A NEW CLASS OF FREQUENTLY HYPERCYCLIC OPERATORS

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Abstract. We study in this paper a hypercyclicity property of linear dynamical systems: a bounded linear operator T acting on a separable infinite-dimensional Banach space X is said to be *hypercyclic* if there exists a vector $x \in X$ such that $\{T^n x \,;\, n \geq 0\}$ is dense in X, and *frequently hypercyclic* if there exists $x \in X$ such that for any non empty open subset U of X, the set $\{n \geq 0 \,;\, T^n x \in U\}$ has positive lower density. We prove in this paper that if $T \in \mathcal{B}(X)$ is an operator which has "sufficiently many" eigenvectors associated to eigenvalues of modulus 1 in the sense that these eigenvectors are perfectly spanning, then T is automatically frequently hypercyclic.

1. Introduction

Let X be a complex infinite-dimensional separable Banach space, and T a bounded linear operator on X. We are concerned in this paper with the dynamics of the operator T, i.e. with the behaviour of the orbits $\mathcal{O}\text{rb}(x,T) = \{T^n x \,;\, n \geq 0\}$, $x \in X$, of the vectors of X under the action of T. Our main interest here will be in strong forms of hypercyclicity: recall that a vector $x \in X$ is said to be *hypercyclic* for T if its orbit under the action of T is dense in X. In this case the operator T itself is said to be hypercyclic. This notion of hypercyclicity as well as related matters in linear dynamics have been intensively studied in the past years. We refer the reader to the recent book [6] for more information on these topics.

Our starting point for this work are the papers [4], [3] and [5], which study the role of the unimodular point spectrum in linear dynamics. By unimodular point spectrum of the operator T, we mean the set of eigenvalues of T which are of modulus 1. That the behaviour of the eigenvectors of an operator has an influence on its hypercyclicity properties was first discovered by Godefroy and Shapiro in [14]: their work deals with eigenvectors associated to eigenvalues of modulus strictly larger than 1 and strictly smaller than 1. The eigenvectors associated to eigenvalues of modulus 1 first appeared in the works of Flytzanis [13] and Bourdon and Shapiro [11]. Then it was shown in [4] that if T has "sufficiently many eigenvectors associated to unimodular eigenvalues" (precise definitions will be given later on) then T is hypercyclic. In [3] and [5] this study is pushed further on in the direction of ergodic theory: under some assumptions bearing either on the geometry of the underlying space X or on the regularity of the eigenvector fields of the operator T, it is proved that T admits a non-degenerate invariant Gaussian measure with respect to which it is ergodic (even weak-mixing). Then a straightforward

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1991 *Mathematics Subject Classification.* 47A16, 37A05, 47A35, 46B09, 46B15.

Key words and phrases. Linear dynamical systems, hypercyclic and frequently hypercyclic operators, measure-preserving and ergodic transformations.

This work was partially supported by ANR-Projet Blanc DYNOP.

application of Birkhoff's ergodic theorem shows that T is "more than hypercyclic": it is *frequently hypercyclic*, i.e. there exists a vector $x \in X$ such that for every non-empty open subset U of X, the set $\{n \geq 0 \; ; \; T^n x \in U\}$ of instants when the iterates of x under T visit U has positive lower density. Such a vector x is called a frequently hypercyclic vector for T. Frequent hypercyclicity is a much stronger notion than hypercyclicity, and some operators are hypercyclic without being frequently hypercyclic: an example is the Bergman backward shift [3], and then it was proved in [18] that no hypercyclic operator whose spectrum has an isolated point can be frequently hypercyclic. Thus, although every infinite-dimensional separable Banach space supports a hypercyclic operator ([1],[8]), there are spaces on which there are no frequently hypercyclic operators. Nonetheless, quite a large number of hypercyclic operators are frequently hypercyclic, at least on Hilbert spaces (see for instance [3], [10]). One of the tools which are used to prove the frequent hypercyclicity of an operator is the ergodic-theoretic argument mentioned above: it shows that as soon as T has sufficiently many eigenvectors associated to unimodular eigenvalues, T is frequently hypercyclic.

More precisely, let us recall the following definition from [4] and [3], which quantifies the fact that T admits "plenty" eigenvectors associated to eigenvalues lying on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} \; ; \; |\lambda| = 1\}$:

Definition 1.1. We say that a bounded operator T on X has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for every σ-measurable subset A of $\mathbb{T}$ which is of σ-measure 1, $\mathrm{sp}[\ker(T - \lambda) \; ; \; \lambda \in A]$ is dense in X.

In other words if we take out from the unit circle a set of σ-measure 0 of eigenvalues, the eigenvectors associated to the remaining eigenvalues still span X.

The following result is proved in [3]:

Theorem 1.2. [3] *If T is a bounded operator acting on a separable infinite dimensional complex Hilbert space H, and if T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, then T is frequently hypercyclic.*

As mentioned, above, the method of proof of this statement is rather complicated, since it involves the construction of an invariant ergodic Gaussian measure for the operator T. Moreover Gaussian measures are much easier to deal with on Hilbert spaces than on general Banach spaces, because a complete description of the covariance operators of Gaussian measures is available on Hilbert spaces. We refer the reader to [7, Ch. 6, Section 2] for a study of Gaussian measures in the Hilbertian setting, and to [20] for a presentation in the Banach space case. This explains why, when trying to prove a Banach space version of Theorem 1.2, we were compelled in [5] to add some assumption concerning either the geometry of the space (that X is of type 2, for instance) or the regularity of the eigenvector fields of the operator (that they can be parametrized in a "smooth", i.e. α-Hölderian way for some suitable α). See the book [6] for more details on these results.

Thus the following question remained open in [5]:

Question 1.3. [5] *If X is a general separable complex infinite-dimensional Banach space and T is a bounded operator on X which has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, must T be frequently hypercyclic?*

It is proved in [3] that if T has perfectly spanning unimodular eigenvectors, then T must already be hypercyclic. The main result of this paper is an affirmative answer to Question 1.3:

Theorem 1.4. *Let T be a bounded operator acting on a complex Banach space X. If the eigenvectors of T associated to eigenvalues of modulus 1 are perfectly spanning, then T is frequently hypercyclic.*

The proof of Theorem 1.4 is the object of the first three sections of the paper. It relies on the construction of an explicit invariant measure and on the use of Birkhoff's ergodic theorem, as in [17] where a "Random Frequent Hypercyclicity Criterion" is proved using somewhat similar tools. One interesting point is that this measure is constructed using independent Steinhaus variables, instead of Gaussian ones as in the previous constructions of [5] and [17]. We obtain on our way (in Section 4) several characterizations, which are of interest in themselves, of operators having perfectly spanning unimodular eigenvectors.

It is also interesting to note that the operator T of Theorem 1.4 will never be ergodic with respect to one of the invariant measures constructed in the proof: this result is proved in Section 5.

In the last section of the paper we collect miscellaneous remarks and open questions. In particular we mention how Theorem 1.4 can be applied to retrieve the main result of [12], namely that any infinite-dimensional separable complex Banach space with an unconditional Schauder decomposition supports a frequently hypercyclic operator.

Acknowledgement: I wish to thank the referee for his/her very careful reading of the paper, and for his/her numerous comments which greatly improved the presentation of this work.

2. Strategy for the proof of Theorem 1.4

We are going to derive Theorem 1.4 from our forthcoming Theorem 2.3, which states that if T is a bounded hypercyclic operator on a separable infinite-dimensional complex Banach space X whose eigenvectors associates to eigenvalues of modulus 1 span a dense subspace of X, then T is frequently hypercyclic provided the unimodular eigenvectors of T satisfy some additional assumption (H). Assumption (H) is a priori weaker than the assumption that T has perfectly spanning unimodular eigenvectors, although it will turn out to be actually equivalent to it (see Section 4).

Before stating assumption (H), let us start with two elementary lemmas. Let T be a hypercyclic operator on X whose eigenvectors associated to unimodular eigenvalues span a dense subspace of X. We denote by $\sigma_p(T) \cap \mathbb{T}$ the set of eigenvalues of T of modulus 1.

