Unbounded Extensions and Operator Moment Problems

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Abstract

Extension results, expressed in terms of complete boundedness, leading to necessary and sufficient conditions for the solvability of power moment problems with unbounded operator data are given. As an application, necessary and sufficient conditions for the existence of selfadjoint and normal extensions for some classes of commuting tuples of unbounded linear operators are obtained.

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1 Introduction.

Let Ω be a nonempty set, and let Σ be a σ-algebra of subsets of Ω. Let also $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ be the algebra of all bounded, linear operators on $\mathcal{H}$. A fundamental concept in functional analysis, connecting the objects above, is that of *spectral measure* (sometimes designed as a *resolution of the identity*, see [Rud], Definition 12.17). Given a spectral measure $E : \Sigma \to B(\mathcal{H})$, one can associate to each measurable function $f : \Omega \to \mathbb{C}$ a densely defined, closed operator in $\mathcal{H}$, say $f(T)$, as a result of an integration, and

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the assignment $f \to f(T)$ enjoys a long list of useful properties (see [Rud], Theorem 13.24).

The construction of a spectral measure is not always an easy matter. An important tool to perform such a construction is offered by Naimark's dilation theorem (see, for instance, [Pau], Theorem 4.6). Naimark's theorem shows that the construction a spectral measure, associated to a certain problem, can be often reduced to the construction of a (somewhat more accessible object called) *positive measure*, i.e., an operator-valued map $F : \Sigma \to B(\mathcal{H})$, assuming positive values, such that $F_{x,y} := \langle F(*)x, y\rangle$ is a complex-valued measure for all $x, y \in \mathcal{H}$, where $\langle *, *\rangle$ is the inner product of $\mathcal{H}$.

Given a positive measure $F : \Sigma \to B(\mathcal{H})$, and a linear space $\mathcal{S}$ consisting of Σ-measurable complex-valued functions on Ω, we may consider the following subset of $\mathcal{H}$:

$$\mathcal{D}_{F,\mathcal{S}} := \{x \in \mathcal{H}, f \in L^2(F_{x,x}),\ f \in \mathcal{S}\}.$$

It is easily seen that $\mathcal{D}_{F,\mathcal{S}}$ is a subspace of $\mathcal{H}$. Moreover, by replacing, if necessary, the measure F by its compression to the closure of $\mathcal{D}_{F,\mathcal{S}}$, we may assume, with no loss of generality, that the space $\mathcal{D}_{F,\mathcal{S}}$ is dense in $\mathcal{H}$. In that case, we have a map assigning to each function $f \in \mathcal{S}$ a sesquilinear form s_f on $\mathcal{D} = \mathcal{D}_{F,\mathcal{S}}$, given by $s_f(x, y) := \int f\, dF_{x,y}$, $x, y \in \mathcal{D}$.

Conversely, given an inner product space $\mathcal{D}$, a linear space $\mathcal{S}$ consisting of Σ-measurable complex-valued functions on Ω, and a map assigning to each function $f \in \mathcal{S}$ a sesquilinear form s_f on $\mathcal{D}$, a positive measure $F : \Sigma \to B(\mathcal{H})$ such that $s_f(x, y) = \int f\, dF_{x,y}$ for all $x, y \in \mathcal{D}$ is sought, where $\mathcal{H}$ is the Hilbert space obtained by completing the inner product space $\mathcal{D}$. This is a special type of a moment problem which is described in [Vas1] (where other details and references concerning such problems can be found).

The main purpose of this paper is to characterize, in terms of complete boundedness and complete positivity (fruitful concepts introduced by Arveson [Arv], and extended to more general conditions by Powers [Pow]), the existence of extensions of some linear maps, defined on subspaces of fractions of continuous functions, whose values are sesquilinear forms on inner product spaces (see Theorem 2.5), following the scalar model initiated in [Vas2].

In the third section, a solution to a moment problem as described above (see Theorem 3.2) is given. This result is then applied to obtain necessary and sufficient conditions for the existence of selfadjoint or normal extensions for certain commuting families of unbounded operators.

As a sample, we present a particular case of Theorem 3.3. Let $T_1, T_2 : \mathcal{D} \mapsto \mathcal{D}$ be commuting symmetric operators in the inner product space $\mathcal{D}$. The operators T_1, T_2 admit commuting selfadjoint extensions if and only if for all

$l_1, l_2 \in \mathbb{Z}_+$, $m \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{D}$ with

$$\sum_{j=1}^{m} \langle (1+T_1^2)^{l_1}(1+T_2^2)^{l_2} x_j, x_j \rangle \leq 1, \quad \sum_{j=1}^{m} \langle (1+T_1^2)^{l_1}(1+T_2^2)^{l_2} y_j, y_j \rangle \leq 1,$$

and for all $m \times m$ two variable polynomial matrices $p = (p_{j,k})$ with the property $\sup_{t_1,t_2} \|(1+t_1^2)^{-l_1}(1+t_2^2)^{-l_2} p(t_1,t_2)\|_m \leq 1$, we have

$$\left| \sum_{j,k=1}^{m} \langle p_{j,k}(T_1,T_2) x_k, y_j \rangle \right| \leq 1,$$

where $\| * \|_m$ is the norm in the algebra of $m \times m$ complex matrices.

Theorem 3.3 gives a characterization of the existence of selfadjoint extensions whose joint spectral measure is supported by a given set, for some finite families of commuting symmetric operators.

The existence of normal extensions is characterized by Theorem 3.4. We note that the present normal extension results are essentially different from the corresponding ones in [StSz2] or [Vas2].

Now, let us describe some of the main tools used in this work. Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the algebra of all continuous, complex-valued functions, endowed with the sup-norm $\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|$. As before, let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ be the algebra of all bounded, linear operators on $\mathcal{H}$. We shall use the following results:

Theorem A. *Let $\Psi : C(\Omega) \to B(\mathcal{H})$ be linear, positive and unital. Then Ψ is completely positive and completely contractive.*

This assertion is essentially Theorem 4 in [Sti], see also [Pau], Theorem 3.11 and Proposition 3.6.

Theorem B. *Let $\mathcal{M} \subset C(\Omega)$ be a subspace with $1 \in \mathcal{M}$. If $\Phi : \mathcal{M} \to B(\mathcal{H})$ is a unital, complete contraction, then there exists a (completely) positive map $\Psi : C(\Omega) \to B(\mathcal{H})$ extending Φ.*

This is a consequence of Arveson's extension theorem (see [Arv] or [Pau], Corollary 7.6).

Let $\mathcal{Q}$ be a family of non-null positive elements of $C(\Omega)$. We say that $\mathcal{Q}$ is a *multiplicative family* if (i) $1 \in \mathcal{Q}$, (ii) $q', q'' \in \mathcal{Q}$ implies $q'q'' \in \mathcal{Q}$, and (iii) if $qh = 0$ for some $q \in \mathcal{Q}$ and $h \in C(\Omega)$, then $h = 0$.

Let $C(\Omega)/\mathcal{Q}$ denote the algebra of fractions with numerators in $C(\Omega)$, and with denominators in the multiplicative family $\mathcal{Q}$, which is a unital $\mathbb{C}$-algebra (see, for instance, [Wae] for details). This algebra has a natural involution $f \to \bar{f}$, induced by the natural involution of $C(\Omega)$.

To define a natural topological structure on $C(\Omega)/\mathcal{Q}$, for every $q \in \mathcal{Q}$ we define the space

$$C(\Omega)/q := \{f \in C(\Omega)/\mathcal{Q}; qf \in C(\Omega)\}.$$

Obviously, $C(\Omega)/q \supset C(\Omega)$. Setting $\|f\|_{\infty,q} := \|qf\|_\infty$ for each $f \in C(\Omega)/q$, the pair $(C(\Omega)/q, \| * \|_{\infty,q})$ becomes a Banach space (see also [Vas2]).

Remark 1.1 *In the family $\mathcal{Q}$ there is a natural partial ordering, which is reflexive and transitive but not necessarily symmetric, written as $q'|q''$ for $q', q'' \in \mathcal{Q}$, meaning q' divides q'', that is, there exists a $q \in \mathcal{Q}$ such that $q'' = q'q$.*

Assuming that the constant function 1 has no divisor in $\mathcal{Q} \setminus \{1\}$, the relation $q'|q''$ becomes symmetric too, but this hypothesis is not necessary for further development.

Note also that if $q', q'' \in \mathcal{Q}$ and $q'|q''$, then $C(\Omega)/q' \subset C(\Omega)/q''$ with continuous inclusion mapping $i_{q',q''} : C(\Omega)/q' \hookrightarrow C(\Omega)/q''$.

Indeed, if $q'' = q'q$ and $f \in C(\Omega)/q'$, then

$$\|f\|_{\infty,q''} = \|q'qf\|_\infty \le \|q\|_\infty \|q'f\|_\infty = \|q\|_\infty \|f\|_{\infty,q'} \,.$$

For this reason, $C(\Omega)/\mathcal{Q} = \bigcup_{q\in\mathcal{Q}} C(\Omega)/q$ can be naturally regarded as an inductive limit of Banach spaces.

As noticed in [Vas2], the algebras of fractions of continuous functions provide an appropriate framework for the study of positive measures having a certain decay related to the given multiplicative family.

2 Positive maps on spaces of fractions

Let Ω be a compact Hausdorff space, let $\mathcal{Q} \subset C(\Omega)$ be a multiplicative family, and let $C(\Omega)/\mathcal{Q}$ be the algebra of fractions with numerators in $C(\Omega)$, and with denominators in $\mathcal{Q}$.

We use throughout the text the notation q^{-1} to designate the fraction $1/q$ for any $q \in \mathcal{Q}$. In each space $C(\Omega)/q$ we have a positive cone $(C(\Omega)/q)^+$ consisting of those elements $f \in C(\Omega)/q$ such that $qf \ge 0$ as a continuous function.

Let $\mathcal{D}$ be an inner product space (whose inner product will be denoted by $\langle *, * \rangle$), and let $SF(\mathcal{D})$ be the vector space of all sesquilinear forms on $\mathcal{D}$. The Hilbert space completion of $\mathcal{D}$ will be denoted by $\mathcal{H}$.

Definition 2.1 *Fix a $q \in \mathcal{Q}$. A linear map $\psi : C(\Omega)/q \to SF(\mathcal{D})$ will be called unital if $\psi(1)(x,y) = \langle x, y \rangle$, $x, y \in \mathcal{D}$. We say that ψ is positive if $\psi(f)$ is positive semidefinite for all $f \in (C(\Omega)/q)^+$.*

More generally, let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty. Let $\mathcal{F} = \sum_{q \in \mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{F} \to SF(\mathcal{D})$ be linear. The map ψ is said to be unital (resp. positive) if $\psi|C(\Omega)/q$ is unital (resp. positive) for all $q \in \mathcal{Q}_0$.

