TRUNCATED MOMENTS AND INTERPOLATION

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Outline

DEDICATION

This talk is dedicated to my successive professors of analysis at the Faculty of Mathematics of the University of Bucharest

Nicolae DINCULEANU,
Solomon MARCUS,
Romulus CRISTESCU,
Miron NICOLESCU (in memoriam),

as well as to

Ciprian FOIAŞ,

the supervisor of my PhD thesis.

Abstract

The aim of this talk is to present a new approach to truncated moment problems, based on the use of the space of characters of some associated finite dimensional commutative Banach algebras.

Some Notation

We fix an integer $n \geq 1$ associated with the euclidean space $\mathbb{R}^n$, and for every integer $m \geq 0$ we denote by $\mathcal{P}_m$ the vector space of all polynomials in n real variables, with complex coefficients, of total degree less or equal to m. The vector space of all polynomials in n real variables, with complex coefficients, will be denoted by $\mathcal{P}$. The vector space (over $\mathbb{R}$) of all polynomials from $\mathcal{P}_m$ with real coefficients will be denoted by $\mathcal{RP}_m$.

Whenever it is necessary to specify the value of n, we write $\mathcal{P}_m^n = \mathcal{P}_m$, respectively $\mathcal{P}^n = \mathcal{P}$.

Interpolation Spaces

Let $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$ a finite family of distinct points in $\mathbb{R}^n$, and let $C(\Xi)$ the set of all maps defined on Ξ, with complex values, regarded as a C^*-algebra, endowed with the natural operations, and with the norm *sup*.

A vector subspace $\mathcal{S} \subset C(\Xi)$ is said to be an *interpolation space* for the set Ξ if for every $f \in C(\Xi)$ we can find an element $p \in \mathcal{S}$ such that

$$f(\xi^{(k)}) = p(\xi^{(k)}),\ k = 1, \ldots, d.$$

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Lagrange's Polynomials

Denote by $\mathcal{P}_m(\Xi)$ the set of all restrictions of polynomials from $\mathcal{P}_m$ to Ξ. If $m \geq 2(d-1)$, then the space $\mathcal{P}_m(\Xi)$ is an interpolation space on Ξ, via *Lagrange's polynomials*:

$$\pi_k(x) = \frac{\prod_{j\neq k} \|x - \xi^{(j)}\|^2}{\prod_{j\neq k} \|\xi^{(k)} - \xi^{(j)}\|^2}, \; x \in \mathbb{R}^n, \; k = 1, \ldots, d,$$

whose degree is equal to $2(d-1)$.

Interpolation Degree

The *interpolation degree* of the set Ξ is the number g_Ξ equal to the smallest integer $m \geq 1$ such that $\mathcal{P}_m(\Xi)$ is an interpolation space for Ξ. Obviously, $g_\Xi \leq 2(d-1)$, and in special cases the inequality is strict (because there exist other interpolation polynomials, depending on the geometric configuration of the set Ξ).

Associated Probability Measures

Let μ be a probability measure concentrated on the set $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. In other words, $\mu = \sum_{j=1}^{d} \lambda_j \delta_j$, with $\sum_{j=1}^{d} \lambda_j = 1$, $\lambda_j > 0$ and δ_j is the Dirac measure at the point $\xi^{(j)}$, $j = 1, \ldots, d$.

We consider the Hilbert space $L^2(\Xi, \mu)$, endowed with the scalar product induced by the measure μ, which coincides, as a vector space, with $C(\Xi)$.

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Associated Maps

The linear map Λ_μ, given by

$$\Lambda_\mu : L^2(\Xi, \mu) \mapsto \mathbb{C}, \ \Lambda_\mu(f) = \int_\Xi f(\xi) d\mu(\xi),$$

has the properties

(i) $\Lambda_\mu(\bar{f}) = \overline{\Lambda_\mu(f)}, \ f \in L^2(\Xi, \mu);$
(ii) $\Lambda_\mu(|f|^2) \geq 0, \ f \in L^2(\Xi, \mu);$
(iii) $\Lambda_\mu(1) = 1$.

