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SPECTRAL THEORY FOR LINEAR RELATIONS VIA LINEAR OPERATORS

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We develop a spectral theory for closed linear operators of the form $T : D(T) \subset X \mapsto X/X_0$, where X is a complex Banach space and X_0 a closed vector subspace of it. This approach, essentially expressed in terms of linear operators, provides a better understanding of the spectral theory for closed linear relations.

1. Introduction

As in the case of linear operators, the spectral theory of linear relations, including the associated analytic functional calculus, is an important tool for studying various properties of these objects and for deriving some of their applications. Results related to the spectral theory of linear relations and its applications can be found in [Baskakov and Chernyshov 2002; Baskakov and Zagorskiĭ 2007; Cross 1998; Favini and Yagi 1993; Favini and Yagi 1999] and elsewhere.

In this paper we emphasize the strong connection between the spectral theory of closed linear relations and that of some closed linear operators. As a matter of fact, we develop a spectral theory for a certain class of linear operators, obtaining as consequences most of the main spectral properties of linear relations.

Our concept of spectrum is equivalent to that of *extended spectrum* of a linear relation, as given by [Baskakov and Chernyshov 2002, Definition 1.5]; see also [Cross 1998, Section VI.4], where it is called *augmented spectrum*. In particular, the point ∞ is in the spectrum unless the quotient range operator is an ordinary everywhere-defined bounded operator (see Proposition 11).

Let us introduce some notation and definitions.

Let X be a complex Banach space and let $\mathscr{B}(X)$ denote the Banach algebra of all bounded linear operators from X into X. Let also $X_0 \subset X$ be a closed vector subspace, and let $J_0 : X \mapsto X/X_0$ be the canonical projection. The identity operator on X will be usually denoted by I.

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We are interested in linear operators of the form $T : D(T) \subset X \mapsto X/X_0$, where $D(T)$ is, of course, the domain of T. (The use of such operators is inspired by the works [Albrecht and Vasilescu 1986; Waelbroeck 1982]; see also [Gheorghe and Vasilescu 2009].) Such an operator is said to be a *quotient range operator*. Although X/X_0 is itself a Banach space, its quotient space form plays an important role in what follows. As a matter of fact, the class of closed quotient range operators is in one-to-one correspondence with the class of closed linear relations (see the definition below), and they have important similar properties. Note that the formula $T : D(T) \subset X \mapsto X/X_0$ implies that T is a quotient range operator, and in such situations the expression "quotient range" will be often omitted.

If $T : D(T) \subset X \mapsto X/X_0$, we denote, as usual, by $N(T)$, $R(T)$ and $G(T)$ the null-space, the range and the graph of T. Let $R_0(T)$ be given by $R(T) = R_0(T)/X_0$, and $G_0(T) = \{(x, y) \in X \times X; x \in D(T), J_0(y) = T(x)\}$, which are called, with the terminology of [Albrecht and Vasilescu 1986], the *lifted range* and *lifted graph*, respectively.

Following Arens [1961], any linear subspace Z of $X \times X$ is called a *linear relation* in X. Given a linear relation $Z \subset X \times X$, we associate it, as usual (see [Arens 1961; Cross 1998]), with the following subspaces:

$$D(Z) = \{u \in X; (u, v) \in Z \text{ for some } v \in X\}, \quad N(Z) = \{u \in D(Z); (u, 0) \in Z\},$$
$$R(Z) = \{v \in X; (u, v) \in Z \text{ for some } u \in X\}, \quad M(Z) = \{v \in R(Z); (0, v) \in Z\}.$$

The left two are called the domain of Z, the range of Z; the right are the kernel of Z and the multivalued part of Z. When $M(Z) = \{0\}$, then Z is the graph of a linear operator. We often identify the relation given by the graph of an operator with the operator itself.

Given an arbitrary relation $Z \subset X \times X$, to avoid any confusion with the inverse of an operator, we will denote the *reverse relation* $\{(y, x) \in X \times X; (x, y) \in Z\}$ by $Z^\dagger$.

The strong connection between linear relations and quotient range operators is well known and easily explained; see [Cross 1998; Gheorghe and Vasilescu 2009] for example. Namely, given an operator $T : D(T) \subset X \mapsto X/X_0$, the space $Z_T = G_0(T) \subset X \times X$ is a linear relation. Conversely, given a linear relation $Z \subset X \times X$, with $M(T)$ closed (which is automatic in the framework which will be used in the sequel), the linear operator $Q_Z : D(Z) \mapsto X/M(Z)$, given by $Q_Z(x) = y + M(Z)$ whenever $(x, y) \in Z$, is a quotient range operator. Moreover, this correspondence is one-to-one. This connection will be exploited to develop a spectral theory for linear relations. The simple but crucial remark leading to this development is that for a closed relation $Z \subset X \times X$, the reverse relation $Z^\dagger$ is (the graph of) a bounded operator if and only if the operator $Q_Z : D(Z) \mapsto X/M(Z)$ has a bounded inverse.

Given a linear relation Z and a complex number $\lambda \in \mathbb{C}$, we consider the linear relations $\lambda I - Z = \{(u, \lambda u - v);\ (u, v) \in Z\}$ and $(\lambda I - Z)^{\dagger} = \{(\lambda u - v, u);\ (u, v) \in Z\}$ (see Section 5). If we assume that Z is closed, $N(\lambda I - Z) = \{0\}$ and $R(\lambda I - Z) = X$, then we have that $(\lambda I - Z)^{\dagger}$ is (the graph of) a closed everywhere-defined linear operator (which is, in general, neither surjective nor injective; see Example 32), and hence $(\lambda I - Z)^{\dagger} \in \mathcal{B}(X)$. Because the bounded operator $(\lambda I - Z)^{\dagger}$ exists if and only if the operator $\lambda J_Z - Q_Z : D(Z) \subset X \mapsto X/M(Z)$ has a bounded inverse (see Remark 4(ii)), where $J_Z : X \mapsto X/M(Z)$ is the canonical projection, the spectral theory of these objects can be simultaneously developed. However, in our opinion, the spectral theory of quotient range operators is easier to handle.

Our main tool is an analytic functional calculus for quotient range operators, defined in Section 2 by using the classical Riesz–Dunford–Waelbroeck integral formula; see [Dunford and Schwartz 1958; Waelbroeck 1954]. A similar formula, valid for linear relations, is also used in [Baskakov and Chernyshov 2002]. Nevertheless, an analytic functional calculus in its full generality seems to appear only in the present work.

The analytic functional calculus allows us to recapture, in terms of operators, most of the main spectral properties known for linear relations; see especially [Cross 1998; Baskakov and Chernyshov 2002]. Among some simplifications, we mention that our approach avoids the use of the concept of *pseudoresolvent*, as well as that of *invariant subspace*, as done in [Baskakov and Chernyshov 2002]. Other differences between our approach and that of the quoted works will be discussed in due course. We should also mention that a calculus with the exponential function and with fractional powers has been already used in [Favini and Yagi 1993] to obtain a Hille–Yoshida–Phillips-type theorem for linear relations.

The paper is organized as follows. In Section 2, we introduce a notion of spectrum for quotient range operators (equivalent to that for linear relations) in the Riemann sphere $\mathbb{C}_{\infty}$, and construct a functional calculus with analytic functions in neighborhoods of this spectrum. As mentioned above, our Theorem 16, asserting in particular the multiplicativity of the analytic functional calculus, seems to be new in this context (as well as in that of linear relations). In Section 3, we study quotient range operators with unbounded spectrum and nonempty resolvent set. The existence of a spectral decomposition corresponding to separate parts of the spectrum as well as a spectral mapping theorem are presented herein. In Section 4, we study the class of quotient range operators for which the point ∞ is an isolated point of the spectrum. In Section 5, we investigate some connections between the analytic functional calculus and the Arens polynomial calculus [Arens 1961].

2. Spectrum and analytic functional calculus for closed quotient range operators

As in the introduction, X denotes a complex Banach space, X_0 a closed vector subspace of it, and $J_0 : X \mapsto X/X_0$ the canonical projection. The symbol $\mathbb{C}_\infty$ denotes the one-point compactification of $\mathbb{C}$. We designate by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators from X into another Banach space Y. As usually, $\mathcal{B}(X, X)$ is denoted by $\mathcal{B}(X)$.

Let $T : D(T) \subset X \mapsto X/X_0$ be a closed linear operator. We denote by $\rho_A(T)$ the *Arens resolvent set* of T, that is, the set of those $\lambda \in \mathbb{C}$ such that $(\lambda J_0 - T)^{-1} \in \mathcal{B}(X/X_0, X)$. The *Arens spectrum* of T is the set $\sigma_A(T) := \mathbb{C} \setminus \rho_A(T)$. Because $\lambda J_0 - T : D(T) \subset X \mapsto X/X_0$ is closed, we have $\lambda \in \rho_A(T)$ if and only if $\lambda J_0 - T$ is bijective.

Remark 1. Given two complex Banach spaces X_1, X_2, we denote by $X_1 \oplus X_2$ their direct sum, endowed with a convenient norm, compatible with the norms of X_1, X_2.

