Functor homology

Birgit Richter

A biased overview
Lille, October 2012

Motivation

Why do we want functor homology interpretations?

Motivation

Why do we want functor homology interpretations?
Combinatorial features of the parametrizing category can be used in order to get extra structure, additional spectral sequences and more, e.g. as in the case of the Hodge decomposition of Hochschild homology.

Motivation

Why do we want functor homology interpretations?
Combinatorial features of the parametrizing category can be used in order to get extra structure, additional spectral sequences and more, e.g. as in the case of the Hodge decomposition of Hochschild homology.
Tor- and Ext-functors have universal properties, and this helps to obtain uniqueness results.

Motivation

Why do we want functor homology interpretations?
Combinatorial features of the parametrizing category can be used in order to get extra structure, additional spectral sequences and more, e.g. as in the case of the Hodge decomposition of Hochschild homology.
Tor- and Ext-functors have universal properties, and this helps to obtain uniqueness results.
In order to get functor homology interpretations we have to understand what something really is...

Some small categories

Let $\mathcal{C}$ be a small category, *i.e.*, $\mathcal{C}$ has a *set* of objects.

Some small categories

Let $\mathcal{C}$ be a small category, *i.e.*, $\mathcal{C}$ has a *set* of objects.
In our context, some important examples are:

Some small categories

Let $\mathcal{C}$ be a small category, *i.e.*, $\mathcal{C}$ has a *set* of objects.
In our context, some important examples are:

1. Fin, the small category of finite sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$. Morphisms are arbitrary functions of finite sets.

Some small categories

Let $\mathcal{C}$ be a small category, *i.e.*, $\mathcal{C}$ has a *set* of objects.
In our context, some important examples are:

1. Fin, the small category of finite sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$. Morphisms are arbitrary functions of finite sets.
2. Γ, the small category of finite pointed sets. Objects are again the sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$ but 0 is interpreted as a basepoint of $[n]$ and morphisms have to send 0 to 0.

Some small categories

Let $\mathcal{C}$ be a small category, *i.e.*, $\mathcal{C}$ has a *set* of objects.
In our context, some important examples are:

1. Fin, the small category of finite sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$. Morphisms are arbitrary functions of finite sets.
2. Γ, the small category of finite pointed sets. Objects are again the sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$ but 0 is interpreted as a basepoint of $[n]$ and morphisms have to send 0 to 0.
3. Δ, the small category of finite ordered sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$ considered as an ordered set with the standard ordering $0 < 1 < \ldots < n$. Morphisms are order preserving, *i.e.*, for $f \in \Delta([n], [m])$ and $i < j$ in $[n]$ we require $f(i) \leq f(j)$.

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Definition

1. A *left* $\mathcal{C}$*-module* is a functor $F\colon \mathcal{C} \to R$-mod.

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Definition

1. A *left* $\mathcal{C}$*-module* is a functor $F\colon \mathcal{C} \longrightarrow R\text{-mod}$.
2. A *right* $\mathcal{C}$*-module* is a functor $F\colon \mathcal{C}^{op} \longrightarrow R\text{-mod}$.

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Definition

1. A *left $\mathcal{C}$-module* is a functor $F\colon \mathcal{C} \to R$-mod.
2. A *right $\mathcal{C}$-module* is a functor $F\colon \mathcal{C}^{op} \to R$-mod.

We write $\mathcal{C}$-mod and mod-$\mathcal{C}$ for the corresponding categories of functors (with natural transformations as morphisms).

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Definition

1. A *left $\mathcal{C}$-module* is a functor $F\colon \mathcal{C} \to R$-mod.
2. A *right $\mathcal{C}$-module* is a functor $F\colon \mathcal{C}^{op} \to R$-mod.

We write $\mathcal{C}$-mod and mod-$\mathcal{C}$ for the corresponding categories of functors (with natural transformations as morphisms).
Examples:

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Definition

1. A *left $\mathcal{C}$-module* is a functor $F\colon \mathcal{C} \to R$-mod.
2. A *right $\mathcal{C}$-module* is a functor $F\colon \mathcal{C}^{op} \to R$-mod.

We write $\mathcal{C}$-mod and mod-$\mathcal{C}$ for the corresponding categories of functors (with natural transformations as morphisms).
Examples:
A simplicial R-module is a right Δ-module.

Left and right modules

Let $\mathcal{C}$ be a small category and let R be a commutative ring with unit.

Definition

1. A *left $\mathcal{C}$-module* is a functor $F\colon \mathcal{C} \to R$-mod.
2. A *right $\mathcal{C}$-module* is a functor $F\colon \mathcal{C}^{op} \to R$-mod.

We write $\mathcal{C}$-mod and mod-$\mathcal{C}$ for the corresponding categories of functors (with natural transformations as morphisms).
Examples:
A simplicial R-module is a right Δ-module.
A covariant functor $F\colon \Gamma \to R$-mod is a Γ-module.