Lemma 2.1. *Let F be a finite subset of $\sigma_p(T) \cap \mathbb{T}$. Then $sp[\ker(T-\lambda) \; ; \lambda \in \mathbb{T} \setminus F]$ is dense in X.*

Proof. Suppose that $X_0 = \overline{\mathrm{sp}}[\ker(T-\lambda) \; ; \; \lambda \in \mathbb{T} \setminus F]$ is not equal to X, and let $\overline{T}$ be the operator induced by T on the quotient space $\overline{X} = X/X_0$. Then $\overline{T}$ is hypercyclic on $\overline{X}$. Let $(x_n)_{n\geq 1}$ be a sequence of elements of $\bigcup_{\lambda\in\mathbb{T}\setminus F} \ker(T-\lambda)$ which span X_0, and $(y_n)_{n\geq 1}$ a sequence of elements of $\bigcup_{\lambda\in F} \ker(T-\lambda)$ such that the set $\{x_n, \, y_n \; ; \; n \geq 1\}$ span a dense subspace of X: then $\{\overline{x_n}, \, \overline{y_n} \; ; \; n \geq 1\}$ span a dense subspace of $\overline{X}$, i.e. $\{\overline{y_n} \; ; \; n \geq 1\}$ span a dense subspace of $\overline{X}$. Hence the eigenvectors associated to the eigenvalues of $\overline{T}$ belonging to the finite set F span a dense subspace of $\overline{X}$, so that $\prod_{\lambda\in F}(\overline{T}-\lambda) = 0$, which contradicts the hypercyclicity of $\overline{T}$. Hence $X_0 = X$. □

The proof of Lemma 2.1 actually shows:

Lemma 2.2. *Let $(x_n)_{n\geq 1}$ be a sequence of eigenvectors of T, $Tx_n = \lambda_n x_n$, $|\lambda_n| = 1$, such that $sp[x_n \; ; \; n \geq 1]$ is dense in X. If F is any finite subset of $\sigma_p(T) \cap \mathbb{T}$, then $sp[x_n \; ; \; n \in A_F]$ is dense in X, where $A_F = \{n \geq 0 \; ; \; \lambda_n \notin F\}$.*

Suppose now that T satisfies the following assumption (H):

There exists a sequence $(x_n)_{n\geq 1}$ of eigenvectors of T, $Tx_n = \lambda_n x_n$, $|\lambda_n| = 1$, $\lambda_n = e^{2i\pi\theta_n}$, $\theta_n \in]0,1]$, $||x_n|| = 1$, having the following properties:

1. *whenever $(\lambda_{n_1}, \ldots, \lambda_{n_k})$ is a finite family of distinct elements of the set $\{\lambda_n \; ; \; n \geq 1\}$, the family $(\theta_{n_1}, \ldots, \theta_{n_k})$ consists of $\mathbb{Q}$-independent irrational numbers;*
2. *$sp[x_n \; ; \; n \geq 1]$ is dense in X;*
3. *for any finite subset F of $\sigma_p(T) \cap \mathbb{T}$ we have $\overline{\{x_n \; ; \; n \geq 1\}} = \overline{\{x_n \; ; \; n \in A_F\}}$, where $A_F = \{k \geq 0 \; ; \; \lambda_k \notin F\}$.*

Assertion (3) of assumption (H) states that given any finite set F of eigenvalues of T, any x_n can be approximated as closely as we wish by eigenvectors associated to eigenvalues not belonging to F. Assertion (1) ensures that we have some “independence” of the eigenvalues λ_n; this will turn out to be necessary in the proof of Theorem 1.4. It is not difficult to see already (more details will be given in Section 4 later on) that assumption (H) will be satisfied provided the unimodular eigenvectors of T can be parametrized via countably many continuous eigenvector fields. As will also be seen in Section 4, this seemingly weaker assumption is in fact equivalent to the requirement that the unimodular eigenvectors of T be perfectly spanning.

The first step in the proof of Theorem 1.4 is to prove the following statement:

Theorem 2.3. *If T is a bounded operator on X which is hypercyclic and satisfies assumption (H), then T is frequently hypercyclic.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space, and $(\chi_n)_{n\geq 1}$ a sequence of independent Steinhaus variables on $(\Omega, \mathcal{F}, \mathbb{P})$: $\chi_n : \Omega \longrightarrow \mathbb{T}$, and for any subarc I of $\mathbb{T}$,

$$\mathbb{P}(\chi_n \in I) = \frac{|I|}{2\pi},$$

where $|I|$ is the length of I. We have $\mathbb{E}(f(\chi_n)) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$ for any continuous function f on $\mathbb{T}$, so that $\mathbb{E}(\chi_n) = 0$ and $\mathbb{E}|\chi_n|^2 = 1$ for any $n \geq 1$. One important feature of these Steinhaus variables is that for any unimodular numbers λ_n, $\lambda_n\chi_n$ and χ_n have the same law. This makes these variables quite useful for constructing invariant measures for linear operators.

Suppose that $(y_n)_{n\geq 1}$ is a sequence of eigenvectors of T, $Ty_n = \lambda_n y_n$, $|\lambda_n| = 1$, such that the random series

$$\Phi(\omega) = \sum_{n\geq 1} \chi_n(\omega) y_n$$

is convergent almost everywhere. Then it is possible to define a measure m on the Banach space X by setting for any Borel subset A of X

$$m(A) = \mathbb{P}(\{\omega \in \Omega \ ; \ \sum_{n\geq 1} \chi_n(\omega) y_n \in A\}).$$

The measure m is invariant by T:

$$\begin{aligned} m(T^{-1}(A)) &= \mathbb{P}(\{\omega \in \Omega \ ; \ \sum_{n\geq 1} \chi_n(\omega) T y_n \in A\}) \\ &= \mathbb{P}(\{\omega \in \Omega \ ; \ \sum_{n\geq 1} \chi_n(\omega) \lambda_n y_n \in A\}). \end{aligned}$$

Since $|\lambda_n| = 1$, $\lambda_n \chi_n$ and χ_n have the same law, and thus $m(T^{-1}(A)) = m(A)$.

Our strategy to prove Theorem 1.4 is to construct a sequence $(y_n)_{n\geq 1}$ of unimodular eigenvectors of T which is such that

(a) the associated random series $\Phi(\omega)$ converges a.e. on Ω;
(b) for almost every $\omega \in \Omega$, $\Phi(\omega)$ is hypercyclic for T.

Once the sequence $(y_n)_{n\geq 1}$ satisfying (a) and (b) is constructed, it is not difficult to see that $\Phi(\omega)$ is frequently hypercyclic for T for almost every $\omega \in \Omega$: this is proved in [17, Prop. 3.1] under the assumption that the measure m associated to Φ is non-degenerate, i.e. that $m(U) > 0$ for any non-empty open subset U of X. This a priori assumption that m be non-degenerate is in fact not necessary:

Proposition 2.4. *Suppose that there exists a measure m which is invariant by T and such that $m(HC(T)) = 1$, where $HC(T)$ denotes the set of hypercyclic vectors for T. Then the set $FHC(T)$ of frequently hypercyclic vectors for T also satisfies $m(FHC(T)) = 1$. In particular T is frequently hypercyclic.*

Proof. For any non-empty open subset U of X,

$$HC(T) \subseteq \bigcup_{n\geq 0} T^{-n}(U)$$

so that $m(\bigcup_{n\geq 0} T^{-n}(U)) = 1$. Since $m(U) = m(T^{-n}(U))$ for any $n \geq 1$, it is impossible that $m(U) = 0$. So $m(U) > 0$, and m actually has full support. The rest of the proof then goes exactly as in [17, Prop. 3.1]. We recall the argument for completeness's sake: since m is T-invariant, Birkhoff's theorem implies that for m-almost every x in X,

$$\frac{1}{N} \#\{n \leq N \ ; \ T^n x \in U\} \longrightarrow \mathbb{E}(\chi_U | \mathcal{I})(x),$$

where χ_U is the characteristic function of the set U and $\mathcal{I}$ the σ-algebra of T-invariant subsets of $(X, \mathcal{B}, m)$. Now $\mathbb{E}(\chi_U | \mathcal{I})$ is a T-invariant function which it is positive almost everywhere on the set $\bigcup_{n\geq 0} T^{-n}(U)$, which has measure 1. So $\mathbb{E}(\chi_U|\mathcal{I})$ is positive almost everywhere, and it follows that m-almost every x is frequently hypercyclic for T. $\square$

In the works [3], [5], [17], invariant measures were constructed using sums of independent Gaussian variables $\sum g_n(\omega)x_n$, and taking advantage of the rotational invariance of the law of g_n. It is important here that we consider Steinhaus variables instead of Gaussian variables, as will be seen shortly.

Let us summarize: we are looking for a sequence $(y_n)_{n\geq 1}$ of eigenvectors of T, such that $\Phi(\omega) = \sum_{n\geq 1} \chi_n(\omega)y_n$ defines an invariant measure m such that $m(HC(T)) = 1$. The construction of such a sequence $(y_n)_{n\geq 1}$ will be done by induction, and by blocks: at step k we construct the vectors y_n for $n \in [s_{k-1}, s_k - 1]$, where (s_k) is a certain fast increasing sequence of integers with $s_0 = 1$.

Before beginning the construction we state separately one obvious fact, which will be used repeatedly in the forthcoming proof:

Lemma 2.5. *Let a be a complex number, and $\varepsilon > 0$. There exists a finite family $(a_1, \ldots, a_N)$ of complex numbers such that*

(i) $a_1 + \ldots + a_N = a$
(ii) $|a_1|^2 + \ldots + |a_N|^2 < \varepsilon$.

Proof. Just take N so large that $\frac{a^2}{N} < \varepsilon$ and set $a_i = \frac{a}{N}$ for any $i = 1, \ldots, N$. $\square$

3. Proof of Theorem 2.3: frequent hypercyclicity of T under assumption (H)

Let $(U_n)_{n\geq 1}$ be a countable basis of open subsets of X, and let $(x_n)_{n\geq}$ be a sequence of eigenvectors of T, $||x_n|| = 1$, $Tx_n = \lambda_n x_n$, satisfying assumption (H).