Let $T_j : D(T_j) \subset X_j \mapsto X_j/X_{0j}$ for $j = 1, 2$ be quotient range operators. Then the map

$$T_1 \oplus T_2 : D(T_1) \oplus D(T_2) \subset X_1 \oplus X_2 \mapsto (X_1/X_{01}) \oplus (X_2/X_{02})$$

may be regarded as a quotient range operator, provided we identify the Banach space $(X_1/X_{01}) \oplus (X_2/X_{02})$ with the Banach space $(X_1 \oplus X_2)/(X_{01} \oplus X_{02})$, using the natural isomorphism

$$V : (X_1/X_{01}) \oplus (X_2/X_{02}) \mapsto (X_1 \oplus X_2)/(X_{01} \oplus X_{02}) \tag{1}$$

given by the assignment

$$(X_1/X_{01}) \oplus (X_2/X_{02}) \ni (x_1 + X_{01}) \oplus (x_2 + X_{02}) \mapsto$$

$$x_1 \oplus x_2 + X_{01} \oplus X_{02} \in (X_1 \oplus X_2)/(X_{01} \oplus X_{02}).$$

We write

$$T_1 \oplus_q T_2 := V(T_1 \oplus T_2).$$

In particular, given $T : D(T) \subset X \mapsto X/X_0$ closed such that there are closed vector subspaces X_1, X_2 of X and X_{01}, X_{02} of X_0 with $X = X_1 \oplus X_2$, $X_0 = X_{01} \oplus X_{02}$, $D(T) = (D(T) \cap X_1) \oplus (D(T) \cap X_2)$, and closed operators $T_j : D(T_j) \subset X_j \mapsto X_j/X_{0j}$ with $D(T_j) = D(T) \cap X_j$ for $j = 1, 2$ and $T(x_1 \oplus x_2) = V(T_1x_1 \oplus T_2x_2)$ for all $x_1 \oplus x_2 \in D(T_1) \oplus D(T_2)$, we have $T = T_1 \oplus_q T_2$.

Definition 2. Let $T : D(T) \subset X \mapsto X/X_0$ be closed.

(1) Assume $\sigma_A(T)$ bounded, and let $m \geq 0$ be an integer. The point ∞ is said to be *m-regular* for T if the set $\{\lambda^{1-m}(\lambda J_0 - T)^{-1} J_0;\ |\lambda| \geq r\}$ is bounded in $\mathcal{B}(X)$ for some $r > \sup_{\lambda \in \sigma_A(T)} |\lambda|$.

(2) If ∞ is not 0-regular we put $\sigma(T) = \sigma_A(T) \cup \{\infty\}$.

(3) Assume ∞ to be 0-regular and $X_0 \neq \{0\}$. If $T = T_0 \oplus_q T_1$, $T_0 : \{0\} \subset X_0 \mapsto X_0/X_0 = \{0\}$, we put $\sigma(T) = \sigma_A(T) \cup \{\infty\}$; otherwise, $\sigma(T) = \sigma_A(T)$.

(4) If ∞ is 0-regular and $X_0 = \{0\}$, we put $\sigma(T) = \sigma_A(T)$.

The set $\sigma(T)$ is called the *spectrum* of T, and the set $\rho(T) = \mathbb{C}_\infty \setminus \sigma(T)$ is called the *resolvent set* of T.

The set $\sigma(T)$ is nonempty except for $X_0 = X = \{0\}$ (see Proposition 7), but it may be equal to $\mathbb{C}_\infty$. For practical reasons, in this paper we work only with (quotient range) operators with nonempty resolvent set.

Example 3. The well-known fact that any continuous linear operator on a Banach space X has a bounded spectrum is no longer true in the case of quotient range operators, as we can see in the following example.

Let X be the Hilbert space of all square-summable complex sequences, let $A \in \mathcal{B}(X)$ be the shift

$$A((x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

and let

$$X_0 = \{(x_1, x_2, 0, 0, \dots) : x_1, x_2 \in \mathbb{C}\}.$$

Consider the operator T defined by $Tx = Ax + X_0$ for $x \in X$. Clearly T is continuous and thus closed. We will show that $\sigma(T)$ is unbounded. Let $\lambda \in \mathbb{C}$. We have

$$x \in N(\lambda J_0 - T) \Longleftrightarrow \lambda x - Ax + y = 0 \qquad \text{for some } y \in X_0,$$

$$\Longleftrightarrow \left.\begin{aligned} -\lambda x_1 &= y_1, \\ x_1 - \lambda x_2 &= y_2, \\ x_k - \lambda x_{k+1} &= 0 \quad \text{for } k \geq 2 \end{aligned}\right\} \quad \text{for some } y_1, y_2 \in \mathbb{C}.$$

For $|\lambda| > 1$ and $x_1, x_2 \in \mathbb{C}$ consider $x_3 = -x_1/\lambda^2 - x_2/\lambda$. Then

$$(-x_1/\lambda, x_3, x_3/\lambda, \dots, x_3/\lambda^k, \dots) \in N(\lambda J_0 - T),$$

which implies that $N(\lambda J_0 - T) \neq \{0\}$ for $|\lambda| > 1$. Consequently,

$$\{\lambda \in \mathbb{C} : |\lambda| > 1\} \subset \sigma(T),$$

so $\sigma(T)$ is unbounded.

Remark 4. (i) For a closed operator $T : D(T) \subset X \mapsto X$ — in particular for an everywhere defined bounded operator on X — Definition 2 provides the usual definition of the spectrum. Note that the density of $D(T)$ in X is not required. For instance, if A is the operator from Example 3, which is injective, and $T = A^{-1}$, then T is not densely defined but $0 \notin \sigma(T)$, so $\rho(T) \neq \varnothing$.

Note also that if $0 : \{0\} \subset X \mapsto X$, we have $\sigma(0) = \mathbb{C}_\infty$ if $X \neq \{0\}$ and $\sigma(0) = \varnothing$ if $X = \{0\}$, by Definition 2.

If $X_0 = X \neq \{0\}$ and $0 : \{0\} \subset X \mapsto X/X_0 = \{0\}$, then 0 is a quotient range operator, whose Arens spectrum is empty, and $\sigma(T) = \{\infty\}$, by Definition 2.

(ii) Let $Z \subset X \times X$ be a closed relation. It is clear that the subspace $M(Z) \subset X$ is closed. As in the introduction, we consider the (quotient range) operator $Q_Z : D(Z) \mapsto X/M(Z)$ given by $Q_Z(x) = y + M(Z)$ whenever $(x, y) \in Z$, which is closed.

Let $J_Z : X \mapsto X/M(Z)$ be the canonical projection. Given $\lambda \in \mathbb{C}$, the operator $\lambda J_Z - Q_Z$ is again closed. If $(\lambda I - Z)^\dagger \in \mathcal{B}(X)$, then $\lambda J_Z - Q_Z$ has an everywhere-defined, and hence bounded, inverse. Indeed, $(\lambda I - Z)^\dagger$ exists if and only if for every $u \in X$ we can find a unique $x \in X$ such that $(x, y) \in Z$ and $\lambda x - y = u$ for some $y \in X$. Moreover, $x = 0$ if and only if $u \in M(Z)$. Hence $\lambda J_Z x - Q_Z x = J_Z u$, showing that $\lambda J_Z - Q_Z$ is bijective.

Conversely, if Q_Z is closed, then Z is closed. In addition, if $\lambda J_Z - Q_Z$ is bijective, for every $u \in X$ we put $x = (\lambda J_Z - Q_Z)^{-1} J_Z u$. Then we have $\lambda x - y = u$ for some $y \in X$ with $(x, y) \in Z$, and so $(\lambda I - Z)^\dagger$ does exist. Evidently, $(\lambda I - Z)^\dagger = (\lambda J_Z - Q_Z)^{-1} J_Z$.

From this discussion it clearly follows that we may define the *Arens resolvent set* and *Arens spectrum* of a closed relation $Z \subset X \times X$ by the equalities $\rho_A(Z) = \rho_A(Q_Z)$ and $\sigma_A(Z) = \sigma_A(Q_Z)$, respectively. Similarly, we may define the *resolvent set* and *spectrum* of a closed relation $Z \subset X \times X$ via the equalities $\rho(Z) = \rho(Q_Z)$ and $\sigma(Z) = \sigma(Q_Z)$. Consequently, most of the spectral properties obtained for a quotient range operator can be translated into properties for linear relations. This definition of the spectrum of a linear relation coincides with the corresponding definition [Cross 1998, Definition VI.4.1] or [Baskakov and Chernyshov 2002, Definition 1.5], because the condition $\lim_{|\lambda| \to \infty} (\lambda I - Z)^\dagger = 0$ is equivalent to the fact that ∞ is a 0-regular point for Q_Z.

In fact, given an integer $m \geq 0$, we may say that the point ∞ is *m-regular* for the closed linear relation Z if ∞ is m-regular for the operator Q_Z.

As an example, if $Z = \{0\} \times X$ ($X \neq \{0\}$), then $D(Z) = \{0\}$, $M(Z) = X$ and $Q_Z : \{0\} \subset X \mapsto X/X = \{0\}$. Therefore, $\sigma(Z) = \sigma(Q_Z) = \{\infty\}$, as in (i).

Definition 5. Let $T : D(T) \subset X \mapsto X/X_0$ be closed, with $\rho_A(T) \neq \varnothing$. The function

$$\rho_A(T) \ni \lambda \mapsto (\lambda J_0 - T)^{-1} J_0 \in \mathcal{B}(X)$$

is called the *resolvent (function)* of T. We also put $R(\lambda, T) = (\lambda J_0 - T)^{-1} J_0$.

As in the case of linear relations (see [Cross 1998; Favini and Yagi 1993]), we have a *resolvent equation*, which is very useful for the construction of the analytic functional calculus.

Lemma 6. *If* $\lambda, \mu \in \rho_A(T)$, *then*

$$R(\mu, T) - R(\lambda - T) = (\lambda - \mu) R(\mu, T) R(\lambda, T).$$

Proof. Indeed, for all $\lambda, \mu \in \rho_A(T)$, we have the identity

$$(\mu J_0 - T)^{-1} J_0 - (\lambda J_0 - T)^{-1} J_0 = (\lambda - \mu)(\mu J_0 - T)^{-1} J_0 (\lambda J_0 - T)^{-1} J_0,$$

which is easily checked. □

As in the case of ordinary operators, the resolvent set is open and the resolvent function is holomorphic on it.