Algebraic properties I

As the category of R-modules is abelian, so are $\mathcal{C}$-mod and mod-$\mathcal{C}$ for every small $\mathcal{C}$.

Algebraic properties I

As the category of R-modules is abelian, so are $\mathcal{C}$-mod and mod-$\mathcal{C}$ for every small $\mathcal{C}$.
Representable objects play an important role:

Algebraic properties I

As the category of R-modules is abelian, so are $\mathcal{C}$-mod and mod-$\mathcal{C}$ for every small $\mathcal{C}$.
Representable objects play an important role:
Consider a fixed object C in $\mathcal{C}$, then

$$R\{\mathcal{C}(C,-)\}\colon \mathcal{C} \to R\text{-mod}$$

and

$$R\{\mathcal{C}(-,C)\}\colon \mathcal{C}^{op} \to R\text{-mod}$$

are left- and right $\mathcal{C}$-modules.

Algebraic properties I

As the category of R-modules is abelian, so are $\mathcal{C}$-mod and mod-$\mathcal{C}$ for every small $\mathcal{C}$.
Representable objects play an important role:
Consider a fixed object C in $\mathcal{C}$, then

$$R\{\mathcal{C}(C,-)\}\colon \mathcal{C} \to R\text{-mod}$$

and

$$R\{\mathcal{C}(-,C)\}\colon \mathcal{C}^{op} \to R\text{-mod}$$

are left- and right $\mathcal{C}$-modules.
The Yoneda-Lemma implies that

$$\mathrm{Hom}_{\mathcal{C}\text{-mod}}(R\{\mathcal{C}(C,-)\}, F) \cong F(C)$$

for all $F \in \mathcal{C}$-mod and

Algebraic properties I

As the category of R-modules is abelian, so are $\mathcal{C}$-mod and mod-$\mathcal{C}$ for every small $\mathcal{C}$.
Representable objects play an important role:
Consider a fixed object C in $\mathcal{C}$, then

$$R\{\mathcal{C}(C,-)\}\colon \mathcal{C} \to R\text{-mod}$$

and

$$R\{\mathcal{C}(-,C)\}\colon \mathcal{C}^{op} \to R\text{-mod}$$

are left- and right $\mathcal{C}$-modules.
The Yoneda-Lemma implies that

$$\mathrm{Hom}_{\mathcal{C}\text{-mod}}(R\{\mathcal{C}(C,-)\}, F) \cong F(C)$$

for all $F \in \mathcal{C}$-mod and $\mathrm{Hom}_{\text{mod-}\mathcal{C}}(R\{\mathcal{C}(-,C)\}, G) \cong G(C)$ for all G in mod-$\mathcal{C}$.

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.
Examples

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.

Examples

$R\{\Gamma([0], -)\}$ is the constant functor.

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.
Examples
$R\{\Gamma([0],-)\}$ is the constant functor.
$R\{\Gamma(-,[0])\}$ is constant, too.

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.

Examples

$R\{\Gamma([0], -)\}$ is the constant functor.

$R\{\Gamma(-, [0])\}$ is constant, too.

$R\{\Gamma([n], [1])\}$ is the free R-module generated by subsets $S \subset \{1, \ldots, n\}$.

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.
Examples
$R\{\Gamma([0], -)\}$ is the constant functor.
$R\{\Gamma(-, [0])\}$ is constant, too.
$R\{\Gamma([n], [1])\}$ is the free R-module generated by subsets $S \subset \{1, \ldots, n\}$.
Let $t \colon \Gamma^{op} \to R$-mod be the functor with $t[n] = \mathrm{Hom}_{\mathsf{Sets}_*}([n], R)$.

Algebraic properties II

The representables are projective generators of $\mathcal{C}$-mod and mod-$\mathcal{C}$.
Examples
$R\{\Gamma([0], -)\}$ is the constant functor.
$R\{\Gamma(-, [0])\}$ is constant, too.
$R\{\Gamma([n], [1])\}$ is the free R-module generated by subsets $S \subset \{1, \ldots, n\}$.
Let $t \colon \Gamma^{op} \to R$-mod be the functor with $t[n] = \mathrm{Hom}_{\mathsf{Sets}_*}([n], R)$.
Then t can be written as the cokernel

$$R\{\Gamma(-, [2])\} \to R\{\Gamma(-, [1])\} \to t \to 0$$

where the map from $R\{\Gamma(-, [2])\}$ to $R\{\Gamma(-, [1])\}$ is induced by $f - p_1 - p_2$ with $f \colon [2] \to [1]$ being the fold map, sending $1, 2$ to 1 and $p_i(i) = 1$ and $p_i(j) = 0$ otherwise.

Tensor products

We can build a tensor product of a left $\mathcal{C}$-module with a right $\mathcal{C}$-module, analogous to the tensor product of modules over a non-commutative ring.

