_z \in L^2(\mathbb{R}^n)$ for every $z \in Z_1$, and $x_z \in \mathbb{C}$ for every $x \in Z_2$. We now observe the following:

Lemma 6.12. — *A Kazhdan subset of $\mathbb{R}^{2n}$ contains elements $(\boldsymbol{q},\boldsymbol{p})$ such that the Euclidean norm $|\boldsymbol{p}|$ of $\boldsymbol{p}$ is arbitrarily large.*

Proof. — Let Q_1 be a Kazhdan subset of $\mathbb{R}^{2n}$, with Kazhdan constant $\varepsilon > 0$, and suppose that there exists a constant $M > 0$ such that $|\boldsymbol{p}| \leqslant M$ for every $(\boldsymbol{q},\boldsymbol{p}) \in Q_1$. Let $\delta > 0$ be such that $2M\delta < \varepsilon$ and consider the probability measure on $\mathbb{R}^{2n}$ defined by

$$\sigma = \delta_{\mathbf{0}} \times \mathbb{1}_{B(\mathbf{0},\delta)} \frac{d\boldsymbol{p}}{|B(\mathbf{0},\delta)|}.$$

For every $(\boldsymbol{q},\boldsymbol{p}) \in Q_1$,

$$|\widehat{\sigma}(\boldsymbol{q},\boldsymbol{p}) - 1| = \Big|\int_{B(\mathbf{0},\delta)} e^{i\boldsymbol{s}\cdot\boldsymbol{p}} \frac{d\boldsymbol{s}}{|B(\mathbf{0},\delta)|} - 1\Big| \leqslant 2\delta|\boldsymbol{p}| \leqslant 2M\delta < \varepsilon.$$

But $\sigma(\{(\mathbf{0},\mathbf{0})\}) = 0$, and it follows that Q_1 is not a Kazhdan set in $\mathbb{R}^{2n}$, which is a contradiction. □

By Fact 6.10, we have for every $x \in Z_1$

$$\langle \pi_{\pm\lambda_z}(t,\boldsymbol{q},\boldsymbol{p})x_z, x_z\rangle \longrightarrow 0 \quad \text{as } |\boldsymbol{p}| \to +\infty\ ,\ (t,\boldsymbol{q},\boldsymbol{p}) \in H_n.$$

Since $|\langle \pi_{\pm\lambda}(t,\boldsymbol{q},\boldsymbol{p})x_z, x_z\rangle| \leqslant ||x_z||^2$ for every $z \in Z_1$, and $\int_Z ||x_z||^2 d\mu(z) = 1$, the dominated convergence theorem implies that

$$\int_{Z_1} \langle \pi_{\pm\lambda_z}(t,\boldsymbol{q},\boldsymbol{p})x_z, x_z\rangle d\mu(z) \longrightarrow 0 \quad \text{as } |\boldsymbol{p}| \to +\infty\ ,\ (t,\boldsymbol{q},\boldsymbol{p}) \in H_n.$$

By Lemma 6.12, there exists an element $(t_0,\boldsymbol{q}_0,\boldsymbol{p}_0)$ of Q with $|\boldsymbol{p}_0|$ so large that

$$\Big|\int_{Z_1} \langle \pi_{\pm\lambda_z}(t_0,\boldsymbol{q}_0,\boldsymbol{p}_0)x_z, x_z\rangle d\mu(z)\Big| < \frac{\varepsilon}{8}.$$

Property (6.2) implies then that

$$\Big|1 - \int_{Z_2} \pi_{\boldsymbol{y}_z,\boldsymbol{\eta}_z}(t_0,\boldsymbol{q}_0,\boldsymbol{p}_0)|x_z|^2 d\mu(z)\Big| < \frac{\varepsilon}{4}$$

from which it follows that $\int_{Z_2} |x_z|^2 d\mu(z) > 1 - \dfrac{\varepsilon}{4}$, so that $\int_{Z_1} ||x_z||^2 d\mu(z) < \dfrac{\varepsilon}{4}$. Plugging this into (6.2) yields that

$$\sup_{(t,\boldsymbol{q},\boldsymbol{p})\in Q} \left|1 - \int_{Z_2} \pi_{\boldsymbol{y}_z,\boldsymbol{\eta}_z}(t,\boldsymbol{q},\boldsymbol{p})|x_z|^2 d\mu(z)\right| < \frac{3\varepsilon}{8}.$$