Proposition 7. *The resolvent sets* $\rho_A(T)$ *and* $\rho(T)$ *are open subsets of* $\mathbb{C}$ *and* $\mathbb{C}_\infty$ *respectively, and the resolvent function* $\lambda \mapsto R(\lambda, T)$ *is holomorphic on* $\rho_A(T)$, *with values in* $\mathcal{B}(X)$, *having an analytic extension to* $\rho(T)$, *whenever* $\infty \in \rho(T)$. *In particular, the spectrum* $\sigma(T)$ *is a closed subset of* $\mathbb{C}_\infty$, *which is nonempty provided* $X_0 \neq X$ *or* $X = X_0 \neq \{0\}$.

Proof. We may assume $\rho(T) \neq \varnothing$. The proof is similar to the corresponding one for linear relations; see for instance [Cross 1998, Section VI.1]. Because of some differences, we shall sketch an appropriate proof.

Let $\lambda_0 \in \rho(T)$. We show that there exists a neighborhood $V \subset \mathbb{C}_\infty$ of λ_0 such that $V \subset \rho(T)$. We have the following situations.

First, if $\lambda_0 = \infty$, it follows from Definition 2 that there exists $r > 0$ such that $\{|\lambda| > r\} \subset \rho(T)$.

Second, assume $\lambda_0 \in \rho_A(T)$ and that $R(\lambda_0, T) \neq 0$. Then, if $|\lambda - \lambda_0| < \|R(\lambda_0, T)\|^{-1}$, then $\lambda \in \rho(T)$ and

$$R(\lambda, T) = R(\lambda_0, T)(I + (\lambda - \lambda_0) R(\lambda_0, T))^{-1},$$

implying, in particular, the holomorphy of $R(\lambda, T)$ in this open disc.

Third, next assume $R(\lambda_0, T) = 0$. Then $J_0 = 0$, and so $X = X_0$. Moreover, $R(\lambda, T) = 0$ for all $\lambda \in \mathbb{C}$.

If $X = X_0 = \{0\}$, then $\rho_A(T) = \mathbb{C}$, $\rho(T) = \mathbb{C}_\infty$ by Definition 2.

If $X = X_0 \neq \{0\}$, then $D(T) = \{0\}$ (otherwise $\rho(T) = \varnothing$) and $\rho_A(T) = \rho(T) = \mathbb{C}$, again by Definition 2.

Note that the assumption $X = X_0 \neq \{0\}$ implies $\sigma(T) \ni \{\infty\}$. Finally, suppose that $X_0 \neq X$ and $\sigma(T) = \varnothing$. Then $R(\lambda, T)$ is analytic in $\mathbb{C}$ and has an analytic extension at ∞. By Liouville's theorem, it follows that $R(\lambda, T)$ is a constant operator, say C_0. Since ∞ is a 0-regular point of T, we must have $C_0 = 0$. Therefore, as above, $X = X_0$, which is not possible. □

Remark 8. If $\sigma_A(T)$ is bounded, according to Proposition 7 we have a development in $\mathcal{B}(X)$ of the form

$$R(\lambda, T) = \sum_{k=-\infty}^{\infty} \lambda^k C_k,$$

where the series is uniformly convergent when $r_1 \le |\lambda| \le r_2$ for fixed $r_2 \ge r_1 > \sup_{\lambda\in\sigma_A(T)} |\lambda|$. This representation shows that ∞ is m-regular for some integer $m \ge 0$ if and only if $C_k = 0$ for all $k \ge m$. In particular, if $m \ge 2$, the point ∞ is m-regular for T if and only if ∞ is a pole of $R(\lambda, T)$ of order $\le m-1$. As already noted, ∞ is a 0-regular point of T if and only if $\lim_{\lambda\to\infty} R(\lambda, T) = 0$, while ∞ is a 1-regular point if and only if $\lim_{\lambda\to\infty} R(\lambda, T)$ exists in $\mathcal{B}(X)$.

Henceforth, to avoid quotient range operators with empty spectrum, we assume that either $X_0 \ne X$ or $X = X_0 \ne \{0\}$, if not otherwise specified.

Definition 9. Let $T : D(T) \subset X \mapsto X/X_0$ be closed, with $\varnothing \ne \rho(T)$.

(i) We denote by $\mathcal{O}(T)$ the set of all complex-valued functions f, each of them defined and analytic in an open set containing $\sigma(T)$ and depending on f. By identifying any two functions equal in a neighborhood of $\sigma(T)$ (that is, considering $\mathcal{O}(T)$ as the set of germs of analytic functions in neighborhoods of $\sigma(T)$), we may and will regard $\mathcal{O}(T)$ as an algebra.

(ii) Let $F \subset \mathbb{C}_\infty$ be closed and let U be an open neighborhood of F. An *admissible contour* surrounding F in U is a finite system of rectifiable Jordan curves Γ, positively oriented, which is the boundary of an open set $\Delta \subset \overline{\Delta} \subset U$, with $\Delta \supset F$. Note that $\Gamma \cap F = \varnothing$ and that Γ is a compact set in $\mathbb{C}$.

(iii) We define the *analytic functional calculus* for the quotient range operator T as follows. Let $f \in \mathcal{O}(T)$. We set

$$f(T) := \begin{cases} (2\pi i)^{-1} \int_\Gamma f(\lambda) R(\lambda, T) d\lambda & \text{if } \infty \notin \sigma(T), \\ f(\infty) I + (2\pi i)^{-1} \int_\Gamma f(\lambda) R(\lambda, T) d\lambda & \text{if } \infty \in \sigma(T), \end{cases}$$

where Γ is an admissible contour surrounding $\sigma(T)$ in the domain of definition of f.

Remark. Via Proposition 7, $f(T)$ is a continuous linear operator on X that does not depend on Γ.

The next result seems to be new even in the context of linear relations.

Proposition 10. *For every quotient range closed operator T with $\varnothing \neq \rho(T)$, the map $f \mapsto f(T)$ of $\mathcal{O}(T)$ into $\mathcal{B}(X)$ is an algebra morphism. If $\sigma(T) \ni \infty$, this morphism is unital.*

Proof. Clearly the map $f \mapsto f(T)$ is linear. To prove the multiplicativity of the application $f \mapsto f(T)$, we follow the lines of [Vasilescu 1982, Proposition III.3.4], via Lemma 6.

Consider first the case $\infty \in \sigma(T)$.

Let $f, g \in \mathcal{O}(T)$ and let $U \subset \mathbb{C}_\infty$ be open in the domain of definition of both f, g, with $\sigma(T) \subset U$. Let Δ and Δ_1 be open sets such that their boundaries Γ and Γ_1, respectively, are admissible contours surrounding $\sigma(T)$ in U, and such that $\sigma(T) \subset \Delta \subset \overline{\Delta} \subset \Delta_1 \subset \overline{\Delta}_1 \subset U$. Then we have

$$\begin{aligned}
&f(T)g(T)\\
&= f(\infty)g(\infty)I + f(\infty)\frac{1}{2\pi i}\int_{\Gamma_1} g(\mu)R(\mu,T)d\mu + g(\infty)\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)R(\lambda,T)d\lambda\\
&\qquad + \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)R(\lambda,T)d\lambda \frac{1}{2\pi i}\int_{\Gamma_1} g(\mu)R(\mu,T)d\mu\\
&= f(\infty)g(\infty)I + f(\infty)\frac{1}{2\pi i}\int_{\Gamma_1} g(\mu)R(\mu,T)d\mu + g(\infty)\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)R(\lambda,T)d\lambda\\
&\qquad + \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\Big(\frac{1}{2\pi i}\int_{\Gamma_1}(\mu-\lambda)^{-1}g(\mu)(R(\lambda,T)-R(\mu,T))d\mu\Big)\\
&= f(\infty)g(\infty)I + \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)g(\lambda)R(\lambda,T)d\lambda = (fg)(T),
\end{aligned}$$

via Lemma 6 and the Cauchy formula at infinity for analytic functions.

If $\infty \notin \sigma(T)$, the proof is similar and will be omitted.

If $\sigma(T) \ni \infty$, by letting p_0 be the constant polynomial equal to 1, we may take as Γ the boundary of a closed disc in $\rho(T)$ (negatively oriented). Since $R(\lambda, T)$ is analytic in $\rho(T)$, it follows that $\int_\Gamma R(\lambda, T)d\lambda = 0$, so $p_0(T) = I$. □

The next result corresponds to [Baskakov and Chernyshov 2002, Lemma 2.2], whose proof uses an ergodic theorem from [Hille and Phillips 1957]. We give a direct proof based on Proposition 10.

Proposition 11. *Given a closed operator $T : D(T) \subset X \mapsto X/X_0$, the spectrum $\sigma(T)$ is a bounded subset of $\mathbb{C}$ if and only if $X_0 = 0$ and $T \in \mathcal{B}(X)$.*

Proof. We use some ideas from [Vasilescu 1982, Lemma III.3.5]; see also [Hille and Phillips 1957].