Tensor products

We can build a tensor product of a left $\mathcal{C}$-module with a right $\mathcal{C}$-module, analogous to the tensor product of modules over a non-commutative ring.

Definition For any left $\mathcal{C}$-module F and any right $\mathcal{C}$-module G we define

$$G \otimes_{\mathcal{C}} F := \bigoplus_{C \in \mathcal{C}} G(C) \otimes_R F(C) / \sim$$

where we have $x \otimes F(f)(y) \sim G(f)(x) \otimes y$ for all $f \colon C \to C'$, $x \in G(C')$, $y \in F(C)$.

Tensor products

We can build a tensor product of a left $\mathcal{C}$-module with a right $\mathcal{C}$-module, analogous to the tensor product of modules over a non-commutative ring.

Definition For any left $\mathcal{C}$-module F and any right $\mathcal{C}$-module G we define

$$G \otimes_{\mathcal{C}} F := \bigoplus_{C \in \mathcal{C}} G(C) \otimes_R F(C) / \sim$$

where we have $x \otimes F(f)(y) \sim G(f)(x) \otimes y$ for all $f\colon C \to C'$, $x \in G(C')$, $y \in F(C)$.

Proposition The natural evaluation map induces isomorphisms

$$R\{\mathcal{C}(-, C)\} \otimes_{\mathcal{C}} F \cong F(C), \quad G \otimes_{\mathcal{C}} R\{\mathcal{C}(C, -)\} \cong G(C).$$

Tor- and Ext-functors

$(-) \otimes_{\mathcal{C}} (-)$ is right exact in both variables.

Tor- and Ext-functors

$(-) \otimes_{\mathcal{C}} (-)$ is right exact in both variables.
Definition For $G \in$ mod-C and $F \in \mathcal{C}$-mod we define

$$\mathrm{Tor}_i^{\mathcal{C}}(G, F) := H_i(P_* \otimes_{\mathcal{C}} F)$$

where $\ldots \to P_1 \to P_0$ is a projective resolution of G in mod-$\mathcal{C}$.

Tor- and Ext-functors

$(-) \otimes_{\mathcal{C}} (-)$ is right exact in both variables.
Definition For $G \in$ mod-C and $F \in \mathcal{C}$-mod we define

$$\mathrm{Tor}_i^{\mathcal{C}}(G, F) := H_i(P_* \otimes_{\mathcal{C}} F)$$

where $\ldots \to P_1 \to P_0$ is a projective resolution of G in mod-$\mathcal{C}$.
Dually for two right $\mathcal{C}$-modules G and H, $\mathrm{Ext}^i_{\mathcal{C}}(G, H)$ is defined as

$$\mathrm{Ext}^i_{\mathcal{C}}(G, H) = H^i \mathrm{Hom}_{\mathrm{mod}-\mathcal{C}}(P_*, H).$$

Tor- and Ext-functors

$(-) \otimes_{\mathcal{C}} (-)$ is right exact in both variables.
Definition For $G \in$ mod-C and $F \in \mathcal{C}$-mod we define

$$\mathrm{Tor}_i^{\mathcal{C}}(G, F) := H_i(P_* \otimes_{\mathcal{C}} F)$$

where $\ldots \to P_1 \to P_0$ is a projective resolution of G in mod-$\mathcal{C}$.
Dually for two right $\mathcal{C}$-modules G and H, $\mathrm{Ext}^i_{\mathcal{C}}(G, H)$ is defined as

$$\mathrm{Ext}^i_{\mathcal{C}}(G, H) = H^i \mathrm{Hom}_{\mathrm{mod}-\mathcal{C}}(P_*, H).$$

We could equally well resolve F or H. In particular $\mathrm{Tor}_i^{\mathcal{C}}(G, F)$ vanishes for projective F and $i > 0$ and $\mathrm{Ext}^i_{\mathcal{C}}(G, H) = 0$ for injective H and $i > 0$.

Tor- and Ext-functors

$(-) \otimes_{\mathcal{C}} (-)$ is right exact in both variables.
Definition For $G \in$ mod-C and $F \in \mathcal{C}$-mod we define

$$\mathrm{Tor}_i^{\mathcal{C}}(G, F) := H_i(P_* \otimes_{\mathcal{C}} F)$$

where $\ldots \to P_1 \to P_0$ is a projective resolution of G in mod-$\mathcal{C}$.
Dually for two right $\mathcal{C}$-modules G and H, $\mathrm{Ext}^i_{\mathcal{C}}(G, H)$ is defined as

$$\mathrm{Ext}^i_{\mathcal{C}}(G, H) = H^i \mathrm{Hom}_{\mathrm{mod}-\mathcal{C}}(P_*, H).$$