Since $\int_{Z_2} |x_z|^2 d\mu(z) > 1 - \varepsilon/4$ and $0 < \varepsilon < 3$, we can, by normalizing the family $(x_z)_{z\in Z_2}$, suppose without loss of generality that $Z = Z_2$, $\int_Z |x_z|^2 d\mu(z) = 1$ and that

$$\sup_{(t,\boldsymbol{q},\boldsymbol{p})\in Q} \left|1 - \int_{Z} e^{i(\boldsymbol{y}_z\cdot\boldsymbol{q}+\boldsymbol{\eta}_z\cdot\boldsymbol{p})}\, |x_z|^2\, d\mu(z)\right| < \varepsilon. \tag{6.3}$$

Consider now the unitary representation ρ of $\mathbb{R}^{2n}$ on $L^2(Z,\mu)$ defined by

$$\rho(\boldsymbol{q},\boldsymbol{p})\, f \;:\; z \longmapsto e^{i(\boldsymbol{y}_z\cdot\boldsymbol{q}+\boldsymbol{\eta}_z\cdot\boldsymbol{p})} f_z$$

for every $(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{R}^{2n}$ and every $f = (f_z)_{z\in Z} \in L^2(Z,\mu)$. Then (6.3) can be rewritten as

$$\sup_{(t,\boldsymbol{q},\boldsymbol{p})\in Q} \left|1 - \langle\rho(\boldsymbol{q},\boldsymbol{p})x, x\rangle\right| < \varepsilon, \quad \text{i.e.} \quad \sup_{(\boldsymbol{q},\boldsymbol{p})\in Q_0} \left|1 - \langle\rho(\boldsymbol{q},\boldsymbol{p})x, x\rangle\right| < \varepsilon.$$

Since ε is a Kazhdan constant for Q_0, the representation ρ admits a non-zero $\mathbb{R}^{2n}$-invariant vector $f \in L^2(Z,\mu)$. Proceeding as in the proof of (1) $\Longrightarrow$ (2), we see that for every $(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{R}^{2n}$, $e^{i(\boldsymbol{y}_z\cdot\boldsymbol{q}+\boldsymbol{\eta}_z\cdot\boldsymbol{p})} f(z) = f(z)$ μ-almost everywhere on Z, so that there exists a subset Z_0 of Z with $\mu(Z_0) > 0$ such that f does not vanish on Z_0 and, for every $z \in Z_0$, $\boldsymbol{y}_z \cdot \boldsymbol{q} + \boldsymbol{\eta}_z \cdot \boldsymbol{p}$ belongs to $2\pi\mathbb{Z}$ for every $(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{Q}^{2n}$. By continuity, $\boldsymbol{y}_z \cdot \boldsymbol{q} + \boldsymbol{\eta}_z \cdot \boldsymbol{p}$ belongs to $2\pi\mathbb{Z}$ for every $z \in Z_0$ and every $(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{R}^{2n}$, so that $(\boldsymbol{y}_z, \boldsymbol{\eta}_z) = (\boldsymbol{0},\boldsymbol{0})$ for every $z \in Z_0$. So if we set $Z_0' = \{z \in Z\,;\, (\boldsymbol{y}_z,\boldsymbol{\eta}_z) = (\boldsymbol{0},\boldsymbol{0})\}$, we have $\mu(Z_0') > 0$. The function $f = \mathbb{1}_{Z_0'}$ is hence a non-zero element of $L^2(Z,\mu)$, which is clearly an H_n-invariant vector for the representation π. So $(Q, \frac{\varepsilon}{8})$ is a Kazhdan pair in H_n, and Theorem 6.11 is proved. □

6.3. Kazhdan sets in the group $\mathrm{Aff}_+(\mathbb{R})$. — The underlying space of the group $\mathrm{Aff}_+(\mathbb{R})$ of orientation-preserving affine homeomorphisms of $\mathbb{R}$ is $(0,+\infty)\times\mathbb{R}$, and the group law is given by $(a,b)(a',b') = (aa', b + ab')$, where (a,b) and (a',b') belong to $(0,+\infty)\times\mathbb{R}$. As in the case of the Heisenberg groups, the irreducible unitary representations of $\mathrm{Aff}_+(\mathbb{R})$ are completely classified (see [**15**, Sec. 6.7]) and fall within two classes:

– the class $(\mathcal{F}_1)$ consists of two infinite dimensional representations π_+ and π_- of $\mathrm{Aff}_+(\mathbb{R})$, which act respectively on the Hilbert spaces $L^2((0,+\infty),ds)$ and $L^2((-\infty,0),ds)$. They are both defined by the formula

$$\pi_\pm(a,b)f \;:\; s \longmapsto \sqrt{a}\, e^{2i\pi bs} f(as)$$

where $(a,b) \in (0,+\infty)\times\mathbb{R}$, $f \in L^2((0,+\infty),ds)$ in the case of π_+, and $f \in L^2((-\infty,0),ds)$ in the case of π_-. It is a direct consequence of the Riemann-Lebesgue lemma that the analogue of Fact 6.10 holds true for the two representations π_+ and π_- of G:

Fact 6.13. — For every $f \in L^2((0,+\infty),ds)$ and every $g \in L^2((-\infty,0),ds)$, we have

$$\langle\pi_+(a,b)f, f\rangle \longrightarrow 0 \quad \text{and} \quad \langle\pi_-(a,b)g, g\rangle \longrightarrow 0 \quad \text{as} \quad |b| \longrightarrow +\infty.$$

– the representations of $\mathrm{Aff}_+(\mathbb{R})$ belonging to the family $(\mathcal{F}_2)$ are one-dimensional. They are parametrized by $\mathbb{R}$, and π_λ is defined for every $\lambda \in \mathbb{R}$ by the formula

$$\pi_\lambda(a,b) = a^{i\lambda} \quad \text{for every } (a,b) \in (0,+\infty) \times \mathbb{R}.$$

Proceeding as in the proof of Theorem 6.11, we characterize the Kazhdan subsets of the group $\mathrm{Aff}_+(\mathbb{R})$ in the following way:

Theorem 6.14. — *Let Q be a subset of $\mathrm{Aff}_+(\mathbb{R})$. The following assertions are equivalent:*
(1) Q is a Kazhdan set in $\mathrm{Aff}_+(\mathbb{R})$;
(2) the set $Q_0 = \{t \in \mathbb{R} \,;\, \exists\, b \in \mathbb{R}\ (e^t, b) \in Q\}$ is a Kazhdan set in $\mathbb{R}$.

Proof. — The proof is similar to that of Theorem 6.11, and we will not give it in full detail here. Let us first sketch briefly a proof of the implication (1) $\Longrightarrow$ (2). Suppose that Q is a Kazhdan set in $\mathrm{Aff}_+(\mathbb{R})$, and let $\varepsilon > 0$ be a Kazhdan constant for Q. Consider a probability measure σ on $\mathbb{R}$ such that $\sup_{t\in Q_0} \left|\widehat{\sigma}(t) - 1\right| < \varepsilon^2/2$. We associate to σ a representation ρ of $\mathrm{Aff}_+(\mathbb{R})$ on $L^2(\mathbb{R},\sigma)$ by setting, for every $(a,b) \in (0,+\infty) \times \mathbb{R}$ and every $f \in L^2(\mathbb{R},\sigma)$, $\rho(a,b)f \,:\, s \longmapsto e^{is(\ln a)} f(s)$. Since

$$||\rho(a,b)\mathbb{1} - \mathbb{1}||^2 \leqslant 2\Big|\int_{\mathbb{R}} \big(e^{is(\ln a)} - 1\big) d\sigma(s)\Big| \quad \text{for every } (a,b) \in (0,+\infty) \times \mathbb{R},$$

we have $\sup_{\{(a,b)\,;\,\ln a\in Q_0\}} ||\rho(a,b)\mathbb{1} - \mathbb{1}|| < \varepsilon$, i.e. $\sup_{(a,b)\in Q} ||\rho(a,b)\mathbb{1} - \mathbb{1}|| < \varepsilon$. Hence ρ admits a non-zero $\mathrm{Aff}_+(\mathbb{R})$-invariant function $f \in L^2(\mathbb{R},\sigma)$, and the same argument as in the proof of Theorem 6.11 shows then that $\sigma(\{0\}) > 0$. The converse implication (2) $\Longrightarrow$ (1) is proved in exactly the same way as in Theorem 6.11, using the same modifications as those outlined above. The group $\mathbb{R}^{2n}$ has to be replaced by the multiplicative group $((0,+\infty),\times)$ and the analogue of Lemma 6.12 is that Kazhdan subsets of this group contain elements of arbitrarily large absolute value. If Q_0 is a Kazhdan set in $\mathbb{R}$, with Kazhdan constant ε small enough, the same argument as in the proof of Theorem 6.11 (involving the same notation) shows that it suffices to prove the following statement: let μ be a finite Borel measure on a Borel space Z, $x = (x_z)_{z\in Z}$ a scalar-valued function of $L^2(Z,\mu)$ with $\int_Z |x_z|^2 d\mu(z) = 1$, and π a representation of G of the form $\pi = \int_Z^{\oplus} \pi_{\lambda_z} d\mu(z)$ with