Square Positive Functionals

Let us fix an integer $m \geq 0$, and let us consider the map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with the properties
(1) $\Lambda(\bar{p}) = \overline{\Lambda(p)},\ p \in \mathcal{P}_{2m}$;
(2) $\Lambda(|p|^2) \geq 0,\ p \in \mathcal{P}_m$;
(3) $\Lambda(1) = 1$,
clearly similar to (i)-(iii).

A map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with the properties (1)-(3) will be designated as a *square positive functional* (briefly, a *spf*).

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Truncated Moment Problem

Given a map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with the properties (1)-(3), the *truncated moment problem* means to find necessary and sufficient conditions for the existence of a finite set Ξ and a probability measure μ on Ξ such that $\Lambda(p) = \int_{\Xi} p(\xi) d\mu(\xi)$ for all $p \in \mathcal{P}_{2m}$.

The measure μ, when exists is said to be a *representing measure* for Λ.

In virtue of a result by Tchakaloff (revised and improved by many authors), the measure μ may be always supposed to be atomic.

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Example

Let $\mathcal{P}_4^1$ be the space of polynomials in one real variable, denoted by t, of degree at most 4. We set $\Lambda(1) = \Lambda(t) = \Lambda(t^2) = \Lambda t^3) = 1$, $\Lambda(t^4) = 2$, and extend Λ to the space $\mathcal{P}_4^1$ by linearity. The properties (1) and (3) are obvious. Moreover, if $p(t) = x_0 + x_1 t + x_2 t^2 \in \mathcal{P}_2^1$, then

$$\Lambda(|p|^2) = |x_0 + x_1 + x_2|^2 + |x_2|^2 \geq 0,$$

showing that Λ also satisfies (2). Nevertheless, one can see that Λ has no representing measure.

This also shows that the properties (1)-(3) are not sufficient to find a solution for the truncated moment problem. It is therefore necessary to find supplementary conditions to approach the existence of a solution of this problem.

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Introducing Idempotents

Let again $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$, and let $g = g_\Xi$ be the interpolation degree of the set Ξ. In particular, there exists a family $\mathcal{B} = \{b_1, \ldots, b_d\} \subset \mathcal{RP}_g$ such that $b_j(\xi^{(k)}) = 1$ if $j = k$, and $b_j(\xi^{(k)}) = 0$ if $j \neq k,\ j, k = 1, \ldots, d$.
Let μ be a probability measure concentrated on Ξ, and so $\mu = \sum_{j=1}^{d} \lambda_j \delta_j$, with $\sum_{j=1}^{d} \lambda_j = 1,\ \lambda_j > 0$ and δ_j is the Dirac measure at the point $\xi^{(j)}$, $j = 1, \ldots, d$. Defining $\Lambda_\mu(p) := \int_\Xi p(\xi) d\mu(\xi), p \in \mathcal{P}_g$, we obtain

$$\Lambda_\mu(b_j^2) = \Lambda_\mu(b_j) = \lambda_j,\ \Lambda_\mu(b_j b_k) = 0,\ j, k = 1, \ldots, d,\ j \neq k.$$

Writing $p = \sum_{j=1}^{d} p(\xi^{(j)}) b_j + r_p$, $p \in \mathcal{P}_g$, $r_p|\Xi = 0$, we get the equality

$$\Lambda_\mu(p) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}),\ \ p \in \mathcal{P}_g.$$

Particularly, if $t_1, \ldots, t_n$ are the independent variables from $\mathbb{R}^n$,

$$\xi^{(j)} = (\lambda_j^{-1}\Lambda_\mu(t_1 b_j), \ldots, \lambda_j^{-1}\Lambda_\mu(t_n b_j)) \in \mathbb{R}^n,\ \ j = 1, \ldots, d,$$

expressing the points $\xi^{(j)}$ in terms of Λ_μ and b_j, $j = 1, \ldots, d$.

Elements similar $b_1, \ldots, b_d$ will later called *idempotents* with respect to Λ_μ.

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Moments and Interpolation

The next result presents a connexion between the moment problem and the interpolation.