We could equally well resolve F or H. In particular $\mathrm{Tor}_i^{\mathcal{C}}(G, F)$ vanishes for projective F and $i > 0$ and $\mathrm{Ext}^i_{\mathcal{C}}(G, H) = 0$ for injective H and $i > 0$.
More is true:

Universal property

If H_* is a functor from $\mathcal{C}$-mod to the category of graded R-modules such that

Universal property

If H_* is a functor from $\mathcal{C}$-mod to the category of graded R-modules such that

- $H_0(F)$ is canonically isomorphic to $G \otimes_C F$ for all $F \in \mathcal{C}$-mod,
- $H_*(-)$ maps short exact sequences of $\mathcal{C}$-modules to long exact sequences and
- $H_i(F) = 0$ for all projective F and $i > 0$,

Universal property

If H_* is a functor from $\mathcal{C}$-mod to the category of graded R-modules such that

- $H_0(F)$ is canonically isomorphic to $G \otimes_C F$ for all $F \in \mathcal{C}$-mod,
- $H_*(-)$ maps short exact sequences of $\mathcal{C}$-modules to long exact sequences and
- $H_i(F) = 0$ for all projective F and $i > 0$,

then $H_i(F) \cong \mathrm{Tor}_i^{\mathcal{C}}(G, F)$ for all F.

Hochschild homology

Assume that A is an associative and unital R-algebra whose underlying module is projective and let M be an A-bimodule.

Hochschild homology

Assume that A is an associative and unital R-algebra whose underlying module is projective and let M be an A-bimodule. Then the ith Hochschild homology group of A with coefficients in M, $HH_i(A; M)$ is defined as

Hochschild homology

Assume that A is an associative and unital R-algebra whose underlying module is projective and let M be an A-bimodule. Then the ith Hochschild homology group of A with coefficients in M, $HH_i(A; M)$ is defined as

$$H_i(\cdots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M).$$

Hochschild homology

Assume that A is an associative and unital R-algebra whose underlying module is projective and let M be an A-bimodule. Then the ith Hochschild homology group of A with coefficients in M, $HH_i(A;M)$ is defined as

$$H_i(\cdots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M).$$

Here, $b = \sum_{i=0}^{n}(-1)^i d_i$ where
$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n$ for $i < n$ and
$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1}$.

Hochschild homology

Assume that A is an associative and unital R-algebra whose underlying module is projective and let M be an A-bimodule. Then the ith Hochschild homology group of A with coefficients in M, $HH_i(A; M)$ is defined as

$$H_i(\cdots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M).$$

Here, $b = \sum_{i=0}^{n} (-1)^i d_i$ where
$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n$ for $i < n$ and
$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1}$.
Hochschild homology is André-Quillen homology for associative algebras up to a shift of degree. For a free algebra (a tensor algebra) it vanishes in degrees higher than one.

Via the simplicial category

Loday:

$$HH_n(A; M) = \mathrm{Tor}_n^{\Delta^{op}}(R, C(A; M))$$

Via the simplicial category

Loday:

$$HH_n(A; M) = \mathrm{Tor}_n^{\Delta^{op}}(R, C(A; M))$$

Here $C(A; M)$ is the simplicial R-module with $C_n(A; M) = M \otimes A^{\otimes n}$. As we assume that A is unital, we have degeneracy maps $s_i \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes (n+1)}$ given by inserting the unit of A.

Via the simplicial category

Loday:

$$HH_n(A; M) = \mathrm{Tor}_n^{\Delta^{op}}(R, C(A; M))$$

Here $C(A; M)$ is the simplicial R-module with $C_n(A; M) = M \otimes A^{\otimes n}$. As we assume that A is unital, we have degeneracy maps $s_i \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes(n+1)}$ given by inserting the unit of A.

R is short for the constant functor R.

Via the simplicial category

Loday:

$$HH_n(A; M) = \mathrm{Tor}_n^{\Delta^{op}}(R, C(A; M))$$

Here $C(A; M)$ is the simplicial R-module with $C_n(A; M) = M \otimes A^{\otimes n}$. As we assume that A is unital, we have degeneracy maps $s_i \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes(n+1)}$ given by inserting the unit of A.

R is short for the constant functor R.

Alternatively: Let $\mathbb{S}^1$ be the simplicial model of the unit circle with $\mathbb{S}^1_n = [n]$ and face and degeneracy maps d_i, s_i as follows

Via the simplicial category

Loday:

$$HH_n(A; M) = \mathrm{Tor}_n^{\Delta^{op}}(R, C(A; M))$$

Here $C(A; M)$ is the simplicial R-module with $C_n(A; M) = M \otimes A^{\otimes n}$. As we assume that A is unital, we have degeneracy maps $s_i \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes (n+1)}$ given by inserting the unit of A.

R is short for the constant functor R.

Alternatively: Let $\mathbb{S}^1$ be the simplicial model of the unit circle with $\mathbb{S}^1_n = [n]$ and face and degeneracy maps d_i, s_i as follows

$s_i \colon [n] \to [n+1]$ is the unique monotone injection that does not contain $i+1$.