Following [Pow], we shall also use a stronger positivity definition. A linear subspace $\mathcal{F} = \sum_{q \in \mathcal{Q}_0} \mathcal{F}_q$ of $C(\Omega)/\mathcal{Q}$ (where $\mathcal{Q}_0 \subset \mathcal{Q}$ and $\mathcal{F}_q \subset C(\Omega)/q$ for all $q \in \mathcal{Q}_0$) will be called *symmetric*, if for all $q \in \mathcal{Q}_0$ and $f \in \mathcal{F}_q$ we have $\bar{f} \in \mathcal{F}_q$. We denote by $M(\mathcal{F})$ the linear space of all finite matrices over $\mathcal{F}$, i.e. all matrices $(f_{j,k})_{j,k \in \mathbb{N}}$ such that $f_{j,k} \neq 0$ for at most finitely many $(j,k) \in \mathbb{N}^2$. The set $M(C(\Omega)/\mathcal{Q})$ has the structure of a $*$-algebra in an obvious manner, and it can be identified with $\bigcup_{q \in \mathcal{Q}} M(C(\Omega)/q)$. For $q \in \mathcal{Q}$, let $\mathcal{K}_q$ denote the set of all $f = (f_{j,k})_{j,k \in \mathbb{N}}$ in $M(C(\Omega)/q)$ such that for all $\omega \in \Omega$ the matrix $(q(\omega) f_{j,k}(\omega))_{j,k \in \mathbb{N}}$ is positive semidefinite. Then an easy calculation shows that $\mathcal{K} := \bigcup_{q \in \mathcal{Q}} \mathcal{K}_q$ is a cone, which is admissible in the sense of Powers (see [Pow], Definition 3.1).

Let $\phi : \mathcal{F} \to SF(\mathcal{D})$ be linear. We say that ϕ is *completely positive* (in the sense of Powers [Pow]), if for all matrices $f = (f_{j,k})_{j,k \in \mathbb{N}} \in M(\mathcal{F}) \cap \mathcal{K}$ we have

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \phi(f_{j,k})(x_k, x_j) \geq 0, \quad (x_j)_{j \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}}. \tag{1}$$

Theorem 2.2 *Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty, let $\mathcal{F} = \sum_{q \in \mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{F} \to SF(\mathcal{D})$ be linear and unital. The map ψ is positive if and only if*

$$\sup\{|\psi(hq^{-1})(x,x)|;\ h \in C(\Omega),\ \|h\|_\infty \leq 1\} = \psi(q^{-1})(x,x), \quad q \in \mathcal{Q}_0,\ x \in \mathcal{D}.$$

If $\psi : \mathcal{F} \to SF(\mathcal{D})$ is positive, there exists a unique positive $B(\mathcal{H})$-valued measure F on the Borel subsets of Ω such that

$$\psi(f)(x,y) = \int_\Omega f \, dF_{x,y}, \quad f \in \mathcal{F},\ x, y \in \mathcal{D}.$$

Proof. Let $\psi : \mathcal{F} \to SF(\mathcal{D})$ be linear and unital. Set $\psi_1 = \psi|C(\Omega)$. The map ψ_1 is positive if and only if there exists a positive, unital, linear map $\Psi_1 : C(\Omega) \to B(\mathcal{H})$ such that $\psi_1(h)(x,y) = \langle \Psi_1(h)x, y \rangle$, $x, y \in \mathcal{D}$. This can be obtained by standard extension arguments, which will be briefly presented for the convenience of the reader.

Assuming ψ_1 positive, if $h \in C(\Omega)$ is positive, then

$$0 \leq \psi_1(h)(x,x) \leq \|h\|_\infty \|x\|^2, \quad x \in \mathcal{D},$$

because ψ_1 is also unital. From this estimate we derive

$$|\psi_1(h)(x,y)| \leq \|h\|_\infty \|x\| \, \|y\|, \quad x,y \in \mathcal{D},$$

via the Cauchy-Schwarz inequality. Using the density of $\mathcal{D}$ in $\mathcal{H}$ and the Riesz theorem concerning the dual of $\mathcal{H}$, we derive the existence of a positive operator $\Psi_1(h) \in B(\mathcal{H})$ such that $\psi_1(x,y) = \langle \Psi_1(h)x, y\rangle$, $x,y \in \mathcal{D}$. Moreover, the assignment $h \to \Psi_1(h)$, $h \geq 0$, is additive and positively homogeneous. As every function $h \in C(\Omega)$ is an algebraic combination of four positive functions, we derive easily the general assertion.

Conversely, the existence of a positive, unital, linear map $\Psi_1 : C(\Omega) \to B(\mathcal{H})$ such that $\psi_1(h)(x,y) = \langle \Psi_1(x), y\rangle$, $x,y \in \mathcal{D}$ clearly implies that ψ_1 is positive.

We use the fact that a linear functional $\theta : C(\Omega) \to \mathbb{C}$ is positive if and only if it is continuous and $\|\theta\| = \theta(1)$

Set $\tilde{\psi}_q(h) = \psi(hq^{-1})$, $h \in C(\Omega)$, $q \in \mathcal{Q}_0$.

Suppose ψ positive and fix an $x \in \mathcal{D}$. As $C(\Omega) \subset \mathcal{F}$ and each positive function $h \in C(\Omega)$ is also positive in $C(\Omega)/q$, the map $\tilde{\psi}_q$ is positive on $C(\Omega)$. Put $\tilde{\psi}_{q,x}(h) = \tilde{\psi}_q(h)(x,x)$, $h \in C(\Omega)$, which is a positive functional on $C(\Omega)$. Hence

$$\begin{aligned}\|\tilde{\psi}_{q,x}\| &= \sup\{|\psi(hq^{-1})(x,x)|;\, h \in C(\Omega),\, \|h\|_\infty \leq 1\} \\ &= \tilde{\psi}_q(1)(x,x) = \psi(q^{-1})(x,x),\, q \in \mathcal{Q}_0,\end{aligned}$$

which is the stated condition.

Conversely, the equality $\|\tilde{\psi}_{q,x}\| = \psi(q^{-1})(x,x) = \tilde{\psi}_q(1)(x,x)$ shows that $\tilde{\psi}_{q,x}$ is positive on $C(\Omega)$. Then there exists a positive (Borel) measure $\mu_{q,x}$ on Ω such that $\tilde{\psi}_{q,x}(h) = \int_\Omega h \, d\mu_{q,x}$, $h \in C(\Omega)$, for all $q \in \mathcal{Q}_0$.

The relation $\tilde{\psi}_{q_1,x}(hq_1) = \psi(h)(x,x) = \tilde{\psi}_{q_2,x}(hq_2)$ for all $q_1, q_2 \in \mathcal{Q}_0$ and $h \in C(\Omega)$ implies the equality $q_1\mu_{q_1,x} = q_2\mu_{q_2,x}$. Therefore, there exists a positive measure μ_x such that $\mu_x = q\mu_{q,x}$ for all $q \in \mathcal{Q}_0$.

The equality $\mu_x = q\mu_{q,x}$ shows the set $\{\omega; q(\omega) = 0\}$ must be μ_x-null. Consequently, $\mu_{q,x} = q^{-1}\mu_x$, and the function q^{-1} is μ_x-integrable for all $q \in \mathcal{Q}_0$. Moreover, the measure μ_x is uniquely determined because of the equality $\psi(h)(x,x) = \int_\Omega h \, d\mu_x$, $h \in C(\Omega)$. Setting $4\mu_{x,y} = \mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}$, $x,y \in \mathcal{D}$, we have the representation $\psi(h)(x,y) = \int_\Omega h \, d\mu_{x,y}$ for all $h \in C(\Omega)$ and $x,y \in \mathcal{D}$, via the polarization formula. This shows, in particular, that the map $\psi_1 = \psi|C(\Omega)$ is (unital and) positive. Therefore, there exists a unital positive map $\Psi_1 : C(\Omega) \to B(\mathcal{H})$ such that $\psi_1(h)(x,y) = \langle \Psi_1(h)x, y\rangle$, $x, y \in \mathcal{D}$. It is well known that the map Ψ_1 has an integral representation $\Psi_1(h) = \int_\Omega h\, dF$, where F is a positive $B(\mathcal{H})$-valued measure F on the Borel subsets of Ω. As $F_{x,y} = \mu_{x,y}$ for all $x, y \in \mathcal{D}$, and $\mathcal{D}$ is dense in $\mathcal{H}$, the measure F is uniquely determined.

If $f \in \mathcal{F}$ is arbitrary, then $f = \sum_{j\in J} h_j q_j^{-1}$, with $h_j \in C(\Omega)$, $q_j \in \mathcal{Q}_0$ for all $j \in J$, J finite. We can write

$$\psi(f)(x,x) = \sum_{j\in J} \tilde{\psi}_{q_j}(h_j)(x,x) = \sum_{j\in J} \int_\Omega h_j\, d\mu_{q_j,x} = \int_\Omega f d\mu_x = \int_\Omega f\, dF_{x,x}$$

for all $x \in \mathcal{D}$, from which we easily derive the formula in the statement. The measure F being positive, the map ψ must be also positive. □

Remark 2.3 *Let* $\mathcal{F} := \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$ *for a nonempty* $\mathcal{Q}_0 \subset \mathcal{Q}$, *and let* $\psi : \mathcal{F} \to SF(\mathcal{D})$ *be a unital positive map on* $\mathcal{F}$. *Set* $\psi_q = \psi|C(\Omega)/q$ *and* $\psi_{q,x}(f) = \psi_q(f)(x,x)$ *for all* $q \in \mathcal{Q}_0$, $h \in C(\Omega)/q$ *and* $x \in \mathcal{D}$. *If* $\tilde{\psi}_q$ *is defined as in the proof of Theorem 2.2, we have the equality*

$$\|\psi_{q,x}\| = \sup_{\|f\|_{\infty,q}\le 1} |\psi_{q,x}(f)| = \sup_{\|h\|_\infty \le 1} |\tilde{\psi}_{q,x}(h)| = \|\tilde{\psi}_{q,x}\| = \psi(q^{-1})(x,x)$$

for all $q \in \mathcal{Q}_0$ *and* $x \in \mathcal{D}$.