Assume $\sigma(T)$ bounded, and fix an $r > 0$ such that $\sigma(T) \subset \{\lambda \in \mathbb{C}; |\lambda| < r\}$. From the analyticity of the resolvent function (Proposition 7), it follows that there exists a sequence $(C_n)_{n\geq 0}\subset \mathcal{B}(X)$ such that

$$R(\lambda,T)=\sum_{n=0}^{\infty}\lambda^{-n}C_n \quad \text{uniformly with respect to} \quad |\lambda|\geq r.$$

The operator C_0, given by the equality $C_0=\lim_{\lambda\to\infty}R(\lambda,T)$, is necessarily 0 because ∞ is 0-regular

We define the bounded linear operators

$$E=\frac{1}{2\pi i}\int_{|\lambda|=r}R(\lambda,T)d\lambda \quad \text{and} \quad A=\frac{1}{2\pi i}\int_{|\lambda|=r}\lambda R(\lambda,T)d\lambda.$$

Because

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\lambda^n d\lambda=\begin{cases}0 & \text{if } n\neq -1,\\ 1 & \text{if } n=-1,\end{cases}$$

we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\lambda|=r}\lambda^k R(\lambda,T)d\lambda&=\frac{1}{2\pi i}\int_{|\lambda|=r}\lambda^k\sum_{n=0}^{\infty}\lambda^{-n}C_n d\lambda\\ &=\sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\int_{|\lambda|=r}\lambda^{k-n}d\lambda\Big)C_n r=C_{k+1}\end{aligned}$$

for all integers $k\geq 0$. Consequently $C_1=E$ and $A^n=C_{n+1}$. The same proposition shows that $E^2=E$ and $A^nE=EA^n=A^n$. On the other hand, if $|\lambda|\geq r$, then

$$\lambda^{-1}I+\lambda^{-2}A+\cdots=(\lambda I-A)^{-1},$$

which implies that

$$R(\lambda,T)=E(\lambda I-A)^{-1}. \tag{2}$$

Let

$$X_1=(I-E)(X) \quad \text{and} \quad X_2=E(X).$$

Hence $X=X_1\oplus X_2$ because E is a projection. Setting $A_2=A|_{X_2}$ and using the fact that $AE=EA$, we have

$$(\lambda I-A)^{-1}|_{X_2}=(\lambda I_2-A_2)^{-1},$$

whenever $|\lambda|\geq r$, where I_2 is the identity on X_2. This together with (2) implies

$$\begin{aligned}&R(\lambda,T)(X_1)\subset X_1, \quad R(\lambda,T)|_{X_1}=0|_{X_1},\\ &R(\lambda,T)(X_2)\subset X_2, \quad R(\lambda,T)|_{X_2}=(\lambda I_2-A_2)^{-1}|_{X_2},\end{aligned} \tag{3}$$

whenever $|\lambda|\geq r$.

Set $0_1 = 0|_{X_1}$. Let $u \in D(T)$ and let $v \in X$ with $J_0 v = Tu$. Then we have $R(\lambda, T)(\lambda u - v) = u$ for a fixed λ with $|\lambda| \geq r$. Write $u = u_1 + u_2$, $v = v_1 + v_2$, with $u_j, v_j \in X_j$ for $j = 1, 2$. Using (3), we have in fact that

$$R(\lambda, T)(\lambda u_1 - v_1) = 0 = u_1 \quad \text{and} \quad R(\lambda, T)(\lambda u_2 - v_2) = u_2.$$

These relations imply that

$$v_1 \in N(0_1) = X_1, \tag{4}$$

$$(\lambda I_2 - A_2)^{-1}(\lambda u_2 - v_2) = u_2, \tag{5}$$

From (5) we obtain that $A_2 u_2 = v_2$. This calculation shows that $D(T) \subset \{0\} \oplus X_2$, and that $T(0 \oplus u_2) = v_1 + A_2 u_2 + X_0$ whenever $0 \oplus u_2 \in D(T)$.

If $u = 0$, then we may take as $v \in X$ with $J_0 v = Tu = 0$ any vector $v \in X_0$. The decomposition $0 = u_1 + u_2$ shows that $u_1 = u_2 = 0$. Then, from (4) and (5) we derive $v_1 \in X_1$ and $v_2 = 0$. Therefore, $X_0 \subset X_1$. As $T(0_1 v_1 \oplus 0) = 0 = v_1 + X_0$ for every $v_1 \in X_1$, we must have $X_0 = X_1$.

In fact, $D(T) = \{0\} \oplus X_2$. Indeed, if $A_2 u_2 = v_2$ for some $u_2 \in X_2$, taking into account (5), we have

$$(\lambda I_2 - A_2)^{-1}(\lambda u_2 - v_2) = u_2 = R(\lambda, T)(\lambda u_2 - v_2) \in D(T).$$

In summary, we have now two closed vector subspaces X_1 and X_2 of X with $X = X_1 \oplus X_2$, the operator $0_1 \in \mathscr{B}(X_1)$, an operator $A_2 \in \mathscr{B}(X_2)$, $X_0 = X_1$, $D(T) = \{0\} \oplus X_2$, and $T : \{0\} \oplus X_2 \mapsto (X_1 \oplus X_2)/X_1$ is given by $T(0 \oplus x_2) = 0 \oplus A_2 x_2 + X_1$ for all $0 \oplus x_2 \in \{0\} \oplus X_2$. Setting $T_1 : \{0\} \subset X_1 \mapsto X_1/X_1 = \{0\}$ and $T_2 = A_2 : X_2 \mapsto X_2$, we obtain $T = T_1 \oplus_q T_2$. Assuming $X_1 \neq \{0\}$, we must have $\sigma(T) \ni \{\infty\}$ via Definition 2, which is not possible. Therefore, which is not possible. Therefore, $X_1 = \{0\}$, and so $T = A_2 \in \mathscr{B}(X_2) = \mathscr{B}(X)$.

Conversely, the conditions in the statement from above are obviously sufficient to insure the boundedness of the spectrum of T. □

Remark 12. From the previous proof it follows that if ∞ is 0-regular for T, then $T = T_1 \oplus_q T_2$, where $T_1 : \{0\} \subset X_1 \mapsto +X_1/X_1 = \{0\}$, and $T_2 : X_2 \mapsto X_2$ is bounded.

Corollary 13. *Let $T : D(T) \subset X \mapsto X/X_0$ be closed. Then $\sigma(T) = \sigma_A(T)$ if and only if $T \in \mathscr{B}(X)$, and $\sigma(T) = \sigma_A(T) \cup \{\infty\}$ otherwise.*

In particular, if $T : D(T) \subset X \mapsto X$ is a closed operator, the spectrum of T is a bounded subset of $\mathbb{C}$ if and only if $T \in \mathscr{B}(X)$.

The next result is related to [Baskakov and Chernyshov 2002, Lemma 2.2].

Corollary 14. *Let $Z \subset X \times X$ be a closed relation. The spectrum of Z is a bounded subset of $\mathbb{C}$ if and only if Z is the graph of an operator in $\mathscr{B}(X)$.*

The spectrum of a direct sum of two quotient range operators behaves as one expects (see also Lemma 2.1 from [Baskakov and Chernyshov 2002], in the context of linear relations):

Corollary 15. *If* $T : D(T) \subset X \mapsto X/X_0$ *is closed and has the form* $T = T_1 \oplus_q T_2$, *then* $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$.

Proof. Note that $J_0 = J_{01} \oplus_q J_{02}$, where $J_{0j} : X_j \mapsto X_j/X_{0j}$ are the canonical projections for $j = 1, 2$. We have to show that $\rho(T) = \rho(T_1) \cap \rho(T_2)$. We have the following cases.

First, fix $\lambda \in \rho(T) \cap \mathbb{C}$. Setting $S = \lambda J_0 - T,\ S_j = \lambda J_{0j} - T_j,\ j = 1, 2$, we have to show that $S = S_1 \oplus_q S_2$ is bijective if and only if both S_1, S_2 are bijective, which is routine and is left to the reader. In fact, we obtain that

(6) $$(\lambda J_0 - T)^{-1} = ((\lambda J_{01} - T_1)^{-1} \oplus (\lambda J_{02} - T_2)^{-1}) V^{-1},$$

where V is given by (1). Therefore,

(7) $$R(\lambda, T) = R(\lambda, T_1) \oplus R(\lambda, T_2).$$

This clearly shows that $\sigma_A(T) = \sigma_A(T_1) \cup \sigma_A(T_2)$.

Second, we have only to note that $T \in \mathcal{B}(X)$ if and only if $T_j \in \mathcal{B}(X_j)$ $(j = 1, 2)$, which easily leads to the equality $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, via Corollary 13. □

A general result concerning the existence of an analytic functional calculus for quotient range closed operators is the following.

Theorem 16. *For every quotient range closed operator* T *with* $\varnothing \neq \rho(T)$, *the map* $f \mapsto f(T)$ *of* $\mathcal{O}(T)$ *into* $\mathcal{B}(X)$ *is a unital algebra morphism. If* $\sigma(T)$ *is bounded, then* $T \in \mathcal{B}(X)$ *and* $p_1(T) = T$, *where* $p_1(\lambda) = \lambda$ *for all* $\lambda \in \mathbb{C}$.

Proof. If $\sigma(T)$ is unbounded, the assertion follows from Proposition 10. If $\sigma(T)$ is bounded, then $T \in \mathcal{B}(X)$ by Proposition 11, and the assertion is classical. □

Remark 17. (i) For every $f \in \mathcal{O}(T)$, we have

$$f(T)|_{X_0} = \begin{cases} f(T)|_{\{0\}} = 0 & \text{if } \infty \notin \sigma(T), \\ f(\infty)I & \text{if } \infty \in \sigma(T). \end{cases}$$

Indeed, we clearly have

$$\int_\Gamma f(\lambda) R(\lambda, T)\, d\lambda = \Big(\int_\Gamma f(\lambda)(\lambda J_0 - T)^{-1} d\lambda\Big) J_0,$$

for each admissible contour Γ surrounding $\sigma(T)$, which in turn implies the desired equalities.

This remark also shows that $f(T)(X_0) \subset X_0$ for every $f \in \mathcal{O}(T)$. This allows us to define an operator $f^\circ(T) \in \mathcal{B}(X/X_0)$, given by

$$f^\circ(T)(x + X_0) := f(T)x + X_0 \quad \text{for } x \in X,$$

for all $f \in \mathcal{O}(T)$. In other words, $f^\circ(T)J_0 = J_0 f(T)$ for all $f \in \mathcal{O}(T)$.