$$\sup_{(a,b)\in Q} \big|1 - \langle \pi(a,b)x, x\rangle\big| = \sup_{\{a\,;\,\ln a\,\in\, Q_0\}} \Big|1 - \int_Z e^{i(\ln a)\lambda_z} |x_z|^2 d\mu(z)\Big| < \varepsilon.$$

Then the set $Z_0 = \{z \in Z \,;\, \lambda_z = 0\}$ satisfies $\mu(Z_0) > 0$. The proof of this statement uses the same argument as the one employed in the proof of Theorem 6.11. It involves the representation ρ of the group $((0,+\infty),\times)$ on $L^2(Z,\mu)$ defined by $\rho(a)f : z \longmapsto e^{i(\ln a)\lambda_z} f_z$ for every $a > 0$ and every $(f_z)_{z\in Z} \in L^2(Z,\mu)$, and uses the obvious fact that since Q_0 is a Kazhdan set in $\mathbb{R}$, $\{a\,;\, \ln a \in Q_0\}$ is a Kazhdan set in $((0,+\infty),\times)$. □

Remark 6.15. — Facts 6.10 and 6.13 have played a crucial role in the proofs of Theorems 6.11 and 6.14 respectively, as they allowed us to discard all irreducible representations except the one-dimensional ones in inequalities of the form (6.2). In groups with the Howe-Moore property (see for instance [**23**], [**39**] or [**7**] for the definition and for more about this property), all non-trivial irreducible representations have the vanishing property of the matrix coefficients stated in Facts 6.10 or 6.13. It easily follows from this observation that all subsets with non-compact closure are Kazhdan sets in groups with the Howe-Moore property, and that if the group is additionally supposed not to have Property (T), the Kazhdan sets are exactly the sets with non-compact closure. As $SL_2(\mathbb{R})$ is a non-compact connected simple real Lie group with finite center, it has the Howe-Moore property. But it does not have Property (T), and so we have:

Example 6.16. — The Kazhdan sets in $SL_2(\mathbb{R})$ are exactly the subsets of $SL_2(\mathbb{R})$ with non-compact closure.

These observations testify of the rigidity of the structure of groups with the Howe-Moore property, and stand in sharp contrast with all the examples we have presented in the rest of this section.

Appendix A
Infinite tensor products of Hilbert spaces

We briefly describe in this appendix some constructions of tensor products of infinite families of Hilbert spaces, and of tensor products of infinite families of unitary representations. These last objects play an important role in the proof of Theorem 2.3. We review here the properties and results which we need, following the original works of von Neumann [**32**] and Guichardet [**18**].

A.1. The complete and incomplete tensor products of Hilbert spaces. — The original construction of the complete and incomplete tensor products of a family $(H_\alpha)_{\alpha\in I}$ of Hilbert spaces is due to von Neumann [**32**]. It was later on taken up by Guichardet in [**18**] under a somewhat different point of view, and the incomplete tensor products of von Neumann are rather known today as the Guichardet tensor products of Hilbert spaces. Although these constructions can be carried out starting from an arbitrary family $(H_\alpha)_{\alpha\in I}$ of Hilbert spaces, we will present them here only in the case of a countable family $(H_n)_{n\geqslant 1}$ of (complex) Hilbert spaces.