Via the simplicial category

Loday:

$$HH_n(A; M) = \mathrm{Tor}_n^{\Delta^{op}}(R, C(A; M))$$

Here $C(A; M)$ is the simplicial R-module with $C_n(A; M) = M \otimes A^{\otimes n}$. As we assume that A is unital, we have degeneracy maps $s_i \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes (n+1)}$ given by inserting the unit of A.

R is short for the constant functor R.

Alternatively: Let $\mathbb{S}^1$ be the simplicial model of the unit circle with $\mathbb{S}^1_n = [n]$ and face and degeneracy maps d_i, s_i as follows

$s_i \colon [n] \to [n+1]$ is the unique monotone injection that does not contain $i+1$.

$d_i \colon [n] \to [n-1]$,

$$d_i(j) = \begin{cases} j, & j < i \\ i, & j = i < n, \quad (0, \quad j = i = n), \\ j-1, & j > i. \end{cases}$$

Via finite 'associative sets' I

$\mathbb{S}^1$:

$$[0] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; [1] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; [2] \quad \cdots$$

Via finite 'associative sets' I

$\mathbb{S}^1$:

$$[0] \leftrightarrows [1] \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; [2] \quad \cdots$$

If we want to interpret Hochschild homology via functor homology on finite sets, A has to be commutative and M has to be a symmetric A-bimodule. Then we can define $\mathcal{L}(A; M)$ which sends $\Gamma \ni [n] \mapsto M \otimes A^{\otimes n}$.

Via finite 'associative sets' I

$\mathbb{S}^1$:

$$[0] \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; [1] \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \; [2] \quad \cdots$$

If we want to interpret Hochschild homology via functor homology on finite sets, A has to be commutative and M has to be a symmetric A-bimodule. Then we can define $\mathcal{L}(A; M)$ which sends $\Gamma \ni [n] \mapsto M \otimes A^{\otimes n}$.
Interpreting $\mathbb{S}^1$ as a functor $\Delta^{op} \to \Gamma$ we get by composition $\mathcal{L}(A; M) \circ \mathbb{S}^1 \colon \Delta^{op} \to R$-mod and

$$HH_*(A; M) = \pi_* \mathcal{L}(A; M)(\mathbb{S}^1).$$

Via finite 'associative sets' II

If we want to allow non-commutative A, we have to change the category!

Via finite 'associative sets' II

If we want to allow non-commutative A, we have to change the category!
Let $\Gamma(as)$ be the category of finite pointed associative sets.

Via finite 'associative sets' II

If we want to allow non-commutative A, we have to change the category!
Let $\Gamma(as)$ be the category of finite pointed associative sets.
$\mathrm{Ob}(\Gamma(as)) = \mathrm{Ob}(\Gamma))$.

Via finite 'associative sets' II

If we want to allow non-commutative A, we have to change the category!
Let $\Gamma(as)$ be the category of finite pointed associative sets.
Ob($\Gamma(as)$) =Ob(Γ)).
A morphism $[n] \to [m]$ is a pointed map $f : [n] \to [m]$ together with a total ordering on the preimages $f^{-1}(j)$ for all $j \in [m]$.
Theorem [Pirashvili-R 2002] For any associative unital R-algebra A and any A-bimodule M

$$HH_*(A; M) \cong \mathrm{Tor}_*^{\Gamma(as)}(\bar{b}, \mathcal{L}(A; M)).$$

Via finite 'associative sets' II

If we want to allow non-commutative A, we have to change the category!
Let $\Gamma(as)$ be the category of finite pointed associative sets.
$\mathrm{Ob}(\Gamma(as)) = \mathrm{Ob}(\Gamma))$.
A morphism $[n] \to [m]$ is a pointed map $f : [n] \to [m]$ together with a total ordering on the preimages $f^{-1}(j)$ for all $j \in [m]$.
Theorem [Pirashvili-R 2002] For any associative unital R-algebra A and any A-bimodule M

$$HH_*(A; M) \cong \mathrm{Tor}_*^{\Gamma(as)}(\bar{b}, \mathcal{L}(A; M)).$$

Here, $\bar{b}$ is $\bar{b}(-) = \mathrm{coker}(R\{\Gamma(as)(-, [1])\} \to R\{\Gamma(as)(-, [0])\})$ where the map is induced by $d_0 - d_1$ where d_0 and d_1 send $0, 1$ to 0 but d_0 has $0 < 1$ as ordering on the preimage whereas d_1 has the ordering $1 < 0$ on $[1]$.

Cyclic homology

Cyclic homology has a similar functor homology interpretation.