(ii) If ∞ is an isolated point of $\sigma(T)$, then $E = (2\pi i)^{-1} \int_\Gamma f(\lambda)R(\lambda, T)d\lambda$ is a projection, where Γ is a contour surrounding $\sigma_A(T)$.

(iii) If $Z \subset X \times X$ is a closed relation with nonempty resolvent set, we may define the operator $f(Z) := f(Q_Z)$ for every analytic function from $\mathcal{O}(Z) := \mathcal{O}(Q_Z)$ (see Remark 4(ii)). This provides an analytic functional calculus for Z, whose properties are easily derived from those valid for Q_Z (see also [Baskakov and Chernyshov 2002, formula (2.8)] for a similar but partial approach.)

3. Quotient range operators with unbounded spectrum

As before, let X be a complex Banach space, let X_0 be a closed vector subspace of X, and let $J_0 : X \mapsto X/X_0$ be the canonical projection. Let also $T : D(T) \subset X \mapsto X/X_0$ be closed. We may consider on $D(T)$ the *graph norm* given by

$$\|x\|_T := \|x\| + \inf_{J_0 y = Tx} \|y\| \quad \text{for } x \in D(T).$$

It is well known that when endowed with this norm, the vector space $D(T)$ becomes a Banach space; see for instance [Cross 1998, Section IV.3]. With the terminology from [Waelbroeck 1982], $(D(T), \|\cdot\|_T)$ becomes a *Banach subspace* of X, which will be occasionally denoted by D_T.

It is obvious that the maps $T : D_T \mapsto X/X_0$ and $J_T : D_T \mapsto X/X_0$, with $J_T = J_0|_{D_T}$, are continuous.

Throughout this section, $T : D(T) \subset X \mapsto X/X_0$ will be a closed (quotient range) operator, with $\infty \in \sigma(T)$ and a nonempty resolvent set.

Lemma 18. *For every function $f \in \mathcal{O}(T)$ and each admissible contour Γ surrounding $\sigma(T)$, the map*

$$X \ni x \mapsto \int_\Gamma f(\lambda)R(\lambda, T)x d\lambda$$

has values into the Banach space D_T and is continuous.

In particular, if $f(\infty) = 0$, then $f(T)$ is a continuous operator from X into D_T.

Proof. Indeed, $R(\lambda, T) = (\lambda J_0 - T)^{-1} J_0 : X \mapsto D_T$ is in $\mathcal{B}(X, D_T)$ for all $\lambda \in \rho(T)$, and hence

$$2\pi i(f(T) - f(\infty)) = \int_\Gamma f(\lambda)(\lambda J_0 - T)^{-1} J_0 d\lambda \in \mathcal{B}(X, D_T),$$

which implies the assertions. □

We recall that for any quotient range operator T and each function $f \in \mathcal{O}(T)$, we denote by $f^\circ(T)$ the operator induced by $f(T)$ in X/X_0; see Remark 17(i).

Lemma 19. *Let $f \in \mathcal{O}(T)$ be such that $f_1(\lambda) = \lambda f(\lambda) \in \mathcal{O}(T)$. Then $Tf(T) = J_0 f_1(T) = f_1^\circ(T) J_0$.*

Proof. It is clear that $f(\infty) = 0$. Let Γ be an admissible contour surrounding $\sigma(T)$ in the domain of definition of f. We have

$$\begin{aligned} T(f(T)x) &= \frac{1}{2\pi i}\int_\Gamma f(\lambda) T(\lambda J_0 - T)^{-1} J_0 x \, d\lambda \\ &= J_0\Big(-\frac{1}{2\pi i}\int_\Gamma f(\lambda) x \, d\lambda + \frac{1}{2\pi i}\int_\Gamma \lambda f(\lambda)(\lambda J_0 - T)^{-1} J_0 x \, d\lambda\Big) \\ &= J_0(f_1(T)x) = f_1^\circ(T) J_0 x, \end{aligned}$$

because $-(1/2\pi i)\int_\Gamma f(\lambda)d\lambda = f_1(\infty)$. □

Remark 20. With the notation from the previous lemma, if $x \in D(T)$ and $y \in X$ satisfy $J_0 y = Tx$, then $f_1(T)x = f(T)y$. Indeed,

$$\begin{aligned} f_1(T)x &= f_1(\infty)x + \frac{1}{2\pi i}\int_\Gamma \lambda f(\lambda)(\lambda J_0 - T)^{-1} J_0 x \, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda J_0 - T)^{-1} J_0 y \, d\lambda = f(T)y, \end{aligned}$$

because $(1/2\pi i)\int_\Gamma f(\lambda) x \, d\lambda = -f_1(\infty)x$, as noticed before.

Lemma 21. *For all $f \in \mathcal{O}(T)$ and $x \in D(T)$, we have $Tf(T)x = f^\circ(T)Tx$.*

Proof. Because the function $\lambda f(\lambda)$ is not necessarily in $\mathcal{O}(T)$, we need an argument different from that in the proof of Lemma 19.

If $(x, y) \in G_0(T)$, then $J_0 y = Tx$. Therefore, for a fixed $\lambda \in \rho(T)$,

$$TR(\lambda, T)x = -J_0 x + \lambda J_0(\lambda J_0 - T)^{-1} J_0 x = J_0 R(\lambda, T)y. \tag{8}$$

Let Γ be an admissible contour surrounding $\sigma(T)$ in the domain of $f \in \mathcal{O}(T)$, positively oriented. We have, via (8), that

$$T\int_\Gamma f(\lambda) R(\lambda, T) x \, d\lambda = J_0 \int_\Gamma f(\lambda) R(\lambda, T) y \, d\lambda,$$

implying $Tf(T)x = J_0 f(T)y$. Consequently,

$$Tf(T)x = J_0 f(T)y = f^\circ(T) J_0 y = f^\circ(T)Tx \quad \text{for all } x \in D(T). \qquad \square$$

The next result is a version of the idempotent theorem in the context of quotient range operators. For a similar result in the context of linear relations, see [Baskakov and Chernyshov 2002, Theorem 2.3]. Unlike the result there, our proof uses essentially Theorem 16.

Theorem 22. *Let $T : D(T) \subset X \mapsto X/X_0$ be a quotient range operator with $\sigma(T) \ni \infty$ and assume that there are two nonempty disjoint closed sets $F, H \subset \mathbb{C}_\infty$ such that $\sigma(T) = F \cup H$. Then there exist closed vector subspaces X_F and X_H with $X = X_F \oplus X_H$, and operators $T_F : D(T_F) \subset X_F \mapsto X_F/X_{0F}$ and $T_H : D(T_H) \subset X_H \mapsto X_H/X_{0H}$, where $X_{0F} \subset X_F$, $X_{0H} \subset X_H$ and $X_0 = X_{0F} \oplus X_{0H}$, such that $D(T) = D(T_F) \oplus D(T_H)$ and $T = T_F \oplus_q T_H$.*

In addition, $\sigma(T_F) = F$ and $\sigma(T_H) = H$.

Proof. To fix the ideas, assume that $\infty \in F$. We choose open sets U and V in $\mathbb{C}_\infty$ such that $U \supset F$, $V \supset H$ and $U \cap V = \varnothing$. Then the characteristic functions χ_U and χ_V of the sets U and V respectively, restricted to $U \cup V$, are analytic. We put $P_F = \chi_U(T)$ and $P_H = \chi_V(T)$. Since $\chi_U^2 = \chi_U$, and by a similar relation for χ_V, the operators P_F and P_H are projections via Proposition 10. Moreover, $P_F P_H = P_H P_F = 0$ and $P_F + P_H = I$.

In fact, since $\infty \in F$, we have

$$P_F = I + \frac{1}{2\pi i}\int_{\Gamma_F} R(\lambda, T)d\lambda, \quad \text{and} \quad P_H = \frac{1}{2\pi i}\int_{\Gamma_H} R(\lambda, T)d\lambda,$$

where Γ_F and Γ_H are admissible contours surrounding F and H in U and V, respectively.

Note that $P_H|_{X_0} = 0$ and $P_F|_{X_0}$ is the identity on X_0; see Remark 17(i).

Lemma 21 shows that if $x \in D(T)$, then $P_F x \in D(T)$, and $T P_F x = P_F^\circ T x$, where $P_F^\circ = \chi_U^\circ(T)$. Similarly, $P_H x \in D(T)$ and $T P_H x = P_H^\circ T x$. This also shows that $D(T) = (D(T) \cap P_F(X)) \oplus (D(T) \cap P_F(H))$.

Let $X_F = P_F(X)$ and $X_H = P_H(X)$. Obviously, $X = X_F \oplus X_H$. We have $X_0 \subset X_F$, and we put $X_{0F} = X_0$ and $X_{0H} = \{0\}$.

Let $T_F = T|_{(D(T)\cap X_F)}$. For each $x \in D(T_F) := D(T) \cap X_F$, Lemma 21 gives $T_F x \in X_F/X_{0F}$. Similarly, if $T_H = T|_{(D(T)\cap X_H)}$ for each $x \in D(T_H) := D(T) \cap X_H$, we have $T_H x \in X_H/X_{0H} = X_H$. Consequently,

$$T(x_F \oplus x_H) = T_F(x_F) \oplus T_H(x_H) \in (X_F/(X_{0F}) \oplus (X_H/(X_{0H})$$

for all $x_F \in D(T_F)$ and $x_H \in D(T_H)$, and so $T = T_F \oplus_q T_H$.

Let us show that $\sigma(T_F) \subset F$.