We shall need the following fact:

Lemma 2.4 *Let* $\mathcal{Q}$ *be a multiplicative system on* Ω *and let* $q, q_1, q_2 \in \mathcal{Q}$ *be such that* $q = q_1 q_2$. *Suppose that* $\psi : C(\Omega)/q \to SF(\mathcal{D})$ *is a positive, linear map satisfying*

$$\psi(q^{-1})(x,x) > 0 \text{ and } \psi(q_1^{-1})(x,x) > 0\,, \text{ for all } x \in \mathcal{D}\setminus\{0\},$$

so that $\langle *, *\rangle_q := \psi(q^{-1})(*,*)$ *and* $\langle *, *\rangle_{q_1} := \psi(q_1^{-1})(*,*)$ *are scalar products on* $\mathcal{D}$. *Let* $\mathcal{D}_q$ *and* $\mathcal{D}_{q_1}$ *denote the completions of* $\mathcal{D}$ *with respect to* $\langle *, *\rangle_q$ *and* $\langle *, *\rangle_{q_1}$, *respectively. Then there exist uniquely determined linear maps* $\Psi_q : C(\Omega) \to B(\mathcal{D}_q)$ *and* $\Psi_{q,q_1} : C(\Omega) \to B(\mathcal{D}_{q_1})$ *such that, for all* $h \in C(\Omega)$,

$$\langle \Psi_q(h)x, y\rangle_q = \psi(h/q)(x,y),\ x, y \in \mathcal{D}, \tag{2}$$

and

$$\langle \Psi_{q,q_1}(h)x, y\rangle_{q_1} = \psi(h/q_1)(x,y) = \langle \Psi_q(hq_2)x, y\rangle_q,\ x, y \in \mathcal{D}. \tag{3}$$

Moreover, the maps Ψ_q *and* Ψ_{q,q_1} *are unital, completely contractive and completely positive.*

Proof. By the positivity of ψ we obtain, for all $h \in (C(\Omega))^+$ and $x \in \mathcal{D}$,

$$\psi(h/q)(x,x) \le \|h\|_\infty \psi(q^{-1})(x,x) = \|h\|_\infty \langle x, x\rangle_q . \tag{4}$$

If h is an arbitrary continuous function on Ω, it can be written in the form $h = h_1 - h_2 + i(h_3 - h_4)$ with continuous functions satisfying $0 \le h_j \le \|h\|_\infty$ on Ω. Representing the sesquilinear forms $(x,y) \mapsto \psi(h_j/q)(x,y)$ in polar form, we conclude from (4) that we have for all $x, y \in \mathcal{D}$ with $\|x\|_q \le 1$ and $\|y\|_q \le 1$

$$|\psi(h/q)(x,y)| \le 16\|h\|_\infty .$$

Therefore, there exists a unique operator $\Psi_q(h) \in B(\mathcal{D}_q)$, such that (2) holds. Moreover, the linearity and positivity of ψ imply the linearity and positivity of Ψ_q. Because of

$$\langle \Psi_q(1)x, y\rangle_q = \psi(q^{-1})(x,y) = \langle x, y\rangle_q \, , \; x, y \in \mathcal{D},$$

we see that $\Psi_q(1)$ is the identity operator on $\mathcal{D}_q$. Hence, by Theorem A in the Introduction, Ψ_q is completely positive and completely contractive.

As q_1 divides q, we have $C(\Omega)/q_1 \subset C(\Omega)/q$ and the restriction of ψ to $C(\Omega)/q_1$ is positive. Hence, replacing in the preceding arguments q by q_1, we obtain a unique linear map $\Psi_{q,q_1} : C(\Omega) \to B(\mathcal{D}_{q_1})$ such that

$$\begin{aligned}\langle \Psi_{q,q_1}(h)x, y\rangle_{q_1} =& \psi(h/q_1)(x,y) = \psi(hq_2/q)(x,y)\\ =& \langle \Psi_q(hq_2)x, y\rangle_q\end{aligned}$$

for all $h \in C(\Omega)$ and $x, y \in \mathcal{D}$, and as before, Ψ_{q,q_1} is completely positive and completely contractive. □

Let " $\prec$ " be another partial ordering on the set $\mathcal{Q}$. We say that " $\prec$ " is *multiplicative* if $q' \prec q''$ implies $q'|q''$ for all $q', q'' \in \mathcal{Q}$.

As usually, a subset $\mathcal{Q}_0 \subset \mathcal{Q}$ is said to be *cofinal* if for every $q \in \mathcal{Q}$ we can find a $q' \in \mathcal{Q}_0$ such that $q \prec q'$.

In the next statement we shall use the notation $\| * \|_{n,\infty}$ to designate the canonical norm of the C^*–algebra $M_n(C(\Omega))$ of $n \times n$ matrices with entries from $C(\Omega)$. Similarly, for further use, the symbol $\| * \|_n$ denotes the canonical norm in the C^*–algebra $M_n(\mathbb{C})$. The norms used in the statement (a) of the next theorem were introduced in Remark 2.3.

Theorem 2.5 *Let $\mathcal{Q}$ be a multiplicative system on Ω endowed with a multiplicative partial ordering " $\prec$ ", and let $\mathcal{Q}_0$ be a cofinal subset of $\mathcal{Q}$ with $1 \in \mathcal{Q}_0$.*

Let $\mathcal{F} = \sum_{q \in \mathcal{Q}_0} \mathcal{F}_q$, where $\mathcal{F}_q$ is a vector subspace of $C(\Omega)/q$ such that $q'^{-1} \in \mathcal{F}_{q'} \subset \mathcal{F}_q$ for all $q' \in \mathcal{Q}_0$ and $q \in \mathcal{Q}_0$, with $q' \prec q$. Let also $\phi : \mathcal{F} \to SF(\mathcal{D})$ be *linear and unital, and set* $\phi_q = \phi|\mathcal{F}_q$, $\phi_{q,x}(*) = \phi_q(*)(x,x)$ *for all* $q \in \mathcal{Q}_0$ *and* $x \in \mathcal{D}$.

The following two statements are equivalent:

(a) *The map* ϕ *extends to a unital, positive, linear map* ψ *on* $C(\Omega)/\mathcal{Q}$ *such that, for all* $x \in \mathcal{D}$ *and* $q \in \mathcal{Q}_0$, *we have:*

$$\|\psi_{q,x}\| = \|\phi_{q,x}\|, \text{ where } \psi_q = \psi|C(\Omega)/q,\ \psi_{q,x}(*) = \psi_q(*)(x,x).$$

(b) (i) $\phi(q^{-1})(x,x) > 0$ *for all* $x \in \mathcal{D} \setminus \{0\}$ *and* $q \in \mathcal{Q}_0$.
(ii) *For all* $q \in \mathcal{Q}_0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{D}$ *with*

$$\sum_{j=1}^{n} \phi(q^{-1})(x_j, x_j) \leq 1, \quad \sum_{j=1}^{n} \phi(q^{-1})(y_j, y_j) \leq 1,$$

and for all $(f_{j,k}) \in M_n(\mathcal{F}_q)$ *with* $\|(qf_{j,k})\|_{n,\infty} \leq 1$, *we have*

$$\left| \sum_{j,k=1}^{n} \phi(f_{j,k})(x_k, y_j) \right| \leq 1.$$

If $\mathcal{F}$ *is a symmetric subspace of* $C(\Omega)/\mathcal{Q}$, *then* (a) *and* (b) *are equivalent to*

(c) ϕ *is completely positive.*

Proof. If $\phi : \mathcal{F} \to SF(\mathcal{D})$ extends to a unital, positive $\psi : C(\Omega)/\mathcal{Q} \to SF(\mathcal{D})$ such that $\|\psi_{q,x}\| = \|\phi_{q,x}\|$ for all $q \in \mathcal{Q}_0$ and $x \in \mathcal{D}$, then $\|\psi_{q,x}\| = \|\phi_{q,x}\| = \phi(q^{-1})(x,x) = \psi(q^{-1})(x,x)$, by Theorem 2.2 and Remark 2.3. Because of $\|q\|_\infty^{-1} \leq q^{-1}$ for all $q \in \mathcal{Q}_0$ and the positivity of ψ, we obtain for all $x \in \mathcal{D}\setminus\{0\}$:

$$\phi(q^{-1})(x,x) = \psi(q^{-1})(x,x) \geq \psi(\|q\|_\infty^{-1})(x,x) = \|q\|_\infty^{-1}\langle x, x\rangle > 0,$$

and thus we have (i).

Applying Lemma 2.4, we obtain, for all $q \in \mathcal{Q}$, a uniquely determined linear map $\Psi_q : C(\Omega) \to B(\mathcal{D}_q)$ satisfying (2), that is completely positive and completely contractive. In particular, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{D}$ with

$$\sum_{j=1}^{n} \|x_j\|_q^2 = \sum_{j=1}^{n} \phi(q^{-1})(x_j, x_j) \leq 1, \quad \sum_{j=1}^{n} \|y_j\|_q^2 = \sum_{j=1}^{n} \phi(q^{-1})(y_j, y_j) \leq 1,$$

and for all $(f_{j,k}) \in M_n(\mathcal{F}_q)$ with $\|(qf_{j,k})\|_{n,\infty} \leq 1$, we have

$$\left| \sum_{j,k=1}^{n} \phi(f_{j,k})(x_k, y_j) \right| = \left| \sum_{j,k=1}^{n} \langle \Psi_q(qf_{j,k})x_k, y_j \rangle_q \right| \leq 1$$

which proves (ii). Hence (a) implies (b).

Moreover, if $\mathcal{F}$ is a symmetric subspace of $C(\Omega)/\mathcal{Q}$ and $F = (f_{j,k})_{j,k=1,\ldots,n} \in M_n(\mathcal{F})$ is positive in the natural order of $M_n(C(\Omega)/\mathcal{Q})$, then $F \in M_n(\mathcal{F}_q)$ for some $q \in \mathcal{Q}_0$ and F is positive in $M_n(C(\Omega)/q)$, i.e. qF is positive in $M_n(C(\Omega))$. By the complete positivity of Ψ_q, we obtain for all $x_1, \ldots, x_n \in \mathcal{D}$,

$$0 \leq \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \Psi_q(qf_{j,k})x_k, x_j \rangle_q = \sum_{j=1}^{n} \sum_{k=1}^{n} \phi(f_{j,k})(x_k, x_j) \, .$$

Hence, we have shown that, in this case, (a) implies (c).