The *complete infinite tensor product* $\bigotimes_{n\geqslant 1} H_n$ of the Hilbert spaces H_n is defined in [**32**, Part II, Ch. 3] in the following way: the elementary infinite tensor products are the elements $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$, where x_n belongs to H_n for each $n \geqslant 1$ and the infinite product $\prod_{n\geqslant 1} ||x_n||$ is convergent in the sense of [**32**, Def. 2.2.1], which by [**32**, Lem. 2.4.1] is equivalent to the fact that either $x_n = 0$ for some $n \geqslant 1$ or the series $\sum_{n\geqslant 1} \max(||x_n||-1, 0)$ is convergent. Sequences $(x_n)_{n\geqslant 1}$ with this property are called by von Neumann in [**32**] *C-sequences*. A scalar product is then defined on the set of finite linear combinations of elementary tensor products by setting

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \prod_{n\geqslant 1} \langle x_n, y_n\rangle$$

for any elementary tensor products $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$ and $\boldsymbol{y} = \otimes_{n\geqslant 1} y_n$, and extending the definition by linearity to finite linear combinations of such elements. The product defining $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ for two elementary vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is quasi-convergent in the sense of [**32**, Def. 2.5.1], i.e. $\prod_{n\geqslant 1} |\langle x_n, y_n\rangle|$ is convergent. The value of this quasi-convergent product is $\prod_{n\geqslant 1} \langle x_n, y_n\rangle$ if the product is convergent in the usual sense, and 0 if it is not.

That this is indeed a scalar product which turns the set of finite linear combinations of elementary tensor products into a complex prehilbertian space is proved in [**32**, Lem. 3.21 and Theorem II]. For any elementary tensor product $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$, $||\boldsymbol{x}|| = \prod_{n\geqslant 1} ||x_n||$. The space $\bigotimes_{n\geqslant 1} H_n$ is the completion of this space for the topology induced by the scalar product. It is always non-separable.

The *incomplete tensor products* are closed subspaces of the complete tensor product. They are defined by von Neumann using an equivalence relation between sequences $(x_n)_{n\geqslant 1}$ of vectors with $x_n \in H_n$ for each $n \geqslant 1$ and such that the series $\sum_{n\geqslant 1}\big|1 - ||x_n||\big|$ is convergent. Such sequences are called *C_0-sequences.* They are C-sequences, and if $(x_n)_{n\geqslant 1}$ is a C-sequence such that $\prod_{n\geqslant 1} ||x_n|| > 0$ (i.e. $(x_n)_{n\geqslant 1}$ is non-zero in $\bigotimes_{n\geqslant 1} H_n$) then $(x_n)_{n\geqslant 1}$ is a C_0-sequence. If $(x_n)_{n\geqslant 1}$ is a C_0-sequence, $(x_n)_{n\geqslant 1}$ is bounded, and the series $\sum_{n\geqslant 1}\big|1 - ||x_n||^2\big|$ is convergent.

Two C_0-sequences $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ are *equivalent* if the series $\sum_{n\geqslant 1}\big|1 - \langle x_n, y_n\rangle\big|$ is convergent. If $\mathcal{A}$ denotes an equivalence class of C_0-sequences for this equivalence relation, the *incomplete tensor product* $\bigotimes_{n\geqslant 1}^{\mathcal{A}} H_n$ *associated to* $\mathcal{A}$ is the closed linear span in $\bigotimes_{n\geqslant 1} H_n$ of the vectors $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$, where $(x_n)_{n\geqslant 1}$ belongs to $\mathcal{A}$ [**32**, Def. 4.1.1]. If $\mathcal{A}$ and $\mathcal{A}'$ are two different equivalence classes, the spaces $\bigotimes_{n\geqslant 1}^{\mathcal{A}} H_n$ and $\bigotimes_{n\geqslant 1}^{\mathcal{A}'} H_n$ are orthogonal, and the linear span of the incomplete tensor products $\bigotimes_{n\geqslant 1}^{\mathcal{A}} H_n$, where $\mathcal{A}$ runs over all equivalence classes of C_0-sequences, is dense in the complete tensor product $\bigotimes_{n\geqslant 1} H_n$.