Let $\mu \in \mathbb{C} \setminus F$. With no loss of generality we may suppose that $\mu \notin U$. Then the function $f_\mu(\lambda) = (\mu - \lambda)^{-1}\chi_U(\lambda)$ is analytic in $U \cup V$, null at infinity, and we may define the operator

$$f_\mu(T) = \frac{1}{2\pi i}\int_{\Gamma_F} f_\mu(\lambda)R(\lambda, T)d\lambda = P_F f_\mu(T).$$

Because we have $(\mu-\lambda)f_\mu(\lambda) = \chi_U(\lambda)$, it follows that $\mu f_\mu(T) - f_{1,\mu}(T) = P_F$, where $f_{1,\mu}(\lambda) = \lambda f_\mu(\lambda) \in \mathcal{O}(T)$.

Let us show that $\mu J_F - T_F$ is injective, where $J_F : X_F \mapsto X_F/X_0$ is $J_0|_{X_F}$. Assuming that for an $x \in D(T_F)$ one has $\mu J_F x = T_F x$, and fixing an $y \in X_F$ with $J_F y = T_F x$, we have $\mu x - y \in X_0$. Because $f_\mu(\infty) = 0$, we infer that

$$\begin{aligned} 0 = f_\mu(T)(\mu x - y) &= \frac{1}{2\pi i}\int_{\Gamma_F} f_\mu(\lambda)(\lambda J_0 - T)^{-1}J_0(\mu x - y)d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_F}((\chi_U(\lambda)R(\lambda, T)x + f_{1,\mu}(\lambda)R(\lambda, T)x - f_\mu(\lambda)(\lambda J_0 - T)^{-1}J_0 y)d\lambda \\ &= P_F x = x, \end{aligned}$$

where we have used the equality $f_{1,\mu}(T)x = f_\mu(T)y$, via Remark 28.

Let us show that $\mu J_F - T_F$ is surjective. Let $y = P_F y \in X_F$. Note that $y = \mu f_\mu(T)y - f_{1,\mu}(T)y$, as we have seen above. Moreover, by Lemma 19 $J_0 y = \mu J_0 f_\mu(T)y - T f_\mu(T)y$. Therefore, $(\mu J_F - T_F)^{-1}$ exists for all $\mu \notin U$. Since U is an arbitrary open neighborhood of F, it follows that $(\mu J_F - T_F)^{-1}J_F = f_\mu(T)|_{X_F}$ for all $\mu \notin F$.

We show now that $\sigma(T_H) \subset H$. First of all, we identify the space $(X_H + X_0)/X_0$ with X_H, and so $J_0|_{X_H} = I_H$, where I_H is the identity on X_H. Note also that $T_H : D(T_H) \mapsto X_H$ is a simply closed operator.

Fixing $\mu \in \mathbb{C} \setminus H$, we may suppose that $\mu \notin V$. Then the function $g_\mu(\lambda) = (\mu - \lambda)^{-1}\chi_V(\lambda)$ is analytic in $U \cup V$, null at infinity, and we can consider the operator $g_\mu(T) = P_H g_\mu(T)$.

Because we have $(\mu-\lambda)g_\mu(\lambda) = \chi_V(\lambda)$, it follows that $\mu g_\mu(T) - g_{1,\mu}(T) = P_H$, where $g_{1,\mu}(\lambda) = \lambda g_\mu(\lambda) \in \mathcal{O}(T)$.

Proceeding as in the previous case, we derive that $\mu I_H - T_H : D(T_H) \mapsto X_H$ is bijective. In fact, $(\mu I_H - T_H)^{-1} = g_\mu(T)|_{X_H}$ for all $\mu \notin H$. We omit the details.

We have only to note that

$$\|g_\mu(T)|X_H\| \le \frac{1}{2\pi\,\mathrm{dist}(\mu, \Gamma_H)}\int_{\Gamma_H}\|R(\lambda, T)\||d\lambda|,$$

implying that ∞ is 0-regular for T_H. In other words, $\sigma(T_H) \subset H$.

Since we already have $\sigma(T_F) \subset F$ and $\sigma(T_H) \subset H$, it suffices to prove that $\sigma(T_F) \cup \sigma(T_H) = \sigma(T)$. Indeed, this follows from Corollary 15, showing that we must have $\sigma(T_F) = F$ and $\sigma(T_H) = H$. □

A result similar to [Baskakov and Chernyshov 2002, Theorem 2.3] follows directly from the previous theorem:

Corollary 23. *Let $Z \subset X \times X$ be a closed relation with $\sigma(Z) \ni \infty$. Assume that there are two nonempty disjoint closed sets $F, H \subset \mathbb{C}_\infty$ such that $\sigma(Z) = F \cup H$. Then we have a decomposition $Z = Z_F \oplus Z_H$ with Z_F and Z_H closed relations and $\sigma(Z_F) = F$ and $\sigma(Z_H) = H$.*

We end this section with a version of the *spectral mapping theorem*. A similar result valid for linear relations can be found in [Baskakov and Chernyshov 2002, Theorem 2.5], whose proof uses Gelfand's theory (see also Corollary 10 there). Our proof is different and is based on Theorems 16 and 22.

Theorem 24. *For every $f \in \mathbb{O}(T)$, we have $\sigma(f(T)) = f(\sigma(T))$.*

Proof. Fix an $f \in \mathbb{O}(T)$. Let $\mu \notin f(\sigma(T))$ with $\mu \neq \infty$. Then the function $g_\mu(\lambda) = (\mu - f(\lambda))^{-1}$ is in $\mathbb{O}(T)$. It is plain that $(\mu I - f(T))g_\mu(T) = I$, showing that $g_\mu(T) = (\mu I - f(T))^{-1}$, and so $\sigma(f(T)) \subset f(\sigma(T))$ (that it is 0-regular for $f(T)$ is obvious).

Conversely, let $\mu_0 \in f(\sigma(T))$, so $\mu_0 = f(\lambda_0)$ for some $\lambda_0 \in \sigma(T)$. Assume that $\mu_0 \notin \sigma(f(T))$.

In the case $\lambda_0 \neq \infty$, we consider the function $h(\lambda) = (\lambda_0 - \lambda)^{-1}(\mu_0 - f(\lambda))$, which can be clearly extended at $\lambda = \lambda_0$, and this extension belongs to $\mathbb{O}(T)$. Note that $\lambda_0 h(T) - h_1(T) = \mu_0 I - f(T)$, where $h_1(\lambda) = \lambda h(\lambda) \in \mathbb{O}(T)$. Therefore,

$$\lambda_0 h(T)(\mu_0 I - f(T))^{-1} - h_1(T)(\mu_0 I - f(T))^{-1} = I. \tag{9}$$

This shows that for each $v \in X$ we have

$$(\lambda_0 J_0 - T)h(T)(\mu_0 I - f(T))^{-1} v = J_0 v$$

via Lemma 19. Therefore, $\lambda_0 J_0 - T$ is surjective.

Further, let $x \in X$ be such that $(\lambda_0 J_0 - T)x = 0$, and let $y \in X$ with $J_0 y = Tx$. Using (9), we have

$$x = (\mu_0 I - f(T))^{-1}(\lambda_0 h(T) - h_1(T)x) = (\mu_0 I - f(T))^{-1} h(T)(\lambda_0 x - y) = 0$$

via Remark 20, and that $J_0(\lambda_0 x - y) = 0$ and $h(\infty) = 0$; see also Remark 17(i). This shows that $\lambda_0 J_0 - T$ is injective too. Consequently, $\lambda_0 J_0 - T$ is invertible, which is not possible.

In the case that $\lambda_0 = \infty$, and there exists a sequence $(\lambda_m)_{m \geq 1}$ in $\sigma_A(T)$ such that $\lim_{m\to\infty} \lambda_m = \lambda_0$, then $f(\lambda_m) \in \sigma(f(T))$ for all $m \geq 1$ by the first part of the proof, implying $f(\infty) \in \sigma(f(T))$.

Finally, if ∞ is an isolated point of $\sigma(T)$, then, according to Theorem 22, there is a decomposition $X = X_1 \oplus X_\infty$, and setting $T_\infty = T|_{D(T)\cap X_\infty}$, we have $\sigma(T_\infty) = \{\infty\}$. Because we have $\sigma(f(T_\infty)) \subset f(\sigma(T_\infty)) = \{f(\infty)\}$ by the first part of the proof, we must actually have $\sigma(f(T_\infty)) = \{f(\infty)\}$ since $\sigma(f(T_\infty))$ is nonempty. Consequently, $f(\infty) \in \sigma(f(T))$, as a consequence of Corollary 15 and of the equality $f(T) = f(T_1) \oplus_q f(T_\infty)$, where $T_1 = T|_{X_1} \in \mathscr{B}(X_1)$. □

Theorem 2.5 from [Baskakov and Chernyshov 2002] is then a consequence of the preceding theorem:

Corollary 25. *If Z is a closed relation with nonempty resolvent set and unbounded spectrum, we have $\sigma(f(Z)) = f(\sigma(Z))$ for all $f \in \mathbb{O}(Z)$.*

Using Theorem 24, we get the superposition of the analytic functional calculus:

Proposition 26. *Let $f \in \mathbb{O}(T)$ and let $g \in \mathbb{O}(f(T))$. Then we have $g \circ f \in \mathbb{O}(T)$ and $(g \circ f)(T) = g(f(T))$.*