Step 1: Since T is hypercyclic, there exists an integer p_1 such that $T^{p_1}(B(0, \frac{1}{2})) \cap U_1$ is non-empty. As the vectors x_k, $k \geq 1$, span a dense subspace of X, there exists a finite linear combination u_1 of the vectors x_k such that $||u_1|| < \frac{1}{2}$ and $T^{p_1}u_1 \in U_1$. Let us write u_1 as

$$u_1 = \sum_{k\in I_1} \alpha_k x_k$$

where $I_1 = [1, r_1]$ is a certain finite interval of $[1, +\infty[$. Since the linear space $\mathrm{sp}[x_k \ ; \ k \in I_1]$ is finite-dimensional, there exists a positive constant M_1 such that for every family $(\beta_k)_{k\in I_1}$ of complex numbers,

$$||\sum_{k\in I_1} \beta_k x_k|| \leq M_1 \left(\sum_{k\in I_1} |\beta_k|^2\right)^{\frac{1}{2}}.$$

Let δ_1 be a very small positive number. By Lemma 2.5, we can write each α_k, $k \in I_1$, as

$$\alpha_k = \sum_{j\in J_k^1} a_j^{(k)},$$

where the sets J_k^1, $k \in I_1$, are successive intervals of $[1, +\infty[$ and

$$\sum_{k \in I_1} \left(\sum_{j \in J_k^1} |a_j^{(k)}|^2 \right)^{\frac{1}{2}} < \delta_1.$$

Thus u_1 can be rewritten as

$$u_1 = \sum_{k \in I_1} \left(\sum_{j \in J_k^1} a_j^{(k)} \right) x_k.$$

Let γ_1 be a very small positive number, to be chosen later on in the proof. Assumption (H) implies that there exists a family $x_j^{(k)}$, $k \in I_1$, $j \in J_k^1$, of elements of the set $\{x_n \ ; \ n \geq 1\}$ such that for any $k \in I_1$ and $j \in J_k^1$,

$$||x_k - x_j^{(k)}|| < \gamma_1$$

and the eigenvalues $\lambda_j^{(k)}$ associated to the eigenvectors $x_j^{(k)}$ are all distinct. Hence the arguments $\theta_j^{(k)}$ of the eigenvalues $\lambda_j^{(k)} = e^{2i\pi\theta_j^{(k)}}$ form a $\mathbb{Q}$-independent sequence of irrational numbers. Set

$$v_1 = \sum_{k \in I_1} \sum_{j \in J_k^1} a_j^{(k)} x_j^{(k)}.$$

We have

$$||u_1 - v_1|| \leq \sum_{k \in I_1} \sum_{j \in J_k^1} |a_j^{(k)}| \, ||x_j^{(k)} - x_k|| \leq \gamma_1 \sum_{k \in I_1} \sum_{j \in J_k^1} |a_j^{(k)}|$$

so that $||u_1 - v_1||$ can be made arbitrarily small if γ_1 is small enough. Hence taking γ_1 very small we can ensure that $T^{p_1} v_1$ belongs to U_1, i.e. that

$$\sum_{k \in I_1} \sum_{j \in J_k^1} a_j^{(k)} (\lambda_j^{(k)})^{p_1} x_j^{(k)} \in U_1.$$

Let $(\chi_j^{(k)})_{k \in I_1, j \in J_k^1}$ be a family of independent Steinhaus variables, and define on $(\Omega, \mathcal{F}, \mathbb{P})$ a random function Φ_1 by

$$\Phi_1(\omega) = \sum_{k \in I_1} \sum_{j \in J_k^1} \chi_j^{(k)}(\omega) \, a_j^{(k)} x_j^{(k)}.$$

Our aim is now to estimate the expectation $\mathbb{E}||\Phi_1(\omega)||$. In order to do this, let us consider the auxiliary random function

$$\Psi_1(\omega) = \sum_{k \in I_1} \left(\sum_{j \in J_k^1} \chi_j^{(k)}(\omega) \, a_j^{(k)} \right) x_k.$$

Writing

$$\beta_k(\omega) = \sum_{j \in J_k^1} \chi_j^{(k)}(\omega) \, a_j^{(k)},$$

we have

$$||\Psi_1(\omega)|| \leq M_1 \left(\sum_{k\in I_1} |\beta_k(\omega)|^2\right)^{\frac{1}{2}} \leq M_1 \sum_{k\in I_1} |\beta_k(\omega)|$$

so that

$$\mathbb{E}||\Psi_1(\omega)|| \leq M_1 \sum_{k\in I_1} \mathbb{E}\left|\sum_{j\in J_k^1} \chi_j^{(k)}(\omega)\, a_j^{(k)}\right|.$$

Now the "Steinhaus version" of Khinchine inequalities states that there is a universal constant $C > 0$ such that for any sequence $(a_n)_{n\geq 1}$ of complex numbers, we have for any $N \geq 1$

$$\frac{1}{C}\left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}} \leq \mathbb{E}\left|\sum_{n=1}^{N} \chi_n(\omega) a_n\right| \leq C\left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}}.$$

Hence

$$\mathbb{E}||\Psi_1(\omega)|| \leq M_1\, C \sum_{k\in I_1} \left(\sum_{j\in J_k^1} |a_j^{(k)}|^2\right)^{\frac{1}{2}} < M_1\, C\, \delta_1.$$

Hence if δ_1 is chosen very small with respect to M_1, we can ensure that $\mathbb{E}||\Psi_1(\omega)|| < 4^{-1}$ for instance. Now

$$||\Phi_1(\omega) - \Psi_1(\omega)|| \leq \sum_{k\in I_1} \sum_{j\in J_k^1} |a_j^{(k)}|\, ||x_j^{(k)} - x_k|| \leq \gamma_1 \sum_{k\in I_1} \sum_{j\in J_k^1} |a_j^{(k)}|.$$

Thus if γ_1 is small enough, $\mathbb{E}||\Phi_1(\omega) - \Psi_1(\omega)||$ is so small that $\mathbb{E}||\Phi_1(\omega)|| < 4^{-1}$ too (recall that M_1 is chosen first, then δ_1 is chosen very small with respect to M_1, and lastly γ_1 is chosen very small with respect to δ_1).

Our next goal is to show that there exists a finite family $\mathcal{P}_1$ of integers such that for almost every $\omega \in \Omega$, there exists an integer $p_1(\omega) \in \mathcal{P}_1$ such that $T^{p_1(\omega)}\Phi_1(\omega)$ belongs to U_1.

We have for any $p \geq 0$

$$T^p\Phi_1(\omega) = \sum_{k\in I_1} \sum_{j\in J_k^1} \chi_j^{(k)}(\omega)\, (\lambda_j^{(k)})^p\, a_j^{(k)} x_j^{(k)}.$$

Let $(\mu_j^{(k)})_{k\in I_1, j\in J_k^1}$ be any family of unimodular numbers indexed by the sets I_1 and J_k^1, $k \in I_1$. Since the arguments of the $\lambda_j^{(k)}$ are $\mathbb{Q}$-independent irrational numbers, there exists for any $\eta_1 > 0$ an integer $p \geq 1$ such that for any $k \in I_1$ and any $j \in J_k^1$

$$|(\lambda_j^{(k)})^p - \mu_j^{(k)}| < \frac{\eta_1}{2}.$$

Considering a finite $\eta_1/2$-net of the set $\mathbb{T}^{\sum |J_k^1|}$, we obtain that there exists a finite family $\mathcal{Q}_1$ of integers such that for almost every $\omega \in \Omega$ there exists an integer $p(\omega) \in \mathcal{Q}_1$ such that for any $k \in I_1$ and any $j \in J_k^1$,

$$|(\lambda_j^{(k)})^{p(\omega)} - \overline{\chi_j^{(k)}(\omega)}| < \eta_1.$$

Now if ρ_1 is any positive number, there exists an $\eta_1 > 0$ such that if p is such that $|\chi_j^{(k)}(\omega)(\lambda_j^{(k)})^p - 1| < \eta_1$ for any $k \in I_1$ and $j \in J_k^1$, then $||T^p\Phi_1(\omega) - v_1|| < \rho_1$. Indeed in this case

$$\begin{aligned} T^p\Phi_1(\omega) - v_1 &= ||\sum_{k\in I_1}\sum_{j\in J_k^1}\left(\chi_j^{(k)}(\omega)\,(\lambda_j^{(k)})^p - 1\right) a_j^{(k)}x_j^{(k)}|| \\ &\le \eta_1 \sum_{k\in I_1}\sum_{j\in J_k^1}|a_j^{(k)}| < \rho_1 \end{aligned}$$

if η_1 is sufficiently small with respect to ρ_1. Choose ρ_1 such that

$$T^{p_1}v_1 + B(0, \rho_1||T||^{p_1}) \subseteq U_1,$$

then η_1 as above, and take $\mathcal{P}_1 = p_1 + \mathcal{Q}_1$: for almost every $\omega \in \Omega$, there exists a $p(\omega) \in \mathcal{Q}_1$ such that $||T^{p(\omega)}\Phi_1(\omega) - v_1|| < \rho_1$. Thus

$$||T^{p_1+p(\omega)}\Phi_1(\omega) - T^{p_1}v_1|| < \rho_1\,||T||^{p_1}$$

so that $T^{p_1+p(\omega)}\Phi_1(\omega)$ belongs to U_1.

Let us summarize what has been done in this first step: we have constructed a function $\Phi_1(\omega)$ which is a finite Steinhaus sum of eigenvectors of T associated to distinct eigenvalues, such that

• $\mathbb{E}||\Phi_1(\omega)|| < 4^{-1}$

• there exists a finite set $\mathcal{P}_1$ of integers such that for almost every $\omega \in \Omega$, there exists an integer $p_1(\omega) \in \mathcal{P}_1$ such that $T^{p_1(\omega)}\Phi_1(\omega)$ belongs to U_1. Let π_1 denote the maximum of the set $\mathcal{P}_1$.