If $\mathcal{A}$ is an equivalence class of C_0-sequences, $\bigotimes_{n\geqslant 1}^{\mathcal{A}} H_n$ admits another, more transparent description, which runs as follows [**32**, Lem. 4.1.2], see also [**18**, Rem. 1.1]: let $(a_n)_{n\geqslant 1}$ be a sequence with $a_n \in H_n$ and $||a_n|| = 1$ for every $n \geqslant 1$, such that the equivalence class of $(a_n)_{n\geqslant 1}$ is $\mathcal{A}$ (such a sequence $(a_n)_{n\geqslant 1}$ does exist: if $(x_n)_{n\geqslant 1}$ is any non-zero C_0-sequence belonging to $\mathcal{A}$, x_n is non-zero for every $n \geqslant 1$, and we can define a C_0-sequence $(a_n)_{n\geqslant 1}$ by setting $a_n = x_n/||x_n||$ for every $n \geqslant 1$. It is not difficult to check that $(a_n)_{n\geqslant 1}$ is equivalent to $(x_n)_{n\geqslant 1}$, and so belongs to $\mathcal{A}$). Then $\bigotimes_{n\geqslant 1}^{\mathcal{A}} H_n$ coincides with the closed linear span in $\bigotimes_{n\geqslant 1} H_n$ of vectors $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$, where $x_n = a_n$ for all but finitely many integers $n \geqslant 1$. Denoting the vector $\otimes_{n\geqslant 1} a_n$ by $\boldsymbol{a}$, we write this closed linear span as $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ (see [**18**]), and thus $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n = \bigotimes_{n\geqslant 1}^{\mathcal{A}} H_n$, where $\mathcal{A}$ is the equivalence class of $\boldsymbol{a}$. The space $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ is usually called *the Guichardet tensor product* of the spaces H_n *associated to the sequence* $(a_n)_{n\geqslant 1}$. Proposition 1.1 of [**18**] states the following, which is a direct consequence of the discussion above: if $\boldsymbol{x} = (x_n)_{n\geqslant 1}$ is a C_0-sequence which is equivalent to $\boldsymbol{a}$, $\boldsymbol{x}$ belongs to $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$. Vectors $\boldsymbol{x}$ of this form are also said to be *decomposable with respect to* $\boldsymbol{a}$, while vectors $\boldsymbol{x} = (x_n)_{n\geqslant 1}$ with $x_n = a_n$ for all but finitely many indices n are called *elementary vectors* of $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$.

Suppose that all the spaces H_n, $n \geqslant 1$, are separable. For each $n \geqslant 1$, let $(e_{p,n})_{1\leqslant p\leqslant p_n}$ be a Hilbertian basis of H_n, with $1 \leqslant p_n \leqslant +\infty$ and $e_{1,n} = a_n$. The family of all elementary vectors $\boldsymbol{e}_\beta = \otimes_{n\geqslant 1} e_{\beta(n),n}$ of $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$, where β is a map from $\mathbb{N}$ into itself such that $1 \leqslant \beta(n) \leqslant p_n$ for every $n \geqslant 1$ and $\beta(n) = 1$ for all but finitely many integers $n \geqslant 1$, forms a Hilbertian basis of $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ [**32**, Lem. 4.1.4]. In particular, $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ is a separable complex Hilbert space.

A.2. Tensor products of unitary representations. — Let G be a topological group, and let $(H_n)_{n\geqslant 1}$ be a sequence of complex separable Hilbert spaces. Let $(a_n)_{n\geqslant 1}$ be a sequence of vectors with $a_n \in H_n$ and $||a_n|| = 1$ for every $n \geqslant 1$. We are looking for conditions under which one can define a unitary representation $\boldsymbol{\pi}$ of G on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ which satisfies

$$\boldsymbol{\pi}(g) \otimes_{n\geqslant 1} x_n = \otimes_{n\geqslant 1} \pi_n(g) x_n \tag{A.1}$$

for every $g \in G$ and every decomposable vector $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$ with respect to $\boldsymbol{a}$. Observe that without any assumption, the equality $\boldsymbol{\pi}(g) \otimes_{n\geqslant 1} x_n = \otimes_{n\geqslant 1} \pi_n(g) x_n$ does not make any sense, since $(\pi_n(g)x_n)_{n\geqslant 1}$, which is a C_0-sequence, may not be equivalent to $\boldsymbol{a}$, and thus may not belong to $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$.