Cyclic homology

Cyclic homology has a similar functor homology interpretation.
Theorem [Pirashvili-R 2002]

$$HC_*(A) \cong \mathrm{Tor}_*^{\mathcal{F}(as)}(b, \mathcal{L}(A; A)).$$

Cyclic homology

Cyclic homology has a similar functor homology interpretation.
Theorem [Pirashvili-R 2002]

$$HC_*(A) \cong \mathrm{Tor}_*^{\mathcal{F}(as)}(b, \mathcal{L}(A; A)).$$

Here, $\mathcal{F}(as)$ is the category of associative (unpointed) sets and b is the cokernel

$$b = \mathrm{coker}(R\{\mathcal{F}(as)(-, [1])\} \to R\{\mathcal{F}(as)(-, [0])\}).$$

What is E_n?

Let C_n denote the operad of little n-cubes.

What is E_n?

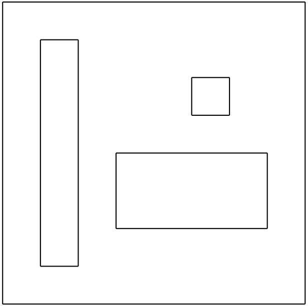
Let C_n denote the operad of little n-cubes. $C_n(r)$, $r \geq 0$.

What is E_n?

Let C_n denote the operad of little n-cubes. $C_n(r)$, $r \geq 0$.
$n = 2, r = 3$:

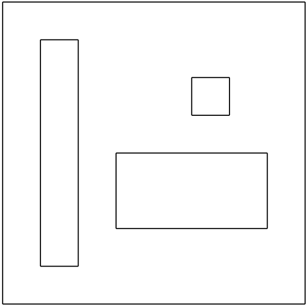
What is E_n?

Let C_n denote the operad of little n-cubes. $C_n(r)$, $r \geq 0$.
$n = 2, r = 3$:



What is E_n?

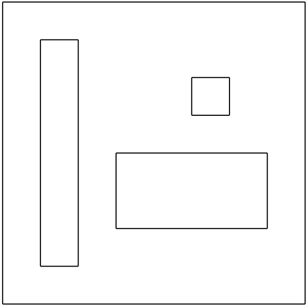
Let C_n denote the operad of little n-cubes. $C_n(r)$, $r \geq 0$.
$n = 2, r = 3$:



C_n acts on and detects n-fold based loop spaces.

What is E_n?

Let C_n denote the operad of little n-cubes. $C_n(r)$, $r \geq 0$.
$n = 2, r = 3$:



C_n acts on and detects n-fold based loop spaces.
$(C_* C_n(r))_r$, $r \geq 1$ is an operad in the category of chain complexes.
Let E_n be a cofibrant replacement of $C_* C_n$.

E_n-homology

For any (cofibrant) operad, there is a notion of André-Quillen homology for its algebras.

E_n-homology

For any (cofibrant) operad, there is a notion of André-Quillen homology for its algebras.
This measures the derived functors of indecomposables.

E_n-homology

For any (cofibrant) operad, there is a notion of André-Quillen homology for its algebras.
This measures the derived functors of indecomposables.
For E_n: Benoit Fresse gave a description of E_n-homology, $H_*^{E_n}$, as

E_n-homology

For any (cofibrant) operad, there is a notion of André-Quillen homology for its algebras.
This measures the derived functors of indecomposables.
For E_n: Benoit Fresse gave a description of E_n-homology, $H_*^{E_n}$, as

1. the homology of an explicit chain complex,

E_n-homology

For any (cofibrant) operad, there is a notion of André-Quillen homology for its algebras.
This measures the derived functors of indecomposables.
For E_n: Benoit Fresse gave a description of E_n-homology, $H_*^{E_n}$, as

1. the homology of an explicit chain complex,
2. the homology of the n-fold desuspension of a suitably defined n-fold bar construction.

E_n-homology

For any (cofibrant) operad, there is a notion of André-Quillen homology for its algebras.
This measures the derived functors of indecomposables.
For E_n: Benoit Fresse gave a description of E_n-homology, $H_*^{E_n}$, as

1. the homology of an explicit chain complex,
2. the homology of the n-fold desuspension of a suitably defined n-fold bar construction.

For simplicity, let $A \to R$ be an augmented commutative R-algebra and $\bar{A}$ its augmentation ideal.

From E_n to E_{n+1}

We can extend a little n-cubes configuration to a little $(n+1)$-cube configuration.

From E_n to E_{n+1}

We can extend a little n-cubes configuration to a little $(n+1)$-cube configuration.
By forgetting structure, we can view an E_{n+1}-algebra as an E_n-algebra. Commutative algebras are E_n-algebras for every n. Similarly for E_∞-algebras.