Step 2: Let V_2 be a non-empty open subset of X and κ_2 be a positive number such that $V_2 + B(0, 2\kappa_2) \subseteq U_2$. For any $p \ge 0$ and almost every $\omega \in \Omega$ we have

$$T^p\Phi_1(\omega) - \Phi_1(\omega) = \sum_{k\in I_1}\sum_{j\in J_k^1}\chi_j^{(k)}(\omega)\,\left((\lambda_j^{(k)})^p - 1\right) a_j^{(k)}x_j^{(k)}.$$

There exists $\eta_2 > 0$ such that if p is in the set D_2 of integers such that $|(\lambda_j^{(k)})^p - 1| < \eta_2$ for every $k \in I_1$ and every $j \in J_k^1$, then for almost every $\omega \in \Omega$

$$||T^p\Phi_1(\omega) - \Phi_1(\omega)|| < \kappa_2.$$

Observe that this set D_2 has bounded gaps. Indeed there exists a set D_2' of positive density such that for any $k \in I_1$ and any $j \in J_k^1$, and for any $p \in D_2'$, $|(\lambda_j^{(k)})^p - 1| < \eta_2/2$. Then for any pair (p, p') of elements of D_2' we have

$$|(\lambda_j^{(k)})^{p-p'} - 1| \le |(\lambda_j^{(k)})^p - 1| + |(\lambda_j^{(k)})^p - 1| < \eta_2.$$

Thus $(D_2' - D_2') \cap \mathbb{N}$ is contained in D_2. Since D_2' has positive lower density, $(D_2' - D_2') \cap \mathbb{N}$ has bounded gaps by a result of [19]. Hence D_2 has bounded gaps too. Let r_2 be such that any interval of $\mathbb{N}$ of length strictly larger than r_2 contains an element of D_2.

Now consider the set $E_2 = \{p \ge 0 \,;\, T^p(B(0, 2^{-2})) \cap V_2 \neq \varnothing\}$. Since T is hypercyclic, E_2 is non-empty. But we can actually say more about E_2: as T is hypercyclic and has spanning unimodular eigenvectors, T satisfies the Hypercyclicity Criterion by [16]. Hence for any $r \ge 1$, the operator T_r which is a direct sum of r copies of T on the direct sum X_r of r

copies of X is hypercyclic. In particular T_{r_2+1} is topologically transitive, which implies that there exists an integer p such that $T^p(B(0,2^{-2}))\cap V_2\neq\varnothing$, $T^p(B(0,2^{-2}))\cap T^{-1}(V_2)\neq\varnothing,\ldots,T^p(B(0,2^{-2}))\cap T^{-r_2}(V_2)\neq\varnothing$. In other words $p,p+1,\ldots,p+r_2$ belong to E_2. Hence $E_2\cap D_2$ is non-empty. Let $p_2\in E_2\cap D_2$:

$$||T^{p_2}\Phi_1(\omega)-\Phi_1(\omega)||<\kappa_2 \quad \text{for almost every } \omega\in\Omega,$$

and

$$T^{p_2}(B(0,2^{-2}))\cap V_2\neq\varnothing.$$

Let $F_1=\{\lambda_j^{(k)}\ ;\ k\in I_1, j\in J_k^1\}$ be the set of eigenvalues which appear in Step 1 of the construction, and $A_{F_1}=\{k\geq 1\ ;\ \lambda_k\notin F_1\}$. As $\mathrm{sp}[x_k\ ;\ k\in A_{F_1}]$ is dense in X, there exists a vector u_2 which is a finite linear combination of vectors x_k, $k\in A_{F_1}$, such that $T^{p_2}u_2\in V_2$. We write

$$u_2=\sum_{k\in I_2}\alpha_k x_k,$$

where I_2 is a suitably chosen interval of $\mathbb{N}$. Let $M_2>0$ be such that for every family $(\beta_k)_{k\in I_2}$ of complex numbers,

$$||\sum_{k\in I_2}\beta_k x_k||\leq M_2\left(\sum_{k\in I_2}|\beta_k|^2\right)^{\frac{1}{2}}.$$

Then as in Step 1 we decompose each α_k, $k\in I_2$, as

$$\alpha_k=\sum_{j\in J_k^2}a_j^{(k)},$$

where

$$\sum_{k\in I_2}\left(\sum_{j\in J_k^{(2)}}|a_j^{(k)}|^2\right)^{\frac{1}{2}}<\delta_2$$

and δ_2 is a very small positive number, determined later on in the construction. Thus

$$u_2=\sum_{k\in I_2}\left(\sum_{j\in J_k^2}a_j^{(k)}\right)x_k.$$

For any $\gamma_2>0$, there exists a family $x_j^{(k)}$, $k\in I_2$, $j\in J_k^2$ of elements of the set $\{x_n\ ;\ n\geq 1\}$ such that $||x_k-x_j^{(k)}||<\gamma_2$ for any $k\in I_2$ and $j\in J_k^2$ and the eigenvalues $\lambda_j^{(k)}$ associated to the eigenvectors $x_j^{(k)}$ are all distinct and distinct from the elements of F_1 (i.e. the eigenvalues involved at Step 1 of the construction). Hence all the arguments $\theta_j^{(k)}$ of the eigenvalues $\lambda_j^{(k)}=e^{2i\pi\theta_j^{(k)}}$, $k\in I_1$ and $j\in J_k^1$, $k\in I_2$ and $j\in J_k^2$, form a $\mathbb{Q}$-independent sequence of irrational numbers. Set

$$v_2=\sum_{k\in I_2}\left(\sum_{j\in J_k^2}a_j^{(k)}\right)x_j^{(k)}.$$

If γ_2 is small enough, we have $T^{p_2}v_2 \in V_2$. Let $(\chi_j^{(k)})_{k\in I_2, j\in J_k^2}$ be a family of independent Steinhaus variables which are independent from the family $(\chi_j^{(k)})_{k\in I_1, j\in J_k^1}$, and set

$$\Phi_2(\omega) = \sum_{k\in I_2}\sum_{j\in J_k^2} \chi_j^{(k)}(\omega)\, a_j^{(k)} x_j^{(k)}.$$

The same reasoning as in Step 1 shows that if we take first δ_2 very small with respect to M_2, and then γ_2 very small with respect to δ_2, we can ensure that $\mathbb{E}||\Phi_2(\omega)||$ is as small as we want, namely that

$$\mathbb{E}||\Phi_2(\omega)|| < \frac{4^{-2}}{||T||^{\pi_1}}.$$

We are now going to show that there exists a finite family $\mathcal{P}_2$ of integers such that for almost every $\omega \in \Omega$, there exists $p_2(\omega) \in \mathcal{P}_2$ such that

$$T^{p_2(\omega)}(\Phi_1(\omega) + \Phi_2(\omega)) - \Phi_1(\omega) \in U_2.$$

Indeed for any $p \geq 0$ we have

$$\begin{aligned} T^p(\Phi_1(\omega) + \Phi_2(\omega)) - \Phi_1(\omega) - v_2 \quad &= \quad \sum_{k\in I_1}\sum_{j\in J_k^1} \chi_j^{(k)}(\omega) \, \left((\lambda_j^{(k)})^p - 1 \right) \, a_j^{(k)} x_j^{(k)} \\ &+ \quad \sum_{k\in I_2}\sum_{j\in J_k^2} \left(\chi_j^{(k)}(\omega) \, (\lambda_j^{(k)})^p - 1 \right) \, a_j^{(k)} x_j^{(k)}. \end{aligned}$$

Let $\eta_2 > 0$. By the irrationality and the $\mathbb{Q}$-independence of the arguments of all the $\lambda_j^{(k)}$ involved in the expression above, there exists a finite family $\mathcal{Q}_2$ of integers such that for almost every $\omega \in \Omega$ there exists an integer $p(\omega) \in \mathcal{Q}_2$ such that

– for every $k \in I_1$ and $j \in J_k^1$, $|(\lambda_j^{(k)})^{p(\omega)} - 1| < \eta_2$,

and

– for every $k \in I_2$ and $j \in J_k^2$, $|(\lambda_j^{(k)})^{p(\omega)} - \overline{\chi_j^{(k)}(\omega)}| < \eta_2$.

Thus if η_2 is small enough,

$$||T^{p(\omega)}(\Phi_1(\omega) + \Phi_2(\omega)) - \Phi_1(\omega) - v_2|| < \frac{\kappa_2}{||T||^{p_2}}.$$

Then

$$||T^{p(\omega)+p_2}(\Phi_1(\omega) + \Phi_2(\omega)) - T^{p_2}\Phi_1(\omega) - T^{p_2}v_2|| < \kappa_2.$$

But

$$||T^{p_2}\Phi_1(\omega) - \Phi_1(\omega)|| < \kappa_2,$$

so that

$$||T^{p(\omega)+p_2}(\Phi_1(\omega) + \Phi_2(\omega)) - \Phi_1(\omega) - v_2|| < 2\kappa_2.$$

Hence if $\mathcal{P}_2 = p_2 + \mathcal{Q}_2$, using the fact that $T^{p_2}v_2 \in V_2$ and $V_2 + B(0, 2\kappa_2) \subseteq U_2$, we get that for almost every $\omega \in \Omega$ there exists $p_2(\omega) \in \mathcal{P}_2$ such that

$$T^{p_2}(\Phi_1(\omega) + \Phi_2(\omega)) - \Phi_1(\omega) \in U_2.$$

Let π_2 denote the maximum of the set $\mathcal{P}_2$.