Suppose now that $\phi : \mathcal{F} \to SF(\mathcal{D})$ is a unital, linear map satisfying conditions (i) and (ii). In particular, for all $q \in \mathcal{Q}_0$, the sesquilinear form $\langle *, * \rangle_q := \phi(q^{-1})$ defines a scalar product on $\mathcal{D}$. We write $\mathcal{D}_q$ for the completion of $\mathcal{D}$ with respect to the corresponding norm $\| * \|_q$ (see also Lemma 2.4) and still denote the extended scalar product by $\langle *, * \rangle_q$. As ϕ is unital, we have $\mathcal{D}_1 = \mathcal{H}$. By (ii), for all $h \in q\mathcal{F}_q$, the sesquilinear form $\phi(h/q)$ extends to a uniquely determined bounded sesquilinear form on $\mathcal{D}_q$. Hence there exists a unique operator $\Phi_q(h) \in B(\mathcal{D}_q)$, satisfying

$$\phi(h/q)(x, y) = \phi(q^{-1})(\Phi_q(h)x, y) = \langle \Phi_q(h)x, y \rangle_q$$

for all $x, y \in \mathcal{D}$. Note that $\Phi_q(1)$ is the identity operator on $\mathcal{D}_q$. From condition (ii), we conclude that the unital linear map

$$\Phi_q : q\mathcal{F}_q \to B(\mathcal{D}_q)$$

is a complete contraction and hence, by Theorem B in the Introduction, it extends to a completely positive unital map $\Psi_q : C(\Omega) \to B(\mathcal{D}_q)$. Thus,

$$\langle \Psi_q(qf)x, y \rangle_q = \phi(f)(x, y), \quad f \in \mathcal{F}_q, \ x, y \in \mathcal{D}. \tag{5}$$

Applying Lemma 2.4 to the map $f \mapsto \langle \Psi_q(qf) *, * \rangle_q$ from $C(\Omega)/q$ to $SF(\mathcal{D})$, we obtain a unique, unital, completely contractive and completely positive linear map $\Psi_{q,q_1} : C(\Omega) \to B(\mathcal{D}_{q_1})$, satisfying

$$\langle \Psi_{q,q_1}(h)x, y \rangle_{q_1} = \langle \Psi_q(hq_2)x, y \rangle_q \tag{6}$$

and

$$|\langle \Psi_{q,q_1}(h)x, y \rangle_{q_1}| = |\langle \Psi_q(hq_2)x, y \rangle_q| \leq \|h\|_\infty \|x\|_{q_1} \|y\|_{q_1}, \tag{7}$$

for all $h \in C(\Omega)$, $x, y \in \mathcal{D}$, $q_1, q \in \mathcal{Q}_0$, $q_2 \in \mathcal{Q}$, $q_1 q_2 = q$.

For every $q \in \mathcal{Q}_0$, we denote by K_q the set of all those families

$$a = (a(f, x, y))_{f \in C(\Omega)/q, \ x, y \in \mathcal{D}} \in \mathbb{C}^{C(\Omega)/q \times \mathcal{D} \times \mathcal{D}}$$

satisfying

$$|a(f,x,y)| \leq \|qf\|_\infty \|x\|_q \|y\|_q, \quad f \in C(\Omega)/q,\ x,y \in \mathcal{D}. \tag{8}$$

Endowed with the product topologies, the topological spaces K_q, $q \in \mathcal{Q}_0$, and hence $K := \prod_{q\in\mathcal{Q}_0} K_q$ are compact.

For each $q \in \mathcal{Q}_0$, let now H_q be the set of all $a = (a_{q'})_{q'\in\mathcal{Q}_0} \in K$ such that for all $q' \in \mathcal{Q}_0$ with $q' \prec q$: the map $f \mapsto a_{q'}(f,*,*)$ is a positive linear map from $C(\Omega)/q'$ to $SF(\mathcal{D})$ extending $\phi|\mathcal{F}_{q'}$ and satisfying

$$a_{q'}(f,x,y) = a_q(f,x,y), \quad f \in C(\Omega)/q',\ x,y \in \mathcal{D}. \tag{9}$$

Clearly, the sets H_q are closed in K and hence compact. In order to prove $\bigcap_{q\in\mathcal{Q}_0} H_q \neq \emptyset$, it therefore suffices to show that all finite intersections $H_{q_1} \cap \cdots \cap H_{q_n}$ with $q_1,\dots,q_n \in \mathcal{Q}_0$ are not empty. As $\mathcal{Q}_0$ is cofinal in $\mathcal{Q}$, there exists some $q \in \mathcal{Q}_0$ such that $q_j \prec q$ for $j = 1,\dots,n$. Let the operator $\Psi_q \in B(\mathcal{D}_q)$ and, for all divisors $q' \in \mathcal{Q}_0$ of q, the operators $\Psi_{q,q'} \in B(\mathcal{D}_{q'})$ be as constructed above. We define for all $x,y \in \mathcal{D}$, $q' \in \mathcal{Q}_0$, and all $f \in C(\Omega)/q'$:

$$a_{q'}(f,x,y) := \begin{cases} 0 & \text{if } q' \text{ does not divide } q \\ \langle \Psi_{q,q'}(q'f)x,y\rangle_{q'} & \text{if } q' \text{ divides } q \end{cases}$$

for all $f \in C(\Omega)/q'$, $x,y \in \mathcal{D}$. Notice, that $a := (a_{q'})_{q'\in\mathcal{Q}_0} \in K$ by (7).

Fix an index $j \in \{1,\dots,n\}$ and let $q' \in \mathcal{Q}_0$ satisfy $q' \prec q_j$. Then there are $q'_j, \tilde{q}_j \in \mathcal{Q}$ such that $q_j = q'q'_j$ and $q = q_j\tilde{q}_j$. We conclude from (6) that for all $f \in C(\Omega)/q'$ and $x,y \in \mathcal{D}$ we have:

$$\begin{aligned} a_{q'}(f,x,y) =& \langle \Psi_{q,q'}(q'f)x,y\rangle_{q'} = \langle \Psi_q(q'q'_j\tilde{q}_jf)x,y\rangle_q = \langle \Psi_q(qf)x,y\rangle_q \\ =& \langle \Psi_{q,q_j}(q_jf)x,y\rangle_{q_j} = a_{q_j}(f,x,y)\,, \end{aligned}$$

so that $f \mapsto a_{q'}(f,*,*)$ is a linear map from $C(\Omega)/q'$ to $SF(\mathcal{D})$ which is positive by the positivity of $\Psi_{q,q'}$ and extends $\phi|\mathcal{F}_{q'}$ because of $\mathcal{F}_{q'} \subset \mathcal{F}_q$ and (5). Hence, $a := (a_{q'})_{q'\in\mathcal{Q}_0} \in H_{q_1} \cap \cdots \cap H_{q_n}$.

It follows that there exists some $b = (b_q)_{q\in\mathcal{Q}_0} \in \bigcap_{q\in\mathcal{Q}_0} H_q$. We define now, for all $f \in C(\Omega)/\mathcal{Q} = \bigcup_{q\in\mathcal{Q}_0} C(\Omega)/q$, $x,y \in \mathcal{D}$:

$$\psi(f)(x,y) := b_q(f,x,y) \text{ if } f \in C(\Omega)/q\,.$$

To see that this is well defined, suppose that $f \in C(\Omega)/q_j$, $j = 1,2$ with $q_1, q_2 \in \mathcal{Q}_0$. As $\mathcal{Q}_0$ is cofinal in $\mathcal{Q}$ there exists some $q \in \mathcal{Q}_0$ such that $q_1 \prec q$ and $q_2 \prec q$, and so both divide q. As $b \in H_q$, we have for all $x,y \in \mathcal{D}$,

$$b_{q_1}(f,x,y) = b_q(f,x,y) = b_{q_2}(f,x,y),$$

and the map $(x,y) \mapsto b_q(f,x,y)$ is a sesquilinear form on $\mathcal{D}$. Thus, $\psi : C(\Omega)/\mathcal{Q} \to SF(\mathcal{D})$ is a well defined linear map which is easily seen to be positive and extends ϕ.

Given $q \in \mathcal{Q}_0$, we see from the fact that b satisfies (8) that

$$\|\phi_{q,x}\| \le \|\psi_{q,x}\| \le \|x\|_q^2 = \phi(q^{-1})(x,x) = \phi_{q,x}(q^{-1}) \le \|\phi_{q,x}\|,$$

for all $x \in \mathcal{D}$, which completes the proof of (a).

Finally, suppose that $\mathcal{F}$ is a symmetric subspace of $C(\Omega)/\mathcal{Q}$ and that the condition in (c) is satisfied. Then ϕ is a completely positive map on $\mathcal{F}$ in the sense of [Pow]. Note also that every $f = \bar{f} \in C(\Omega)/\mathcal{Q}$ can be represented as $f = h/q$ with $h \in C(\Omega)$ real valued and $q \in \mathcal{Q}_0$ (via the fact that $\mathcal{Q}_0$ is cofinal in $\mathcal{Q}$). Setting $g = \|h\|_\infty/q$, we have $g \in \mathcal{F}_q$, and the difference $g - f$ is positive in $\mathcal{F}_q$ (even when $g - f$ is regarded as a matrix). In other words, with the terminology of [Pow], the space $\mathcal{F}$ is cofinal in $C(\Omega)/\mathcal{Q}$ with respect to the admissible cone $\mathcal{K}$ (see Definition 2.1). We conclude from Theorem 3.7 in [Pow] that ϕ extends to a completely positive map ψ on $C(\Omega)/\mathcal{Q}$, showing that (c) implies (a), via Theorem 2.2. □

In the scalar case $\mathcal{D} = \mathbb{C}$, we identify $SF(\mathcal{D})$ with $\mathbb{C}$. Using the fact that, in this situation, bounded linear functionals on $\mathcal{F}_q$ are automatically completely bounded and the *cb*-norm coincides with the norm ([Pau], Theorem 3.9), we obtain, as a particular case, Theorem 3.7 in [Vas2].

Corollary 2.6 *Let $\mathcal{Q}$ be a multiplicative system on Ω endowed with a multiplicative partial ordering " $\prec$ " and let $\mathcal{Q}_0$ be a cofinal subset of $\mathcal{Q}$ with $1 \in \mathcal{Q}_0$.*

Let $\mathcal{F} = \sum_{q \in \mathcal{Q}_0} \mathcal{F}_q$, where $\mathcal{F}_q$ is a vector subspace of $C(\Omega)/q$ such that $q'^{-1} \in \mathcal{F}_{q'} \subset \mathcal{F}_q$ for all $q' \in \mathcal{Q}_0$ and $q \in \mathcal{Q}_0$, with $q' \prec q$. A linear functional $\phi : \mathcal{F} \to \mathbb{C}$ with $\phi(1) > 0$ extends to a positive linear functional $\psi : C(\Omega)/\mathcal{Q} \to \mathbb{C}$ with $\|\psi|C(\Omega)/q\| = \|\phi|\mathcal{F}_q\|$ for all $q \in \mathcal{Q}_0$ if and only if $\|\phi|\mathcal{F}_q\| = \phi(q) > 0$ for all $q \in \mathcal{Q}_0$.