Proof. The property $g \circ f \in \mathbb{O}(T)$ follows easily from Theorem 24. The proof of the equality $(g \circ f)(T) = g(f(T))$ follows the lines of the similar assertion in [Vasilescu 1982, Theorem III.3.10(4)]. Specifically, we may choose an admissible contour Γ surrounding $\sigma(T)$ such that $\Gamma_1 = f(\Gamma)$ surrounds $\sigma(f(T))$. Then

$$\begin{aligned} g(f(T)) &= \frac{1}{2\pi i}\int_{\Gamma_1} g(\mu)R(\mu, f(T))d\mu \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} g(\mu)\Big((\mu - f(\infty))^{-1}I + \frac{1}{2\pi i}\int_{\Gamma}(\mu - f(\lambda)^{-1})R(\lambda, T)d\lambda\Big)d\mu \\ &= g(f(\infty))I + \frac{1}{2\pi i}\int_{\Gamma}\Big(\frac{1}{2\pi i}\int_{\Gamma_1} g(\mu)(\mu - f(\lambda)^{-1})d\mu\Big)R(\lambda, T)d\lambda \\ &= g(f(\infty))I + \frac{1}{2\pi i}\int_{\Gamma} g(f(\lambda))R(\lambda, T)d\lambda = (g\circ f)(T), \end{aligned}$$

which proves the result. □

A result similar to [Baskakov and Chernyshov 2002, Corollary 2.4] can be also obtained with our techniques:

Proposition 27. *We have $\sigma(T) = \{\infty\}$ if and only if there is a quasinilpotent operator $Q \in \mathscr{B}(X)$ such that $T : R(Q) \mapsto X/N(Q)$, $T(Qx) = x + N(Q)$ for $x \in X$.*

Proof. Assume $\sigma(T) = \{\infty\}$. If $h(\lambda) = \lambda^{-1}$ $(\lambda \neq 0)$, we have $h \in \mathbb{O}(T)$ and $h(\infty) = 0$. Therefore, by Lemma 19, $h(T)x \in D(T)$ for all $x \in X$, and $Th(T)x = J_0 h_1(T)x = J_0 x$, where $h_1(\lambda) = 1$ for all λ. Hence $h(T) = T^{-1}J_0$, showing that $D(T) = R(h(T))$ and $N(h(T)) = X_0$. We have only to remark that $\sigma(h(T)) = h(\{\infty\}) = \{0\}$, showing that $Q = h(T)$ is quasinilpotent.

Conversely, if there is a quasinilpotent operator $Q \in \mathscr{B}(X)$ such that $T : R(Q) \mapsto X/N(Q)$, $T(Qx) = x + N(Q)$ for $x \in X$, then one has $(\lambda J_0 - T)^{-1}(y + N(Q)) = (\lambda Q - I)^{-1}Qy$ for all $y \in X$ and $\lambda \in \mathbb{C}$. Hence, $\sigma(T) = \{\infty\}$.

Note also that $R(\lambda, T) = (\lambda Q - I)^{-1}Q$, $\lambda \in \mathbb{C}$. □

Remark 28. The spectrum of the relation $Z \subset X \times X$ is equal to $\{\infty\}$ if and only if Z is the reverse of the graph of a quasinilpotent operator $Q \in \mathcal{B}(X)$. This can be deduced either from the previous result or directly, from the fact that $Z^\dagger$ is a bounded operator and the equality

$$(\lambda^{-1}I - Z^\dagger)^\dagger = \lambda I + \lambda^2(\lambda I - Z)^\dagger \quad \text{for } \lambda \neq 0;$$

see for instance [Sandovici 2006, (2.1.2)] or [Baskakov and Chernyshov 2002, Corollary 2.4].

4. Quotient range operators with bounded Arens spectrum

In this section we study those quotient range operators for which the point ∞ is isolated and m-regular, for some integer $m \geq 1$. We discuss the case $m = 0$ in Remark 12. Similar results for linear relations can be also found in [Baskakov and Chernyshov 2002, Section 3]. We start with a version of Proposition 27.

Proposition 29. *Let $T : D(T) \subset X \mapsto X/X_0$ be closed with $\sigma(T) = \{\infty\}$. The point ∞ is m-regular for T for some integer $m \geq 1$ if and only if there exists $Q \in \mathcal{B}(X)$ such that $Q^{m+1} = 0$, and $T : R(Q) \mapsto X/N(Q)$ is given by $T(Qx) = x + N(Q)$ for all $x \in X$.*

Proof. The condition is sufficient by Proposition 27. Let us prove its necessity.

With the notation from Remark 8, because $\sigma(T) = \{\infty\}$ and so $R(\lambda, T)$ should be of the form $-\sum_{k=0}^{\infty} \lambda^k C_k$ for all $\lambda \in \mathbb{C}$, we must have $C_k = 0$ for all $k \geq m$. Therefore, $R(\lambda, T) = -\sum_{k=0}^{m-1} \lambda^k C_k$. For the rest of the proof, we sketch an algebraic argument.

For any two distinct points λ and μ in $\mathbb{C}$, the resolvent equation shows that

$$(\mu - \lambda) \sum_{k=0}^{m-1} \sum_{p+q=k} \lambda^p \mu^q C_p C_q = -\sum_{k=0}^{m-1} (\lambda^k - \mu^k) C_k.$$

Hence

$$\sum_{p+q=k-1} (\lambda - \mu) \lambda^p \mu^q C_p C_q = (\lambda^k - \mu^k) C_k$$

whenever $1 \leq k \leq m-1$, implying by recurrence $C_0 C_{k-1} = C_k$, and so $C_k = C_0^{k+1}$. Therefore, taking $Q = C_0$, we must have $Q^{m+1} = C_m = 0$.

Finally, since $R(\lambda, T) = Q(\lambda Q - I)^{-1}$, we infer the equality, $T^{-1} J_0 = Q$, showing that $X_0 = N(Q)$, $D(T) = R(Q)$, and $TQx = x + N(Q)$ for all $x \in X$. □

The next result is related to [Baskakov and Chernyshov 2002, Theorem 3.1].

Theorem 30. *Let $T : D(T) \subset X \mapsto X/X_0$ be closed, with $\sigma_A(T)$ bounded and $\infty \in \sigma(T)$. The point ∞ is m-regular for some integer $m \geq 1$ if and only if there* *are closed vector subspaces* X_1 *and* X_2 *of* X *with* $X = X_1 \oplus X_2$, *an operator* $A_1 \in \mathcal{B}(X_1)$ *with* $A_1^{m+1} = 0$, *another operator* $A_2 \in \mathcal{B}(X_2)$, *with* $X_0 = N(A_1) \oplus \{0\}$, $D(T) = R(A_1) \oplus X_2$, *and* $T = T_1 \oplus_q T_2$, *where* $T_1(A_1 x_1) = x_1 + N(A_1)$ *for all* $x_1 \in X_1$, *and* $T_2 = A_2$.

In addition, $\sigma_A(T) = \sigma(A_2)$.

Proof. Assume that T is closed, with $\sigma_A(T)$ bounded, such that the point ∞ is m-regular for some integer $m \geq 1$. Then $\sigma(T) = F \cup \{\infty\}$, where $F := \sigma_A(T)$. Since F is bounded, according to Theorem 22 and Proposition 27, there exist closed vector subspaces X_F and X_∞ with $X = X_F \oplus X_\infty$, and operators $T_F : X_F \mapsto X_F$ and $T_\infty : D(T_\infty) \subset X_\infty \mapsto X_\infty / X_{0\infty}$, with $\sigma(T_F) = F$ and $\sigma(T_\infty) = \{\infty\}$, where $X_{0\infty} = N(Q_\infty) = X_0$, $D(T_\infty) = R(Q_\infty) \oplus X_F$, and $Q_\infty \in \mathcal{B}(X_\infty)$ is quasinilpotent. Moreover, $T_\infty(Q_\infty x) = x + N(Q_\infty)$ for all $x \in X_\infty$, and $T = T_\infty \oplus_q T_F$. In fact, since ∞ is m-regular for T, it is also m-regular for T_∞. Therefore, $Q_\infty^{m+1} = 0$ by Proposition 29. The assertion from the statement is obtained for $A_1 = Q_\infty$ and $A_2 = T_F$.

Conversely, if $T = T_1 \oplus_q T_2$ with the stated properties, then $\sigma(T_1) = \{\infty\}$ and ∞ is m-regular for T_2 by Proposition 29, and so $\sigma_A(T) = \sigma(T_2)$ is bounded and ∞ is m-regular also for T, by (7). □

A part of [Baskakov and Chernyshov 2002, Theorem 3.1] is now obtained as a consequence of the previous theorem.

Corollary 31. *Given a closed linear relation* $Z \subset X \times X$ *with* $\sigma_A(Z)$ *a bounded subset of* $\mathbb{C}$ *and* ∞ *not* 0*-regular, the set* $\{|\lambda|^{1-m} \|(\lambda - Z)^\dagger\|; |\lambda| \geq r\}$ *is bounded for an integer* $m \geq 1$ *and some* $r > \sup\{|\lambda|; \lambda \in \sigma_A(Z)\}$ *if and only if there exist closed linear subspaces* X_1 *and* X_2 *with* $X_1 \oplus X_2 = X$, *and operators* $A_1 \in \mathcal{B}(X_1)$ *with* $A_1^{m+1} = 0$, *and* $A_2 \in \mathcal{B}(X_2)$, *such that*

$$Z = G(A_1)^\dagger \oplus G(A_2).$$

In this case, *one has* $\sigma_A(Z) = \sigma(A_2)$.

Example 32. Let $P \in \mathcal{B}(X)$ be a proper projection, and let $Z = G(P)^\dagger$. Clearly $Z^\dagger = P$ and thus $0 \in \rho(Z)$, and so $Z^\dagger$ is neither injective nor surjective. In fact, we can now easily compute the spectrum of Z. Setting $X_1 = N(P)$ and $X_2 = R(P)$, we have that $X = X_1 \oplus X_2$. Therefore $Z = G(0_1)^\dagger \oplus G(I_2)$, where 0_1 is the null operator on X_1 and I_2 is the identity on X_2. Using Corollary 31, it follows that $\sigma(Z) = \sigma(\{0\}) \cup \sigma(I_2) = \{\infty\} \cup \{1\}$.