Step n: Continuing in this way, we construct at step n a random Steinhaus function

$$\Phi_n(\omega) = \sum_{k \in I_n} \sum_{j \in J_k^n} \chi_j^{(k)}(\omega)\, a_j^{(k)} x_j^{(k)}$$

such that

• we have

$$\mathbb{E}||\Phi_n(\omega)|| < \frac{4^{-n}}{||T||^{\max(\pi_1,\ldots,\pi_{n-1})}} \quad \text{in particular} \quad \mathbb{E}||\Phi_n(\omega)|| < 4^{-n}$$

• there exists a finite family $\mathcal{P}_n$ of integers such that for almost every $\omega \in \Omega$, there exists $p_n(\omega) \in \mathcal{P}_n$ such that

$$T^{p_n(\omega)}\left(\Phi_1(\omega) + \Phi_2(\omega) + \ldots + \Phi_n(\omega)\right) - \left(\Phi_1(\omega) + \ldots + \Phi_{n-1}(\omega)\right) \in U_n.$$

We denote by π_n the maximum of the set $\mathcal{P}_n$.

All the Steinhaus variables $\chi_j^{(k)}$, $k \in I_m$, $j \in J_k^m$ with $m \le n$ are independent, and the numbers $p_n(\omega)$ depend only on the construction until step n. In other words, p_n is $\mathcal{F}_n$-measurable, where $\mathcal{F}_n$ denotes the σ-algebra generated by the variables $\chi_j^{(k)}$, $k \in I_m$, $j \in J_k^m$, $m \le n$.

Construction of the invariant measure: We are now ready to construct our function Φ. Set

$$\Phi(\omega) = \sum_{n \ge 1} \Phi_n(\omega) = \sum_{n \ge 1} \left(\sum_{k \in I_n} \sum_{j \in J_k^n} \chi_j^{(k)}(\omega)\, a_j^{(k)} x_j^{(k)} \right)$$

Since

$$\mathbb{E}||\Phi(\omega)|| \le \sum_{n \ge 1} \mathbb{E}||\Phi_n(\omega)|| \le \sum_{n \ge 1} 4^{-n} < +\infty,$$

the series of Steinhaus variables written above has a subsequence of partial sums which converges in $L^1(\Omega, \mathcal{F}, \mathbb{P}; X)$, and hence by Lévy's inequalities the series defining Φ converges almost everywhere.

Recall that if we define m by $m(A) = \mathbb{P}(\Phi \in A)$ for any Borel subset A of X, m is T-invariant since all the vectors $x_j^{(k)}$ are unimodular eigenvectors for T. We are going to show that $\Phi(\omega)$ is hypercyclic for T for almost every $\omega \in \Omega$, and this will conclude the proof of Theorem 2.3.

For almost every $\omega \in \Omega$ we can write for every $n \ge 1$

$$\begin{aligned} T^{p_n(\omega)}\Phi(\omega) - \Phi(\omega) &= \left(T^{p_n(\omega)} \left(\sum_{m \le n} \Phi_m(\omega) \right) - \sum_{m < n} \Phi_m(\omega) \right) \\ &+ T^{p_n(\omega)} \left(\sum_{m > n} \Phi_m(\omega) \right) - \sum_{m \ge n} \Phi_m(\omega). \end{aligned}$$

We know that for almost every $\omega \in \Omega$, the first term in this expression belongs to U_n. So we have to estimate the second and third terms. Let us begin with the third one:

$$\mathbb{E}||\sum_{m \ge n} \Phi_m(\omega)|| \le \sum_{m \ge n} 4^{-m} = \frac{4}{3} 4^{-n}.$$

By Markov's inequality

$$\mathbb{P}\left(||\sum_{m\geq n}\Phi_m(\omega)||>2^{-n}\right)\leq\frac{4}{3}\,2^{-n},\quad\text{i.e.}\quad\mathbb{P}\left(||\sum_{m\geq n}\Phi_m(\omega)||\leq 2^{-n}\right)\geq 1-\frac{4}{3}\,2^{-n}.$$

Hence the third term in the display above is small with large probability. As for the second term, we have

$$\begin{aligned}\mathbb{E}\left|\left|\sum_{m>n}T^{p_n(\omega)}\Phi_m(\omega)\right|\right| &\leq \sum_{m>n}\mathbb{E}\left|\left|T^{p_n(\omega)}\Phi_m(\omega)\right|\right|\leq\sum_{m>n}\mathbb{E}(||T||^{p_n(\omega)}.||\Phi_m(\omega)||)\\ &\leq \sum_{m>n}||T||^{\pi_n}\mathbb{E}||\Phi_m(\omega)||\end{aligned}$$

since $\pi_n=\sup\{p_n(\omega)\ ;\ \omega\in\Omega\}$. Now since $m\geq n+1$,

$$\mathbb{E}||\Phi_m(\omega)||\leq\frac{4^{-m}}{||T||^{\max(\pi_1,\ldots,\pi_{m-1})}}\leq\frac{4^{-m}}{||T||^{\pi_n}}$$

so that

$$\mathbb{E}\left|\left|\sum_{m>n}T^{p_n(\omega)}\Phi_m(\omega)\right|\right|\leq\sum_{m>n}4^{-m}=\frac{1}{3}4^{-n}.$$

Thus

$$\mathbb{P}\left(||\sum_{m>n}T^{p_n(\omega)}\Phi_m(\omega)||\leq 2^{-n}\right)\geq 1-\frac{1}{3}\,2^{-n}.$$

Putting everything together yields that for every $n\geq 1$,

$$\mathbb{P}\left(T^{p_n(\omega)}\Phi(\omega)-\Phi(\omega)\in U_n+B(0,2^{-(n-1)})\right)\geq 1-\frac{5}{3}\,2^{-n}.$$

We are now done: let U be any non-empty open subset of X, and $(n_l)_{l\geq 1}$ a sequence of integers such that $U_{n_l}+B(0,2^{-(n_l-1)})\subseteq U$. Let $A_{n_l}=\{\omega\in\Omega\ ;\ T^{p_{n_l}(\omega)}\Phi(\omega)-\Phi(\omega)\in U\}$: we have seen that $\mathbb{P}(A_{n_l})\geq 1-\frac{5}{3}\,2^{-n_l}$. If

$$A=\{\omega\in\Omega\ ;\ \text{there exists } l\geq 1 \text{ such that } T^{p_{n_l}(\omega)}\Phi(\omega)-\Phi(\omega)\in U\}=\bigcup_{l\geq 1}A_{n_l},$$

then $\mathbb{P}(A)\geq\sup_{l\geq 1}\mathbb{P}(A_{n_l})$ and thus $\mathbb{P}(A)=1$. This being true for any non empty open subset of X, by considering a countable basis of open subsets of X we obtain that for almost every $\omega\in\Omega$ the set $\{T^p\Phi(\omega)-\Phi(\omega)\ ;\ p\geq 1\}$ is dense in X. This means that $\Phi(\omega)$ is hypercyclic for almost every $\omega\in\Omega$, and this concludes the proof of Theorem 2.3.

Remark 3.1. Suppose that X is a Hilbert space, and for $n\geq 1$ and $k\in I_n$, $j\in J_k^n$, denote by $y_j^{(k)}$ the vector $y_j^{(k)}=a_j^{(k)}x_j^{(k)}$. Then $\sum_{n\geq 1}\sum_{k\in I_n}\sum_{j\in J_k^n}||y_j^{(k)}||^2$ is finite, and the proof above shows that the set of finite linear combinations $\sum_n\sum_{k\in I_n}\sum_{j\in J_k^n}c_j^{(k)}y_j^{(k)}$ where $|c_j^{(k)}|=1$ is dense in X. This can be related to the following result of [2], which gives conditions on a sequence (x_n) of vectors implying that the set of its linear combinations with unimodular coefficients is dense in X: if $\sum||x_n||^2$ is finite and $\sum|\langle x,x_n\rangle|=+\infty$ for any non-zero x in X, then $\{\sum c_nx_n\ ;\ |c_n|=1\}$ is dense in X. See [6] for an elegant proof of this fact. The simplest way to construct such a sequence (x_n) is to take $x_n = \frac{1}{n}x_0$ with $x_0 \neq 0$ for a large number of n, let us say $n < n_1$, then $x_n = \frac{1}{n}x_{n_1}$ for a large number of n's with another suitable x_{n_1}, etc... A look at the proof of Theorem 2.3 shows that this is exactly what we do there: we "duplicate" each vector x_k in a family of eigenvectors $x_j^{(k)}$, $j \in J_k^n$, associated to eigenvalues which are very close to the initial one but all distinct, with multiplicative coefficients $a_j^{(k)}$, and $\sum_{j\in J_k} |a_j^{(k)}|^2$ small but $\sum_{j\in J_k} |a_j^{(k)}|$ large.

4. Proof of Theorem 1.4: frequent hypercyclicity of operators with perfectly spanning unimodular eigenvectors

In order to prove Theorem 1.4, it remains to show that assumption (H) is satisfied when the unimodular eigenvectors of T are perfectly spanning. We are going to show that this follows from the (seemingly) weaker assumption that whenever D is a countable subset of $\mathbb{T}$, sp$[\ker(T-\lambda)\ ;\ \lambda \in \mathbb{T}\setminus D]$ is dense in X. This assumption comes from the pioneering work of Flytzanis [13], where the ergodic theory of bounded operators on Hilbert spaces was first studied. We prove that this condition is equivalent to the property that T has perfectly spanning unimodular eigenvectors, and even to the stronger property that the unimodular eigenvectors of T can be parametrized via countably many continuous eigenvector fields. In the statement of Theorem 4.1, $S_X = \{x \in X\ ;\ ||x|| = 1\}$ denotes the unit sphere of X.