Proof. Without loss of generality we may assume $\phi(1) = 1$. With the remarks above, the statement follows directly from Theorem 2.5. □

For every $q \in \mathcal{Q}$ we denote by $Z(q)$ the set $\{\omega \in \Omega; q(\omega) = 0\}$, that is the zeros of q on Ω. For subsets $\mathcal{Q}_1$ of $\mathcal{Q}$ we write $Z(\mathcal{Q}_1) := \bigcup_{q \in \mathcal{Q}_1} Z(q)$.

Combining Theorem 2.5 with Theorem 2.2, we show now:

Corollary 2.7 *Suppose that, with the hypotheses of Theorem 2.5, condition (b) is satisfied. Then there exists a positive $B(\mathcal{H})$–valued measure F on the*

Borel subsets of Ω *such that*

$$\phi(f)(x,y) = \int_\Omega f\, dF_{x,y},\ f \in \mathcal{F},\ x,y \in \mathcal{D}. \tag{10}$$

For every such measure F *and every* $q \in \mathcal{Q}$*, we have* $F(Z(q)) = 0$*. Hence, if* $\mathcal{Q}$ *contains a countable subset* $\mathcal{Q}_1$ *with* $Z(\mathcal{Q}_1) = Z(\mathcal{Q})$*, then* $F(Z(\mathcal{Q})) = 0$*.*

Proof. The existence of F with (10) follows from Theorems 2.5 and 2.2. If F is any such measure and $q \in \mathcal{Q}$, since $\mathcal{Q}_0$ is cofinal in $\mathcal{Q}$, there exists some $q_0 \in \mathcal{Q}_0$ with $q \prec q_0$, and hence $Z(q) \subset Z(q_0)$. Fix an $x \in \mathcal{D} \setminus \{0\}$ and an arbitrary $\varepsilon > 0$. With $A_\varepsilon := \{\omega \in \Omega; 0 \le q_0(\omega) \le \varepsilon/\phi(q_0^{-1})(x,x)\}$, we obtain

$$\begin{aligned} F_{x,x}(Z(q)) \le & \int_{A_\varepsilon} \frac{q_0(\omega)}{q_0(\omega)} dF_{x,x}(\omega) \le \int_{A_\varepsilon} \frac{\varepsilon}{\phi(q_0^{-1})(x,x) q_0(\omega)} dF_{x,x}(\omega) \\ \le & \frac{\varepsilon}{\phi(q_0^{-1})(x,x)} \int_\Omega q_0^{-1} dF_{x,x} = \varepsilon \end{aligned}$$

and thus $F_{x,x}(Z(f)) = 0$ for all $x \in \mathcal{D}$, which implies $F(Z(q)) = 0$. By means of the σ–additivity of the scalar measures, we also obtain the last statement of the Corollary. $\square$

3 Selfadjoint and normal extensions

In a classical paper by Fuglede (see [Fug]) dealing with the multidimensional power moment problem, an operator theoretic characterization of moment multi-sequences in terms of existence of some commuting selfadjoint extensions is given. This is an important motivation to study selfadjoint or normal extensions of some given linear transformations.

To fix the terminology, let $T_1, \ldots, T_n$ be linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under $T_1, \ldots, T_n$ and that $T_1, \ldots, T_n$ commute on $\mathcal{D}$. We say that the tuple $T = (T_1, \ldots, T_n)$ has a *selfadjoint* (resp. *normal*) *extension* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and a tuple $A = (A_1, \ldots, A_n)$ consisting of commuting (in the sense, that the corresponding spectral measures commute) selfadjoint (resp. normal) operators in $\mathcal{K}$ such that $\mathcal{D} \subset \cap_{j=1}^n D(A_j)$ and $T_j x = A_j x$, $x \in \mathcal{D}$, for all $j = 1, \ldots, n$.

Remark 3.1 *Our methods give primarily some "dilations" but, as in the proofs of Theorem 3.3 in* [Bi] *and Lemme 2 in* [Fo]*, we can prove that these are actually extensions. Let us explain the meaning of this assertion, giving some direct arguments.*

Note that if $S : D(S) \subset \mathcal{H} \mapsto \mathcal{H}$ *is a symmetric operator with* $SD(S) \subseteq D(S)$*,* *and if $B : D(B) \subset \mathcal{K} \mapsto \mathcal{K}$ is a selfadjoint operator such that $\mathcal{H} \subset \mathcal{K}$, $D(S) \subset D(B^2)$ and $S^k x = PB^k x$, $x \in D(S)$, $k = 1, 2$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then $Sx = Bx$ for all $x \in D(S)$. Indeed, we have*

$$\langle Sx, Sx\rangle = \langle Sx, PBx\rangle = \langle Sx, Bx\rangle$$

and

$$\langle Bx, Sx\rangle = \langle PBx, Sx\rangle = \langle S^2x, x\rangle = \langle B^2x, x\rangle = \langle Bx, Bx\rangle,$$

for all $x \in D(S)$. Therefore

$$\|Sx - Bx\|^2 = \langle Sx, Sx\rangle - \langle Sx, Bx\rangle - \langle Bx, Sx\rangle + \langle Bx, Bx\rangle = 0,$$

for all $x \in D(S)$.

Similarly, if $S : D(S) \subset \mathcal{H} \mapsto \mathcal{H}$ is an arbitrary linear operator and if $B : D(B) \subset \mathcal{K} \mapsto \mathcal{K}$ is a normal operator such that $\mathcal{H} \subset \mathcal{K}$, $D(S) \subset D(B)$, $Sx = PBx$ and $\|Sx\| = \|Bx\|$ for all $x \in D(S)$, then $Sx = Bx$ for all $x \in D(S)$.

Let $\mathcal{D}$ be a complex inner product space and let $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ be a linear unital map. We are interested to find a positive measure F on the Borel subsets of $\mathbb{R}^n$, with values in $B(\mathcal{H})$, where $\mathcal{H}$ denotes the completion of $\mathcal{D}$, such that $\phi(p)(x, y) = \int p\, dF_{x,y}$ for all $p \in \mathcal{P}_n$ and $x, y \in \mathcal{D}$, which is, in fact, an operator moment problem (see, for instance, [Vas1]). When such a positive measure F exists, we say that $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ is a *moment form* and the measure F is said to be a *representing measure* for ϕ. When the representing measure F of ϕ vanishes on the complement of the closed subset K in $\mathbb{R}^n$, we say that ϕ is a K-*moment form*.

We intend to apply the characterization given by Theorem 2.5. As in [Vas2], we shall use the following framework.

Let $\mathbb{Z}_+^n$ be the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, i.e., $\alpha_j \in \mathbb{Z}_+$ for all $j = 1, \ldots, n$.

Let $\mathcal{P}_n$ be the algebra of all polynomial functions on $\mathbb{R}^n$, with complex coefficients. We shall denote by t^α the monomial $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, where $t = (t_1, \ldots, t_n)$ is the current variable in $\mathbb{R}^n$, and $\alpha \in \mathbb{Z}_+^n$.

Let $(\mathbb{R}_\infty)^n = (\mathbb{R} \cup \{\infty\})^n$, i.e., the Cartesian product of n copies of the one point compactification $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ of the real line $\mathbb{R}$. We consider the family $\mathcal{Q}_n$ consisting of all rational functions of the form $q_\alpha(t) = (1 + t_1^2)^{-\alpha_1} \cdots (1 + t_n^2)^{-\alpha_n}$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is arbitrary. The function q_α can be continuously extended to $(\mathbb{R}_\infty)^n \setminus \mathbb{R}^n$ for all $\alpha \in \mathbb{Z}_+^n$. Moreover, the set $\mathcal{Q}_n$ becomes a multiplicative family in $C((\mathbb{R}_\infty)^n)$. Set also $p_\alpha(t) = q_\alpha(t)^{-1}$, $t \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}_+^n$.

Let $\mathcal{P}_{n,\alpha}$ be the vector space generated by the monomials $t^\beta = t_1^{\beta_1} \cdots t_n^{\beta_n}$, with $\beta_j \leq 2\alpha_j$, $j = 1, \ldots, n$, $\alpha \in \mathbb{Z}_+^n$. For each $p \in \mathcal{P}_{n,\alpha}$, the rational function p/p_α can be continuously extended to $(\mathbb{R}_\infty)^n \setminus \mathbb{R}^n$, and so it can be regarded as an element of $C((\mathbb{R}_\infty)^n)$. Therefore, $\mathcal{P}_{n,\alpha}$ is a subspace of $C((\mathbb{R}_\infty)^n)/q_\alpha = p_\alpha C((\mathbb{R}_\infty)^n)$ for all $\alpha \in \mathbb{Z}_+^n$.

Fix a closed set $K \subset \mathbb{R}^n$. As we are primarily interested in the unbounded case, we will assume, in general, that K is unbounded. We write $\hat{K}$ for the closure of K in $(\mathbb{R}_\infty)^n$, which is a compact space. Let $\mathcal{Q}_n(\hat{K})$ be the set of all functions from $\mathcal{Q}_n$, (extended to $(\mathbb{R}_\infty)^n$ and) restricted to $\hat{K}$. This is a multiplicative family in $C(\hat{K})$. For $q \in \mathcal{Q}_n(\hat{K})$ let $A(q) := \{\alpha \in \mathbb{Z}_+^n \,;\, q_\alpha | K \equiv q\}$. We introduce a partial ordering in $\mathcal{Q}_n(\hat{K})$ by defining $q' \prec q''$ if for all $\alpha \in A(q')$ there exists some $\beta \in A(q'')$ such that $\beta - \alpha \in \mathbb{Z}_+^n$. In this case we have $q'' = q' q_{\beta-\alpha} | \hat{K}$, which shows that the partial ordering "$\prec$" is multiplicative. Notice, that the space $\mathcal{P}_n(K)$ of all restrictions to K of polynomials in $\mathcal{P}_n$ may be regarded as a subspace of the algebra of fractions $C(\hat{K})/\mathcal{Q}_n(\hat{K})$. Indeed, with $\mathcal{P}_q := \sum_{\alpha \in A(q)} \mathcal{P}_{n,\alpha}|K$, we have $q^{-1} \in \mathcal{P}_q \subset C(\hat{K})/q$ and $\mathcal{P}_n(K) = \bigcup_{q \in \mathcal{Q}_n(\hat{K})} \mathcal{P}_q$. Moreover, because of $\mathcal{P}_{n,\alpha} \subset \mathcal{P}_{n,\beta}$ if $\beta - \alpha \in \mathbb{Z}_+^n$, we see that $\mathcal{P}_{q'} \subset \mathcal{P}_{q''}$ whenever $q' \prec q''$.