Theorem 1

Fie $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ a *spf* such that $\Lambda(|p|^2) = 0$ implies $p = 0$. Assume that there exists a family $\mathcal{B} = \{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the following properties:

$$\Lambda(b_j^2) = \Lambda(b_j) > 0,\ \Lambda(b_j b_k) = 0,\ j, k = 1, \ldots, d,\ j \neq k.$$

Put

$$\xi^{(j)} = ((\Lambda(b_j))^{-1}\Lambda(t_1 b_j), \ldots, (\Lambda(b_j))^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n,\ j = 1, \ldots, d,$$

and $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$.

Continuation

If the family $\mathcal{B}$ is maximal with respect to the inclusion and if the points $\xi^{(1)}, \ldots, \xi^{(d)}$ are distinct, then every polynomial $p \in \mathcal{P}_m$ can be written under the form

$$p = \sum_{j=1}^{d} p(\xi^{(j)}) b_j.$$

Moreover, Λ has a representing measure with support in Ξ given by

$$\Lambda(p) = \int_{\Xi} p(\xi) d\mu(\xi), \; p \in \mathcal{P}_{2m},$$

where μ is the probability measure concentrated on the set Ξ, with weights $\lambda_j = \Lambda(b_j)$ at points $\xi^{(j)}$, $j = 1, \ldots, d$, respectively.

Comments

- The conditions from Theorem 1 are also necessary.
- To find a maximal family $\mathcal{B} = \{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the properties

$$\Lambda(b_j^2) = \Lambda(b_j) > 0,\ \Lambda(b_j b_k) = 0,\ j, k = 1, \ldots, d,\ j \neq k,$$

is a solvable problem. Indeed, it is sufficient to endow the real spce $\mathcal{RP}_m$ with the inner product$\langle p, q \rangle = \Lambda(pq)$, to choose an orthonormal basis$\{c_1, \ldots, c_d\}$ with $\Lambda(c_j) \neq 0$; then setting $b_j = \Lambda(c_j)c_j$, $j = 1, \ldots, d$, we obtain the desired family. The condition that the points

$$\xi^{(j)} = ((\Lambda(b_j))^{-1}\Lambda(t_1 b_j), \ldots, (\Lambda(b_j))^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n,\ j = 1, \ldots, d$$

be distinct can be verified but its dependence of the choice of the basis $\{c_1, \ldots, c_d\}$ is not completely explicit.

Again about Square Positive Functionals

Let us remark that every *spf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ satisfies the

Cauchy-Schwarz inequality:

$$|\Lambda(pq)|^2 \leq \Lambda(|p|^2)\Lambda(|q|^2),\ p, q \in \mathcal{P}_m. \tag{1}$$

Setting

$$\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; \Lambda(|p|^2) = 0\},$$

the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace.

Basic Associated Hilbert Space

The quotient space

$$\mathcal{H}_\Lambda = \mathcal{P}_m/\mathcal{I}_\Lambda$$

is a Hilbert space, whose scalar product is given by

$$\langle p + \mathcal{I}_\Lambda, q + \mathcal{I}_\Lambda\rangle = \Lambda(p\bar{q}),\ \ p, q \in \mathcal{P}_m. \tag{2}$$

The symbol $\mathcal{RH}_\Lambda$ designate the space $\{\hat{p} \in \mathcal{H}_\Lambda; p \in \mathcal{RP}_m\}$, which is a real Hilbert space. Fixing an element $\hat{p} \in \mathcal{RH}_\Lambda$, we always suppose that its representative p is in $\mathcal{RP}_m$.
Finally, let us remark that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.

Idempotents with Respect to a *SPF*

Definition 1 An element $\hat{p} \in \mathcal{RH}_\Lambda$ is called Λ-*idempotent* (or simply *idempotent* when Λ is fixed) if it is a solution of the equation

$$\|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle. \tag{3}$$

Remark 1 Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is an idempotent if and only if $\Lambda(p^2) = \Lambda(p)$.

Put

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{4}$$

which is a nonempty family because $\hat{1} \in \mathcal{ID}(\Lambda)$.