Theorem 4.1. *Suppose that T is a bounded operator on X. The following assertions are equivalent:*

1. *for any countable subset D of $\mathbb{T}$, sp$[\ker(T-\lambda)\ ;\ \lambda \in \mathbb{T}\setminus D]$ is dense in X;*
2. *T has perfectly spanning unimodular eigenvectors;*
3. *there exists a sequence $(K_i)_{i\geq 1}$ of subsets of $\mathbb{T}$ which are homeomorphic to the Cantor set 2^ω and a sequence $(E_i)_{i\geq 1}$ of continuous functions $E_i : K_i \longrightarrow S_X$ such that:*
   - *for any $i \geq 1$ and any $\lambda \in K_i$, $TE_i(\lambda) = \lambda E_i(\lambda)$;*
   - *sp$[E_i(\lambda)\ ;\ i \geq 1, \lambda \in K_i]$ is dense in X.*

Assuming for the moment that Theorem 4.1 is proved, let us deduce Theorem 1.4 from it.

Proof of Theorem 1.4. As T has perfectly spanning unimodular vectors, assertion (3) of Theorem 4.1 above is satisfied. Since for each $i \geq 1$ the set K_i is a Cantor-like subset of $\mathbb{T}$, we can construct a family of sequences of unimodular numbers $(\lambda_n^{(i)})_{n\geq 1}$, $i \geq 1$, which have the following properties:
– for each $i \geq 1$, the set $\{\lambda_n^{(i)}\ ;\ n \geq 1\}$ is dense in K_i;
– all the numbers $\lambda_n^{(i)}$, $i, n \geq 1$, are distinct;
– for any finite family $(\lambda_{n_1}^{(i_1)}, \ldots, \lambda_{n_r}^{(i_r)})$ consisting of distinct elements, the arguments of these unimodular numbers consist of $\mathbb{Q}$-independent irrational numbers.

For each $i, n \geq 1$, let $x_n^{(i)} = E_i(\lambda_n^{(i)})$ denote the associated eigenvector via the eigenvector field E_i.

It is now not difficult to see that the family $\{x_n^{(i)}\ ;\ i, n \geq 1\}$ satisfies the requirements of assumption (H). First of all assertion (1) is true by construction. Then for any $i \geq 1$, any

$\lambda \in K_i$ can be written as a limit of a sequence of elements of the set $\{\lambda_n^{(i)} ; n \geq 1\}$. The continuity of the function E_i then implies that $E_i(\lambda)$ can be written as a limit of a sequence of elements of the set of vectors $\{x_n^{(i)} ; n \geq 1\}$. Since the vectors $E_i(\lambda) ; i \geq 1, \lambda \in K_i$, span a dense subspace of X, it follows that $\mathrm{sp}[x_n^{(i)} ; i, n \geq 1]$ is dense in X, hence that assertion (2) of assumption (H) is satisfied. Assertion (3) is again a consequence of the continuity of the eigenvector fields E_i: for any $i, n \geq 1$, $\lambda_n^{(i)}$ can be written as the limit of a sequence of distinct elements $(\lambda_{n_k}^{(i)})_{k \geq 1}$, which can in particular be chosen so as to avoid a given finite subset F of $\mathbb{T}$. Then $x_n^{(i)}$ is the limit of the sequence $(x_{n_k}^{(i)})_{k \geq 1}$.
So T satisfies assumption (H). Since T is hypercyclic [3], it follows from Theorem 2.3 that T is frequently hypercyclic, and thus Theorem 1.4 is proved. □

It remains now to prove Theorem 4.1.

Proof of Theorem 4.1. (3) $\Longrightarrow$ (2) is easy: for any $i \geq 1$ let σ_i be a continuous probability measure supported on the compact set K_i, and let σ be the probability measure $\sigma = \sum_{i \geq 1} 2^{-i} \sigma_i$. Then σ is continuous on $\mathbb{T}$. If B is any σ-measurable subset of $\mathbb{T}$ such that $\sigma(B) = 1$, then $\sigma_i(B) = 1$ for any $i \geq 1$. Suppose now that $x^* \in X^*$ is a functional which vanishes on $E_i(\lambda)$ for any $i \geq 1$ and any $\lambda \in K_i \cap B$: since E_i is continuous on K_i, $\langle x^*, E_i(\lambda) \rangle = 0$ for any $\lambda \in K_i$, and thus $x^* = 0$. Hence the eigenvector fields E_i, $i \geq 1$, are perfectly spanning with respect to σ.
(2) $\Longrightarrow$ (1) is also clear: if the unimodular eigenvectors of T are perfectly spanning with respect to a continuous measure σ on $\mathbb{T}$, then $\sigma(D) = 0$ for any countable subset D of $\mathbb{T}$, so that (1) holds true.
(1) $\Longrightarrow$ (3) is the core of the proof of Theorem 4.1. Let

$$A = S_X \cap \left(\bigcup_{\lambda \in \mathbb{T}} \ker(T - \lambda) \right)$$

be the set of eigenvectors of T of norm 1 associated to unimodular eigenvalues. Since A is separable, there exists a countable basis $(\Omega_n)_{n \geq 1}$ of open subsets of A: $\Omega_n = A \cap U_n$, where U_n is open in X. Consider the set E of integers $n \geq 1$ having the following property: the set of eigenvalues λ such that Ω_n contains an element of $S_X \cap \ker(T - \lambda)$ is at most countable. Then let Δ be the set of eigenvalues of T such that there exists an $n \in E$ such that $S_X \cap \ker(T - \lambda) \cap \Omega_n$ is non-empty. In other words λ belongs to Δ if and only if there is an eigenvector associated to λ belonging to an Ω_n containing only eigenvectors associated to a countable family of eigenvalues:

$$\Delta = \bigcup_{n \in E} \{\lambda \in \mathbb{T} ; S_X \cap \ker(T - \lambda) \cap \Omega_n \neq \varnothing\}.$$

By the definition of E, Δ is at most countable. Let $\lambda \in \mathbb{T} \setminus \Delta$ be an eigenvalue of T, and let x be an associated eigenvector of norm 1. Let V be an open neighborhood of x in A, and let $p \geq 1$ be such that $\Omega_p \subseteq V$ and $x \in \Omega_p$. It is impossible that $p \in E$: if $p \in E$, then $x \in \ker(T - \lambda) \cap S_X \cap \Omega_p$ which is hence non-empty, and thus λ belongs to Δ, which is contrary to our assumption. Hence Ω_p, and so V, contain eigenvectors of norm 1 associated to an uncountable family of unimodular eigenvalues. Let us summarize this as follows: the set

$$\Omega = S_X \cap \left(\bigcup_{\lambda \in \mathbb{T}\setminus\Delta} \ker(T-\lambda) \right)$$

consists of eigenvectors of T of norm 1 such that any neighborhood of a vector $x \in \Omega$ contains eigenvectors of norm 1 associated to uncountably many eigenvalues, in particular eigenvectors of norm 1 associated to uncountably many eigenvalues not belonging to Δ.

Since Δ is countable, $\text{sp}[\ker(T-\lambda)\ ;\ \lambda \in \mathbb{T}\setminus\Delta] = \text{sp}[\Omega]$ is dense in X by assumption (1). We choose a sequence $(u_i)_{i\geq 1}$ of vectors of Ω which is dense in Ω and which is such that the corresponding eigenvalues $\lambda_i, i \geq 1$ are all distinct and belong to $\mathbb{T}\setminus\Delta$. In particular the vectors u_i span a dense subspace of X. Let us now fix $i \geq 1$ and construct K_i and E_i. Let $s = (s_1, \ldots, s_n) \in 2^{<\omega}$ be a finite sequence of 0's and 1's. We associate to each such $s \in 2^{<\omega}$ an eigenvalue $\lambda_s \in \{\lambda_j\ ;\ j \geq 1\}$ and an eigenvector $u_s \in \{u_j\ ;\ j \geq 1\}$ with $Tu_s = \lambda_s u_s$ in the following way:

• Step 1: we start from $u_{(0)} = u_i$ and $\lambda_{(0)} = \lambda_i$. Let $n \neq i$ be such that $||u_n - u_{(0)}|| < 2^{-1}$ and $|\lambda_n - \lambda_{(0)}| < 2^{-1}$ (remark that if $||u_n - u_{(0)}|| < 2^{-1}$ is very small, $|\lambda_n - \lambda_{(0)}| < 2^{-1}$ is automatically very small too). In particular $\lambda_n \neq \lambda_{(0)}$. We set $u_{(1)} = u_n$ and $\lambda_{(1)} = \lambda_n$.