This discussion shows that the required conditions to apply Theorem 2.5 are fulfilled. Note also that if $s = \sum_{\alpha \in A(q)} s_\alpha \in \mathcal{P}_q = \sum_{\alpha \in A(q)} \mathcal{P}_{n,\alpha}|K$ for a fixed $q \in \mathcal{Q}_n(\hat{K})$, we have $qs = \sum_{\alpha \in A(q)} s_\alpha q = \sum_{\alpha \in A(q)} s_\alpha q_\alpha \in C(\hat{K})$. Hence, $s \in C(\hat{K})/q$ and $\|s\|_{\infty,q} := \sup_{t \in \hat{K}} |sq| = \sup_{t \in \hat{K}} |s q_\alpha|$ for all $\alpha \in A(q)$, which provides the natural norm of the space $\mathcal{P}_q$.

Let $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ be a unital, linear map. If ϕ is a K-moment form, then we clearly have $\psi(p) = 0$ for each polynomial p such that $p|K = 0$.

Conversely, the linear map $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ is said to be K-*compatible* if it has the property that $\phi(p) = 0$ whenever $p|K = 0$. In such a case, ϕ induces a linear map on $\mathcal{P}_n(K)$, say $\tilde{\phi}$, given by $\tilde{\phi}(f) = \phi(p)$, for all $f \in \mathcal{P}_n(K)$ and $p \in \mathcal{P}_n$ with $f = p|K$. As the map $\tilde{\phi}$ is unambiguously defined by ϕ, it will be also denoted by ϕ.

Notice, that K-compatibility can only be violated if K is contained in the set of zeros of a polynomial $p \neq 0$. In particular, if $\mathrm{int}K \neq \emptyset$, then every linear map $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ is K-compatible. In the case $n = 1$, every linear map $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ is K-compatible for all unbounded closed subsets K of $\mathbb{R}$

It is clear that if $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ is a positive, linear map such that $|\phi(p)(x,x)| \leq \sup_{t \in K} |q_\alpha(t) p(t)| \phi(p_\alpha)(x,x)$ for all $p \in \mathcal{P}_{n,\alpha}$, $x \in \mathcal{D}$ and $\alpha \in \mathbb{Z}_+^n$, then ϕ is K-compatible. Nevertheless, a stronger condition is necessary in order to derive the existence of a representing measure for such a form.

Theorem 3.2 *Let $K \subset \mathbb{R}^n$ be closed and unbounded, let $\mathcal{D}$ be a complex inner product space and let $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ be a unital, linear map.*

The map ϕ is a K-moment form if and only if

(i) $\phi(p_\alpha)(x,x) > 0$ *for all* $x \in \mathcal{D} \setminus \{0\}$ *and* $\alpha \in \mathbb{Z}_+^n$.
(ii) *The map ϕ is K-compatible and for all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m$, $y_1, \ldots, y_m \in \mathcal{D}$ with*

$$\sum_{j=1}^m \phi(p_\alpha)(x_j, x_j) \leq 1, \quad \sum_{j=1}^m \phi(p_\alpha)(y_j, y_j) \leq 1,$$

and for all $f = (f_{j,k}) \in M_m(\mathcal{P}_q)$ with $\sup_{t\in K} \|q(t)f(t)\|_m \leq 1$, where $q = q_\alpha|\hat{K}$, we have

$$\left| \sum_{j,k=1}^m \phi(f_{j,k})(x_k, y_j) \right| \leq 1.$$

Proof. Assume first that ϕ is a K-moment form and let F be a representing measure of ϕ carried by K. Then we have $\phi(p)(x,y) = \int_K p\, dF_{x,y}$ for all $p \in \mathcal{P}_n$ and $x, y \in \mathcal{D}$. As noticed above, such a map ϕ is K-compatible. Therefore, it induces a linear and unital map $\phi : \mathcal{P}_n(K) \to SF(\mathcal{D})$. Moreover, if $q \in \mathcal{Q}_n(\hat{K})$ and $\alpha \in A(q)$, we have for all $x \in \mathcal{D} \setminus \{0\}$,

$$\phi(q^{-1})(x,x) = \phi(p_\alpha)(x,x) = \int_K p_\alpha\, dF_{x,x} \geq \int_K dF_{x,x} = \|x\|^2 > 0,$$

as $p_\alpha \geq 1$ and F is positive. This shows that condition (i) in Theorem 2.5 holds for $\tilde{\phi}$.

To prove condition (ii), we define a unital, linear map $\psi : C(\hat{K})/\mathcal{Q}_n(\hat{K}) \to SF(\mathcal{D})$ via the equation $\psi(f)(x,y) = \int_K f\, dF_{x,y}$ for all $f \in C(\hat{K})/\mathcal{Q}_n(\hat{K})$ and $x, y \in \mathcal{D}$. This definition is correct, since for each $f \in C(\hat{K})/\mathcal{Q}_n(\hat{K})$ we can find an index $\alpha \in \mathbb{Z}_+^n$ such that $h := fq_\alpha|\hat{K} \in C(\hat{K})$. Then we have

$$\int_K |f|\, dF_{x,x} \leq \|h\|_\infty \int_K p_\alpha\, dF_{x,x} = \|h\|_\infty \phi(p_\alpha)(x,x) < \infty,$$

for all $x \in \mathcal{D}$. This shows that each $f \in C(\hat{K})/\mathcal{Q}_n$ is integrable with respect to each measure $F_{x,x}$, and hence integrable with respect to each measure $F_{x,y}$, for all $x, y \in \mathcal{D}$, by the polarization formula. Moreover, the restriction of ψ to the space $C(\hat{K})/q$ is clearly positive, for all $q \in \mathcal{Q}_n(\hat{K})$. Thus, ψ is a unital, positive extension of ϕ. Setting, as before, $\psi_q = \psi|C(\hat{K})/q$, $\psi_{q,x}(*) = \psi_q(*)(x,x)$, $\phi_q = \phi|\mathcal{P}_q$, $\phi_{q,x}(*) = \psi_q(*)(x,x)$ for all $q \in \mathcal{Q}_n(\hat{K})$ and $x \in \mathcal{D}$, we have:

$$\phi(q^{-1})(x,x) = \psi(q^{-1})(x,x) = \|\psi_{q,x}\| \geq \|\phi_{q,x}\| \geq \phi(q^{-1})(x,x),$$

via Theorem 2.2. This shows that the map $\phi : \mathcal{P}_n(K) \to SF(\mathcal{D})$ satisfies condition (a) in Theorem 2.5.

Fix $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{D}$ such that

$$\sum_{j=1}^m \phi(q^{-1})(x_j, x_j) = \sum_{j=1}^m \phi(p_\alpha)(x_j, x_j) \le 1,$$

$$\sum_{j=1}^m \phi(q^{-1})(y_j, y_j) = \sum_{j=1}^m \phi(p_\alpha)(y_j, y_j) \le 1,$$

where $q = q_\alpha|\hat{K}$. Take $f = (f_{j,k}) \in M_m(\mathcal{P}_q)$ with $\sup_{t\in K} \|q(t)p(t)\|_m \le 1$. Then, by (ii) in Theorem 2.5, we infer that

$$\left| \sum_{j,k=1}^m \phi(f_{j,k})(x_k, y_j) \right| \le 1.$$

Conversely, assume that conditions (i) and (ii) are fulfilled. Because ϕ is K-compatible, it induces the unital, linear map $\phi : \mathcal{P}_n(K) \to SF(\mathcal{D})$, as noticed before. Moreover, conditions (i) and (ii) in the statement above imply conditions (i) and (ii) in Theorem 2.5. Note also that the function $q_{(1,\dots,1)}$ is null on the set $\hat{K} \setminus K$, and this set contains the zeros of any function from $\mathcal{Q}_n(\hat{K})$.

By virtue of Theorem 2.5, and by Corollary 2.7 as well, it follows that the map ϕ extends to a unital, positive, linear map having a representing measure $\tilde{F}$ on the Borel sets of $\hat{K}$, whose support lies in K. Then the measure F given by $F(B) := \tilde{F}(B \cap \hat{K})$ for all Borel sets B in $\mathbb{R}^n$ is a positive $B(\mathcal{H})$–valued measure satisfying

$$\phi(p)(x, y) = \int_K p\, d\tilde{F}_{x,y} = \int_{\mathbb{R}^n} p dF_{x,y}\,, \quad p \in \mathcal{P}_n\,.$$

Hence, ϕ is a K–moment form. □

Remark 1 *In the statements of Theorem 3.2, the conditions (i) and (ii) may be replaced by similar conditions, in which the multi-index α runs only in a cofinal family in $\mathbb{Z}_+^n$ (with respect to the partial ordering $\xi \prec \eta$ for two multi-indices $\xi = (\xi_1, \dots, \xi_n)$, $\eta = (\eta_1, \dots, \eta_n)$, meaning that $\xi_j \le \eta_j$, $j = 1, \dots, n$), which suffices to apply Theorem 2.5.*

Note also that if the map $\mathcal{P}_n \ni p \to p|K \in \mathcal{P}_n(K)$ is injective, then $\mathcal{P}_q = \mathcal{P}_{n,\alpha}|K$ where α is the only multi-index such that $A(q) = \{\alpha\}$, for all $q \in \mathcal{Q}_n(\hat{K})$.

We shall use the following well-known fact: If $S = (S_1, \dots, S_n)$ is a tuple of (not necessarily bounded) commuting normal linear operators in a Hilbert space $\mathcal{K}$, in the sense that their spectral measures $E_1, \ldots, E_n$ commute, then there exists a unique spectral measure E on $\mathbb{C}^n$, satisfying $E(A_1 \times \ldots \times A_n) = E_1(A_1) \cdots E_n(A_n)$ for arbitrary $A_1, \ldots, A_n$ in σ-algebra $Bor(\mathbb{C})$ of all Borel sets in $\mathbb{C}$, and $S_j x = \int_{\mathbb{C}^n} z_j dE(z)x$ for all x in the domain $D(S_j)$ of S_j, $j = 1, \ldots, n$. We call E the *joint spectral measure* of S and say that E *has support in* a closed set $K \subset \mathbb{C}^n$ if $E(\mathbb{C}^n \setminus K) = 0$. In particular, when $S = (S_1, \ldots, S_n)$ consists of commuting selfadjoint operators, the support of their joint spectral measure lies in $\mathbb{R}^n$ (details concerning joint spectral measures and integrals can be found in [B-S]; see also [Rud] for some details).