Two Lemmas

Lemma 1 (1) If $\hat{p}, \hat{q}, \hat{p} - \hat{q} \in \mathcal{ID}(\Lambda)$, then $\hat{q}$ and $\hat{p} - \hat{q}$ are othogonal.
(2) If $\hat{q} \in \mathcal{ID}(\Lambda)$, $\hat{q} \neq \hat{1}$, then $\hat{1} - \hat{q} \in \mathcal{ID}(\Lambda)$, and $\hat{q}$, $\hat{1} - \hat{q}$ are orthogonal.
(3) If $\{\hat{p}_1, \ldots, \hat{p}_d\} \subset \mathcal{ID}(\Lambda)$ are mutually orthogonal, then $\sum_{j=1}^{d} \hat{p}_j \in \mathcal{ID}(\Lambda)$.

Lemma 2 Let $\{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$, be a family of mutually orthogonal elements. This family is maximal with respect to the inclusion if and only if $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$.

Special Orthogonal Bases

Let $\mathcal{B}_\Lambda = \{\hat{v} \in \mathcal{RH}_\Lambda; \|\hat{v}\| = 1\}$, and $\mathcal{B}_\Lambda^1 = \{\hat{v} \in \mathcal{B}_\Lambda; \langle \hat{v}, \hat{1} \rangle \neq 0\}$. The existence of orthogonal bases consisting of idempotents with respect to a fixed *spf* Λ is given by the following.

Proposition 1 We have the following properties:
(1) $\mathcal{ID}(\Lambda) = \{\langle \hat{v}, \hat{1} \rangle \hat{v}; \hat{v} \in \mathcal{B}_\Lambda, \langle \hat{v}, \hat{1} \rangle \neq 0\} = \{\Lambda(v)\hat{v}; \hat{v} \in \mathcal{B}_\Lambda, \Lambda(v) \neq 0\}$.
(2) The map

$$\mathcal{B}_\Lambda^1 \ni \hat{v} \mapsto \langle \hat{v}, \hat{1} \rangle \hat{v} \in \mathcal{ID}(\Lambda) \tag{5}$$

is bijective.
(3) If $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{B}_\Lambda$ is an orthonormal basis in $\mathcal{H}_\Lambda$ with $\langle \hat{v}_j, \hat{1} \rangle \neq 0$, $j = 1, \ldots, d$, then $\{\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1, \ldots \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d\}$ is an orthogonal basis in $\mathcal{H}_\Lambda$ consisting of idempotents. Moreover,

$$\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1 + \cdots + \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d = \hat{1}.$$

Theorem 2

For every *spf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, the space $\mathcal{H}_\Lambda$ has othogonal bases consisting of idempotents.

Corollary 1 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ a *spf*. There exist functions $b_1, \ldots, b_d \in \mathcal{RP}_m$ such that $\Lambda(b_j^2) = \Lambda(b_j) > 0$, $\Lambda(b_j b_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, and every $p \in \mathcal{P}_m$ has a unique representation of the form

$$p = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) b_j + p_0,$$

with $p_0 \in \mathcal{I}_\Lambda$ and $d = \dim \mathcal{H}_\Lambda$.

C^*–Algebra Structures

Given a *spf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, according to Theorem 2 we can choose an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$. With respect to the basis $\mathcal{B}$, we can define on $\mathcal{H}_\Lambda$ a structure of a unital commutative C^*–algebra.

If $\hat{p} = \sum_{j=1}^d \alpha_j \hat{b}_j$, $\hat{q} = \sum_{j=1}^d \beta_j \hat{b}_j$, are from $\mathcal{H}_\mathcal{B}$, we put

$$\hat{p} \cdot \hat{q} = \sum_{j=1}^{d} \alpha_j \beta_j \hat{b}_j.$$

The involution and norm are given respectively by

$$\hat{p}^* = \sum_{j=1}^{d} \overline{\alpha_j} \hat{b}_j, \ \ \|\hat{p}\|_\infty = \max_{1 \le j \le d} |\alpha_j|.$$

To obtain the assertion, we also use the equality $\hat{1} = \sum_{j=1}^d \hat{b}_j$.

The C^*-algebra structure of $\mathcal{H}_\Lambda$ associated to the orthogonal basis $\mathcal{B}$ is referred to as the *C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$*.