• Step 2: we take $u_{(0,0)} = u_{(0)}$, $\lambda_{(0,0)} = \lambda_{(0)}$, and then take $u_{(0,1)}$ in the set $\{u_j\ ;\ j \geq 1\}$ and $\lambda_{(0,1)}$ in the set $\{\lambda_j\ ;\ j \geq 1\}$ so that

$$||u_{(0,0)} - u_{(0,1)}|| < 2^{-1}||u_{(0)} - u_{(1)}|| < 2^{-2} \text{ and } |\lambda_{(0,0)} - \lambda_{(0,1)}| < 2^{-1}|\lambda_{(0)} - \lambda_{(1)}| < 2^{-2},$$

with $\lambda_{(0,1)} \neq \lambda_{(0,0)}$. In the same way we take $u_{(1,0)} = u_{(1)}$ and $\lambda_{(1,0)} = \lambda_{(1)}$ and then choose $u_{(1,1)}$ and $\lambda_{(1,1)}$ very close to $u_{(1,0)}$ and $\lambda_{(1,0)}$ respectively so that

$$||u_{(1,0)} - u_{(1,1)}|| < 2^{-1}||u_{(0)} - u_{(1)}|| < 2^{-2} \text{ and } |\lambda_{(1,0)} - \lambda_{(1,1)}| < 2^{-1}|\lambda_{(0)} - \lambda_{(1)}| < 2^{-2},$$

with $\lambda_{(1,1)}$ not belonging to the set $\{\lambda_{(0,0)}, \lambda_{(0,1)}, \lambda_{(1,0)}\}$.

• Step n: we take $u_{(s_1,\ldots,s_{n-1},0)} = u_{(s_1,\ldots,s_{n-1})}$ and $\lambda_{(s_1,\ldots,s_{n-1},0)} = \lambda_{(s_1,\ldots,s_{n-1})}$, and then $u_{(s_1,\ldots,s_{n-1},1)}$ very close to $u_{(s_1,\ldots,s_{n-1},0)}$ and $\lambda_{(s_1,\ldots,s_{n-1},1)}$ very close to $\lambda_{(s_1,\ldots,s_{n-1},0)}$, so that

$$||u_{(s_1,\ldots,s_{n-1},0)} - u_{(s_1,\ldots,s_{n-1},1)}|| < 2^{-1}||u_{(s_1,\ldots,s_{n-2},0)} - u_{(s_1,\ldots,s_{n-2},1)}|| < 2^{-n}$$

and

$$|\lambda_{(s_1,\ldots,s_{n-1},0)} - \lambda_{(s_1,\ldots,s_{n-1},1)}| < 2^{-1}|\lambda_{(s_1,\ldots,s_{n-2},0)} - \lambda_{(s_1,\ldots,s_{n-2},1)}| < 2^{-n}.$$

We manage the construction in such a way that for all finite sequences $(s_1, \ldots, s_n)$ of 2^ω of length n, the numbers $\lambda_{(s_1,\ldots,s_n)}$ are distinct.

This defines λ_s and u_s for $s \in 2^{<\omega}$. If now $s = (s_1, s_2, \ldots) \in 2^\omega$ is an infinite sequence of 0's and 1's, we define $\lambda_s = \lim_{n\to+\infty} \lambda_{s_{|n}}$ and $u_s = \lim_{n\to+\infty} u_{s_{|n}}$, where $s_{|n} = (s_1, \ldots, s_n)$. These two limits do exist: indeed we have for any $n \geq 1$ that $|\lambda_{s_{|n-1}} - \lambda_{s_{|n}}| < 2^{-n}$ and $||u_{s_{|n-1}} - u_{s_{|n}}|| < 2^{-n}$.

Let $\phi_i : 2^\omega \to \mathbb{T}$ be the map defined by $\phi_i(s) = \lambda_s$. It is continuous and injective: if $s \neq s'$ are two distinct elements of 2^ω, and p is the smallest integer such that $s_n \neq s'_n$ for any

$n < p$, then for any $n \geq p$ we have

$$\begin{aligned}
|\lambda_{(s_1,\ldots,s_n)} - \lambda_{(s'_1,\ldots,s'_n)}| &\geq |\lambda_{(s_1,\ldots,s_{p-1},s_p)} - \lambda_{(s_1,\ldots,s_{p-1},s'_p)}| \\
&- \sum_{m=p+1}^{n} |\lambda_{(s_1,\ldots,s_m)} - \lambda_{(s_1,\ldots,s_{m-1})}| \\
&- \sum_{m=p+1}^{n} |\lambda_{(s'_1,\ldots,s'_m)} - \lambda_{(s'_1,\ldots,s'_{m-1})}| \\
&\geq |\lambda_{(s_1,\ldots,s_{p-1},s_p)} - \lambda_{(s_1,\ldots,s_{p-1},s'_p)}| \\
&- \left(\sum_{m=p+1}^{n} 2^{-(m-p)}\right) |\lambda_{(s_1,\ldots,s_{p-1},s_p)} - \lambda_{(s_1,\ldots,s_{p-1},s'_p)}| \\
&\geq 2^{-1}|\lambda_{(s_1,\ldots,s_{p-1},s_p)} - \lambda_{(s_1,\ldots,s_{p-1},s'_p)}| = \delta_p > 0.
\end{aligned}$$

It follows that $|\lambda_s - \lambda_{s'}| \geq \delta_p > 0$, hence that $\lambda_s \neq \lambda_{s'}$, and ϕ_i is injective. We set $K_i = \phi_i(2^\omega)$, and with this definition K_i is a compact set homeomorphic to the Cantor set 2^ω via the map ϕ_i. Let now $E_i : K_i \to X$ be defined by $E_i(\lambda_s) = u_s$: E_i can be written as $E_i = \psi_i \circ \phi_i^{-1}$, where $\psi_i : 2^\omega \to X$, $\psi_i(s) = u_s$. By the same argument as above ψ_i is continuous on 2^ω, and since ϕ_i is an homeomorphism from 2^ω onto K_i, E_i is a continuous map from K_i into X. Lastly $\phi_i((0,0,\ldots)) = \lambda_i$ belongs to K_i, and $E_i(\lambda_i) = u_i$ so that $\text{sp}[E_i(\lambda)\ ;\ i \geq 1, \lambda \in K_i]$ is dense in X. Thus assertion (3) is satisfied, and this finishes the proof of Theorem 4.1. □

The proof of Theorem 4.1 actually yields the following result, which gives a rather weak condition for an operator to have perfectly spanning unimodular eigenvectors:

Theorem 4.2. *Let X be a complex separable infinite-dimensional Banach space, and let T be a bounded operator on X. Suppose that there exists a sequence $(u_i)_{i\geq 1}$ of vectors of X having the following properties:*

(a) *for each $i \geq 1$, u_i is an eigenvector of T associated to an eigenvalue λ_i of T with $|\lambda_i| = 1$ and the λ_i's all distinct;*
(b) *$sp[u_i\ ;\ i \geq 1]$ is dense in X;*
(c) *for any $i \geq 1$ and any $\varepsilon > 0$, there exists an $n \neq i$ such that $||u_n - u_i|| < \varepsilon$.*

Then T has a perfectly spanning set of unimodular eigenvectors, and hence T is frequently hypercyclic.

In particular T is frequently hypercyclic as soon as the following assumption (H') holds true:

There exists a sequence $(x_n)_{n\geq 1}$ of eigenvectors of T, $Tx_n = \lambda_n x_n$, $|\lambda_n| = 1$, $||x_n|| = 1$, having the following properties:

(2) *$sp[x_n\ ;\ n \geq 1]$ is dense in X;*
(3) *for any finite subset F of $\sigma_p(T) \cap \mathbb{T}$ we have $\overline{\{x_n\ ;\ n \geq 1\}} = \overline{\{x_n\ ;\ n \in A_F\}}$, where $A_F = \{n \geq 0\ ;\ \lambda_n \notin F\}$.*

Asumption (H') is nothing else than Assumption (H) without its first condition (1). Observe that we have proved that Assumptions (H) and (H') are again both equivalent to the fact that T has perfectly spanning unimodular eigenvectors.

5. Ergodicity of operators with perfectly spanning unimodular eigenvectors

Although we now know that any operator on a separable Banach space with perfectly spanning unimodular eigenvectors is frequently hypercyclic, we still do not know whether such an operator admits a non-degenerate invariant Gaussian measure with respect to which it is ergodic. This question was mentioned in [5]. Some examples seem to point out that the answer to this question should be negative, but so far no counter-example has been constructed. In this context it is interesting to note the following:

Theorem 5.1. *If T is a bounded operator on X which has spanning unimodular eigenvectors, then T is not ergodic with respect to the invariant non-degenerate measure m constructed in the proof of Theorem 1.4. More generally, T will never be ergodic with respect to a measure associated to a random function*

$$\Phi(\omega)=\sum_{n=1}^{+\infty}\chi_n(\omega)x_n$$

where the x_n's are spanning eigenvectors of T associated to a family of unimodular eigenvalues λ_n and $(\chi_n)_{n\geq 1}$ a sequence of independent rotation-invariant variables such that $\mathbb{E}(\chi_n)=0$ and $\mathbb{E}(|\chi_n|^2)=1$.

These invariant measures are in a sense the "trivial" ones, i.e. the ones which can be constructed without any additional assumption on the eigenvectors of T (the existence of such an invariant measure does not even imply that T is hypercyclic). When the operator T has perfectly spanning unimodular eigenvectors with respect to a certain continuous measure σ on $\mathbb{T}$, the measures which are used in [3] and [5] to obtain ergodicity results are intrinsically different from these ones.