Theorem 3.3 *Let $K \subset \mathbb{R}^n$ be closed and let $T = (T_1, \ldots, T_n)$ be a tuple of symmetric, linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under $T_1, \ldots, T_n$ and that $T_1, \ldots, T_n$ commute on $\mathcal{D}$. Let $\phi_T : \mathcal{P}_n \to SF(\mathcal{D})$ be the linear unital map given by*

$$\phi_T(p)(x, y) := \langle p(T)x, y\rangle , \quad p \in \mathcal{P}_n, x, y \in \mathcal{D} .$$

The tuple T admits a selfadjoint extension such that its joint spectral measure has support in K if and only if the map ϕ_T is K-compatible and for all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{D}$ with

$$\sum_{j=1}^{m} \phi_T(p_\alpha)(x_j, x_j) \leq 1, \quad \sum_{j=1}^{m} \phi_T(p_\alpha)(y_j, y_j) \leq 1,$$

and for all $p = (p_{j,k}) \in M_m(\mathcal{P}_q)$ with $\sup_{t\in K} \|q(t)p(t)\|_m \leq 1$, where $q = q_\alpha|\hat{K}$, we have

$$\left| \sum_{j,k=1}^{m} \phi_T(p_{j,k})(x_k, y_j) \right| \leq 1.$$

Proof. Assume first that T admits a selfadjoint extension. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a tuple $A = (A_1, \ldots, A_n)$ consisting of commuting selfadjoint operators in $\mathcal{K}$ such that $\mathcal{D} \subset D(A^\alpha)$ and $T^\alpha x = A^\alpha x$, $x \in \mathcal{D}$, for all $\alpha \in \mathbb{Z}_+^n$. Moreover, the joint spectral measure $E : Bor(\mathbb{R}^n)$ of A vanishes on $\mathbb{R}^n \setminus K$. If we define $F : Bor(\mathbb{R}^n) \to B(\mathcal{H})$ by $F(S) := PE(S)|\mathcal{H}$ for all $S \in Bor(\mathbb{R}^n)$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, we have for all $p \in \mathcal{P}_n$, $x, y \in \mathcal{D}$,

$$\begin{aligned}\int_{\mathbb{R}^n} p dF_{x,y} = \int_{\mathbb{R}^n} p dE_{x,y} = \left\langle \int_{\mathbb{R}^n} p(z)dE(z)x, y \right\rangle = \langle p(A)x, y\rangle \\ = \langle p(T)x, y\rangle = \phi_T(p)(x, y) .\end{aligned}$$

Hence, F is a representing measure for ϕ_T which vanishes on $\mathbb{R}^n \setminus K$ and ϕ_T is a K–moment form. It now follows from Theorem 3.2 that the condition in the theorem must be satisfied.

Conversely, suppose that the condition in the statement holds.

First note that, $\phi_T(p_\alpha)(x,x) = \langle p_\alpha(T)x,x\rangle \geq \langle x,x\rangle$ for all $\alpha \in \mathbb{Z}_+^n$. Thus, if the condition in the statement is fulfilled for T, then the map ϕ_T satisfies conditions (i) and (ii) of Theorem 3.2. Hence, there exists a positive measure F on the Borel sets of $\mathbb{R}^n$ with values in $B(\mathcal{H})$, which is a representing measure for ϕ_T, and which vanishes on $\mathbb{R}^n \setminus K$. By the Naimark dilation theorem ([Pau], Theorem 4.6), there exists a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ and a selfadjoint spectral measure E on the Borel sets of $\mathbb{R}^n$ with values in $B(\mathcal{K})$, such that $F(A) = V^*E(A)V$ for all $A \in Bor(\mathbb{R}^n)$. Moreover, the measure E also vanishes on $\mathbb{R}^n \setminus K$.

Because of $F(K) = 1_{\mathcal{H}}$ and $E(K) = 1_{\mathcal{K}}$, the operator V is an isometry. Hence, identifying $\mathcal{H}$ with its isometric image $V(\mathcal{H})$, we see that

$$F(S) = PE(S)|\mathcal{H}\,, \qquad S \in Bor(\mathbb{R}^n),$$

where P denotes the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. We then obtain selfadjoint operators $A_1, \ldots, A_n$ by

$$D(A_j) = \{x \in \mathcal{K}\,;\, \int_K |t_j|^2 d\langle E(t)x,x\rangle\} < \infty\}$$

and

$$A_j x := \int_K t_j dE(t)x\,, \quad x \in D(A_j)\,.$$

As we have $0 \leq \phi_T(|p|^2)(x,x) = \int_K |p(t)|^2 d\langle F(t)x,x\rangle < +\infty$ for all $p \in \mathcal{P}_n$ and $x \in \mathcal{D}$, using a well known argument (see [Bi] or [Fo]), we infer that

$$\langle T^\alpha x,y\rangle = \phi_T(t^\alpha)(x,y) = \int_K t^\alpha d\langle F(t)x,y\rangle$$
$$= \int_K t^\alpha d\langle E(t)x,y\rangle = \langle A^\alpha x,y\rangle = \langle PA^\alpha x,y\rangle,$$

provided $x,y \in \mathcal{D}$. Hence, $PA^\alpha x = T^\alpha x$ for all $x \in \mathcal{D}$, $\alpha \in \mathbb{Z}_+^n$, which shows that the selfadjoint tuple $A := (A_1, \ldots, A_n)$ is a dilation of $T = (T_1, \ldots, T_n)$, and so A is actually a selfadjoint extension of T, via Remark 3.1. □

Remark 2 *When $K = \mathbb{R}^n$, the previous statement becomes much simpler. Indeed, in this case the map ϕ_T is automatically $\mathbb{R}^n$-compatible and we have $\mathcal{P}_q = \mathcal{P}_{n,\alpha}$ where α is the only multi-index such that $q = q_\alpha$. Therefore, with the notation of Theorem 3.3, the tuple $T = (T_1, \ldots, T_n)$ admits a selfadjoint extension if and only if for all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{D}$ with*

$$\sum_{j=1}^m \langle p_\alpha(T)x_j,x_j\rangle \leq 1, \quad \sum_{j=1}^m \langle p_\alpha(T)y_j,y_j\rangle \leq 1,$$

and for all $p = (p_{j,k}) \in M_m(\mathcal{P}_{n,\alpha})$ *with* $\sup_{t\in\mathbb{R}^n} \|q_\alpha(t)p(t)\|_m \le 1$*, we have*

$$\left|\sum_{j,k=1}^{m} \langle p_{j,k}(T)x_k, y_j\rangle\right| \le 1.$$

The particular case $n = 2$ *has been already presented in the Introduction.*

We now consider situations in $(\mathbb{C}_\infty)^n$, where $\mathbb{C}_\infty$ is the one point compactification of the complex plane $\mathbb{C}$. From now on, let $\mathcal{Q}_n$ denote the family of all functions of the form $q_\alpha(z) := (1 + |z_1|^2)^{-\alpha_1} \cdots (1 + |z_n|^2)^{-\alpha_n}$, with $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. As in the real case, the functions q_α extend continuously to $(\mathbb{C}_\infty)^n$ and $\mathcal{Q}_n$ is a multiplicative family in $C((\mathbb{C}_\infty)^n)$.

For all $\alpha \in \mathbb{Z}_+^n$, we denote by $\mathcal{T}_{n,\alpha}$ the linear spaces generated by the monomials $z^\xi \overline{z}^\eta := z_1^{\xi_1} \cdots z_n^{\xi_n} \overline{z_1}^{\eta_1} \cdots \overline{z_n}^{\eta_n}$ such that, for $j = 1, \dots, n$, we have $\xi_j + \eta_j < 2\alpha_j$ or $\xi_j = \eta_j = \alpha_j$.

Note, that for all $f \in \mathcal{T}_{n,\alpha}$ the function $q_\alpha f$ extends continuously to $(\mathbb{C}_\infty)^n$ and that $\mathcal{T}_{n,\alpha} \subset \mathcal{T}_{n,\beta}$ if $\alpha_j \le \beta_j$, $j = 1, \dots, n$. We also consider the linear subspace $\mathcal{T}_n := \sum_{\alpha\in\mathbb{Z}_+^n} \mathcal{T}_{n,\alpha}$ of $C((\mathbb{C}_\infty)^n)/\mathcal{Q}_n$.

Let now K be an unbounded closed subset of $\mathbb{C}^n$ and $\hat{K}$ its closure in $(\mathbb{C}_\infty)^n$. We define (as in the discussion before Theorem 3.2) $\mathcal{T}_n(K) := \{f|K\,;\, f \in \mathcal{T}_n\}$, and let $\mathcal{Q}_n(\hat{K})$ denote the set of restrictions to $\hat{K}$ of all functions in $\mathcal{Q}_n$ (extended to $(\mathbb{C}_\infty)^n$), which is a multiplicative set in $C(\hat{K})$. For $q \in \mathcal{Q}_n(\hat{K})$ let $A(q) := \{\alpha \in \mathbb{Z}_+^n\,;\, q_\alpha|\hat{K} = q\}$. We introduce a partial ordering in this set by defining $q' \prec q''$ if for all $\alpha \in A(q')$ there exists some $\beta \in A(q'')$ such that $\beta - \alpha \in \mathbb{Z}_+^n$. In this case we have $q'' = q' q_{\beta-\alpha}|\hat{K}$, so the partial ordering "$\prec$" is multiplicative. The space $\mathcal{T}_n(K)$ may be regarded as a subspace of the algebra of fractions $C(\hat{K})/\mathcal{Q}_n(\hat{K})$. Indeed, with $\mathcal{T}_q := \sum_{\alpha\in A(q)} \mathcal{T}_{n,\alpha}|K$, we have $q^{-1} \in \mathcal{T}_q \subset C(\hat{K})/q$ and $\mathcal{T}_n(K) = \bigcup_{q\in\mathcal{Q}_n(\hat{K})} \mathcal{T}_q$. Moreover, because of $\mathcal{T}_{n,\alpha} \subset \mathcal{T}_{n,\beta}$ if $\beta - \alpha \in \mathbb{Z}_+^n$, we see that $\mathcal{T}_{q'} \subset \mathcal{T}_{q''}$ whenever $q' \prec q''$. Hence, the conditions to apply Theorem 2.5 are fulfilled.