The space of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$, say $\Delta = \{\delta_1, \ldots, \delta_d\}$, coincides with the dual basis of $\mathcal{B}$. Using also the Hilbert space structure of $\mathcal{H}_\Lambda$, we obtain

$$\delta_j(\hat{p}) = \Lambda(b_j)^{-1}\langle \hat{p}, \hat{b}_j\rangle,\ \hat{p} \in \mathcal{H}_\Lambda,\ j = 1, \ldots, d.$$

Integral Representations

Proposition 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *spf*, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algeba structure induced by an orthogonal basis consisting of idempotents. Also let $\mathcal{H}_\mathcal{C}$ be the sub-C^*-algebra generated by the set $\mathcal{C} = \{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ în $\mathcal{H}_\Lambda$. Then there exists a subset Ξ in $\mathbb{R}^n$, whose cardinal is $\leq \dim \mathcal{H}_\Lambda$, and a linear map $\mathcal{S}_\mathcal{C} \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, such that

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi), \ \ u \in \mathcal{S}_\mathcal{C},$$

where $\mathcal{S}_\mathcal{C} = \{u \in \mathcal{P}_m; \hat{u} \in \mathcal{H}_\mathcal{C}\}$ and μ is a probability measure on Ξ.

Proposition 3 With the conditions of the previous proposition, assume the equality $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\Lambda}$. Then $\mathcal{S}_{\mathcal{C}} = \mathcal{P}_m$ and the map $\mathcal{P}_m \ni u \mapsto u^{\#} \in C(\Xi)$ induces a $*$-isomorphism between the C^*-algebras $\mathcal{H}_{\Lambda}$ and $C(\Xi)$.

If $r(\hat{t}_1, \ldots, \hat{t}_n) = 0$ for every $r \in \mathcal{I}_{\Lambda}$, then $u^{\#} = u|\Xi$ for all $u \in \mathcal{P}_m$.

Definition 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ a spf, and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ an orthogonal basis of the space $\mathcal{H}_\Lambda$ consisting of idempotents. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* if

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j) \tag{6}$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

Theorem 3

The *spf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ with $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a Λ-multiplicative basis of the space $\mathcal{H}_\Lambda$.

Under an explicit form, the previous theorem asserts the following:

Corollary 2 The *spf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ with $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a family of polynomials $\{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the following properties:

(i) $\Lambda(b_j^2) = \Lambda(b_j) > 0,\ j = 1, \ldots, d$;

(ii) $\Lambda(b_j b_k) = 0,\ j, k = 1, \ldots, d,\ j \neq k$;

(iii)

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

Corollary 3 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *spf* with $\mathcal{I}_\Lambda = \{0\}$. Λ has a representing measure in $\mathbb{R}^n$ having $d = \dim \mathcal{P}_m$ atoms if and only if there exists a family of orthogonal idempotents $\{b_1, \ldots, b_d\}$ in $\mathcal{H}_\Lambda = \mathcal{P}_m$ such that

$$p = p(\xi^{(1)})b_1 + \cdots + p(\xi^{(d)})b_d, \ p \in \mathcal{P}_m,$$

where

$$\xi^{(j)} = (\Lambda(b_1)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_d)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \ j = 1, \ldots, d.$$

Theorem 4

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *spf* with $\mathcal{I}_\Lambda = \{0\}$. Also let $\mathcal{B} = \{b_1, \ldots, b_d\} \subset \mathcal{H}_\Lambda = \mathcal{P}_m$ $(d = \dim \mathcal{P}_m)$ an orthogonal basis consisting of idempotents, which induces a C^*-algebra structure on $\mathcal{P}_m$.

The following conditions are equivalent:

(i) $\mathcal{B}$ is Λ-multiplicative.

(ii) The polynomials $\{1, t_1, \ldots, t_n\}$ generate the C^*-algebra $\mathcal{P}_m$.

(iii) The points

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \; j = 1, \ldots, d,$$

are distinct.

Special Case

Theorem 4 implies the fact that every *spf* $\Lambda : \mathcal{P}_2 \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ with $d = \dim \mathcal{H}_\Lambda$ atoms. Indeed, the condition from Definition 2 is automatically fulfilled when $|\alpha| + |\beta| \leq 1$, $j = 1, \ldots, d$.