Remark 33. Let Z be a densely defined closed linear relation such that, for some $r > 0$, we have $\{\lambda; |\lambda| > r\} \subset \rho_A(Z)$ and $\mathfrak{R} = \{(\lambda I - Z)^\dagger; |\lambda| > r\}$ is a bounded subset of $\mathcal{B}(X)$. Then $\sigma_A(Z)$ is bounded, possibly empty. Let us show that $\sigma_A(Z)$ is nonempty. If ∞ is 0-regular, the assertion follows via Corollary 14 (see also

Remark 12). Assuming that $\sigma_A(Z)$ is empty and ∞ is not 0-regular, Corollary 31 shows that $\sigma(A_2)$ is empty, leading to $X_2 = \{0\}$, and $A_1^2 = 0$. Since $D(Z) = R(A_1)$ is dense, the closure of $R(A_1)$ should be equal to X. Therefore $A_1 = 0$ implying $R(A_1) = \{0\}$, and so $X = \{0\}$, which is not possible. Consequently, $\sigma_A(Z)$ is nonempty.

One can see that the conditions from above on Z are more general than those from [Cross 1998, Theorem VI.3.3], leading to the same conclusion.

5. Applications to Arens polynomial calculus

Given the linear relations Z, Z_1, Z_2 in $X \times X$, and $\alpha \in \mathbb{C}$, we may consider, as usual (see e.g., [Arens 1961; Cross 1998]), the following linear relations in X. The *composition* of Z_1 and Z_2:

$$Z_1 \circ Z_2 = \{(u, w) \in X \times X;\ (u, v) \in Z_2,\ (v, w) \in Z_1 \text{ for some } v \in X\},$$

which will be also denoted by $Z_1 Z_2$. The *sum* of Z_1 and Z_2:

$$Z_1 + Z_2 = \{(u, v + w);\ u \in D(Z_1) \cap D(Z_2),\ (u, v) \in Z_1,\ (u, w) \in Z_2\}.$$

The *product* of Z by a number $\alpha \in \mathbb{C}$:

$$\alpha Z = \{(u, \alpha v);\ (u, v) \in Z\} = \alpha I \circ Z,$$

where we identify the operator αI with its graph. Note that $Z_1 + Z_2$ is not an algebraic sum and that $0Z$ is the null operator on $D(Z)$.

For a linear relation $Z \subset X \times X$ we write

$$Z^n := \underbrace{Z \circ Z \circ \cdots \circ Z}_{\text{n}} \quad \text{for } n \in \mathbb{N}^*.$$

If $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$ for $z \in \mathbb{C}$, following Arens [1961] we define the relation

$$p_A(Z) := \alpha_0 I + \alpha_1 Z + \cdots + \alpha_n Z^n.$$

Remark 34. Let Z, Z_1 and Z_2 be linear relations defined on a linear space X. The following assertions, which are well known, follow by a simple calculation.

(i) For any $\xi, \eta \in \mathbb{C}$, one has that $(\xi I - Z)(\eta I - Z) = (\eta I - Z)(\xi I - Z)$.

(ii) $(Z_1 Z_2)^\dagger = Z_2^\dagger Z_1^\dagger$.

We recall that the symbol $\sigma_A(Z)$ denotes the Arens spectrum of the linear relation Z; see Remark 4(ii). We also define $\rho_A(Z) := \mathbb{C} \setminus \sigma_A(Z)$.

The next proposition enables us to apply the results from the previous sections to linear relations of the form $p_A(Z)$; see also [Kascic 1968, Theorem 3.16].

Proposition 35. *If Z is a closed linear relation on the Banach space X such that $\rho_A(Z) \neq \varnothing$ and p is a polynomial, then $p_A(Z)$ is a closed linear relation on X.*

Proof. Fix a $\lambda \in \rho_A(Z)$, so $(\lambda I - Z)^\dagger \in \mathcal{B}(X)$. Using [Brezis 1983, Theorem III.9], we obtain that $(\lambda I - Z)^\dagger$ is continuous from $(X, \sigma(X, X'))$ to $(X, \sigma(X, X'))$. Therefore we can finish by applying [Kascic 1968, Theorem 3.16]. □

The next results show that the functional calculus introduced in Theorem 16 agrees, in some sense, with the Arens polynomial calculus.

Remark 36. Let Z be a closed linear relation in X such that $\sigma(Z) = \{\infty\}$ and the point ∞ is m-regular for some integer $m \geq 1$. Let us compute $p_A(Z)$, where $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$ for $z \in \mathbb{C}$. According to Corollary 31 (see also Remark 12), there exists $Q \in \mathcal{B}(X)$ such that $Q^{m+1} = 0$ and $Z = G(Q)^\dagger$. Hence $Z^k = G(Q^k)^\dagger$ for all integers $k \geq 0$. In particular, $Z^k = G(0)^\dagger$ if $k \geq m+1$. In other words, $p_A(Z) = p_A(G(Q)^\dagger)$. Therefore, if $n = 0$ we have $p_A(Z) = \alpha_0 G(I)$; if $1 \leq n \leq m$ we have

$$\begin{aligned} p_A(Z) &= \alpha_0 G(I) + \alpha_1 G(Q)^\dagger + \cdots + \alpha_n G(Q^n)^\dagger \\ &= \{(x_0, \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n);\, x_0 = Q x_1 = \cdots = Q^n x_n\}; \end{aligned}$$

and if $n \geq m+1$,

$$p_A(Z) = \{(0, \alpha_1 x_1 + \cdots + \alpha_m x_m + y_m);\, Q x_1 = \cdots = Q^n x_n = 0,\, y_m \in X\},$$

Proposition 37. *Let Z be a closed linear relation in X such that the point ∞ is m-regular for Z for some integer $m \geq 1$. Then there exist closed linear subspaces X_1 and X_2 with $X_1 \oplus X_2 = X$, and operators $A_1 \in \mathcal{B}(X_1)$ with $A_1^{m+1} = 0$, and $A_2 \in \mathcal{B}(X_2)$, such that*

$$p_A(Z) = p_A(G(A_1)^\dagger) \oplus G(p_A(A_2)),$$

with $p_A(G(A_1)^\dagger)$ computed as in Remark 36.

Proof. If $Z = Z_1 \oplus Z_2$, then $p_A(Z) = p_A(Z_1) \oplus p_A(Z_2)$. In particular, using $Z_1 = G(A_1)^\dagger$ and $Z_2 = G(A_2)$ obtained by Corollary 31 (see also Remark 12), we deduce the formula from the statement. Clearly, the computation of $p_A(G(A_1)^\dagger)$ is given by Remark 36 for $Q = A_1$. □

Proposition 38. *Let Z be a closed linear relation with $\sigma(Z) \ni \infty$, and let $f \in \mathcal{O}(Z)$. Assume that $f_n(\lambda) = \lambda^n f(\lambda) \in \mathcal{O}(Z)$, where $n \geq 1$ is an integer. Then we have $(f(Z)x, (pf)(Z)x) \in p_A(Z)$ for all polynomials p of degree n and all vectors $x \in X$.*

Proof. Set $f_k(\lambda) = \lambda^k f(\lambda) \in \mathcal{O}(Z)$ for $1 \leq k \leq n$. It follows, as in Lemma 19, that $(f(Z)x, f_1(Z)x) \in Z$. Similarly, $(f_{k-1}(Z)x, f_k(Z)x) \in Z$ for all $k = 2, \ldots, n$. Consequently, $(f(Z)x, f_k(Z)x) \in Z^k$ for all $k = 1, \ldots, n$.

If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ ($z \in \mathbb{C}$), then

$$(f(Z)x, (pf)(Z)x) = (f(Z)x, a_0 f(Z)x + a_1 f_1(Z)x + \cdots + a_n f_n(Z)x \in p_A(Z).$$

□

We have the following *spectral mapping theorem* for polynomials.

Proposition 39. *Let Z be a closed linear relation on the Banach space X such that $\rho(Z) \neq \varnothing$ and let p be a nonconstant polynomial.*

(i) $\sigma_A(p_A(Z)) = p(\sigma_A(Z))$.

(ii) *If $\infty \in \sigma(p_A(Z))$, then $\infty \in \sigma(Z)$. Conversely, if $\infty \in \sigma(Z)$ and ∞ is not isolated in $\sigma(Z)$, then $\infty \in \sigma(p_A(Z))$.*

Proof. (i) This part follows with minor changes as [Arens 1961, Theorem 2.5]. For this reason, we omit the details.

(ii) Assume that $\infty \in \sigma(p_A(Z))$. Assuming $\infty \notin \sigma(Z)$, we deduce that $Z = G(T)$, with $T \in \mathcal{B}(X)$, via Corollary 14. In this case, as we have $p_A(Z) = G(p_A(T))$ and $p_A(T) \in \mathcal{B}(X)$, we infer that $\infty \notin \sigma(p_A(Z))$, which is not possible.

Conversely, assume that $\infty \in \sigma(Z)$ and that ∞ is not isolated in $\sigma(Z)$. Then we can find a sequence $(\lambda_n)_n$ in $\sigma(Z)$ such that $\lim_{n\to\infty} \lambda_n = \infty$. Since $\mu_n = p_A(\lambda_n) \in \sigma(p(Z))$ for all n by (i), it follows that $\infty = \lim_{n\to\infty} \mu_n \in \sigma(p_A(Z))$. □

Remark. If $Z = \{0\} \times X$ and $p(z) = \alpha_0$, then $\sigma(Z) = \{\infty\}$, while $\sigma(p_A(Z)) = \sigma(\alpha_0 I) = \{\alpha_0\}$. In other words, there is a linear relation Z with ∞ isolated in $\sigma(Z)$ such that $\infty \notin \sigma(p_A(Z))$ for some polynomial p_A.

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