From E_n to E_{n+1}

We can extend a little n-cubes configuration to a little $(n+1)$-cube configuration.
By forgetting structure, we can view an E_{n+1}-algebra as an E_n-algebra. Commutative algebras are E_n-algebras for every n.
Similarly for E_∞-algebras.
For commutative algebras there are maps

$$H_*^{E_1}(\bar{A}) \to H_*^{E_2}(\bar{A}) \to \ldots \to H_*^{E_\infty}(\bar{A}).$$

From E_n to E_{n+1}

We can extend a little n-cubes configuration to a little $(n+1)$-cube configuration.
By forgetting structure, we can view an E_{n+1}-algebra as an E_n-algebra. Commutative algebras are E_n-algebras for every n. Similarly for E_∞-algebras.
For commutative algebras there are maps

$$H_*^{E_1}(\bar{A}) \to H_*^{E_2}(\bar{A}) \to \ldots \to H_*^{E_\infty}(\bar{A}).$$

Fresse's description in terms of iterated bar constructions gives a direct identification (in the commutative case over a field k) of $H_*^{E_n}(\bar{A})$ with $HH_{*+n}^{[n]}(A; k)$, that is Pirashvili's Hochschild homology of order n.

Higher order Hochschild homology

In general: Let k be a field and A an augmented commutative k-algebra.

Higher order Hochschild homology

In general: Let k be a field and A an augmented commutative k-algebra.
Definition [Pirashvili] Hochschild homology of order $n \geq 1$ of A with coefficients in k, $HH_*^{[n]}(A; k)$ is $\pi_* \mathcal{L}(A; k)(\mathbb{S}^n)$.

Higher order Hochschild homology

In general: Let k be a field and A an augmented commutative k-algebra.

Definition [Pirashvili] Hochschild homology of order $n \geq 1$ of A with coefficients in k, $HH_*^{[n]}(A; k)$ is $\pi_* \mathcal{L}(A; k)(\mathbb{S}^n)$.

Here, $\mathbb{S}^n = (\mathbb{S}^1)^{\wedge n}$ is a simplicial model of the n-sphere.

Higher order Hochschild homology

In general: Let k be a field and A an augmented commutative k-algebra.

Definition [Pirashvili] Hochschild homology of order $n \geq 1$ of A with coefficients in k, $HH_*^{[n]}(A; k)$ is $\pi_* \mathcal{L}(A; k)(\mathbb{S}^n)$.

Here, $\mathbb{S}^n = (\mathbb{S}^1)^{\wedge n}$ is a simplicial model of the n-sphere.

The case $n = 1$ coincides with the usual definition of Hochschild homology of A with coefficients in k.

Higher order Hochschild homology

In general: Let k be a field and A an augmented commutative k-algebra.
Definition [Pirashvili] Hochschild homology of order $n \geq 1$ of A with coefficients in k, $HH_*^{[n]}(A; k)$ is $\pi_* \mathcal{L}(A; k)(\mathbb{S}^n)$.
Here, $\mathbb{S}^n = (\mathbb{S}^1)^{\wedge n}$ is a simplicial model of the n-sphere.
The case $n = 1$ coincides with the usual definition of Hochschild homology of A with coefficients in k.

'Proof' that $H_*^{E_n}(\bar{A}) \cong HH_{*+n}^{[n]}(A; k)$:

$$\begin{aligned} H_*^{E_n}(\bar{A}) &\cong H_*(\Sigma^{-n} B^n(\bar{A})) \cong H_{*+n} B^n(\bar{A}) \\ &\cong H_{*+n}(\mathbb{S}^n \bar{\otimes} A) \cong HH_{*+n}^{[n]}(A; k). \end{aligned}$$

The limit: Gamma homology

Fresse showed as well, that in the limiting case

$$H^{E_\infty}(\bar{A}) \cong H\Gamma_*(A; k).$$

Here, $H\Gamma_*(A; k)$ denotes Gamma homology of A with coefficients in k, as defined by Alan Robinson and Sarah Whitehouse.

The limit: Gamma homology

Fresse showed as well, that in the limiting case

$$H^{E_\infty}(\bar{A}) \cong H\Gamma_*(A; k).$$

Here, $H\Gamma_*(A; k)$ denotes Gamma homology of A with coefficients in k, as defined by Alan Robinson and Sarah Whitehouse.
For Gamma homology a functor homology description is known:
Theorem [Pirashvili-R, 2000]

$$H\Gamma_*(A; k) \cong \mathrm{Tor}_*^{\Gamma}(t, \mathcal{L}(A; k))$$

The limit: Gamma homology

Fresse showed as well, that in the limiting case

$$H^{E_\infty}(\bar{A}) \cong H\Gamma_*(A; k).$$

Here, $H\Gamma_*(A; k)$ denotes Gamma homology of A with coefficients in k, as defined by Alan Robinson and Sarah Whitehouse.
For Gamma homology a functor homology description is known:
Theorem [Pirashvili-R, 2000]

$$H\Gamma_*(A; k) \cong \mathrm{Tor}_*^{\Gamma}(t, \mathcal{L}(A; k))$$

Here $t[n] = \mathrm{Hom}_{\mathrm{Sets}_*}([n], k)$ as above.