Infinite tensor products of unitary representations have already been studied in various contexts (see for instance [**4**] and the references therein). In [**4**, Prop. 2.3], the following observation is made: suppose that, for each $n \geqslant 1$, U_n is a unitary operator on H_n. Then there exists a unitary operator $\boldsymbol{U} = \otimes_{n\geqslant 1} U_n$ on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ satisfying

$$\boldsymbol{U}\big(\otimes_{n\geqslant 1} x_n\big) = \otimes_{n\geqslant 1} U_n x_n$$

for every decomposable vector $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$ with respect to $\boldsymbol{a}$ if and only if the series $\sum_{n\geqslant 1} \big|1 - \langle U_n a_n, a_n\rangle\big|$ is convergent (which is equivalent to requiring that the C_0-sequence $(U_n a_n)_{n\geqslant 1}$ be equivalent to $(a_n)_{n\geqslant 1}$, i.e. to the fact that $\otimes_{n\geqslant 1} U_n a_n$ be a decomposable vector with respect to $\boldsymbol{a}$). It follows from this result that the formula (A.1) makes sense in $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ if and only if the series

$$\sum_{n\geqslant 1} \big|1 - \langle \pi_n(g) a_n, a_n\rangle\big| \tag{A.2}$$

is convergent for every $g \in G$. Under this condition $\boldsymbol{\pi}(g) = \otimes_{n\geqslant 1} \pi_n(g)$ is a unitary operator on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ for every $g \in G$, and $\boldsymbol{\pi}(gh) = \boldsymbol{\pi}(g)\,\boldsymbol{\pi}(h)$ for every $g, h \in G$.

If the group G is discrete, this tensor product representation is of course automatically strongly continuous. It is also the case if G is supposed to be locally compact.

Proposition A.1. — *Suppose that G is a locally compact group, and that the series $\sum_{n\geqslant 1} |1 - \langle \pi_n(g) a_n, a_n\rangle|$ is convergent for every $g \in G$. Then $\boldsymbol{\pi} = \otimes_{n\geqslant 1} \pi_n$ is strongly continuous, and is hence a unitary representation of G on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$.*

Proof. — Since all the spaces H_n, $n \geqslant 1$, are separable, $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ is separable too, and by [**6**, Lem. A.6.2] it suffices to show that $g \longmapsto \langle \boldsymbol{\pi}(g)\,\boldsymbol{\xi}, \boldsymbol{\xi}\rangle$ is a measurable map from G into $\mathbb{C}$ for every vector $\boldsymbol{\xi} \in \bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ (measurability refers to the Haar measure on G). Since the linear span of the elementary vectors is dense in $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$, standard arguments show that it suffices to prove this for elementary vectors $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$ of $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$. Since each map $g \longmapsto \langle \pi_n(g) x_n, x_n\rangle$ is continuous on G, it is clear that $g \longmapsto \langle \boldsymbol{\pi}(g)\boldsymbol{x}, \boldsymbol{x}\rangle = \prod_{n\geqslant 1} \langle \pi_n(g) x_n, x_n\rangle$ is measurable on G. □

In the general case one needs to impose an additional condition on the representations π_n and on the vectors a_n in order that $\boldsymbol{\pi}$ be a strongly continuous representation of G on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$.

Proposition A.2. — *Suppose that the series $\sum_{n\geqslant 1} \big|1 - \langle \pi_n(g) a_n, a_n\rangle\big|$ is convergent for every $g \in G$ and that the function $g \longmapsto \sum_{n\geqslant 1} \big|1 - \langle \pi_n(g) a_n, a_n\rangle\big|$ is continuous on a neighborhood of the identity element e of G. Then $\boldsymbol{\pi} = \otimes_{n\geqslant 1} \pi_n$ is strongly continuous, and is hence a unitary representation of G on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$.*

Proof of Proposition A.2. — Since the linear span of the elementary vectors is dense in $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$, and the operators $\boldsymbol{\pi}(g)$, $g \in G$, are unitary, it suffices to prove that the map $g \longmapsto \boldsymbol{\pi}(g)\boldsymbol{x}$ is continuous at e for every elementary vector $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$ of norm 1 of $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$. Let $N \geqslant 1$ be such that $x_n = a_n$ for every $n > N$. We have for every $g \in G$:

$$||\boldsymbol{\pi}(g)\boldsymbol{x} - \boldsymbol{x}||^2 = 2(1 - \mathrm{Re}\langle \boldsymbol{\pi}(g)\boldsymbol{x}, \boldsymbol{x}\rangle) = 2\bigg(1 - \prod_{n\geqslant 1} \mathrm{Re}\langle \pi_n(g)\frac{x_n}{||x_n||}, \frac{x_n}{||x_n||}\rangle\bigg)$$