Proof. Let U_T denote the isometric operator defined on $L^2(X,\mathcal{B},m)$ by $U_Tf=f\circ T$, $f\in L^2(X,\mathcal{B},m)$. If x^* and y^* are two elements of X^*, they belong to $L^2(X,\mathcal{B},m)$. For any $n\geq 0$ we have

$$\begin{aligned}
\langle U_T^n|x^*|^2\,,\,|y^*|^2\rangle &= \int_X |\langle x^*,T^nx\rangle\,\langle y^*,x\rangle|^2dm(x)\\
&= \int_\Omega |\sum_{p\geq 0}\chi_p(\omega)\lambda_p^n\langle x^*,x_p\rangle\,.\,\sum_{q\geq 0}\chi_q(\omega)\langle y^*,x_q\rangle|^2d\mathbb{P}(\omega)\\
&= \sum_{p_1,p_2,q_1,q_2\geq 0} I_{p_1,p_2,q_1,q_2}\lambda_{p_1}^n\overline{\lambda}_{p_2}^n\langle x^*,x_{p_1}\rangle\overline{\langle x^*,x_{p_2}\rangle}\langle y^*,x_{q_1}\rangle\overline{\langle y^*,x_{q_2}\rangle}
\end{aligned}$$

where

$$I_{p_1,p_2,q_1,q_2}=\int_\Omega\chi_{p_1}(\omega)\overline{\chi_{p_2}(\omega)}\chi_{q_1}(\omega)\overline{\chi_{q_2}(\omega)}.$$

Now I_{p_1,p_2,q_1,q_2} is non-zero if and only if $p_1 = p_2$ and $q_1 = q_2$ or $p_1 = q_2$ and $p_2 = q_1$. Thus $\langle U_T^n |x^*|^2\,,\ |y^*|^2\rangle$ is equal to

$$\sum_{p_1,q_1\geq 0} |\langle x^*, x_{p_1}\rangle|^2 |\langle y^*, x_{q_1}\rangle|^2 + \sum_{p_1,p_2\geq 0} \lambda_{p_1}^n \overline{\lambda}_{p_2}^n \langle x^*, x_{p_1}\rangle \overline{\langle x^*, x_{p_2}\rangle \langle y^*, x_{p_1}\rangle} \langle y^*, x_{p_2}\rangle$$
$$= \sum_{p\geq 0} |\langle x^*, x_p\rangle|^2 \,.\, \sum_{p\geq 0} |\langle y^*, x_p\rangle|^2 + |\sum_{p\geq 0} \lambda_p^n \langle x^*, x_p\rangle \overline{\langle y^*, x_p\rangle}|^2.$$

Consider now the Cesàro sums

$$\frac{1}{N}\sum_{n=0}^{N-1} \langle U_T^n |x^*|^2\,,\ |y^*|^2\rangle = \sum_{p\geq 0} |\langle x^*, x_p\rangle|^2 \,.\, \sum_{p\geq 0} |\langle y^*, x_p\rangle|^2 + \frac{1}{N}\sum_{n=0}^{N-1} |\sum_{p\geq 0} \lambda_p^n \langle x^*, x_p\rangle \overline{\langle y^*, x_p\rangle}|^2.$$

If T were ergodic with respect to m, this quantity would tend to

$$\int_X |\langle x^*, x\rangle|^2 dm(x) \,.\, \int_X |\langle y^*, x\rangle|^2 dm(x) = \sum_{p\geq 0} |\langle x^*, x_p\rangle|^2 \,.\, \sum_{p\geq 0} |\langle y^*, x_p\rangle|^2$$

as N tends to infinity (see for instance [21] for this standard characterization of ergodicity). Hence

$$\frac{1}{N}\sum_{n=0}^{N-1} |\sum_{p\geq 0} \lambda_p^n \langle x^*, x_p\rangle \overline{\langle y^*, x_p\rangle}|^2$$

would tend to zero as N tends to infinity. This would imply that

$$|\sum_{p\geq 0} \lambda_p^n \langle x^*, x_p\rangle \overline{\langle y^*, x_p\rangle}|^2$$

tends to zero as n tends to infinity along a set D of density 1 (see again [21]). We are going to show that it is not the case if x^* is such that $|\langle x^*, x_0\rangle|^2 = \varepsilon > 0$ and $y^* = x^*$. Since the series $\sum_{p\geq 0} |\langle x^*, x_p\rangle|^2$ is convergent, there exists a p_0 such that for any $n \geq 0$

$$|\sum_{p>p_0} \lambda_p^n |\langle x^*, x_p\rangle|^2| < \varepsilon.$$

Hence

$$|\sum_{p\geq 0} \lambda_p^n |\langle x^*, x_p\rangle|^2| \geq |\sum_{p\leq p_0} \lambda_p^n |\langle x^*, x_p\rangle|^2| - \varepsilon$$

for any $n \geq 0$. Now for any $\delta > 0$ the set $D_\delta = \{n \geq 0 \ ; \text{for every } p \leq p_0\ |\lambda_p^n - 1| < \delta\}$ has positive lower density d_δ. For any $n \in D_\delta$,

$$|\sum_{p\leq p_0} \lambda_p^n |\langle x^*, x_p\rangle|^2| \geq \sum_{p\leq p_0} |\langle x^*, x_p\rangle|^2 - \delta \sum_{p\leq p_0} |\langle x^*, x_p\rangle|^2$$

so that if δ is small enough,

$$|\sum_{p\geq 0} \lambda_p^n |\langle x^*, x_p\rangle|^2|^2 \geq \sum_{p\leq p_0} |\langle x^*, x_p\rangle|^2 - 2\varepsilon \geq |\langle x^*, x_0\rangle|^2 - 2\varepsilon \geq \varepsilon.$$

Hence

$$\frac{1}{N}\#\{n \leq N \ ; \ |\sum_{p\geq 0} \lambda_p^n |\langle x^*, x_p\rangle|^2|^2 \geq \varepsilon\} \geq \frac{1}{2} d_\delta$$

for N large enough, so that

$$\frac{1}{N}\#\{n \le N \; ; \; |\sum_{p\ge 0} \lambda_p^n |\langle x^*, x_p\rangle|^2|^2 < \varepsilon\} \le (1 - \frac{1}{2} d_\delta).$$

Thus

$$|\sum_{p\ge 0} \lambda_p^n \langle x^*, x_p\rangle \overline{\langle y^*, x_p\rangle}|^2$$

does not tend to zero along a set of density 1. This contradiction shows that T is not ergodic with respect to m. □

6. Open questions and remarks

6.1. **Hypercyclic operators with spanning unimodular eigenvectors.** Let T be a bounded hypercyclic operator on X whose eigenvectors associated to eigenvalues of modulus 1 span a dense subspace of X. It is still an open question to know whether such an operator must be frequently hypercyclic. If T is a chaotic operator (i.e. a hypercyclic operator which has a dense set of periodic points), then T falls into this category of operators: T is chaotic if and only if it is hypercyclic and its eigenvectors associated to eigenvalues which are n^{th} roots of 1 span a dense subspace of X. Thus the following question of [4] is still unanswered:

Question 6.1. [4] *Must a chaotic operator be frequently hypercyclic?*

It is an intriguing fact that all operators which are known to be hypercyclic and to have spanning unimodular eigenvectors have in fact perfectly spanning unimodular eigenvectors. Hence a natural way to prove (or disprove) the conjecture that all hypercyclic operators with spanning unimodular eigenvectors are frequently hypercyclic would be to answer the following question:

Question 6.2. *If T is a hypercyclic operator on X whose eigenvectors associated to eigenvalues of modulus 1 span a dense subspace of X, is is true that the unimodular eigenvectors of T are perfectly spanning?*

A related question of [13] is interesting in this context:

Question 6.3. [13] *Does there exist a bounded hypercyclic operator T on X whose unimodular point spectrum consists of a countable set $\{\lambda_n \; ; \; n \ge 1\}$, and which is such that the eigenvectors associated to the eigenvalues λ_n span a dense subspace of X?*

6.2. **Existence of frequently hypercyclic and chaotic operators on complex Banach spaces with an unconditional Schauder decomposition.** Let X be a complex separable infinite-dimensional Banach space X with an unconditional Schauder decomposition. This means that there exists a sequence $(X_n)_{n\ge 0}$ of closed subspaces of X such that any $x \in X$ can be written in a unique way as an unconditionnally convergent series $x = \sum_{n\ge 0} x_n$, where x_n belongs to X_n for any $n \ge 0$, and there is no loss of generality in supposing that all the subspaces X_n are infinite-dimensional. The main result of [12] states that there exists a bounded operator on X which is frequently hypercyclic and chaotic. This result was motivated by the fact that any infinite-dimensional Banach space supports a hypercyclic operator ([1], [8]), but that the corresponding statement for frequently hypercyclic operators is not true [18]: if X is a separable complex hereditarily indecomposable space (like the space of Gowers and Maurey [15]), then there is no frequently hypercyclic operator on X. Recall that a Banach space X is said to be hereditarily indecomposable if no pair of closed infinite-dimensional subspaces Y and Z of X form a topological direct sum $Y \oplus Z$. Also [9] there are no chaotic operators on a complex hereditarily indecomposable Banach space. The operators constructed in [12] are perturbations of a diagonal operator with unimodular coefficients by a vector-valued nuclear backward shift. In [12] we first construct such operators on a Hilbert space, prove that they have perfectly spanning unimodular eigenvectors, and then transfer them to our Banach space X. This result can also be obtained as a consequence of Theorem 1.4: the eigenvectors can be directly computed, and if at each step of the constuction we take the perturbation of the diagonal coefficients to be small enough, the operator satisfies assumption (H). The proof of [12] is, however, much simpler.

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