The limit: Gamma homology

Fresse showed as well, that in the limiting case

$$H^{E_\infty}(\bar{A}) \cong H\Gamma_*(A;k).$$

Here, $H\Gamma_*(A;k)$ denotes Gamma homology of A with coefficients in k, as defined by Alan Robinson and Sarah Whitehouse.
For Gamma homology a functor homology description is known:
Theorem [Pirashvili-R, 2000]

$$H\Gamma_*(A;k) \cong \mathrm{Tor}_*^{\Gamma}(t, \mathcal{L}(A;k))$$

Here $t[n] = \mathrm{Hom}_{\mathrm{Sets}_*}([n], k)$ as above.
Gamma (co)homology plays an important role as the habitat for obstructions to E_∞-ring structures on ring spectra.

A functor homology description

Can we generalize this to $1 < n < \infty$?

A functor homology description

Can we generalize this to $1 < n < \infty$?

Theorem [Livernet-R,2011] For all $1 \leq n < \infty$:

$$H_*^{E_n}(\bar{A}) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n}(b_n^{\mathrm{epi}}, \mathcal{L}(A; k)).$$

A functor homology description

Can we generalize this to $1 < n < \infty$?
Theorem [Livernet-R,2011] For all $1 \leq n < \infty$:

$$H_*^{E_n}(\bar{A}) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n}(b_n^{\mathrm{epi}}, \mathcal{L}(A; k)).$$

Epi_n is a category that captures the combinatorial properties of n-fold bar constructions, a category of trees with n levels.

A functor homology description

Can we generalize this to $1 < n < \infty$?

Theorem [Livernet-R,2011] For all $1 \leq n < \infty$:

$$H_*^{E_n}(\bar{A}) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n}(b_n^{\mathrm{epi}}, \mathcal{L}(A; k)).$$

Epi_n is a category that captures the combinatorial properties of n-fold bar constructions, a category of trees with n levels.
b_n^{epi} is a cokernel $\mathrm{coker}(k\{\mathrm{Epi}_n(-, Y_n)\} \to k\{\mathrm{Epi}_n(-, I_n)\})$.

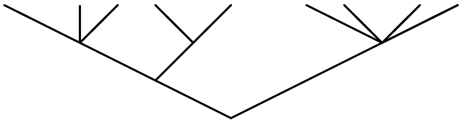
A functor homology description

Can we generalize this to $1 < n < \infty$?
Theorem [Livernet-R,2011] For all $1 \leq n < \infty$:

$$H_*^{E_n}(\bar{A}) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n}(b_n^{\mathrm{epi}}, \mathcal{L}(A; k)).$$

Epi_n is a category that captures the combinatorial properties of n-fold bar constructions, a category of trees with n levels.
b_n^{epi} is a cokernel $\mathrm{coker}(k\{\mathrm{Epi}_n(-, Y_n)\} \to k\{\mathrm{Epi}_n(-, I_n)\})$.
Here, I_n is the n-tree with only one leaf and Y_n is the tree that has two leaves at the top level.

The category Epi_n – an example



The category Epi_n – the definition

Objects are sequences

$$[r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1] \tag{1}$$

where the f_i are surjective and order-preserving.

The category Epi_n – the definition

Objects are sequences

$$[r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} [r_1] \tag{1}$$

where the f_i are surjective and order-preserving.

A morphism to an object $[r'_n] \xrightarrow{f'_n} [r'_{n-1}] \xrightarrow{f'_{n-1}} \ldots \xrightarrow{f'_2} [r'_1]$ consists of surjective maps $\sigma_i \colon [r_i] \to [r'_i]$ for $1 \le i \le n$ such that σ_1 is order-preserving surjective and for all $2 \le i \le n$ the map σ_i is order-preserving on the fibres $f_i^{-1}(j)$ for all $j \in [r_{i-1}]$ and such that the diagram

$$\begin{array}{ccccccc}
[r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\
\downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\
[r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & [r'_1]
\end{array}$$

commutes.

Some references

- B. Fresse, Iterated bar complexes of E-infinity algebras and homology theories, AGT 11 (2011), 747–838
- M. Livernet, B. Richter, An interpretation of E_n-homology as functor homology, Math. Zeitschrift 269 (1) 2011, 193–219.
- J. L. Loday, Cyclic Homology, Grundlehren der math. Wissenschaften, 301, 2nd edition, Springer, 1998.
- T. Pirashvili, Hodge decomposition for higher order Hochschild homology, Ann. Scient. de l'ENS, 33, 2000, 151–179.
- T. Pirashvili, B. Richter, Robinson-Whitehouse complex and stable homotopy, Topology 39, 2000, 525–530. 1
- T. Pirashvili, B. Richter, Hochschild and cyclic homology via functor homology, K-theory 25 (1), 2002, 39–49.