since $||\boldsymbol{x}|| = \prod_{n\geqslant 1} ||x_n|| = 1$. Thus

$$\begin{aligned}||\boldsymbol{\pi}(g)\boldsymbol{x} - \boldsymbol{x}||^2 &\leqslant 2\sum_{n\geqslant 1}\Big|1 - \langle \pi_n(g)\frac{x_n}{||x_n||}, \frac{x_n}{||x_n||}\rangle\Big| \\ &\leqslant 2\sum_{n=1}^{N}\Big|1 - \langle \pi_n(g)\frac{x_n}{||x_n||}, \frac{x_n}{||x_n||}\rangle\Big| + 2\sum_{n\geqslant 1}|1 - \langle \pi_n(g)a_n, a_n\rangle|.\end{aligned}$$

If ε is any positive number, it follows from the assumptions that $||\boldsymbol{\pi}(g)\boldsymbol{x} - \boldsymbol{x}|| < \varepsilon$ if g lies in a suitable neighborhood of e. This proves the continuity of the map $g \longmapsto \boldsymbol{\pi}(g)\boldsymbol{x}$. □

We finish this appendix by giving a sufficient condition for an infinite tensor product representation on a space $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ to be weakly mixing: let, for each $n \geqslant 1$, H_n be a separable Hilbert space, a_n a vector of H_n with $||a_n|| = 1$, and π_n a unitary representation of G on H_n. We suppose that the assumptions of either Proposition A.1 (when G is locally compact) or Proposition A.2 (in the general case) are satisfied, so that $\boldsymbol{\pi} = \otimes_{n\geqslant 1}\pi_n$ is a unitary representation of G on $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$. Then

Proposition A.3. — *In the case where* $\underline{\lim}_{n\to+\infty} m(|\langle \pi_n(\,.\,)a_n, a_n\rangle|^2) = 0$, *the representation* $\boldsymbol{\pi} = \otimes_{n\geqslant 1}\pi_n$ *is weakly mixing.*

Proof. — The proof of Proposition A.3 relies on the same idea as that of Proposition A.2: let $\boldsymbol{x} = \otimes_{n\geqslant 1} x_n$ and $\boldsymbol{y} = \otimes_{n\geqslant 1} y_n$ be two elementary vectors in $\bigotimes_{n\geqslant 1}^{\boldsymbol{a}} H_n$ with $||\boldsymbol{x}|| = ||\boldsymbol{y}|| = 1$. We have

$$|\langle \boldsymbol{\pi}(g)\,\boldsymbol{x}, \boldsymbol{y}\rangle|^2 = \prod_{k\geqslant 1}\Big|\Big\langle \pi_k(g)\,\frac{x_k}{||x_k||}, \frac{y_k}{||y_k||}\Big\rangle\Big|^2 \leqslant \Big|\Big\langle \pi_n(g)\,\frac{x_n}{||x_n||}, \frac{y_n}{||y_n||}\Big\rangle\Big|^2$$

for every $n \geqslant 1$ and every $g \in G$. But

$$\Big|\Big\langle \pi_n(g)\,\frac{x_n}{||x_n||}, \frac{y_n}{||y_n||}\Big\rangle\Big| \leqslant |\langle \pi_n(g)a_n, a_n\rangle| + \Big|\Big|\frac{x_n}{||x_n||} - a_n\Big|\Big| + \Big|\Big|\frac{y_n}{||y_n||} - a_n\Big|\Big|.$$

Squaring and taking the mean on both sides we obtain that

$$m(|\langle \boldsymbol{\pi}(\,.\,)\boldsymbol{x}, \boldsymbol{y}\rangle|^2) \leqslant 4\,m(|\langle \pi_n(\,.\,)\,a_n, a_n\rangle|^2) + 4\Big|\Big|\frac{x_n}{||x_n||} - a_n\Big|\Big|^2 + 4\Big|\Big|\frac{y_n}{||y_n||} - a_n\Big|\Big|^2$$

for every $n \geqslant 1$. Since $\underline{\lim}_{n\to+\infty} m(|\langle \pi_n(\,.\,)a_n, a_n\rangle|^2) = 0$ and the two other terms are equal to zero for n sufficiently large, $m(|\langle \boldsymbol{\pi}(\,.\,)\boldsymbol{x}, \boldsymbol{y}\rangle|^2) = 0$. Weak mixing of $\boldsymbol{\pi}$ now follows from standard density arguments. □

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