In this case, the support of the representing measure, say $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, is given by the equalities

$$\xi^{(j)} = (\Lambda(b_1)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_d)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \; j = 1, \ldots, d,$$

and the corresponding weights are $\Lambda(b_1), \ldots, \Lambda(b_d)$.

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Continuous Point Evaluations

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *spf*. For every point $\xi \in \mathbb{R}^n$, we denote by δ_ξ the point evaluation at ξ, that is, $\delta_\xi(p) = p(\xi)$, for every polynomial $p \in \mathcal{P}$.

Recall that $\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; \Lambda(|p|^2) = 0\}$, while $\mathcal{H}_\Lambda$ is the finite dimensional Hilbert space $\mathcal{P}_m/\mathcal{I}_\Lambda$.

Definition 3 The point evaluation δ_ξ is said to be Λ-*continuous* if there exists a constant $c_\xi > 0$ such that

$$|\delta_\xi(p)| \le c_\xi \Lambda(|p|^2)^{1/2}, \ p \in \mathcal{P}_m.$$

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Let $\mathcal{Z}_\Lambda$ be the subset of those points $\xi \in \mathbb{R}^n$ such that δ_ξ is Λ-continuous. For every polynomial p let us denote by $\mathcal{Z}(p)$ the set of its zeros.

Lemma 3 We have the equality

$$\mathcal{Z}_\Lambda = \cap_{p \in \mathcal{I}_\Lambda} \mathcal{Z}(p).$$

Remark The previous lemma shows that the set $\mathcal{Z}_\Lambda$ coincides with the algebraic variety of the moment sequence associated to Λ (as defined by Curto and Fialkow).

Lemma 4 (Curto & Fialkow) Suppose that the *spf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has an atomic representing measure μ in $\mathbb{R}^n$. Then $\mathrm{supp}(\mu) \subset \mathcal{Z}_\Lambda$.

Remark It follows from the previous lemma that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

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Remark It follows from the previous lemma that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ ($m \geq 1$) be a *spf* with the property $\mathcal{Z}_\Lambda \neq \emptyset$, and let δ_ξ^Λ the linear functional induced by δ_ξ in the Hilbert space $\mathcal{H}_\Lambda$. Then for every $\xi \in \mathcal{Z}_\Lambda$ there exists a vector $\hat{v}_\xi \in \mathcal{H}_\Lambda$ such that

$$\delta_\xi^\Lambda(\hat{p}) = \langle \hat{p}, \hat{v}_\xi \rangle = \Lambda(pv_\xi) = p(\xi), \forall p \in \mathcal{P}_m.$$

Since $m \geq 1$, the assignment $\xi \mapsto \hat{v}_\xi$ is injective. In addition, we may assume that $v_\xi \in \mathcal{RP}_m$, so $\hat{v}_\xi \in \mathcal{RH}_\Lambda$.

Let $\mathcal{V}_\Lambda = \{\hat{v}_\xi; \xi \in \mathcal{Z}_\Lambda\}$.

The next result is an approach to truncated moment problems when the number of the atomes of the representing measures is not necessarily equal to the maximal cardinal of a family of ortogonal idempotents. The basic elements are in this case projections of idempotents.

Theorem 5

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ $(m \geq 1)$ a *spf* with $\mathcal{Z}_\Lambda$ nonempty. Λ has a representing measure in $\mathbb{R}^n$ consisting of d-atoms, where $d \geq \dim \mathcal{H}_\Lambda$, if and only if there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{R}\mathcal{H}_\Lambda$ such that

$$\Lambda(v_j) > 0, \quad \hat{v}_j/\Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d, \tag{7}$$

$$\hat{p} = \Lambda(v_1)^{-1}\Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(v_d)^{-1}\Lambda(pv_d)\hat{v}_d, \; p \in \mathcal{P}_m, \tag{8}$$

and

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \Lambda(v_j)^{-1}\Lambda(v_j v_k)\Lambda(v_j v_l), \; k, l = 1, \ldots, d. \tag{9}$$

Vă mulţumesc pentru atenţie !

(Thank you for your attention !)