Let now $T = (T_1, \dots, T_n)$ be a tuple of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ such that $T_j(\mathcal{D}) \subset \mathcal{D}$ and $T_j T_k x = T_k T_j x$ for all $j, k \in \{1, \dots, n\}$, $x \in \mathcal{D}$. In this situation, we may define a unital linear map $\phi_T : \mathcal{T}_n \to SF(\mathcal{D})$ by

$$\phi_T(z^\xi \overline{z}^\eta)(x, y) := \langle T^\xi x, T^\eta y\rangle\ , \quad x, y \in \mathcal{D},\, \alpha \in \mathbb{Z}_+^n, \tag{11}$$

which extends by linearity to the space $\mathcal{T}_n$ of all polynomials in $z_1, \dots, \overline{z_1}, \dots, \overline{z_n}$ which is generated by these monomials. An easy induction proof shows that, for all α, β in $\mathbb{Z}_+^n$ with $\beta - \alpha \in \mathbb{Z}_+^n$, and $x \in \mathcal{D} \setminus \{0\}$, we have

$$0 < \langle x, x\rangle \leq \phi_T(q_\alpha^{-1})(x,x) \leq \phi_T(q_\beta^{-1})(x,x)\,. \tag{12}$$

When the map $\phi_T : \mathcal{T}_n \mapsto SF(\mathcal{D})$ is K-compatible (that is, $\phi_T(p) = 0$ if $p \in \mathcal{T}_n$ and $p|K = 0$), then it induces a map from $\mathcal{T}_n(K)$ into $SF(\mathcal{D})$, for which we keep the same notation.

Theorem 3.4 *Let $K \subset \mathbb{C}^n$ be closed and let $T = (T_1, \ldots, T_n)$ be a tuple of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under $T_1, \ldots, T_n$ and that $T_1, \ldots, T_n$ commute on $\mathcal{D}$. The tuple T admits a normal extension having a joint spectral measure whose support lies in K if and only if the map $\phi_T : \mathcal{T}_n \mapsto SF(\mathcal{D})$ is K-compatible, and for all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{D}$ with*

$$\sum_{j=1}^m \phi_T(q_\alpha^{-1})(x_j, x_j) \leq 1, \quad \sum_{j=1}^m \phi_T(q_\alpha^{-1})(y_j, y_j) \leq 1,$$

and for all $p = (p_{j,k}) \in M_m(\mathcal{T}_q)$ with $\sup_{t\in K} \|q(t)p(t)\|_m \leq 1$, where $q = q_\alpha|\hat{K}$, we have

$$\left| \sum_{j,k=1}^m \phi_T(p_{j,k})(x_k, y_j) \right| \leq 1.$$

Proof. If the condition of the theorem is fulfilled, and so we have a linear and unital map $\phi_T : \mathcal{T}_n(K) \to SF(\mathcal{D})$ induced by ϕ_T, then conditions (i) (by (12)) and (ii) of Theorem 2.5) are satisfied for ϕ_T. Hence, by that theorem and Corollary 2.7, there exists a regular, positive $B(\mathcal{H})$–valued measure F on the Borel sets of $\hat{K}$, such that (10) holds for ϕ_T and such that $F(\hat{K} \setminus K) = 0$. Because ϕ_T is unital, $F(K)$ is the identity operator on $\mathcal{H}$. As in the proof of the preceding theorem, by the Naimark dilation theorem ([Pau], Theorem 4.6), there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a spectral measure $E : Bor(K) \to B(\mathcal{K})$ such that $F(A) = PE(A)|\mathcal{H}$ for all $A \in Bor(K)$, where P denotes the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. For $j = 1, \ldots, n$, let N_j be the corresponding normal operators with domains

$$D(N_j) := \left\{ x \in \mathcal{K}\,;\, \int_K |z_j|^2 d\langle E(z)x, x\rangle\} < \infty \right\}$$

and

$$N_j x := \int_K z_j dE(z)x\,, \quad x \in D(N_j)\,.$$

Going back to the arguments from [Bi] and [Fo], for all $x, y \in \mathcal{D}$, $j = 1, \ldots, n$, we have

$$\begin{aligned}\langle PN_j x, y\rangle =& \langle N_j x, y\rangle = \int_K z_j d\langle E(z)x, y\rangle = \int_K z_j d\langle F(z)x, y\rangle \\ =& \phi_T(z_j|K)(x,y) = \phi_T(z_j)(x,y) = \langle T_j x, y\rangle\,.\end{aligned}$$

Hence, $PN_jx = T_jx$ for all $x \in \mathcal{D}$, $j = 1, \ldots, n$. Note also that

$$\begin{aligned}\|T_jx\|^2 =&\phi_T(|z_j|^2)(x,x) = \int_K |z_j|^2 d\langle F(z)x, x\rangle \\ =&\int_K |z_j|^2 d\langle E(z)x, x\rangle = \|N_jx\|^2.\end{aligned}$$

for all $x \in \mathcal{D}$, $j = 1, \ldots, n$, which shows that the tuple $N := (N_1, \ldots, N_n)$ is a normal extension of $T = (T_1, \ldots, T_n)$ (see also the proof in Remark 3.1).

Conversely, if $T = (T_1, \ldots, T_n)$ admits a normal extension $N := (N_1, \ldots, N_n)$ with joint spectral measure E having support contained in K, then, for all $\alpha \in \mathbb{Z}_+^n$, the space $\mathcal{D}$ is contained in

$$D(T^\alpha) \subset D(N^\alpha) = \left\{x \in \mathcal{K}\,;\, \int_K |z^\alpha|^2 d\langle E(z)x, x\rangle < \infty\right\}.$$

It follows that, for all $f \in C(\hat{K})/(q_\alpha|\hat{K})$, the function f is integrable on K with respect to the positive scalar measure $dE_{x,x} := d\langle E(*)x, x\rangle$. Using the decomposition $4E_{x,y} = E_{x+y,x+y} - E_{x-y,x-y} + iE_{x+iy,x+iy} - iE_{x-iy,x-iy}$ we see that $\psi : C(\hat{K})/\mathcal{Q}_n(\hat{K}) \mapsto SF(\mathcal{D})$, defined by

$$\psi(f)(x,y) := \int_K f(z) d\langle E(z)x, y\rangle, \qquad x, y \in \mathcal{D},\, f \in C(\hat{K})/\mathcal{Q}_n(\hat{K}),$$

is a linear map which is obviously unital and positive. Moreover, ψ is an extension of ϕ_T because N extends T. As the map ψ has support in K, the map ϕ_T should be K-compatible.

If we set $\phi = \phi_T$, $\psi_q = \psi|C(\hat{K})/q$, $\psi_{q,x}(*) = \psi_q(*)(x,x)$, $\phi_q = \phi|\mathcal{T}_q$, $\phi_{q,x}(*) = \psi_q(*)(x,x)$ for all $q \in \mathcal{Q}_n(\hat{K})$ and $x \in \mathcal{D}$, we have:

$$\phi(q^{-1})(x,x) = \psi(q^{-1})(x,x) = \|\psi_{q,x}\| \geq \|\phi_{q,x}\| \geq \phi(q^{-1})(x,x),$$

via Theorem 2.2. This shows that the map $\phi : \mathcal{T}_n(K) \to SF(\mathcal{D})$ satisfies condition (a) in Theorem 2.5. Consequently, by Theorem 2.5, we infer that the condition in the statement is satisfied. $\square$

For the particular case $n = 1$, the set $\mathcal{Q}_1$ consists of all functions of the form $q_l := (1 + |z|^2)^{-l}$, with $z \in \mathbb{C}$ and $l \in \mathbb{Z}_+$. Because of

$$q_l^{-1}(z) = (1 + z\overline{z})^l = \sum_{k=0}^{l} \binom{l}{k} z^k \overline{z}^k$$

we obtain from Theorem 3.4:

Corollary 3.5 *Let $S : D(S) \subset \mathcal{H} \mapsto \mathcal{H}$ be a linear operator such that $SD(S) \subset D(S)$. The operator S admits a normal extension if and only if* *for all* $l \in \mathbb{Z}^+$, $m \in \mathbb{N}$ *and* $x_1, \ldots, x_m, y_1, \ldots, y_m \in D(S)$ *with*

$$\sum_{j=1}^{m} \sum_{k=0}^{l} \binom{l}{k} \langle S^k x_j, S^k x_j \rangle \leq 1, \quad \sum_{j=1}^{m} \sum_{k=0}^{l} \binom{l}{k} \langle S^k y_j, S^k y_j \rangle \leq 1,$$

and for all $p = (p_{j,k}) \in M_m(\mathcal{T}_1)$, *with* $\sup_{z \in \mathbb{C}} \|(1 + |z|^2)^{-l} p(z)\|_m \leq 1$, *we have*

$$\left| \sum_{j,k=1}^{m} \phi_S(p_{j,k})(x_k, y_j) \right| \leq 1.$$

Instead of working with $(\mathbb{R}_\infty)^n$ and $(\mathbb{C}_\infty)^n$, we could also have taken the one point compactifications $\mathbb{R}^n \cup \{\infty\}$ and $\mathbb{C}^n \cup \{\infty\}$. Instead of $\mathcal{Q}_n$, one then considers the multiplicative family $\{q^k\,;\, k \in \mathbb{Z}_+\}$, where $q(t) := 1 + \|t\|^2$ for all $t \in \mathbb{R}^n$, respectively $t \in \mathbb{C}^n$. Instead of $\mathcal{P}_{n,\alpha}$ and $\mathcal{T}_{n,\alpha}$ the linear hull $\mathcal{T}_{n,k}$ of the space of all polynomials of degree $< 2k - 1$ and the functions q^j, $0 \leq j \leq k$, have to be taken.

The formulations and the proofs of the corresponding variants of Theorems 3.2, 3.3 and 3.4 are left to the reader.

A different characterization of tuples of symmetric operators having selfadjoint extensions can be found in [Vas1]. The actual statement of Theorem 3.4 is more explicit in terms of the given data. The case of one operator, covered by our Corollary 3.5, also occurs in [StSz2], with a completely different approach. For a further characterization for subnormal operators see also Theorem 3 in [StSz1]

Let us finally mention that our main result (Theorem 2.5) has been recently extended to a a noncommutative context in [Dos].

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